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Review Article

Bit and Power Allocation in Constrained Multicarrier Systems: The Single-User Case

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Multicarrier modulation is a powerful transmission technique that provides improved performance in various communication fields. A fundamental topic of multicarrier communication systems is the bit and power loading, which is addressed in this article as a constrained multivariable nonlinear optimization problem. In particular, we present the main classes of loading problems, namely, rate maximization and margin maximization, and we discuss their optimal solutions for the single-user case. Initially, the classical water-filling solution subject to a total power constraint is presented using the Lagrange multipliers optimization approach. Next, the peak-power constraint is included and the concept of cup-limited waterfilling is introduced. The loading problem is also addressed subject to the integer-bit restriction and the optimal discrete solution is examined using combinatorial optimization methods. Furthermore, we investigate the duality conditions of the rate maximization and margin maximization problems and we highlight various ideas for low-complexity loading algorithms. This article surveys and reviews existing results on resource allocation in constrained multicarrier systems and presents new trends in this area.

1. INTRODUCTION

Multicarrier modulation (MCM) [1, 2] is well recognized as an efficient and powerful transmission technique that has been adopted by various standard committees for both wireless [3–6] and wireline [7–9] systems. MCM provides important benefits including, among others, efficient bandwidth optimization, enhanced spectrum utilization, low equalization complexity, and multi-user potentiality. Moreover, MCM is widely used in new application fields, such as powerline communications (PLC) [10, 11], and wireless local area networks (WLANs) [12–14] due to its recognized value to confront various channel impairments, including frequency selectivity, intersymbol interference (ISI), and impulse noise.

The principle of MCM is the spectrum decomposition into a set of orthogonal narrowband subchannels by utilizing complex exponentials as information-bearing carriers. Two important MCM techniques have widespread use: orthogonal frequency-division multiplexing (OFDM) [14] mainly employed in wireless applications and discrete multitone (DMT) [15] used in wireline systems. Both OFDM and DMT employ the fast Fourier transform (FFT) for spectrum decomposition, hence data transmission is performed in blocks. In order to avoid ISI and to preserve orthogonality, a cyclic prefix is introduced at the expense of a data rate loss [16]. Using the cyclic prefix, the system carriers can be viewed as separate independent channels, on which different information rates can be transferred by utilizing constellations of different sizes.

The allocation of bits and power to the subchannels is a fundamental aspect in the design of multicarrier systems. The allocation problem is known as *bit and power loading* and is based on *loading algorithms*, which aim to distribute the total number of bits and the available power over the subchannels in an optimal way that maximizes performance and preserves a target quality of service. In fact, the bit and power loading is a constraint optimization problem and generally two cases are of practical interest [17]: *rate maximization* (RM) and *margin maximization* (MM), where the objective is the maximization of the achievable data rate or the achievable system margin, respectively. In fact, margin maximization is equivalent to power minimization given a target data rate. The loading problem defines a set of constraints imposed either by recommendation rules and specifications [18], or by practical limitations and implementation issues [9]. Such constraints include total available power budget, power spectral density (PSD) mask, integer number of bits per subcarrier, and so forth.

Adaptive loading is possible only when channel state information (CSI) is known both at the transmitter and the receiver. In wireless applications, the channel is time-varying and therefore OFDM systems usually employ the same constellation in all carriers. On the other hand, the wireline channel is treated either as almost constant or as slow time-varying, and therefore CSI can be sent to the transmitter by a feedback link. Thus, in DMT applications, the utilization of different signal constellations per subchannel by adaptive bit and power loading is of great importance, and as the number of subchannels required in commercial applications [9, 17] increases, the development of efficient loading algorithms is a challenging task.

The literature contains several loading algorithms proposed for DMT-based systems. These algorithms consider either the RM or the MM problem, and two general classes can be distinguished. The first class of loading algorithms treats the allocation problem using numerical methods that employ Lagrange optimization, which in general results in real numbers for optimum bit allocation [19–22]. However, for practical applications, the number of bits per subchannel is restricted to integer values, and thus, the above algorithms include a final suboptimum bit-rounding step. The integer-bit constraint imposes a combinatorial structure in the loading optimization problem. The second class of loading algorithms employs discrete greedy-type methods in order to obtain the optimum integer-bit allocation results [23–27].

