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Bishop's property (β) and an elementary operator

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Abstract. A Banach space operator $T \in B(\mathcal{X})$ is hereditarily polaroid, $T \in (\mathcal{HP})$, if the isolated points of the spectrum of every part T_p of the operator are poles of the resolvent of T_p; T is hereditarily normaloid, $T \in (\mathcal{HN})$, if every part T_p of T is normaloid. Let $(\mathcal{HNP})$ denote the class of operators $T \in B(\mathcal{X})$ such that $T \in (\mathcal{HP}) \cap (\mathcal{HN})$. $(\mathcal{HNP})$ operators such that the Berberian–Quigley extension T° of T is also in $(\mathcal{HNP})$ satisfy Bishop's property (β). Given Hilbert space operators $A, B^* \in B(\mathcal{H})$, let $d_{AB} \in B(B(\mathcal{H}))$ stands for either of the elementary operators $\delta_{AB}(X) = AX - XB$ and $\triangle_{AB}(X) = AXB - X$. If $A, B^* \in (\mathcal{HP})$ satisfy property (β), and the eigenspaces corresponding to distinct eigenvalues of A (resp., B^*) are orthogonal, then $f(d_{AB})$ satisfies Weyl's theorem and $f(d_{AB})^*$ satisfies a-Weyl's theorem for every function f which is analytic on a neighbourhood of $\sigma(d_{AB})$. Finally, we characterize perturbations of d_{AB} by quasinilpotent and algebraic operators $A, B \in B(\mathcal{H})$.

Key words: Hilbert space, elementary operator, polaroid operator, SVEP, property (β), Browder's theorem, Weyl's theorem, perturbation

1. Introduction

Let $\mathcal{X}$ (or $\mathcal{H}$) be a complex Banach (Hilbert, respectively) space and $B(\mathcal{X})$ (or $B(\mathcal{H})$) be the set of all bounded linear operators on $\mathcal{X}$ ($\mathcal{H}$, respectively). A Banach space operator $T \in B(\mathcal{X})$ is said to have SVEP, *the single–valued extension property*, at a point λ of the complex plane $\mathbf{C}$ if, for every neighbourhood $\mathcal{O}_\lambda$ of λ, the only analytic function $f : \mathcal{O}_\lambda \longrightarrow \mathcal{X}$ satisfying $(T - \mu)f(\mu) = 0$ for all $\mu \in \mathcal{O}_\lambda$ is the function $f \equiv 0$; we say that T *has SVEP* if it has SVEP at every $\lambda \in \mathbf{C}$. The ascent $\mathrm{asc}(T)$ (resp., descent $\mathrm{dsc}(T)$) of T is the least non–negative integer n such that $T^{-n}(0) = T^{-(n+1)}(0)$ (resp., $T^n\mathcal{X} = T^{n+1}\mathcal{X}$); if no such integer exists, then $\mathrm{asc}(T) = \infty$ (resp., $\mathrm{dsc}(T) = \infty$). A point $\lambda \in \mathrm{iso}\sigma(T)$ is a pole of the resolvent of T if $\mathrm{asc}(T-\lambda) = \mathrm{dsc}(T-\lambda) < \infty$. We say that T is *polaroid* if every $\lambda \in \mathrm{iso}\sigma(T)$ is a pole of the resolvent of T; T is *hereditarily polaroid*, denoted $T \in (\mathcal{HP})$, if every part of T (i.e., its restriction to an invariant subspace) is polaroid. The class of $(\mathcal{HP})$-operators is large; see [10] for examples of $(\mathcal{HP})$

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operators. We say that $T \in B(\mathcal{X})$ is hereditarily normaloid, $T \in (\mathcal{HN})$ if every part T_p of T is normaloid (i.e., $\|T_p\|$ equals the spectral radius of T_p); let $(\mathcal{HNP})$ denote those $T \in B(\mathcal{X})$ for which $T \in (\mathcal{HP}) \cap (\mathcal{HN})$.

Let T° denote the Berberian–Quigley extension of $T \in B(\mathcal{X})$, and let $(\mathcal{HNP})^\circ$ denote the space of $T \in (\mathcal{HNP}) \cap B(\mathcal{X})$ such that $T^\circ \in (\mathcal{HNP})$. (Examples of operators $T \in (\mathcal{HNP})^\circ$ occur naturally, as we shall see in the sequel.) We prove that operators $T \in (\mathcal{HNP})^\circ$ satisfy Bishop's property (β). Let L_A and R_B denote, respectively, the operators of left multiplication by A and right multiplication by B. For $A, B \in B(\mathcal{X})$, let $d_{AB} \in B(B(\mathcal{X}))$ denote either of the operators $\delta_{AB} = L_A - R_B$ and $\triangle_{AB} = L_A R_B - 1$. Choosing the entries $A, B^* \in B(\mathcal{H})$, $\mathcal{H}$ an infinite dimensional complex Hilbert space, to be such that $A, B^* \in (\mathcal{HP})$ satisfy property (β) and the eigenspaces corresponding to their distinct eigenvalues are orthogonal, we prove that $f(d_{AB})$ satisfies Weyl's theorem and $f(d^*_{AB})$ satisfies a-Weyl's theorem for every function f which is analytic on a neighbourhood of the spectrum $\sigma(d_{AB})$ of d_{AB}. Here, in keeping with current terminology, we say that a Banach space operator $T \in B(\mathcal{X})$ satisfies Weyl's theorem, Wt for short, if the complement of the Weyl spectrum $\sigma_w(T) = \{\lambda \in \sigma(T) :$ either $T - \lambda$ is not Fredholm or $\mathrm{ind}(T - \lambda) \neq 0\}$ of T in $\sigma(T)$ is the set $\pi_{00}(T) = \{\lambda \in \mathrm{iso}\sigma(T) : 0 < \dim(T - \lambda)^{-1}(0) < \infty\}$; T satisfies a-Weyl's theorem, or $a - Wt$, if the complement of the a-Weyl spectrum $\sigma_{aw}(T) = \{\lambda \in \sigma_a(T) :$ either $T - \lambda$ is not left Fredholm or $\mathrm{ind}(T - \lambda) \not\leq 0\}$ of T in the approximate point spectrum $\sigma_a(T)$ of T is the set $\pi^a_{00}(T) = \{\lambda \in \mathrm{iso}\sigma_a(T) : 0 < \dim(T - \lambda)^{-1}(0) < \infty\}$. Similar results, but (mostly) for operators $A, B^* \in (\mathcal{HNP}) \cap B(\mathcal{H})$ satisfying the Putnam-Fuglede property $d^{-1}_{AB}(0) \subseteq d^{-1}_{A^*B^*}(0)$, have earlier been considered in [7], [11]. Observe that if $d^{-1}_{AB}(0) \subseteq d^{-1}_{A^*B^*}(0)$ for some operators $A, B \in B(\mathcal{H})$, then 0 is a normal eigenvalue of A and B. Since, in general, 0 is not a normal eigenvalue of operators in $(\mathcal{HNP})$ (for example, $A^{-1}(0) \subseteq A^{*-1}(0)$ fails for paranormal operators), $(\mathcal{HNP}) \cap B(\mathcal{H})$ operators do not in general satisfy the Putnam-Fuglede property.

