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CHAPTER 8

Bispectra of Ocean Waves

Klaus Hasselmann, Walter Munk, Gordon MacDonald,
*Institute of Geophysics and Planetary Physics, University of California**

1. INTRODUCTION

To a first approximation, a random sea surface may be regarded as a linear superposition of statistically independent free waves and is consequently completely described by its two-dimensional power spectrum. For many purposes this approximation is adequate. A number of interesting phenomena, such as surf beats (W. H. Munk, 1949, M. J. Tucker, 1950, and M. S. Longuet-Higgins and R. W. Stewart, 1962), wave breaking (O. M. Phillips, 1958), and the energy transfer between wave components (O. M. Phillips, 1960, K. Hasselmann, 1962) can be explained only by the nonlinearity of the wave motion. To investigate these processes, third- and higher order moments must be analyzed. As a first step, we have evaluated the bispectrum of the wave record of a single station. The wave measurements were taken in shallow water (11 meters) in order to obtain relatively strong nonlinearities. The experimental values agree well with the theoretical bispectrum obtained by perturbation analysis.

2. THE BISPECTRUM

If $\zeta(t)$ is a stationary random function of time, the spectrum $F(\omega)$ and bispectrum $B(\omega_1, \omega_2)$ of $\zeta(t)$ are defined respectively as the Fourier transforms of the mean second- and third-order products:

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R(\tau) e^{-i\omega\tau} \, d\tau, \tag{1}$$

where

$$R(\tau) = \langle \zeta(t) \, \zeta(t + \tau) \rangle, \tag{2}$$

$$B(\omega_1, \omega_2) = \frac{1}{(2\pi)^2} \iint_{-\infty}^{+\infty} S(\tau_1, \tau_2) e^{-i\omega_1\tau_1 - i\omega_2\tau_2} \, d\tau_1 \, d\tau_2, \tag{3}$$

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where

$$S(\tau_1, \tau_2) = \langle \zeta(t)\, \zeta(t + \tau_1)\, \zeta(t + \tau_2) \rangle. \tag{4}$$

The brackets $\langle \quad \rangle$ denote ensemble means. The inverse relations to (1), (3) are

$$R(\tau) = \int_{-\infty}^{+\infty} F(\omega) e^{i\omega\tau}\, d\omega \tag{5}$$

$$S(\tau_1, \tau_2) = \iint_{-\infty}^{+\infty} B(\omega_1, \omega_2) e^{i\omega_1\tau_1 + i\omega_2\tau_2}\, d\omega_1\, d\omega_2. \tag{6}$$

For real $\zeta(t)$

$$F(\omega) = F(-\omega)^* \tag{7}$$

$$B(\omega_1, \omega_2) = B(-\omega_1, -\omega_2)^*. \tag{8}$$

From the stationarity of $\zeta(t)$ the known symmetry relations

$$R(\tau) = R(-\tau) \tag{9}$$

$$\begin{aligned} S(\tau_1, \tau_2) &= S(\tau_2, \tau_1) = S(-\tau_2, \tau_1 - \tau_2) = S(\tau_1 - \tau_2, -\tau_2) \\ &= S(-\tau_1, \tau_2 - \tau_1) = S(\tau_2 - \tau_1, -\tau_1) \end{aligned} \tag{10}$$

follow immediately. Only two of the relations (10) are independent. In terms of the spectra and bispectra, (9) and (10) become

$$F(\omega) = F(-\omega) \tag{11}$$

$$\begin{aligned} B(\omega_1, \omega_2) = B(\omega_2, \omega_1) &= B(\omega_1, -\omega_1 - \omega_2) = B(-\omega_1 - \omega_2, \omega_1) \\ &= B(\omega_2, -\omega_1 - \omega_2) = B(-\omega_1 - \omega_2, \omega_2). \end{aligned} \tag{12}$$

From (7), (8), (11), and (12) it follows that the spectrum is real and is determined by its value on a half line, whereas the bispectrum is determined by its values in an octant; for example, $0 \leqslant \omega_1 < \infty$, $0 \leqslant \omega_2 \leqslant \omega_1$.

The physical significance of the spectra and bispectra becomes clearer when expressed in terms of the components $dZ(\omega)$ of the Fourier-Stieltjes representation of $\zeta(t)$:

$$\zeta(t) = \int_{-\infty}^{+\infty} dZ(\omega) e^{i\omega t}.$$

Then

$$\langle dZ(\omega_1)\, dZ(\omega_2) \rangle = F(\omega_1)\, d\omega_1 \quad \text{if} \quad \omega_1 + \omega_2 = 0, \tag{13}$$

and zero if

$$\omega_1 + \omega_2 \neq 0;$$

$$\langle dZ(\omega_1)\, dZ(\omega_2)\, dZ(\omega_3) \rangle = B(\omega_1, \omega_2)\, d\omega_1\, d\omega_2 \quad \text{if} \quad \omega_1 + \omega_2 + \omega_3 = 0 \tag{14}$$

and zero if

$$\omega_1 + \omega_2 + \omega_3 \neq 0.$$

The spectrum represents the contribution to the mean square $\overline{\zeta^2}$ from the product of two Fourier components whose frequencies add to zero, whereas the bispectrum represents the contribution to the mean cube $\overline{\zeta^3}$ from the product of three Fourier components whose resultant frequency is zero. The symmetry relations are seen to be a direct consequence of the symmetrical form of (13) and (14). The derivations of (13) and (14) proceed along analogous lines: by taking the limits of the approximating Fourier sums and making use of the equality of ensemble and time averages for stationary processes.