This article aims at providing a tutorial survey on the bit and power loading in constrained multicarrier systems and at reviewing the most popular results on the loading algorithms for the RM and MM problems. We examine the single-user communications scenario, that is, a point-to-point link between a DMT-based transmitter and a receiver. We start out with a short introductory overview of the multicarrier basics. Then, the loading problem is considered only subject to a total power constraint and the classical water-filling solution is discussed using the Lagrange multipliers optimization approach. Next, the peak-power constraint is included and the concept of cup-limited water-filling is introduced. The loading problem is also addressed subject to the integer-bit restriction and the optimal discrete solution is examined using combinatorial optimization methods. Moreover, we investigate the duality conditions of the RM and MM loading problems and we highlight some ideas for low-complexity loading algorithms. This article aims to provide the basic knowledge for more complex and challenging problems of bit and power allocation in constrained multicarrier systems under the multi-user context.

2. MULTICARRIER LOADING

MCM decomposes the channel spectrum into a set of N orthogonal narrowband subchannels of equal bandwidths [1]. For each subchannel i, where $1 \leq i \leq N$, a rate function $\mathcal{R}(P_i)$ is defined, which gives the number of bits b_i that can be transmitted using power P_i. The rate function depends on the maximum probability of error that can be tolerated and the applied modulation and coding schemes, which we assume to be shared among all the subchannels. In addition, we assume the existence of the inverse function $\mathcal{R}^{-1}(b_i)$, namely, power function, which gives the power P_i required for the transmission of b_i bits. We consider practical QAM-coded MCM, where the rate function is given by the following logarithmic expression in bits per two-dimensional symbol:

$$b_i = \log_2\left(1 + \frac{P_i \cdot g_i}{\Gamma}\right), \quad (1)$$

where $g_i = |H_i|^2/\mathcal{N}_i$ is the gain-to-noise ratio of subchannel i, H_i is the channel frequency response, $\mathcal{N}_i$ is the noise power, and Γ is the SNR gap expressing the loss, in terms of SNR, between the actual rate b_i conveyed by the used transmission scheme and the theoretical capacity achieved for $\Gamma = 1$ (0 dB).

The SNR gap is calculated according to the "gap-approximation" analysis [15, 28], based on the target error probability $\mathcal{P}_e$, the applied coding gain γ_c, and the system performance margin γ_m. Some useful comments on the validity limits of the "gap-approximation" can be found also in [21]. When QAM transmission is employed, we can write

$$\Gamma = \frac{1}{3} \cdot \left[Q^{-1}\left(\frac{\mathcal{P}_e}{4}\right)\right]^2 \cdot \frac{\gamma_m}{\gamma_c}, \quad (2)$$

where Q^{-1} is the inverse of the well-known Q-function defined as

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2}\, dy. \quad (3)$$

Note that the margin γ_m in (2) expresses the SNR degradation immunity, which the system designer tries to achieve, so that the MCM performance is maintained for the desired probability of error. The higher the system margin is, the more power is required for a given probability of error. On the other hand, as the coding gain γ_c increases, the transmission rate approaches the system capacity.

Since OFDM and DMT systems usually require the same error rate for all subchannels [15], in the rest of this article we will consider that Γ is embedded in the g_is, that is, $g_i \equiv |H_i|^2/(\mathcal{N}_i \cdot \Gamma)$ for $1 \leq i \leq N$. From (1), it is clear that the power function is defined by the exponential expression

$$P_i = \frac{2^{b_i} - 1}{g_i} \quad (4)$$

while the total power and the total data rate of the multicarrier system are, respectively,

$$P = \sum_{i=1}^{N} P_i, \qquad B = \sum_{i=1}^{N} b_i. \quad (5)$$

The loading problem aims at determining the optimal distribution of the available power in all subchannels. Using the rate and power functions, (1) and (4), respectively, the optimal distribution of the available power is transformed into an optimal distribution of the achievable data rate over the subchannels, and vice versa. The loading problem is formulated as a multivariable constraint optimization problem. The optimization objective is the maximization of the rate function (RM case), or the minimization of the power function (MM case), subject to a set of constraint functions that reflect system limitations and restrictions. In the following sections, we formulate the loading problem, initially only subject to the total power budget constraint, and afterwards when the peak-power restriction per subchannel is also included.

3. TOTAL POWER-CONSTRAINED LOADING

Let P_{budget} denote the total power budget and B_{target} denote the desired data rate. The RM and MM loading problems are formulated as follows.

RM loading problem:

$$\begin{aligned} &\max \sum_{i=1}^{N} \log_2 \left(1 + P_i \cdot g_i\right), \\ &\text{subject to} \sum_{i=1}^{N} P_i \leq P_{\text{budget}}, \\ &P_i \geq 0, \quad \forall i : 1 \leq i \leq N. \end{aligned} \tag{6}$$

MM loading problem:

$$\begin{aligned} &\min \sum_{i=1}^{N} \frac{2^{b_i} - 1}{g_i}, \\ &\text{subject to} \sum_{i=1}^{N} b_i = B_{\text{target}}, \\ &b_i \geq 0, \quad \forall i : 1 \leq i \leq N. \end{aligned} \tag{7}$$

We observe that the logarithmic expression in (6) is a strictly increasing and concave function of P_i, while the exponential expression in (7) is a strictly increasing and convex function of b_i [29]. As a result, we recognize that both RM and MM belong to the class of convex optimization problems with convex constraint sets, and therefore a unique global solution exists. Moreover, we observe that both problems are nonlinear. The optimal solution is calculated by forming the corresponding Lagrangian function and applying the Kuhn-Tucker conditions [30].