2. Preliminaries

In addition to the notation and terminology already introduced, we shall use the following further notation and terminology. We remark that even though many of our results are proved in the setting of Hilbert space operators, we introduce our terminology and prove some of the results in their full generality in the setting of a Banach space. A Banach space operator $T \in B(\mathcal{X})$ satisfies (Bishop's) property (β) if, for every open subset $\mathcal{U}$ of $\mathbf{C}$ and every sequence of analytic functions $f_n : \mathcal{U} \longrightarrow \mathcal{X}$ with the property that

$$(T-\lambda)f_n(\lambda) \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

uniformly on all compact subsets of $\mathcal{U}$, $f_n(\lambda) \longrightarrow 0$ as $n \longrightarrow \infty$ locally uniformly on $\mathcal{U}$. Given an open subset $\mathcal{U}$ of $\mathbf{C}$, let $H(\mathcal{U},\mathcal{X})$ denote the Fréchet space of analytic functions from $\mathcal{U}$ to $\mathcal{X}$. Then $T \in B(\mathcal{X})$ satisfies property (β) precisely when the operator $T_{\mathcal{U}} : H(\mathcal{U},\mathcal{X}) \longrightarrow H(\mathcal{U},\mathcal{X})$, $(T_{\mathcal{U}}f)(\lambda) := (T-\lambda)f(\lambda)$, has closed range [21, Proposition 3.3.5]. If T satisfies property (β), then the conjugate operator T^* satisfies (the decomposition) property (δ) [21, Theorem 2.5.18], and both L_T and R_{T^*} satisfy (Dunford's) condition (C) [21, Corollary 3.6.11]. (We refer the reader to [21], Definitions 1.2.18 and 1.2.28, for the definitions of condition (C) and property (δ).)

Proposition 2.1 *If $A, B \in B(\mathcal{X})$ and A, B^* satisfy property (β), then $L_A - R_B$ and $L_A R_B$ have* SVEP.

Proof. Apparently, L_A and R_B commute. Since both L_A and R_B satisfy condition (C), $L_A - R_B$ and $L_A R_B$ have SVEP (see [21, Theorem 3.6.3 and Notes 3.6.19 on page 283]). □

The Browder spectrum (Browder essential approximate point spectrum) of T is the set $\sigma_b(T) = \{\lambda \in \sigma(T) : T-\lambda \text{ is not Fredholm or one of } \operatorname{asc}(T-\lambda) \text{ and } \operatorname{dsc}(T-\lambda) \text{ is not finite}\}$ (resp., $\sigma_{ab}(T) = \{\lambda \in \sigma(T) : T-\lambda \text{ is not upper semi–Fredholm or } \operatorname{asc}(T-\lambda) = \infty\}$). We say that an operator $T \in B(\mathcal{X})$ satisfies Browder's theorem, Bt for short (resp., a-Browder's theorem, $a-Bt$ for short) if $\sigma_w(T) = \sigma_b(T)$ (resp., $\sigma_{aw}(T) = \sigma_{ab}(T)$). Observe that if we let $p_0(T)$ denote the set of $\lambda \in \operatorname{iso}\sigma(T)$ which are finite rank poles of the resolvent of T, then T satisfies Bt if and only if $\sigma(T) \setminus \sigma_w(T) = p_0(T)$; similarly, if we let $p_0^a(T) = \{\lambda \in \operatorname{iso}\sigma_a(T) : \operatorname{asc}(T-\lambda) < \infty, (T-\lambda)\mathcal{X} \text{ is closed and } \dim(T-\lambda)^{-1}(0) < \infty\}$, then T satisfies $a-Bt$ if and only if $\sigma_a(T) \setminus \sigma_{aw}(T) = p_0^a(T)$ (see [12, Theorems 8.3.1 and 8.3.3]). The following implications hold:

T satisfies $a-Wt \Longrightarrow T$ satisfies $a-Bt \Longrightarrow T$ satisfies Bt
$\Longleftrightarrow T^*$ satisfies Bt;

T satisfies $a-Wt \Longrightarrow T$ satisfies $Wt \Longrightarrow T$ satisfies Bt.

Here the forward implications can not in general be reversed.

Proposition 2.2 *A necessary and sufficient condition for $T \in B(\mathcal{X})$ to satisfy $a-Bt$ is that T has* SVEP *at points $\lambda \notin \sigma_{aw}(T)$.*

Proof. See [8, Lemma 2.18]. □

Remark 2.3

(i) Let $\sigma_e(T)$ denote the Fredholm spectrum of the operator T. It is well known, [15], [16], that if $A, B \in B(\mathcal{H})$, then $\sigma_e(L_A - R_B) = \{\sigma(A) - \sigma_e(B)\} \cup \{\sigma_e(A) - \sigma(B)\}$ and $\sigma_e(L_A R_B) = \sigma(A)\sigma_e(B) \cup \sigma_e(A)\sigma(B)$. Hence if the operator A (resp., B) is a quasinilpotent, then $\sigma_e(L_A - R_B) = -\sigma(B)$ (resp., $\sigma_e(L_A - R_B) = \sigma(A)$) and $\sigma_e(L_A R_B) = \{0\}$. Since $\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) \subseteq \sigma(T)$ for every operator T, and since $\sigma(L_A - R_B) = \sigma(A) - \sigma(B)$ and $\sigma(L_A R_B) = \sigma(A)\sigma(B)$ [14], it follows that if either of A or B is quasinilpotent, then $L_A - R_B$ and $L_A R_B$ satisfy Bt.

(ii) Recall that SVEP for operators S and $T \in B(\mathcal{X})$ does not guarantee SVEP for the operators $S+T$ and ST, even when S and T commute. Thus, in general, L_S and R_T satisfy Wt does not guarantee d_{ST} satisfies Wt, even Bt.

We say that $T \in B(\mathcal{X})$ is polaroid (resp., left-polaroid) if every $\lambda \in \mathrm{iso}\sigma(T)$ is a pole of the resolvent of T (resp., if, for every $\lambda \in \mathrm{iso}\sigma_a(T)$, $\mathrm{asc}(T-\lambda) < \infty$ and $(T-\lambda)\mathcal{X}$ is closed). A necessary and sufficient condition for $\lambda \in \sigma(T)$ to be a pole of the resolvent of T is that both $\mathrm{asc}(T)$ and $\mathrm{dsc}(T)$ are finite [1, Theorem 3.81]. We say that $\lambda \in \sigma(T)$ is a simple pole if $\mathrm{asc}(T) = \mathrm{dsc}(T) = 1$; T is said to be polaroid (resp., left-polaroid) on a subset S of $\mathrm{iso}\sigma(T)$ (resp., $\mathrm{iso}\sigma_a(T)$) if every point of the subset is a pole (resp., a left-pole) of T.

Proposition 2.4 *A necessary and sufficient condition for $T \in B(\mathcal{X})$ to satisfy Wt (resp., $a-Wt$) is that T satisfies Bt (resp., $a-Bt$) and is polaroid at points in $\pi_{00}(T)$ (resp., left-polaroid at points in $\pi^a_{00}(T)$).*

Proof. Since T satisfies $a-Wt$ (resp., Wt) implies $\sigma_a(T)\setminus\sigma_{aw}(T) = p_0^a(T) = \pi_{00}^a(T)$ (resp., $\sigma(T)\setminus\sigma_w(T) = p_0(T) = \pi_{00}(T)$), the necessity is evident. For the sufficiency, T satisfies $a-Bt$ (resp., Bt) implies that $\sigma_a(T)\setminus\sigma_{aw}(T) = p_0^a(T) \subseteq \pi_{00}^a(T)$ (resp., $\sigma(T)\setminus\sigma_w(T) = p_0(T) \subseteq \pi_{00}(T)$). Hence, if $\pi_{00}^a(T) \subseteq p_0^a(T)$ (resp., $p_0(T) \subseteq \pi_{00}(T)$), then $p_0^a(T) = \pi_{00}^a(T)$ (resp., $p_0(T) = \pi_{00}(T)$). □

