Birkhoff's Theorem for Electromagnetic Fields in General Relativity

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Exact solutions for combined electromagnetic and gravitational field equations with non-static spherical symmetry are obtained. Besides electric fields, the contributions of the magnetic pole are included. It is shown that the solutions may be transformed to the static form, and thus Birkhoff's theorem[1] is extended from the purely gravitational case to the combined electromagnetic and gravitational case. `

In our units, $c=8\pi G=1$. The Roman indices will stand for space-time components and commas and strokes will denote partial and covariant derivatives respectively.

The combined Maxwell-Einstein field equations in the absence of matter are

$$D^j \equiv F^{ij}{}_{/i}=0,$$

$$E^i \equiv \frac{1}{2}\mathcal{E}^{jkli}F_{kl,j}=F_{[kl,j]}=0 \qquad (1)$$

$$Q^i{}_{\cdot j} \equiv R^i{}_{\cdot j}-\frac{1}{2}\delta^i{}_{\cdot j}R-F^{ik}F_{jk}$$

$$+\frac{1}{4}\delta^i{}_{\cdot j}F^{kl}F_{kl}=0.$$

Now, the most general non-static spherically symmetric metric form can be reduced to[2]

$$\Phi=-e^{\lambda(r,t)}dr^2-r^2 d\theta^2-r^2\sin^2\theta\, d\phi^2$$

$$+e^{\nu(r,t)}dt^2. \qquad (2)$$

From considerations of spherical symmetry we retain only the radial components of electromagnetic fields, and put

$$F_{12}=F_{13}=F_{24}=F_{34}=0. \qquad (3)$$

Without this choice, the transverse components would define a physically distinguishable direction on the surface of a sphere, which would destroy the spherical symmetry of the field. It may be noted that we demand no other

symmetry conditions on the surviving components F_{14} and F_{23}.

We shall briefly outline the main steps of our argument, without showing explicit calculations. By virtue of (2) and (3), the field equations $Q_{12}=Q_{13}=Q_{24}=Q_{34}=Q_{23}=0$ are identically satisfied.

From $Q_4{}^1=0$ and $Q_4{}^4-Q_1{}^1=0$, one gets respectively

$$\lambda_{,4}=0, \qquad (4)$$

$$(\lambda+\nu)_{,1}=0, \quad \lambda+\nu=f(t). \qquad (5)$$

By the choice (3), the electromagnetic field equations $D^1=D^4=E^2=E^3=0$ and $D^2=D^3=E^1=E^4=0$ yield

$$F_{14}=\frac{\mathcal{E}}{r^2}e^{1/2 \cdot f(t)},$$

$$F_{23}=\mu\sin\theta, \qquad (6)$$

where $\mathcal{E}$, μ are constants of integration.

Utilizing (3), (4), (5) and (6), one obtains from the equations $Q_2{}^2=Q_3{}^3=0$,

$$e^\lambda=\left[1-\frac{2m}{r}+\frac{(\mathcal{E}^2+\mu^2)}{2r^2}\right]^{-1},$$

$$e^\nu=\left[1-\frac{2m}{r}+\frac{(\mathcal{E}^2+\mu^2)}{2r^2}\right]e^{f(t)}. \qquad (7)$$

Introducing a transformation $dt'=e^{1/2 \cdot f(t)}dt$, and dropping primes afterwards, we pass over from (6) and (7) to the corresponding static field strengths and metric form

$$F_{14}=\mathcal{E}/r^2, \quad F_{23}=\mu\sin\theta,$$

$$\Phi=-\left[1-\frac{2m}{r}+\frac{(\mathcal{E}^2+\mu^2)}{2r^2}\right]^{-1}dr^2$$

$$-r^2d\theta^2-r^2\sin^2\theta\,d\phi^2$$

$$+\left[1-\frac{2m}{r}+\frac{(\mathcal{E}^2+\mu^2)}{2r^2}\right]dt^2. \qquad (8)$$

The constants $\mathcal{E}$ and μ can be inter-

preted as the electric charge and magnetic pole strength of the point source. It was possible to include the magnetic contribution because we worked directly with the field strengths F_{14}, F_{23} instead of potentials and did not assume that they have any spatial symmetry. When μ vanishes the set (8) go over to the Nordström-Reissner-Jeffrey solutions.

The existence of Birkhoff's theorem, that is, the possibility of transforming away the explicit time-dependence, is due to the fact that an electromagnetic monopole cannot radiate.

In the case of a vector-meson field[3] the corresponding $Q_4{}^1=0$ gives $e^{-\lambda}\cdot\lambda_{,4}/r=\mu^2\phi^1\phi_4\neq0$, so that the present method fails to prove the existence of Birkhoff's theorem. This failure can be understood by the argument that in the vector-meson case even in the absence of transverse fields the longitudinal component can carry out energy. Birkhoff's theorem does not hold also in the case of a scalar field with or without rest mass[4], because $Q_4{}^1=0$ gives $e^{-\lambda}\cdot\lambda_{,4}/r=\phi^1\phi_{,4}\neq0$.

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4) O. Bergmann and R. Leipnik, Phys. Rev. **107** (1958), 1157.