The dimensionless ratio

$$\frac{\langle \zeta^3 \rangle}{\langle \zeta^2 \rangle^{3/2}}$$

is called the skewness and is generally finite. It may be interesting to note that the related ratio

$$\frac{\langle \zeta_1 \zeta_2 \zeta_3 \rangle}{[\langle \zeta_1^2 \rangle \langle \zeta_2^2 \rangle \langle \zeta_3^2 \rangle]^{1/2}} = \frac{B(\omega_1, \omega_2)(\delta\omega)^2}{[F(\omega_1)\, F(\omega_2)\, F(\omega_3)(\delta\omega)^3]^{1/2}}$$

for three records "played" through three filters centered on ω_1, ω_2, $\omega_3 = -\omega_1 - \omega_2$ and of bandwidth $\delta\omega$ is proportional to $(\delta\omega)^{1/2}$. This result can be looked at another way. Narrowing the filter width $\delta\omega$ is equivalent to time averaging over increasing intervals $1/\delta\omega$, and in the limit this leads to a Gaussian joint distribution of the three variables.

3. QUASI-LINEAR, QUASI-GAUSSIAN PROCESS

For a Gaussian process the bispectrum vanishes. $\zeta(t)$ is then a linear superposition of an infinite number of statistically independent Fourier components. We shall be concerned in the following with a process that is almost linear and Gaussian. In particular, we shall consider a process that can be expanded with respect to some perturbation parameter ϵ in a series

$$\zeta(t) = \zeta^{(1)}(t) + \zeta^{(2)}(t) + \zeta^{(3)}(t) + \cdots,$$

where $\zeta^{(n)}(t) = 0(\epsilon^n)$. Without loss of generality, we may take $\langle \zeta \rangle$, hence $\langle \zeta^{(n)} \rangle$, as zero. We assume that the first-order term $\zeta^{(1)}(t)$ is Gaussian and that the higher order terms can be expressed in terms of the first-order term in the form

$$\zeta^{(n)}(t) = \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} G^{(n)}(t - t_1, t - t_2, \cdots, t - t_n)\zeta^{(1)}(t_1)\zeta^{(1)}(t_2) \cdots \zeta^{(1)}(t_n)\, dt_1\, dt_2 \cdots dt_n.$$

Using the Fourier-Stieltjes representations $\zeta^{(n)}(t) = \int_{-\infty}^{+\infty} dZ^{(n)}(\omega)e^{i\omega t}$, (13) transforms to

$$dZ^{(n)}(\omega) = \int_{\omega_1+\omega_2+\cdots+\omega_n=\omega} \cdots \int K^{(n)}(\omega_1, \omega_2, \cdots, \omega_n)\, dZ^{(1)}(\omega_1)\, dZ^{(1)}(\omega_2) \cdots dZ^{(1)}(\omega_n)$$

where $K^{(n)}$ is the Fourier transform of $G^{(n)}$ ($G^{(n)}$ being defined as zero for negative arguments).

Generally, the value of an mth-order mean product $\langle\zeta(t_1)\zeta(t_2)\zeta(t_3)\cdots\zeta(t_m)\rangle$ correct to the nth perturbation order depends on all interaction coefficients up to the $(n-m+1)$th order. We shall consider only the quadratic mean product to the second perturbation order and the cubic mean product to the fourth perturbation order. This limits us to the second-order interaction coefficient. We have then

$$F(\omega)\,d\omega = F^{(2)}(\omega)\,d\omega + \cdots = \langle dZ^{(1)}(\omega)\,dZ^{(1)}(-\omega)\rangle + \cdots$$

$$\begin{aligned} B(\omega_1, \omega_2)\,d\omega_1\,d\omega_2 &= B^{(3)}(\omega_1, \omega_2)\,d\omega_1\,d\omega_2 + B^{(4)}(\omega_1, \omega_2)\,d\omega_1\,d\omega_2 + \cdots \\ &= \langle dZ^{(1)}(\omega_1)\,dZ^{(1)}(\omega_2)\,dZ^{(1)}(-\omega_1-\omega_2)\rangle \\ &+ \langle dZ^{(1)}(\omega_1)\,dZ^{(1)}(\omega_2)\,dZ^{(2)}(-\omega_1-\omega_2)\rangle \\ &+ \langle dZ^{(1)}(\omega_1)\,dZ^{(2)}(\omega_2)\,dZ^{(1)}(-\omega_1-\omega_2)\rangle \\ &\qquad + \langle dZ^{(2)}(\omega_1)\,dZ^{(1)}(\omega_2)\,dZ^{(1)}(-\omega_1-\omega_2)\rangle + \cdots. \end{aligned} \tag{15}$$

Since $\zeta^{(1)}$ is Gaussian, the term $B^{(3)}$ in (15) vanishes. Substituting (14) in the remaining expression and making use again of the Gaussian property of $\zeta^{(1)}$ in determining the fourth-order mean products, we obtain

$$\begin{aligned} B(\omega_1, \omega_2) = 2[&F(\omega_1)\,F(\omega_2)\,K(-\omega_1, -\omega_2) \\ &+ F(\omega_1)\,F(\omega_1+\omega_2)\,K(-\omega_1, +\omega_1+\omega_2) \\ &+ F(\omega_2)\,F(\omega_1+\omega_2)\,K(-\omega_2, +\omega_1+\omega_2)] + \cdots. \end{aligned} \tag{16}$$

Thus, for a weakly nonlinear, non-Gaussian process, the bispectrum is a direct measure of the second-order interaction coefficients. Apparently the coefficients are not completely determined by the bispectrum, since they enter in (16) only in the linear combination corresponding to the symmetry relations (12). However, this has its counterpart in the nonuniqueness of the perturbation expansion. In place of $\zeta^{(1)}$ we could equally well begin the perturbation expansion with a first-order term

$$\tilde{\zeta}^{(1)} = \zeta^{(1)} + \int_{-\infty}^{t} H^{(2)}(t-t_1, t-t_2)\zeta^{(1)}(t_1)\zeta^{(1)}(t_2)\,dt_1\,dt_2 + \cdots,$$

where $H^{(n)}$ are arbitrary kernels satisfying only the condition that $\tilde{\zeta}^{(1)}$ is again Gaussian. This leads to a different set of interaction coefficients, the only invariants of the transformation being the set of all mean products determining the stochastic process $\zeta(t)$. In particular, the only invariant of the second-order interactions is the combination (16) representing the bispectrum.