Operators $T \in B(\mathcal{X})$ with property (β) have SVEP. Hence they satisfy $a-Bt$ (so also Bt). Indeed, if $T \in B(\mathcal{X})$ has SVEP, then T^* also satisfies $a-Bt$. To see this, start by observing that if T has SVEP, then $\sigma(T) = \sigma(T^*) = \sigma_a(T^*)$. Furthermore, $(\sigma_w(T) =)\sigma_w(T^*) = \sigma_{aw}(T^*)$, as the following argument shows. Recall that $\sigma_{aw}(S) \subseteq \sigma_w(S)$ for every $S \in B(\mathcal{X})$. If $\lambda \notin \sigma_{aw}(T^*)$, then $T^* - \lambda I^*$ is upper semi–Fredholm and $\operatorname{ind}(T^*-\lambda I^*) \leq 0$. Since T has SVEP at λ, we also have that $\operatorname{ind}(T^*-\lambda I^*) \geq 0$. Hence $T^* - \lambda I^*$ is Fredholm and $\operatorname{ind}(T^*-\lambda I^*) = 0$, i.e., $\lambda \notin \sigma_w(T^*)$. Hence $\sigma_w(T^*) \subseteq \sigma_{aw}(T^*)$, which implies the claimed equality. Apparently, λ is a pole of the resolvent of T implies λ is a pole of the resolvent of T^*; hence $p_0(T) = p_0(T^*)$. Notice that if $\lambda \in p_0^a(T^*)$, then $T^* - \lambda I^*$ is upper semi-Fredholm of finite ascent; if also T has SVEP at λ, then $\operatorname{dsc}(T^*-\lambda I^*) < \infty \Longrightarrow \lambda \in p_0(T^*)$. Thus $p_0(T) = p_0^a(T^*)$. Putting it all together, $\sigma(T)\setminus\sigma_w(T) = p_0(T) = p_0^a(T^*) = \sigma_a(T^*)\setminus\sigma_{aw}(T^*)$, i.e., T^* satisfies $a-Bt$.

The following proposition is well known, see [24], [25].

Proposition 2.5 *If $T \in B(\mathcal{X})$ has* SVEP*, T satisfies Wt and T^* satisfies $a-Wt$, then $f(T)$ satisfies Wt and $f(T^*) = f(T)^*$ satisfies $a-Wt$ for every f analytic on an open neighbourhood of $\sigma(T)$.*

$(\mathcal{THN})$-operators. An operator $T \in B$ is *normaloid* if its norm equals its spectral radius $r(T)$. An important subclass of the class of $(\mathcal{HNP})$-operators is the class $(\mathcal{THN})$ of *totally hereditarily normaloid operators*, where (for an operator $T \in B(\mathcal{X})$) we say that $T \in (\mathcal{THN})$ if every part, and also the inverse of every invertible part, of T is normaloid. Recall from [8, Proposition 2.1] that $(\mathcal{THN})$-operators are simply polaroid (i.e., isolated points of the spectrum are order one poles of the resolvent of the operator). Evidently, $(\mathcal{THN}) \subset (\mathcal{HNP})$.

A subspace M of the Banach space $\mathcal{X}$ is said to be orthogonal to a subspace N of $\mathcal{X}$ (in the Birkhoff–James sense, [13, p. 93]), denoted $M \perp N$, if $\|x\| \leq \|x+y\|$ for all $x \in M$ and $y \in N$. This asymmetric definition of orthogonality coincides with the usual definition of orthogonality in the case in which $\mathcal{X} = \mathcal{H}$ is a Hilbert space.

Proposition 2.6 *If N and M are eigenspaces corresponding to distinct eigenvalues μ and ν, $|\mu| \leq |\nu|$, of an operator $T \in (\mathcal{THN})$, then $M \perp N$ if $\mu = 0$ and M, N are mutually orthogonal if $\mu \neq 0$.*

Proof. The proof of the proposition is similar to that of [8, Proposition 2.5]. Let $\mathcal{L}$ denote the subspace generated by M and N. Then the operator $T_1 = T|_{\mathcal{L}} \in (\mathcal{THN})$ (being simply polaroid) is normaloid and meromorphic. Apply [17, Proposition 54.4]. □

Translated to Hilbert space operators T, Proposition 2.6 implies the following.

Corollary 2.7 *Eigenspaces corresponding to distinct eigenvalues of $T \in (\mathcal{THN}) \cap B(\mathcal{H})$ are mutually orthogonal.*

For an operator $T \in B(\mathcal{X})$, the *quasinilpotent part* $H_0(T-\lambda)$ and the *analytic core* $K(T-\lambda)$ of $(T-\lambda)$ are defined by

$$H_0(T-\lambda) = \left\{x \in \mathcal{X} : \lim_{n \longrightarrow \infty} \|(T-\lambda)^n x\|^{\frac{1}{n}} = 0\right\}$$

and

$$\begin{aligned} K(T-\lambda) = \{x \in \mathcal{X} :\ & \text{there exists a sequence } \{x_n\} \subset \mathcal{X} \text{ and } \delta > 0 \text{ for} \\ & \text{which } x = x_0,\ (T-\lambda)(x_{n+1}) = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \\ & \text{for all } n = 1, 2, \dots\}. \end{aligned}$$

We note that $H_0(T-\lambda)$ and $K(T-\lambda)$ are (generally) non-closed hyperinvariant subspaces of $(T-\lambda)$ such that $(T-\lambda)^{-q}(0) \subseteq H_0(T-\lambda)$ for all $q = 0, 1, 2, \dots$ and $(T-\lambda)K(T-\lambda) = K(T-\lambda)$ [22].

3. Operators $T \in (\mathcal{HNP}) \cap B(\mathcal{X})$ and property (β)

In this section we prove that operators $T \in (\mathcal{HNP})^\circ$ satisfy property (β). We start by proving that operators $T \in (\mathcal{HNP})$ have SVEP.

Lemma 3.1 *Operators in $(\mathcal{HNP}) \cap B(\mathcal{X})$ have* SVEP*.*

Proof. Suppose to the contrary that T does not have SVEP at a point $\lambda \in \sigma(T)$. Then λ is necessarily an eigenvalue of T, and there exists a disc $\mathcal{D}_\lambda$ centered at λ and a non-trivial analytic function $f : \mathcal{D}_\lambda \to \mathcal{X}$ such that $f(\mu) \in (T-\mu)^{-1}(0)$ for all $\mu \in \mathcal{D}_\lambda$. Let $\beta \in \mathcal{D}_\lambda$, and let $\{\alpha_n\} \subset \mathcal{D}_\lambda$ be a sequence of complex numbers such that $|\beta| \geq |\alpha_n|$ for all n and α_n converges to β. Then $f(\alpha_n), f(\beta)$ are non-zero and $f(\alpha_n)$ converges to $f(\beta)$. Let $\mathcal{X}_0$ be the subspace generated by $(T-\beta)^{-1}(0)$ and $(T-\alpha_n)^{-1}(0)$ $(n = 1, 2, 3, \dots)$. Then $T_0 = T|_{\mathcal{X}_0}$ is a meromorphic normaloid operator. Hence the spectral projection P_β corresponding to the pole β of T_0 has norm 1, and $\|x\| \leq \|x-y\|$ for every $x \in P_\beta\mathcal{X}_0$ and $y \in P_\beta^{-1}(0)$ [17, Proposition 54.4]. Let $\|f(\beta)\| = 1$, and choose an n_0 large enough so that $\|f(\beta) - f(\alpha_{n_0})\| < \epsilon$ for some ϵ $(0 < \epsilon < 1)$. Then, with $x = f(\beta)$ and $y = f(\alpha_{n_0})$, we have $1 = \|f(\beta)\| \leq \|f(\beta) - f(\alpha_{n_0})\| < \epsilon < 1$, a contradiction. □

Let $\ell^\infty(\mathcal{X})$ denote the space of all bounded sequences of elements of $\mathcal{X}$, and let $c_0(\mathcal{X})$ denote the space of all null sequences of $\mathcal{X}$. Endowed with the canonical norm, the quotient space $\mathcal{K} = \ell^\infty(\mathcal{X})/c_0(\mathcal{X})$ is a Banach space into which $\mathcal{X}$ may be isometrically embedded. The Berberian–Quigley extension theorem, [21, p. 255], says that given an operator $T \in B(\mathcal{X})$ there exists an isometric algebra isomorphism $T \longrightarrow T^\circ \in B(\mathcal{K})$ preserving order such that $\sigma(T) = \sigma(T^\circ)$ and $\sigma_a(T) = \sigma_a(T^\circ) = \sigma_p(T^\circ)$. Let $(\mathcal{HNP})^\circ$ denote the class of $T \in (\mathcal{HNP})$ such that $T^\circ \in (\mathcal{HNP})$.

Theorem 3.2 *Operators $T \in (\mathcal{HNP})^\circ \cap B(\mathcal{X})$ satisfy property (β).*

Proof. Let $T \in (\mathcal{HNP})^\circ \cap B(\mathcal{X})$. Let $\mathcal{U}$ be an open subset of $\mathbf{C}$, and let $H(\mathcal{U}, \mathcal{X})$ denote the Fréchet space of analytic functions from $\mathcal{U}$ to $\mathcal{X}$. If

$$(T-\lambda)f_n(\lambda) \longrightarrow 0 \text{ on } H(\mathcal{U}, \mathcal{X}),$$

then, upon letting $[f_n(\lambda)]$ denote the equivalence class of the sequence $\{f_n(\lambda)\}$ in $\mathcal{K}$,

$$(T^\circ - \lambda)[f_n(\lambda)] = 0,$$

for every $\lambda \in \mathcal{U}$. Since T° has SVEP, it follows $[f_n(\lambda)] \equiv 0$ on $\mathcal{U}$. We claim that $f_n(\lambda) \longrightarrow 0$ on $H(\mathcal{U}, \mathcal{X})$. Observe that if $D(\lambda; r) = \{\mu \in \mathbf{C} : |\lambda - \mu| < r\}$ is such that $\overline{D(\lambda; r)} \subset U$, then the analytic sequence $\{f_n(\lambda)\}$ is uniformly bounded on $\overline{D(\lambda;r)}$; also, for every $\epsilon > 0$, there exists a natural number N and $0 < \rho < r$ such that

$$\|f_n(\mu)\| < \frac{\epsilon}{2} \quad \text{and} \quad \|f_n(\lambda) - f_n(\mu)\| < \frac{\epsilon}{2}$$

for all $n > N$ and $\mu \in D(\lambda;\rho)$. Indeed, if need be, considering $\frac{f_n}{1+\|f_n\|}$ instead of f_n we may assume that $\sup_n \|f_n\|_{\overline{D(\lambda;r)}} = M < \infty$. Since the function f_n is analytic, it has a Taylor series $f_n(\mu) - f_n(\lambda) = \sum_{m=1}^{\infty} a_{nm}(\mu - \lambda)^m$. Consequently, $\|f_n(\lambda) - f_n(\mu)\| \leq \frac{M\rho}{r-\rho}$ for all $\mu \in \overline{D(\lambda;\rho)}$ such that $0 < \rho < r$. Now choose N and ρ such that $|f_n(\lambda)| < \frac{\epsilon}{4}$ (recall that the sequence $\{f_n(\lambda)\} \in c_0$) and $\frac{M\rho}{r-\rho} < \frac{\epsilon}{4}$. Then

$$\|f_n(\mu)\| \leq \|f_n(\lambda)\| + \|f_n(\lambda) - f_n(\mu)\| < \frac{\epsilon}{2}$$

for all $n > N$ and $\mu \in D(\lambda;\rho)$, and hence that $f_n(\lambda) \to 0$ in $H(U,\mathcal{H})$, i.e., T satisfies property (β). □

Remark 3.3 For a Banach space operator $T \in B(\mathcal{X})$, T is called *paranormal* if $\|Tx\|^2 \leq \|T^2x\| \cdot \|x\|$ for all $x \in \mathcal{X}$. Evidently, if T is paranormal, then so is its Berberian–Quigley extension T°. Since paranormal operators are $(\mathcal{HNP})$ operators [10], paranormal operators are $(\mathcal{HNP})^\circ$ operators, and so enjoy property (β) (thereby answering a question of Laursen [20]). Tanahashi and Uchiyama [26] have recently proved the following:

Theorem 3.4 *Let $T \in B(\mathcal{H})$, and let μ, ν be (any) two distinct approximate eigenvectors of T. If $\lim_{n\to\infty}(x_n, y_n) = 0$ for all bounded sequences of vectors $\{x_n\}$ and $\{y_n\} \subset \mathcal{H}$ such that $\lim_{n\to\infty} \|(T-\mu)x_n\| = \lim_{n\to\infty} \|(T-\nu)y_n\| = 0$, then T satisfies property (β).*

Theorem 3.4 implies that paranormal Hilbert space operators satisfy property (β): Theorem 3.2 generalizes this result to Banach space paranormal operators.

Theorem 3.2 has a number of consequences: we list below but only a few of them. If $T \in (\mathcal{HNP})^\circ$ and $Q \in B(\mathcal{X})$ is a quasinilpotent operator which commutes with T, then T and $T + Q$ are quasinilpotent equivalent; hence $T + Q$ satisfies property (β) [21, Proposition 3.4.11]. The following corollary says (in particular) that the direct sum of an operator satisfying property (β) with a quasinilpotent satisfies property (β).

Corollary 3.5 *Let $T \in B(\mathcal{X})$. If $T \in (\mathcal{HNP})^{\circ}$ and $Q \in B(\mathcal{X}_0)$ is a quasinilpotent operator, then the operator $A \in B(\mathcal{X} \oplus \mathcal{X}_0)$, $A = \left(\begin{smallmatrix} T & X \\ 0 & Q \end{smallmatrix}\right)$, satisfies property (β).*

Proof. Let $\mathcal{U} \subseteq \mathbf{C}$ be open, and let $f_n = f_{1n} \oplus f_{2n} \in H(\mathcal{U}, \mathcal{X} \oplus \mathcal{X}_0)$ be a sequence such that $(A - \lambda)f_n(\lambda) \longrightarrow 0$ as $n \longrightarrow \infty$ in $H(\mathcal{U}, \mathcal{X} \oplus \mathcal{X}_0)$. Then, since $Q - \lambda$ is invertible for all $\lambda \neq 0$, $(Q - \lambda)f_{2n}(\lambda) \longrightarrow 0$ as $n \longrightarrow \infty$ in $H(\mathcal{U}, \mathcal{X}_0) \implies f_{2n}(\lambda) \longrightarrow 0$ as $n \longrightarrow \infty$ in $H(\mathcal{U}, \mathcal{X}_0)$. This in turn implies that $(T - \lambda)f_{1n}(\lambda) \longrightarrow 0$ as $n \longrightarrow \infty$ in $H(\mathcal{U}, \mathcal{X})$. Since T satisfies property (β), $f_{1n}(\lambda) \longrightarrow 0$ as $n \longrightarrow \infty$ in $H(\mathcal{U}, \mathcal{X})$. Hence $f_n(\lambda) \longrightarrow 0$ as $n \longrightarrow \infty$ in $H(\mathcal{U}, \mathcal{X} \oplus \mathcal{X}_0)$. □