4. THE THEORETICAL BISPECTRUM FOR FINITE-DEPTH OCEAN WAVES

Time records of ocean waves at a fixed position are to a first approximation stationary and Gaussian. Small deviations from normality exist, however, because of small nonlinearities in the equations of wave propagation. We have seen that these can be determined to the first order by measuring the bispectrum. It is also possible to evaluate the nonlinearities theoretically, thus obtaining a mutual check on theory and experiment. A slight extension of our results in Section 3 is needed, however, since the perturbation expansion of the wave height at a fixed point cannot be carried through in terms of the values at that point alone but only for the complete wave field.

The perturbation expansion for a random sea of constant finite depth h has been described in varying degrees of completeness by several authors [O. M. Phillips (1960), L. J. Tick (1961), M. S. Longuet-Higgins and R. W. Stewart (1962), and K. Hasselmann (1962)]. We shall employ Hasselmann's results and notation (with minor changes) to evaluate the bispectrum of the bottom pressure (rather than the wave height, since the instrument used in the experiment was a bottom-pressure recorder). To the first approximation the velocity potential ϕ can be represented as a superposition of statistically independent free waves:

$$\phi(\mathbf{x}, t) = \phi^{(1)}(\mathbf{x}, t) + \cdots$$
$$= \iint_{-\infty}^{+\infty} (d\phi_+^{(1)}(\mathbf{k})e^{-i\omega t} + d\phi_-^{(1)}(\mathbf{k})e^{+i\omega t}) \frac{\cosh k(z+h)}{\cosh kh} e^{i\mathbf{k}\cdot\mathbf{x}} + \cdots$$

where

$$d\phi_+^{(1)}(\mathbf{k}) = (d\phi_-^{(1)}(-\mathbf{k}))^*$$
$$\omega^2 = gk \tanh kh,$$

and $\mathbf{x}$, z are horizontal and vertical coordinates, respectively, the vertical measured upward from the mean surface.

The second-order term of ϕ is then

$$\phi^{(2)}(\mathbf{x}, t) = \iiiint_{-\infty}^{+\infty} \sum_{s_1, s_2} d\phi_{s_1}^{(1)}(\mathbf{k}_1)\, d\phi_{s_2}^{(1)}(\mathbf{k}_2)\, A(\mathbf{k}_1, s_1\omega_1, \mathbf{k}_2, s_2\omega_2)$$
$$\frac{\cosh [|\mathbf{k}_1 + \mathbf{k}_2|(z+h)]}{\cosh (|\mathbf{k}_1 + \mathbf{k}_2|h)} e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x} - i(s_1\omega_1 + s_2\omega_2)t}, \quad (17)$$

where s_1 and s_2 denote signs and

$$A(\mathbf{k}_1, \omega_1, \mathbf{k}_2, \omega_2) = \frac{i}{\omega^2 - (\omega_1+\omega_2)^2} \Big\{ (\omega_1+\omega_2)[k_1k_2 \tan k_2h \tanh k_2h$$
$$- \mathbf{k}_1 \cdot \mathbf{k}_2] - \frac{1}{2}\left(\frac{\omega_1 k_2^2}{\cosh^2 k_2h} + \frac{\omega_2 k_1^2}{\cosh^2 k_1h}\right)\Big\}, \quad (18)$$

with

$$\omega^2 = g|\mathbf{k}_1 + \mathbf{k}_2| \tanh (|\mathbf{k}_1 + \mathbf{k}_2|h).$$

The corresponding expansion for the bottom pressure is then obtained from (17) and Bernoulli's law

$$P + P_m = -\rho \left[\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2\right]_{z=-h} + gh,$$

where P_m is the mean bottom pressure and P, its variation, so that $\langle P \rangle = 0$.

The first-order Fourier component of the pressure is then

$$dP_{\pm}^{(1)}(\mathbf{k}) = \frac{\pm i\omega\rho}{\cosh kh}\, d\phi_{\pm}^{(1)}(\mathbf{k})$$

and the second-order term is given by

$$P^{(2)}(\mathbf{x}, t) = \iiiint_{-\infty}^{+\infty} \sum_{s_1, s_2} dP_{s_1}^{(1)}(\mathbf{k}_1)\, dP_{s_2}^{(1)}(\mathbf{k}_2)\, C(\mathbf{k}_1, s_1\omega_1, \mathbf{k}_2, s_2\omega_2) e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x} - i(s_1\omega_1 + s_2\omega_2)t}, \quad (19)$$

where

$$C(\mathbf{k}_1, \omega_1, \mathbf{k}_2, \omega_2) = -\frac{\cosh k_1h \cosh k_2h(\omega_1 + \omega_2)}{\rho\,\omega_1\,\omega_2 \cosh(|\mathbf{k}_1 + \mathbf{k}_2|h)}\, iA(\mathbf{k}_1, \omega_1, \mathbf{k}_2, \omega_2) - \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)}{2\rho\omega_1\omega_2}. \quad (20)$$