An operator $T \in B(\mathcal{H})$ is a quasi–class $\mathcal{A}$ operator, $T \in \mathcal{QA}$, if $T^*(|T^2| - |T|^2)T \geq 0$ [18]. $\mathcal{QA}$ operators T have an upper triangular matrix representation

$$T = \begin{pmatrix} T_1 & X \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \overline{\operatorname{ran}(T(\mathcal{H}))} \\ \ker T^* \end{pmatrix},$$

where T_1, $|T_1|^2 \leq |T_1^2|$, is a class $\mathcal{A}$ operator [18, Theorem 2.2]. It is well known that class $\mathcal{A}$ operators are paranormal, hence satisfy property (β) (see also [5, Theorem 3.1]). Corollary 3.5 implies that $\mathcal{QA}$ operators satisfy property (β).

An operator T on a separable Banach space $\mathcal{X}$ is said to be supercyclic if the homogeneous orbit $\{\lambda T^n x : \lambda \in \mathbf{C}, n \in \mathbf{N} \cup \mathbf{0}\}$ is dense in $\mathcal{X}$ for some $x \in \mathcal{X}$. It is known that paranormal operators in $B(\mathcal{H})$ are not supercyclic [3]. Recall that an operator $T \in B(\mathcal{X})$ is normaloid if its spectral radius $r(T)$ equals its norm.

Corollary 3.6 *Operators $T \in (\mathcal{THN}) \cap B(\mathcal{X})$ such that $T \in (\mathcal{HNP})^{\circ}$ are not supercyclic.*

Proof. Since T is normaloid, we may assume that $r(T) = \|T\| = 1$. Suppose that T is supercyclic. Since T satisfies property (β), $|\mu| = r(T) = 1$ for every $\mu \in \sigma(T)$ [21, Proposition 3.3.18]. Hence $\sigma(T) \subseteq \partial D$, the boundary of the unit disc in $\mathbf{C}$. But then T is invertible and $r(T^{-1}) = \|T^{-1}\| = 1$, which implies that $\|Tx\| = \|x\|$ for all $x \in \mathcal{X}$. Hence T is an isometry. This is a contradiction, since no isometry on an infinite dimensional Banach space can be supercyclic [21, Proposition 3.3.19]. □

An operator $A \in B(\mathcal{X})$ is said to be algebraic if there exists a non–trivial polynomial $q(\cdot)$ such that $q(A) = 0$. If an $A \in B(\mathcal{X})$ is algebraic, then $\sigma(A) = \{\mu_1, \ldots, \mu_n\}$ for some scalars μ_i, $1 \leq i \leq n$, and $A = \oplus_{i=1}^n A_i$, where $A_i = A|_{H_0(A-\mu_i)}$.

Corollary 3.7 *If $T \in B(\mathcal{X})$ is such that $T \in (\mathcal{HNP})^\circ$, and $A \in B(\mathcal{X})$ is an algebraic operator which commutes with T, then $T + A$ satisfies property (β).*

Proof. Let $T_i = T|_{H_0(A-\mu_i)}$, where the scalars μ_i are as above. The commutativity of T and A then implies that T_i and A_i commute for all $1 \leq i \leq n$. Furthermore, $T + A = \oplus_{i=1}^n (T_i + A_i)$. Recall, [10, Lemma 3.2], that $A_i - \mu_i$ is nilpotent ($\Longrightarrow A_i - \mu_i$ satisfies property (β)) for all $1 \leq i \leq n$. Assume that $(A_i - \mu_i)^{k_i} = 0$ for some positive integer k_i. We prove that $T_i + A_i - \mu_i$ satisfies property (β). Let $\mathcal{U} \subseteq \mathbf{C}$ be open, and assume that

$$(T_i + A_i - \mu_i - \lambda) f_n(\lambda) \longrightarrow 0 \text{ in } H(\mathcal{U}, \mathcal{X}).$$

Then

$$\begin{aligned} &\big\{(A_i - \mu_i)^{k_i - 1}(T_i - \lambda) + (A_i - \mu_i)^{k_i}\big\} f_n(\lambda) \\ &\quad = (T_i - \lambda)(A_i - \mu_i)^{k_i - 1} f_n(\lambda) \longrightarrow 0 \text{ in } H(\mathcal{U}, \mathcal{X}), \end{aligned}$$

which (since $T = \oplus_{i=1}^n T_i$ satisfies property (β) implies T_i satisfies property (β) for all $1 \leq i \leq n$) implies that

$$(A_i - \mu_i)^{k_i - 1} f_n(\lambda) \longrightarrow 0 \text{ in } H(\mathcal{U}, \mathcal{X}).$$

Again, since

$$\big\{(A_i - \mu_i)^{k_i - 2}(T_i - \lambda) + (A_i - \mu_i)^{k_i - 1}\big\} f_n(\lambda) \longrightarrow 0 \text{ in } H(\mathcal{U}, \mathcal{X})$$

implies

$$(T_i - \lambda)(A_i - \mu_i)^{k_i - 2} f_n(\lambda) \longrightarrow 0,$$

we have that

$$(A_i - \mu_i)^{k_i - 2} f_n(\lambda) \longrightarrow 0 \text{ in } H(\mathcal{U}, \mathcal{X}).$$

Repeating this argument a finite number of times, it follows that

$$(T_i - \lambda) f_n(\lambda) \longrightarrow 0 \text{ in } H(\mathcal{U}, \mathcal{X}).$$

Hence

$$f_n(\lambda) \longrightarrow 0 \text{ in } H(\mathcal{U}, \mathcal{X}),$$

i.e., $T_i + A_i - \mu_i$ satisfies property (β) for all $1 \leq i \leq n$. Since an operator $S \in B(\mathcal{X})$ satisfies property (β) if and only if $S - \alpha$ satisfies property (β) for all scalars α, $T_i + A_i$ satisfies property (β) for all $1 \leq i \leq n$. This implies that $T + A = \oplus_{i=1}^n (T_i + A_i)$ satisfies property (β). □

Remark 3.8 A Hilbert space operator $T \in B(\mathcal{H})$ is hyponormal if $|T^*|^2 \leq |T|^2$; p-hyponormal $(0 < p \leq 1)$ if $|T^*|^{2p} \leq |T|^{2p}$; log–hyponormal if T is invertible and $\log |T^*| \leq \log |T|$; w-hyponormal if $|\tilde{T}^*| \leq |T| \leq |\tilde{T}|$, where for $T = U|T|$, $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$; of class $\mathcal{A}$ if $|T|^2 \leq |T^2|$; paranormal if $\|Tx\|^2 \leq \|T^2 x\|$ for all unit vectors $x \in \mathcal{H}$, and *-paranormal if $\|T^* x\|^2 \leq \|T^2 x\|$ for all unit vectors $x \in \mathcal{H}$. Operators T belonging to these classes are known to satisfy the property $T \in (\mathcal{HNP})$; furthermore, the Berberian–Quigley extension T° of an operator T in any one of these (Hilbert space) classes is again of the same class. Hence operators in these classes of operators satisfy property (β). M-hyponormal operators, i.e. operators $T \in B(\mathcal{H})$ such that $\|(T - \lambda)^* x\| \leq M \cdot \|(T - \lambda)x\|$ for some $M \geq 1$, all $x \in \mathcal{H}$ and all $\lambda \in \mathbf{C}$, are well known to satisfy property (β). Since a (p, k)-quasihyponormal operator, i.e. an operator $T \in B(\mathcal{H})$ such that $T^{*k}(|T|^{2p} - |T^*|^{2p}) T^k \geq 0$ for some integer $k \geq 1$ and $0 < p \leq 1$, has the upper triangular representation $T = \left(\begin{smallmatrix} T_1 & T_2 \\ 0 & T_3 \end{smallmatrix}\right) : \overline{T^k(\mathcal{H})} \oplus T^{*-k}(0) \longrightarrow \overline{T^k(\mathcal{H})} \oplus T^{*-k}(0)$, where T_1 is p-hyponormal and T_3 is k-nilpotent [19], Corollary 3.5 implies that (p, k)-quasihyponormal operators satisfy property (β). Note further that an invertible (p, k)-quasihyponormal operator is p-hyponormal, and an M-hyponormal operator with spectrum in $\partial \mathbf{D}$ is unitary. Hence, Corollary 3.6 implies that operators belonging to any one of the above classes are not supercyclic.