For the third-order mean product of the bottom pressure at a fixed position x we thus obtain correct to the fourth perturbation order

$$\begin{aligned}\langle P(\mathbf{x}, t)\, P(\mathbf{x}, t + \tau_1)\, P(\mathbf{x}, t + \tau_2)\rangle &= \langle P^{(2)}(\mathbf{x}, t)\, P^{(1)}(\mathbf{x}, t + \tau_1)\, P^{(1)}(\mathbf{x}, t + \tau_2)\rangle \\ &+ \langle P^{(1)}(\mathbf{x}, t)\, P^{(2)}(\mathbf{x}, t + \tau_1)\, P^{(1)}(\mathbf{x}, t + \tau_2)\rangle \\ &+ \langle P^{(1)}(\mathbf{x}, t)\, P^{(1)}(\mathbf{x}, t + \tau_1) P^{(2)}(\mathbf{x}, t + \tau_2)\rangle, \quad (21)\end{aligned}$$

where

$$\begin{aligned}&\langle P^{(2)}(\mathbf{x}, t)\, P^{(1)}(\mathbf{x}, t + \tau_1)\, P^{(1)}(\mathbf{x}, t + \tau_2)\rangle \\ &= \iiiint_{-\infty}^{+\infty} E(\mathbf{k}_1)\, E(\mathbf{k}_2) \left[\tfrac{1}{2} \sum_{s_1, s_2} C(s_1\mathbf{k}_1, s_1\omega_1, s_2\mathbf{k}_2, s_2\omega_2) e^{+i(s_1\omega_1\tau_1 + s_2\omega_2\tau_2)}\right] d^2k_1\, d^2k_2 \quad (22)\end{aligned}$$

and

$$E(\mathbf{k})\, d^2k = \langle 2dP_{+}(\mathbf{k})\, dP_{-}(-\mathbf{k})\rangle.$$

$E(\mathbf{k})$ is the two-dimensional spectrum of $P^{(1)}$ with respect to waves traveling in the *positive* **k**-direction.

In order to write (22) in the form of a Fourier integral with respect to ω_1 and ω_2 as in expression (3), we transform the spectral density from the **k**-plane to ω and the propagation direction α:

$$E(\mathbf{k})\, d^2k = 2F(\omega)\, S(\omega, \alpha)\, d\omega\, d\alpha,$$

where $S(\omega, \alpha)$ is the spreading factor, normalized so that

$$\int_{-\pi}^{+\pi} S(\omega, \alpha)\, d\alpha = 1,$$

and $E(\omega)$ is the (two-sided) one-dimensional frequency spectrum. Equation (22) then becomes

$$\langle P^{(2)}(\mathbf{x}, t)\, P^{(1)}(\mathbf{x}, t + \tau_1)\, P^{(1)}(\mathbf{x}, t + \tau_2)\rangle = \iint_{-\infty}^{+\infty} 2K(-\omega_1, -\omega_2)\, F(\omega_1)\, F(\omega_2) e^{i(\omega_1\tau_1+\omega_2\tau_2)}\, d\omega_1\, d\omega_2, \quad (23)$$

where

$$K(\omega_1, \omega_2) = \iint_{-\pi}^{+\pi} S(\alpha_1)\, S(\alpha_2) C(-s_1\mathbf{k}_1, \omega_1, -s_2\mathbf{k}_2, -\omega_2)\, d\alpha_1\, d\alpha_2 \quad (24)$$

and here $s_j = \text{sign}\ (\omega_j)$.

Thus the first term on the right-hand side of (21) yields the first term in the bispectrum of P as given by (16). It is easily verified that the remaining terms in (21) correspond to the remaining terms in (16). Hence the expression (16) derived for the case of a weakly nonlinear process depending on one parameter only holds also for the more general case considered here, provided we define $K(\omega_1, \omega_2)$ by (24). We note that $K(\omega_1, \omega_2)$ is real, so that the bispectrum has only a real part.

The net interaction coefficient is seen to depend on the spreading factor, which cannot be determined from the one-dimensional frequency spectrum recorded at a single fixed position. By going to the next order, however, and determining the spreading factor that gives the best agreement between the theoretical and experimental bispectrum, it is now possible to gain some information on the directional spread of the waves from the record of a single station. The method gives only an indication of the relative angular spread and no information on the mean direction of the waves, since our theoretical model assumes constant depth and is independent of the choice of horizontal directions. For a truly adequate theoretical model (ours is not) the bispectral method of determining the angular spread is in principle more powerful than the usual one of correlating a number of records at different positions, since the two-dimensional bispectrum contains more information than a finite number of one-dimensional functions.

The dependence of the interaction coefficient K on the spreading factor and the water depth is shown in Figure 1 for the interactions $(\omega, \omega, -2\omega)$ and $(\omega, -1.2\omega, 0.2\omega)$. These are characteristic of the principal contributions to the positive and negative peaks in Figures 2 and 3 (to be described in more detail later). The water depth is nondimensionalized by the wavelength $\lambda_0 = 2\pi g/\omega^2$ in deep water and the coefficients by the factor $\lambda_0/2\pi = g/\omega^2$. The angular spread refers to the beam in deep water. The effect of collimation as the waves enter shallow water has been allowed for. A spreading factor $S(\alpha) = 1/[\alpha'(2\pi)^{1/2}]e^{-\alpha^2/2\alpha'^2}$ was assumed, where α' is the (rms) angular spread. The values for the angular spread in Figures 2 and 3 refer to the *local* wave field at depth h. The principle interactions in these figures correspond to values of h/λ_0 near 0.015.

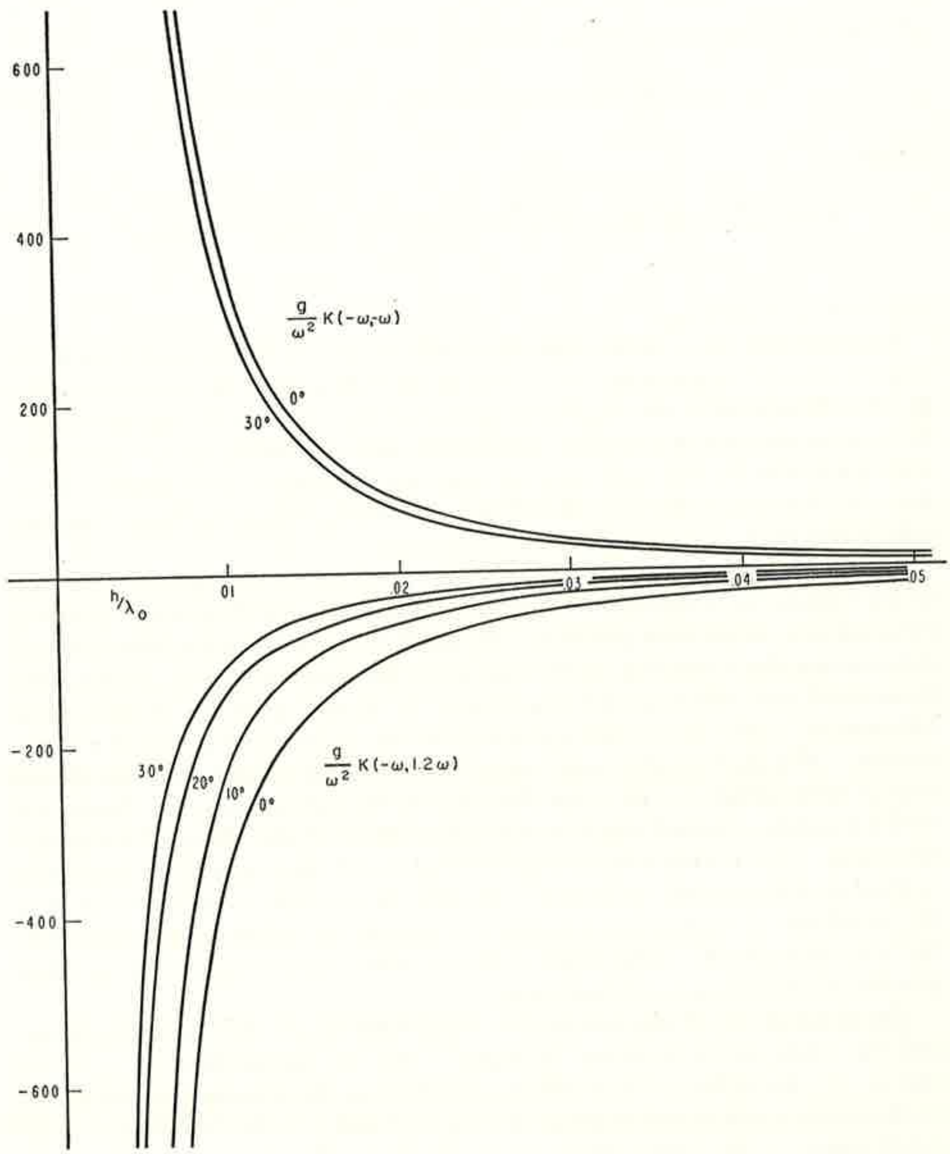

Figure 1. Dependence of the interaction coefficients on spreading angle and water depth. The positive (sum-frequency) coefficient is characteristic for the positive peaks in Figures 2 and 3. The negative (difference-frequency) coefficient is characteristic for the negative peaks.