Let, for convenience, $\mathcal{C}$ denote the class of Hilbert space operators $T \in B(\mathcal{H})$ which are either hyponormal, or p-hyponormal, or log–hyponormal, or w-hyponormal, or M-hyponormal, or paranormal, or *-paranormal, or (p,k)-quasihyponormal. Let $\operatorname{ind}(T)$ denote the Fredholm index of T, and let (as above) $\sigma_e(T)$ denote the (Fredholm) essential spectrum of T. The following corollary has been proved for particular subclasses of the class $\mathcal{C}$ by various authors. Recall that operators A and B are said to be densely intertwined if there exist operators X and Y with dense range such that $AX = XB$ and $BY = YA$.

Corollary 3.9 *If $A, B \in \mathcal{C}$ are densely intertwined operators, all combinations are allowed, then $\sigma(A) = \sigma(B)$, $\sigma_e(A) = \sigma_e(B)$ and $\operatorname{ind}(A-\lambda) = \operatorname{ind}(B-\lambda)$ at every Fredholm point λ of A.*

Proof. Apply [21, Theorem 3.7.15]. □

4. Elementary operator d_{AB} and Weyl's theorem

Unless otherwise stated, we assume in the following that the operators $A, B^* \in B(\mathcal{H})$ are $(\mathcal{HP})$ operators which satisfy property (β) and for which eigenspaces corresponding to distinct eigenvalues are orthogonal. Let, as before, $d_{AB} \in B(B(\mathcal{H}))$ denote either of the operators δ_{AB} and $\triangle_{AB}$. Since [14]

$$\sigma(\delta_{AB}) = \{\alpha - \beta : \alpha \in \sigma(A), \beta \in \sigma(B)\}$$

and

$$\sigma(\triangle_{AB}) = \{\alpha\beta - 1 : \alpha \in \sigma(A), \beta \in \sigma(B)\},$$

for every $\lambda \in \operatorname{iso}\sigma(\delta_{AB})$ (resp., $\lambda \in \operatorname{iso}\sigma(\triangle_{AB})$) there exist finite sequences $\{\alpha_i\}_{i=1}^n \subset \operatorname{iso}\sigma(A)$ and $\{\beta_i\}_{i=1}^n \subset \operatorname{iso}\sigma(B)$ such that $\alpha_i - \beta_i = \lambda$ (resp., $\alpha_i\beta_i - 1 = \lambda$) for all $1 \le i \le n$. Here, the hypothesis $A, B^* \in (\mathcal{HP})$ implies that the points α_i and $\overline{\beta_i}$ are eigenvalues of A and B^*, respectively. If we let

$$\mathcal{M} = \bigvee_{1 \le i \le n} (A - \alpha_i)^{-1}(0), \quad \mathcal{N} = \bigvee_{1 \le i \le n} (B^* - \overline{\beta_i})^{-1}(0),$$

then (our hypothesis on the orthogonality of eigenspaces corresponding to distinct eigenvalues) implies that $A|_{\mathcal{M}}$ and $(B^*|_{\mathcal{N}})^*$ are normal operators.

Let $H(\sigma(d_{AB}))$ denote the set of functions f which are analytic on an open neighbourhood of $\sigma(d_{AB})$. Recall, [21], [4], that a Banach space operator $A \in B(\mathcal{X})$ is generalized scalar if there exists a continuous algebra homomorphism Φ from the space $C^\infty(\mathbf{C})$ of infinitely differentiable complex valued functions into $B(\mathcal{X})$, $\Phi : C^\infty(\mathbf{C}) \longrightarrow B(\mathcal{X})$, such that $\Phi(1) = I$ and $\Phi(z) = A$.

Theorem 4.1 *$f(d_{AB})$ satisfies Wt and $f(d^*_{AB})$ satisfies $a - Wt$ for every $f \in H(\sigma(d_{AB}))$.*

Proof. Combining Propositions 2.1 and 2.2 it follows that d_{AB} satisfies $a - Bt$, hence also Bt. In particular,

$$\sigma(d_{AB}) \setminus \sigma_w(d_{AB}) = p_0(d_{AB}) \subseteq \pi_{00}(d_{AB})$$

and

$$\sigma_a(d_{AB}) \setminus \sigma_{aw}(d_{AB}) = p_0^a(d_{AB}) \subseteq \pi_{00}^a(d_{AB}).$$

The conclusion that d_{AB} has SVEP, Proposition 2.1, implies also that d^*_{AB} satisfies $a - Bt$, i.e.,

$$\sigma_a(d^*_{AB}) \setminus \sigma_{aw}(d^*_{AB}) = p_0^a(d^*_{AB}) = p_0(d^*_{AB}) \subseteq \pi_{00}^a(d^*_{AB}) = \pi_{00}(d^*_{AB}).$$

Observe that d_{AB} is polaroid implies d^*_{AB} is polaroid. Hence, in view of Propositions 2.4 and 2.5, to prove the theorem it would suffice to prove that d_{AB} is polaroid. We consider the cases $d_{AB} = \delta_{AB}$ and $d_{AB} = \triangle_{AB}$ separately.

(1) $d_{AB} = \delta_{AB}$. For $\lambda \in \text{iso}\sigma(\delta_{AB})$, define sequences $\{\alpha_i\}_{1 \le i \le n}$ and $\{\beta_i\}_{1 \le i \le n}$, and subspaces $\mathcal{M}$ and $\mathcal{N}$, as above. Then A and B have representations

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} \mathcal{M} \\ \mathcal{H} \ominus M \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} \mathcal{N} \\ \mathcal{H} \ominus N \end{pmatrix}$$

where the operators A_{11} and B_{11} are normal, $\sigma(A) = \sigma(A_{11}) \cup \sigma(A_{22})$, $\sigma(B) = \sigma(B_{11}) \cup \sigma(B_{22})$ and $0 \notin \sigma(\delta_{A_{ii}B_{jj}} - \lambda)$ for all $1 \le i,\ j \le 2$ except

$i=j=1$. Consider an $X \in H_0(\delta_{AB}-\lambda)$. Letting $X : \mathcal{N} \oplus (\mathcal{H} \ominus N) \longrightarrow \mathcal{M} \oplus (\mathcal{H} \ominus M)$ have the matrix representation $X = [X_{ij}]_{i,j=1}^2$, it follows that

$$(\delta_{AB}-\lambda)^m X = \begin{pmatrix} * & * \\ * & (\delta_{A_{22}B_{22}}-\lambda)^m X_{22} \end{pmatrix}$$