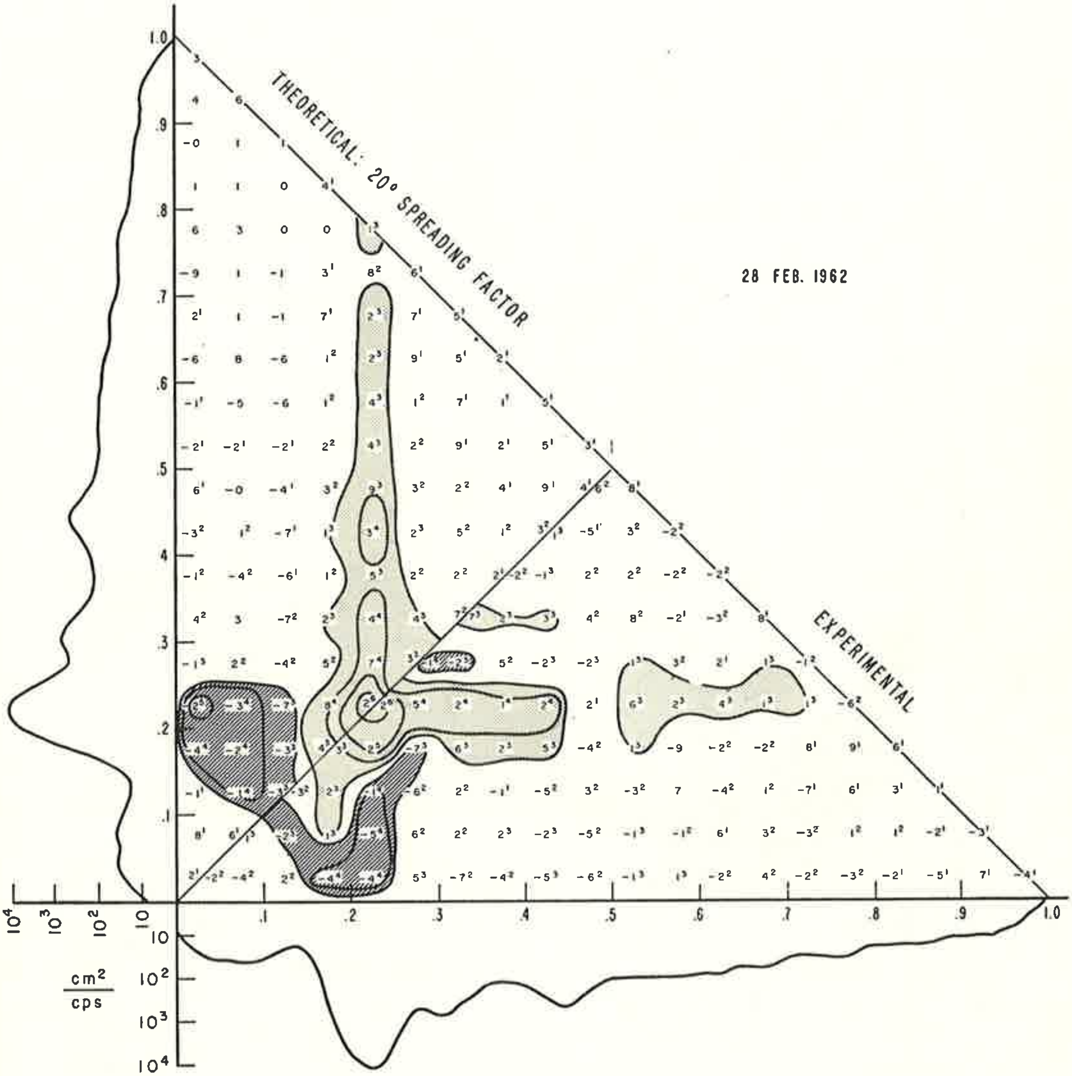

Figure 2. Bispectra of ocean wave record. The numbers give the contributions towards the mean-cubed record (in cm^3) per unit frequency band squared (in cps^2) and are thus in units $cm^3 sec^2$. The number -7^4 denotes -7×10^4. Contours are drawn for -10^3, -10^4, -10^5, -10^6. In the case of perfect agreement between theory and experiment the pattern would be symmetrical about the 45° line. The two axes give frequencies in Nyquist units; 1 Nyquist is 0.25 cps. The (identical) plots along the two axes are the power spectra in cm^2/cps.

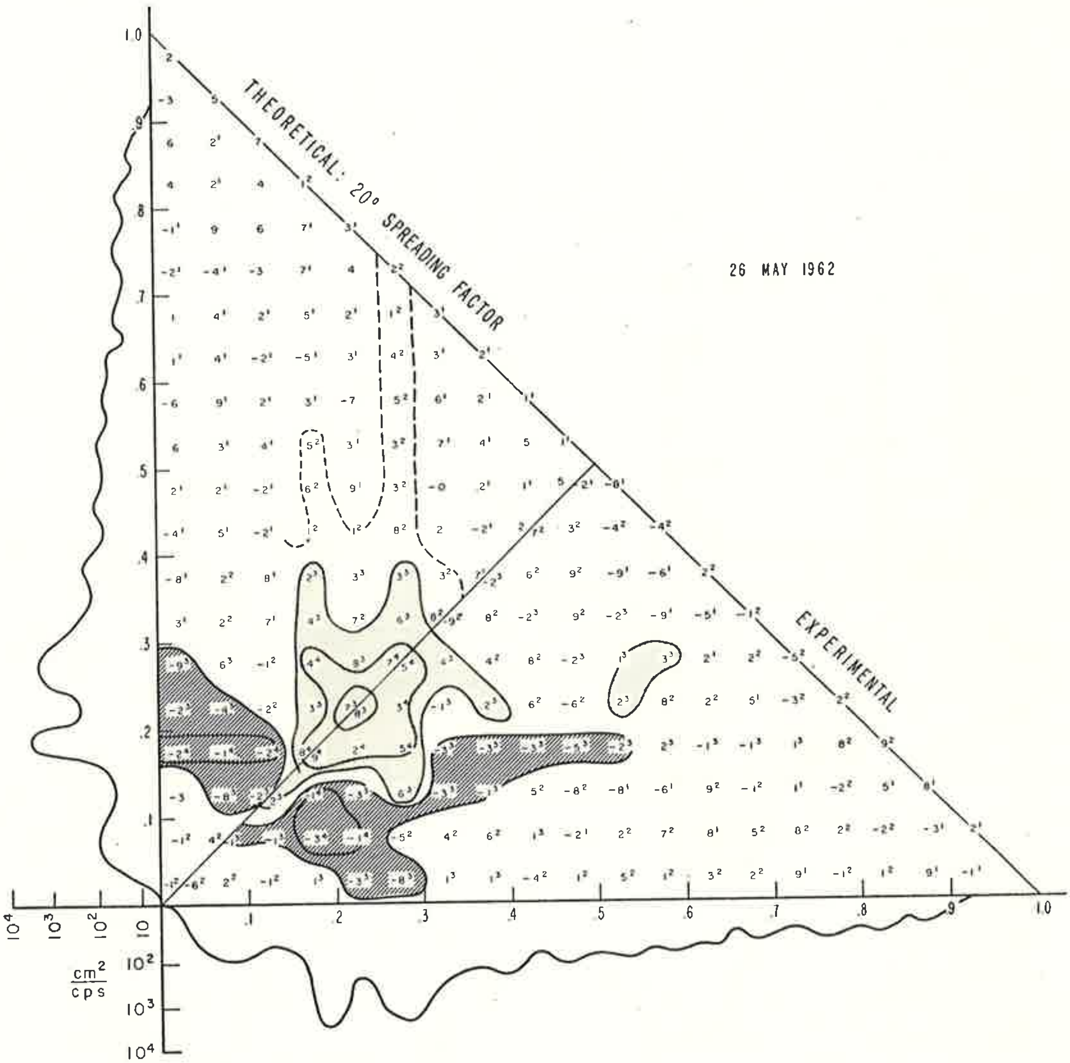

Figure 3. Bispectra of ocean wave record. The numbers give the contributions towards the mean-cubed record (in cm^3) per unit frequency band squared (in cps^2) and are thus in units cm^3sec^2. The number -7^4 denotes -7×10^4. Contours are drawn for -10^3, -10^4, -10^5, -10^6. In the case of perfect agreement between theory and experiment the pattern would be symmetrical about the 45° line. The two axes give frequencies in Nyquist units; 1 Nyquist is 0.25 cps. The (identical) plots along the two axes are the power spectra in cm^2/cps.