(for some as yet to be determined entries $*$). Since

$$\lim_{m \longrightarrow \infty} \|(\delta_{AB}-\lambda)^m X\|^{\frac{1}{m}} = 0$$

implies

$$\lim_{m \longrightarrow \infty} \left\|(\delta_{A_{22}B_{22}}-\lambda)^m X_{22}\right\|^{\frac{1}{m}} = 0,$$

and since $0 \notin \sigma(\delta_{A_{22}B_{22}}-\lambda)$, we have that $X_{22}=0$ and

$$(\delta_{AB}-\lambda)^m X = \begin{pmatrix} * & (\delta_{A_{11}B_{22}}-\lambda)^m X_{12} \\ (\delta_{A_{22}B_{11}}-\lambda)^m X_{21} & 0 \end{pmatrix}.$$

Since

$$\lim_{m \longrightarrow \infty} \left\|(\delta_{A_{ii}B_{jj}}-\lambda)^m X_{ij}\right\|^{\frac{1}{m}} = 0$$

for all $1 \le i,j \le 2$, and since $0 \notin \sigma(\delta_{A_{11}B_{22}}-\lambda)$ and $0 \notin \sigma(\delta_{A_{22}B_{11}}-\lambda)$, $X_{12}=X_{21}=0$. Hence

$$(\delta_{AB}-\lambda)^m X = (\delta_{A_{11}B_{11}}-\lambda)^m X_{11} \oplus 0.$$

The operators A_{11} and B_{11} being normal are generalized scalar operators. Hence $\delta_{A_{11}B_{11}}-\lambda = L_{A_{11}-\lambda} - R_{B_{11}}$ is a generalized scalar operator (see 4.3.3 Theorem, 4.4.2 Proposition and 4.4.3 Theorem of [4]). Recall that a generalized scalar operator satisfies the property that $H_0(T-\lambda) = (T-\lambda)^{-p_\lambda}(0)$ for all $\lambda \in \mathbf{C}$ and for some non–negative integer p dependent upon λ [1, pages 175–176]. Hence there exists a positive integer p such that $H_0(\delta_{A_{11}B_{11}}-\lambda) = (\delta_{A_{11}B_{11}}-\lambda)^{-p}(0)$. (Indeed, $p=1$: argue as in the proof of [7, Proposition 2.3] to prove that $\mathrm{asc}(\delta_{A_{11}B_{11}}-\lambda) \le 1$.) Consequently, $H_0(\delta_{AB}-\lambda) = (\delta_{AB}-\lambda)^{-p}(0)$. To complete the proof, we now observe that if $\lambda \in \operatorname{iso}\sigma(\delta_{AB})$, then

$$B(\mathcal{H}) = H_0(\delta_{AB} - \lambda) \oplus K(\delta_{AB} - \lambda) = (\delta_{AB} - \lambda)^{-p}(0) \oplus K(\delta_{AB} - \lambda).$$

Hence

$$(\delta_{AB} - \lambda)^p B(\mathcal{H}) = 0 \oplus (\delta_{AB} - \lambda)^p K(\delta_{AB} - \lambda) = K(\delta_{AB} - \lambda),$$

which implies that

$$B(\mathcal{H}) = (\delta_{AB} - \lambda)^{-p}(0) \oplus (\delta_{AB} - \lambda)^p B(\mathcal{H}),$$

i.e., λ is a pole of the resolvent of d_{AB}.

(2) $d_{AB} = \triangle_{AB}$. We consider the cases $\lambda \neq -1$ and $\lambda = -1$ separately. The proof for the case $\lambda \neq -1$ is similar to the earlier case, so we shall be economical with our argument. Defining the subspaces $\mathcal{M}$ and $\mathcal{N}$ as before, and letting A, B, X have the representations above, it is seen that

$$(\triangle_{AB} - \lambda)^m X = \begin{pmatrix} * & * \\ * & (\triangle_{A_{22}B_{22}} - \lambda)^m X_{22} \end{pmatrix}$$

(for some as yet to be determined entries $*$). If $X \in H_0(\triangle_{AB} - \lambda)$, then $0 \notin \sigma(\triangle_{A_{22}B_{22}} - \lambda)$ and $\lim_{m\longrightarrow\infty} \|(\triangle_{A_{22}B_{22}} - \lambda)^m X_{22}\|^{\frac{1}{m}} = 0$ imply that $X_{22} = 0$. Consequently,

$$(\triangle_{AB} - \lambda)^m X = \begin{pmatrix} * & (\triangle_{A_{11}B_{22}} - \lambda)^m X_{12} \\ (\triangle_{A_{22}B_{11}} - \lambda)^m X_{21} & 0 \end{pmatrix},$$

where

$$\lim_{m\longrightarrow\infty} \left\|\triangle_{A_{11}B_{22}} - \lambda)^m X_{12}\right\|^{\frac{1}{m}} = \lim_{m\longrightarrow\infty} \left\|\triangle_{A_{22}B_{11}} - \lambda)^m X_{21}\right\|^{\frac{1}{m}} = 0.$$

Since 0 is not in both $\sigma(\triangle_{A_{11}B_{22}} - \lambda)$ and $\sigma(\triangle_{A_{22}B_{11}} - \lambda)$, we have that $X_{12} = X_{21} = 0$, and then

$$(\triangle_{AB} - \lambda)^m X = (\triangle_{A_{11}B_{11}} - \lambda)^m X_{11} \oplus 0.$$

Thus the operator $\triangle_{AB} - \lambda$ is a generalized scalar operator. Hence

$H_0(\triangle_{AB} - \lambda) = (\triangle_{AB} - \lambda)^{-p}(0)$ for some positive integer p. (Indeed, as for the case $d_{AB} = \delta_{AB}$, it can be seen that $p = 1$.) As above, this implies that λ is a pole of the resolvent of $\triangle_{AB}$.

To complete the proof we now let $\lambda = -1$. Then either $0 \in \text{iso}\sigma(A) \cap \text{iso}\sigma(B)$, or $0 \in \text{iso}\sigma(A)$ and $0 \notin \sigma(B)$, or $0 \notin \sigma(A)$ and $0 \in \text{iso}\sigma(B)$. If $0 \in \text{iso}\sigma(A) \cap \text{iso}\sigma(B)$, then upon letting $\mathcal{M} = A^{-1}(0)$ and $\mathcal{N} = B^{*-1}(0)$ we have

$$A = \begin{pmatrix} 0 & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} \mathcal{M} \\ \mathcal{H} \ominus M \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} \mathcal{N} \\ \mathcal{H} \ominus N \end{pmatrix},$$

where the operators A_{22} and B_{22} are invertible (which implies that $\triangle_{A_{22}B_{22}} - \lambda = L_{A_{22}} R_{B_{22}}$ is invertible). Letting $X = [X_{ij}]_{i,j=1}^2$ as above, it follows that $X_{22} = 0$, and hence that $L_A R_B(X) = 0$ for every $X \in H_0(L_A R_B) = H_0(\triangle_{AB} - \lambda)$. Thus $H_0(\triangle_{AB} - \lambda) = (\triangle_{AB} - \lambda)^{-1}(0)$. The proof of the remaining cases is similar: we consider $0 \in \text{iso}\sigma(A)$ and $0 \notin \sigma(B)$. Then $X \in H_0(L_A R_B) \implies \lim_{n\to\infty} \|L_A^n X\|^{\frac{1}{n}} \leq \|B^{-1}\| \lim_{n\to\infty} \|(L_A R_B)^n X\|^{\frac{1}{n}} = 0$; again, if $X \in H_0(L_A)$, then $\lim_{n\to\infty} \|(L_A R_B)^n X\|^{\frac{1}{n}} \leq \|B\| \lim_{n\to\infty} \|L_A^n X\|^{\frac{1}{n}} = 0$. Hence $H_0(\triangle_{AB} - \lambda) = H_0(L_A R_B) = (L_A)^{-1}(0) = (\triangle_{AB} - \lambda)^{-1}(0)$. $\square$

Remark 4.2 Recall from Remark 3.8 that class $\mathcal{C}$ operators satisfy property (β). Operators $T \in \mathcal{C}$ which are either hyponormal or p-hyponormal or log–hyponormal or w-hyponormal or paranormal or $*$-paranormal are $(\mathcal{THN})$ operators (see [6], [8] and [10]); hence, if T is an operator in any one of these classes, then Corollary 2.7 implies that eigenspaces corresponding to distinct eigenvalues of the operator are orthogonal. This property is well known for the class of M-hyponormal operators (which is not a subclass of the class $(\mathcal{THN})$), but fails for the class of (p,k)-quasihyponormal operators. If, however, $T \in B(\mathcal{H})$ is a (p,k)-quasihyponormal operator such that $T^{-1}(0) \subseteq T^{*-1}(0)$, then T does have this property. Thus Theorem 4.1 applies to d_{AB} for operators A, B^* in class $\mathcal{C}$, provided one assumes that $T^{-1}(0) \subseteq T^{*-1}(0)$ whenever $T = A$ or B^* is a (p,k)-quasihyponormal operator.

Perturbations. An operator $C \in B(\mathcal{H})$ is a Riesz operator if $\sigma_e(C) = \{0\}$. If $C, D \in B(\mathcal{H})$ are commuting Riesz operators, and $E \in B(\mathcal{H})$ is any operator which commutes with C, then $C - D$, CD, CE and EC are

Riesz operators [1, Theorem 3.112]. Observe that for operators C, D, E and $F \in B(\mathcal{X})$,

$$\delta_{(E+C)(F+D)} = \delta_{EF} + \delta_{CD}$$

and

$$\triangle_{(E+C)(F+D)} = \triangle_{EF} + L_E R_D + L_C R_D + L_C R_F.$$

Hence perturbation of the operators E and F in d_{EF} by operators C and D, respectively, such that C commutes with E and D commutes with F results in a perturbation of d_{EF} by an operator which commutes with d_{EF}.

Lemma 4.3 *L_C (resp., R_C), $C \in B(\mathcal{H})$, is a Riesz operator if and only if C is quasinilpotent.*

Proof. Recall, [15, Theorem 3.1], $\sigma_e(\delta_{CD}) = \{\sigma_e(C) - \sigma(D)\} \cup \{\sigma(C) - \sigma_e(D)\}$ for all $C, D \in B(\mathcal{H})$. Taking D (resp., C) to be the trivial operator 0, it follows that $\sigma_e(L_C) = \{0\}$ (resp., $\sigma_e(R_D) = \{0\}$) if and only if $\sigma(C) = \{0\}$ (resp., $\sigma(D) = \{0\}$).

Theorem 4.4 *If $C, D \in B(\mathcal{H})$ are quasinilpotent operators such that A commutes with C and B commutes with D, then $d_{(A+C)(B+D)}$ and $d^*_{(A+C)(B+D)}$ satisfy $a - Bt$ (hence also Bt).*

Proof. It is known, see the proof of [9, Theorem 1.5(iii)] and [23], that if $S \in B(\mathcal{X})$ is a Riesz operator which commutes with $T \in B(\mathcal{X})$, then $\sigma_x(T + S) = \sigma_x(T)$ for $\sigma_x = \sigma_b$ or σ_w or σ_{ab} or σ_{aw}, and $\sigma_x(T^* + S^*) = \sigma_x(T^*)$ for $\sigma_x = \sigma_{ab}$ or σ_{aw}. Thus if T and T^* satisfy $a - Bt$, then $\sigma_{ab}(T + S) = \sigma_{ab}(T) = \sigma_{aw}(T) = \sigma_{aw}(T + S)$ and $\sigma_{ab}(T^* + S^*) = \sigma_{ab}(T^*) = \sigma_{aw}(T^*) = \sigma_{aw}(T^* + S^*)$, i.e., $T + S$ and $(T + S)^*$ satisfy $a - Bt$. Hence the proof of the theorem follows from Theorem 4.1 and the argument above.

It is known that the null spaces of the operators L_E, R_F and $L_E R_F$ are either trivial or infinite dimensional [2, Theorem 3.3] for all operators E and F. If δ_{CD} and $L_A R_D + L_C R_D + L_C R_B$ are injective quasinilpotent operators, then an argument from the proof of [9, Theorem 1.5(iii)], or [10, Theorem 3.10(ii)], shows that ($p_0(d_{(A+C)(B+D)}) = \pi_{00}(d_{(A+C)(B+D)}) = \emptyset$, and hence that) $d_{(A+C)(B+D)}$ satisfies Wt and $d^*_{(A+C)(B+D)}$ satisfies $a - Wt$.

Recall from [10, Theorem 3.8] that if $T \in B(\mathcal{X})$ is a polaroid operator with SVEP, and $S \in B(\mathcal{X})$ is an algebraic operator which commutes with T, then $f(T+S)$ satisfies Wt and $f(T+S)^*$ satisfies $a-Wt$ for every $f \in H(\sigma(T+A))$. Does a similar result hold for the operator d_{AB}? We have a partial result.

Theorem 4.5

(i) *If $C, D \in B(\mathcal{H})$ are algebraic operators such that C commutes with A and D commutes with B, then $d_{(A+C)(B+D)}$ and $d^*_{(A+C)(B+D)}$ satisfy $a-Bt$.*

(ii) *Let $C, D \in B(\mathcal{H})$ be generalized scalar operators such that $\sigma(C)$ and $\sigma(D)$ are finite subsets of $\mathbf{C}$. If C commutes with A and D commutes with B, then $f(\delta_{(A+C)(B+D)})$ satisfies Wt and $f(\delta^*_{(A+C)(B+D)})$ satisfies $a-Wt$ for every $f \in H(\sigma(\delta_{(A+C)(B+D)}))$.*

Proof.

(i) Recall from Corollary 3.7 that $A+C$ and $(B+D)^*$ satisfy property (β). Hence $d_{(A+C)(B+D)}$ has SVEP (see Proposition 2.1), which implies that $d_{(A+C)(B+D)}$ and $d^*_{(A+C)(B+D)}$ satisfy $a-Bt$.

(ii) L_C and R_D being generalized scalar operators with finite spectrum, δ_{CD} is a generalized scalar operator with finite spectrum. Hence δ_{CD} is an algebraic operator [21, Proposition 1.5.10], and $\delta_{(A+C)(B+D)} = \delta_{AB} + \delta_{CD}$ is the perturbation of δ_{AB} by an algebraic operator. Since δ_{AB} is polaroid, see the proof of Theorem 4.1, and since δ_{AB} has SVEP, $\delta_{(A+C)(B+D)}$ is polaroid and has SVEP (see the proof of [10, Theorem 3.6] or [12, Theorem 8.4.12]. Hence, by Propositions 2.4 and 2.5, $f(\delta_{(A+C)(B+D)})$ satisfies Wt and $f(\delta^*_{(A+C)(B+D)})$ satisfies $a-Wt$ for every $f \in H(\sigma(\delta_{(A+C)(B+D)}))$.

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