5. MEASUREMENT OF THE BISPECTRUM

Notation

There is a problem of normalization. The most convenient definitions of power spectrum and bispectrum for theoretical work are those we have already used. Setting all τ's equal to zero in (5) and (6) yields

$$\int_{-\infty}^{\infty} F(\omega)\, d\omega = R(0) = \langle\zeta^2\rangle, \qquad \iint_{-\infty}^{\infty} B(\omega_1, \omega_2)\, d\omega_1\, d\omega_2 = S(0, 0) = \langle\zeta^3\rangle.$$

Either of the following definitions is more convenient for numerical analysis:

$$\int_{-\infty}^{\infty} F'(f)\, df = \int_0^{\infty} F''(f)\, df = \langle\zeta^2\rangle,$$

where $\omega = 2\pi f$. In view of (5), the connections are

$$F' = 2\pi F, \qquad F'' = 4\pi F.$$

We shall use F'; in engineering practice F'' is better established, but it leads to awkward conventions when the generalization is made to the bispectrum. The best choice appears to be

$$\iint_{-\infty}^{+\infty} B'(f_1, f_2)\, df_1\, df_2 = \langle\zeta^3\rangle,$$

which yields

$$B' = 4\pi^2 B.$$

We shall plot F' for $f > 0$ and B' for $0 \leqslant f_1 < \infty$, $0 \leqslant f_2 \leqslant f_1$. The entire field is then determined by the relations (7), (8), (11), and (12), and it can be shown that

$$\langle\zeta^2\rangle = 2\int_0^{\infty} F'(f)\, df, \qquad \langle\zeta^3\rangle = 12\int_0^{\infty} df_1 \int_0^{f_1} df_2\, \mathrm{Re}\,[B'(f_1, f_2)]$$

Aliasing

For digital sampling at intervals Δt, any energy associated with frequencies above the Nyquist frequency $f_N = (2\Delta t)^{-1}$ appears in the power spectrum in the "alias" of a lower frequency. It is convenient to refer all frequencies to the dimensionless Nyquist units f/f_N. To avoid aliasing of the energy spectrum, we must sample at a rate sufficient to make certain that the energy density above unit Nyquist frequency is low compared to the energy below unit frequency. Similarily, to avoid aliasing the bispectrum, the bispectrum must be small for all frequencies f_1, f_2 outside the region $|f_1| \leqslant 1$, $|f_2| \leqslant 1$, $|f_1 + f_2| \leqslant 1$. The product of this region with the octant $0 \leqslant f_1 < \infty$, $0 \leqslant f_2 \leqslant f_1$ yields a triangular region with the interval $0 \leqslant f_1 \leqslant 1$ as base line and the point $(\frac{1}{2}, \frac{1}{2})$ as apex. An inspection of Figures 2 and 3 indicates that aliasing has been successfully avoided for both spectra and bispectra.

Recording

A "vibratron" pressure transducer on the sea bottom off Oceanside, California, at a depth of 11 meters converted water pressure into a fm electric signal, which was brought ashore through a submarine cable and digitally recorded on punched paper tape [Snodgrass (1958)]. The data were automatically scanned for errors, and the power spectra was computed according to the Tukey method. Details of the analysis have been reported elsewhere [Munk, Miller, Snodgrass, and Barber (1963)].

Bispectral analysis

The bispectra were computed directly and *not* by a Fourier transform of the two-dimensional correlation. The procedure is somewhat wasteful of computer time, but it had the advantage that it could be coded and performed in one day. Extensive use could be made of the BOMM method of time series analysis (Bullard et al, 1963).

First the tides were removed by a numerical high-pass filter. Then records were "played" through 20 successive numerical low-pass filters, with cutoff frequencies (half-amplitude points) at 0.05, 0.10, · · · 0.95 Nyquists. Subtraction of successive records led to 20 bandpassed time series, each covering a frequency band of 0.05 Nyquists. The set of *amplitude* factors for the band 0.50 to 0.55 Nyquists is shown below:

Nyquists:	0.48	0.49	0.50	0.51	0.52	0.53	0.54	0.55	0.56	0.57
	−0.00	0.04	0.49	0.95	1.01	1.00	0.96	0.51	0.05	−0.00

so that there is only slight overlap between neighboring bands.

We denoted the series whose center frequency is f_i by $\zeta_i(t)$. Subsequently, we computed 20 high-passed series, $\zeta_{>i}(t)$, with cutoff frequencies at 0.025, 0.075, · · · , 0.975 Nyquists. We then formed the triple products, term-per-term,

$$b_{ij}(t) = \zeta_i(t)\zeta_{>i}(t)\zeta_j(t) \tag{25}$$

and computed the mean value, $\langle b_{ij} \rangle$. This contained two equal interactions between the frequencies f_i, f_j, f_{-i-j} and $-f_i$, $-f_j$, f_{i+j}, and the sum yielded twice the real part of each interaction. Thus, dividing by 2 and by

$$(\Delta f)^2 = (0.05 \text{ Nyquists})^2 = (0.05 \times 0.25)^2 \text{ cps}^2 = \left(\frac{1}{80}\right)^2 \text{ cps}^2$$

to obtain bispectral density, we had

$$\text{Re}\,[B'(f_i, f_j)] = \tfrac{1}{2} \times (80)^2 \langle b_{ij} \rangle.$$

6. COMPARISON BETWEEN THEORY AND EXPERIMENT

Figures 2 and 3 show the comparisons for two such cases. The experimental and theoretical results are contoured in neighboring octants; perfect agreement would call for symmetry about the 45° axis. The figures have the appearance of Rorschach ink blots,* thus indicating some degree of symmetry and some agreement between theory and experiment.

For the case of February 28 the power spectrum shows a predominant peak at 0.22 Nyquists (0.055 cps). The bispectra reveal a positive ridge for this frequency, highest at the 45° line and then tapering off. Theory and experiment are in satisfactory accord. The result implies a strong interaction of the peak with itself, which leads to the double frequency peak that is observed in the power spectrum, and appreciable interaction with all other frequencies. Interactions with frequencies equal to or greater than that of the peak leads to positive bispectra. This may be interpreted physically as the peaking of the wave crests with resultant harmonics that are in phase with the fundamental.†

Interaction of the principal peak with low frequencies leads to negative bispectra. We can interpret the theoretical value $B'(0.075, 0.225) = -3 \times 10^4$ cm^{3}sec^2 as due to the interaction between the main peak at 0.225 Nyquists and the side peak at 0.30 Nyquists in producing the difference frequency 0.075. The theoretical value $B'(0.025, 0.225) = -2 \times 10^5$ cm^3 sec^2, and the corresponding observed value $B'(0.225, 0.025) = -4 \times 10^4$ cm^3 sec^2 is probably the result of difference frequencies produced by interactions *within* the main peak, but the resolution is not adequate to separate this effect from the interaction between the two peaks. The negative sign implies that a group of high waves leads to a *lowering* of the sea level; this is in accord with the findings of Longuet-Higgins and Stewart (1962). Bernoulli's equation $p + \frac{1}{2}\rho u^2 =$ constant would lead to this effect of lowering the pressure with the increased mean-square velocity of the interacting waves. However, this is only part of the story, because comparable nonlinear interactions arise from the surface boundary conditions.

The theoretical calculation was carried out for several spreading factors, and it is found that the selected factor of 20° leads to better accord than either a much narrower or much broader beam. As we have already pointed out, we have a means of obtaining information on directional spreading (but not on the mean direction) from a record at a single point. Offhand, we might have thought it impossible to obtain two-dimensional information from a one-dimensional array!

The case of May 26 was purposely selected to portray a more complicated case. There are two peaks of comparable energy at 0.19 and 0.27 Nyquists, and the bispectrum reveals two ridges, as expected, separated by a bispectral

* This observation is due to Freeman Gilbert.

† *Peaking* of the wave crests leads to a *skewness* in the distribution of $\zeta(t)$ and a nonzero value of $\langle\zeta^3\rangle$. Assymetry of the wave crests prior to breaking (analogous to shock-wave formation) contributes nothing to $\langle\zeta^3\rangle$ but something to $\langle\zeta^4\rangle$ and is therefore a problem that involves the trispectral interaction between four frequencies.

trough down by a factor of 10. The interaction of the peaks with themselves and with one another lead to the following values:

Interacting frequencies, f_1, f_2:	0.19, 0.19	0.27, 0.27	0.19, 0.27	Nyquists
$B'(f_1, f_2)$ theory	8×10^4	7×10^4	4×10^4	$\text{cm}^3\ \text{sec}^2$
$B'(f_1, f_2)$ experiment	9×10^4	5×10^4	5×10^4	$\text{cm}^3\ \text{sec}^2$
The resulting sum frequencies	0.38	0.54	0.46	Nyquists

can barely be made out in the power spectrum. The difference frequency yields roughly -10^4 cm^3 sec^2 for both theory and experiment.

So far we have considered only the co-bispectrum. There is also a quadrature component, which can be derived by shifting $\zeta_{>i}(t)$ in (25) by 90°. A calculation equivalent to this operation indicated no important interaction in accordance with theory.

For the skewness we find

$$\langle \zeta^3 \rangle / \langle \zeta^2 \rangle^{3/2} = \frac{2.2 \times 10^3 \text{ cm}^3}{(3.3 \times 10^2 \text{ cm}^2)^{3/2}} = 0.352$$

$$= \frac{2.5 \times 10^2 \text{ cm}^3}{(1.6 \times 10^2 \text{ cm}^2)^{3/2}} = 0.124$$

for February 28 and May 26, respectively. Kinsman (1960, Table 5.11) has measured skewness for a large number of observations of surface elevation in water of 20-ft depth. He finds average values of 0.336 and 0.090 for July and November observations* of the same order as our results, and he points out that such values can lead to sizable corrections to the Gaussian distribution.

7. CONCLUSIONS

We find pleasing agreement between observed and derived bispectra, which demonstrates the validity of a perturbation scheme based on the Navier-Stokes equation—but who ever doubted it in the first place? So, in a sense, the result is disappointing, for we have learned little new (except for the information on directional distribution). However, we have gained some experience with higher order spectra that will prove rewarding when applied to processes for which the interaction theory is not known or is thought to be known but is found to be inapplicable.

For many problems a step further to the trispectrum will probably be necessary. For example, the simple question, is immediately suggested by our analysis, "What fraction of the spectral density at the low frequencies (surf beat) is the result of nonlinear interaction between higher frequencies?" can be answered only by a third-order analysis.

* Kinsman's definition of the skewness differs from ours by a factor of 2.

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