From the MM problem formulation in (7), it is clear that margin maximization is equivalent to power minimization. In fact, given a target data rate, the MM objective is to determine (from all possible bit allocations that correspond to a data rate equal to the target one) the optimum bit allocation, which requires the least total power. The additional power, that is, the difference between the available power budget and the total power of the optimum bit allocation, is used in order to increase the system margin γ_m in (2).

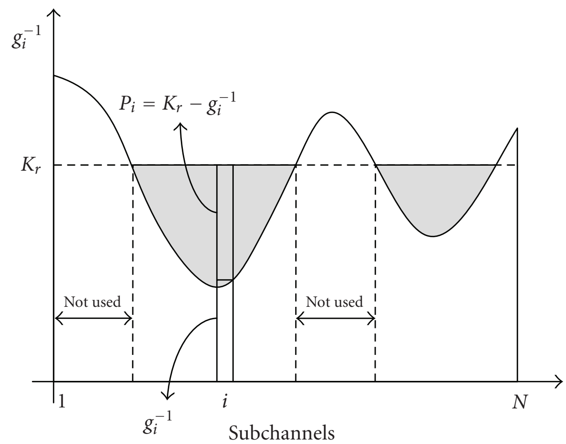

Figure 1: Water-filling rate maximization. The shaded area represents the total available power.

Note that for the MM problem, we assume that the available power budget is sufficient to support the desired target rate, otherwise the problem in (7) has no solution. Based on this assumption, a total power budget constraint is not included in (7).

3.1. Rate maximization water-filling

The optimal solution to the RM problem is given by the following relation, which is known as the *water-filling* formula:

$$P_i = \begin{cases} K_r - g_i^{-1} & \text{for all used subchannels}, \\ 0 & \text{otherwise}, \end{cases} \tag{8}$$

where K_r is a constant value depending on the total power budget.

The solution in (8) can be best described using Figure 1. The spectrum can be considered as a vessel and the shape of the bottom of this vessel is determined by the inverse of g_i values. We can say that the available power is *poured* over the spectrum vessel, so that the subchannels covered by the *water-level* K_r are assigned power, while the remaining subchannels are not used at all (water-filling is also referred to as *water-pouring*). Assuming that the subchannels are sorted,[1] the water-level K_r is

$$K_r = \frac{P_{\text{budget}}}{M_r} + \frac{1}{M_r} \sum_{i=1}^{M_r} g_i^{-1}, \tag{9}$$

[1] The subchannels are said to be sorted (in descending order) when $g_1 \geq g_i \geq g_N$ for $1 \leq i \leq N$.

where M_r is the total number of the used subchannels determined according to the following criteria:

$$\frac{P_{\text{budget}}}{M_r} + \frac{1}{M_r}\sum_{i=1}^{M_r} g_i^{-1} \geq g_{M_r}^{-1},$$
$$\frac{P_{\text{budget}}}{M_r+1} + \frac{1}{M_r+1}\sum_{i=1}^{M_r+1} g_i^{-1} < g_{M_r+1}^{-1}. \tag{10}$$

An iterative algorithm that determines the water-filling RM solution by using an initial sorting of the subchannels' gain-to-noise ratio values is described in [20]. When the subchannels are sorted, the objective of the loading algorithm is to determine the *cut-off* subchannel M_r and the constant K_r. Sorting is not a trivial task when the number of subchannels is large. In general, this task dominates the computational complexity of all practical algorithms for water-filling, so the complexity is $\mathcal{O}(N \log_2 N)$.

The optimum bit allocation is derived by (1) and (8), which results in the following compact formula:

$$b_i = \begin{cases} \log_2(K_r g_i) & \forall i: 1 \leq i \leq M_r, \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

while the total data rate is

$$B = \sum_{i=1}^{N} b_i = M_r \log_2(K_r) - \sum_{i=1}^{M_r} \log_2(g_i^{-1}). \tag{12}$$

Remark 1. By combining (8) and (9), we derive that the optimal solution uses all the available power budget, that is, the total power constraint in (6) is met with equality. Moreover, we observe that as more power budget is available, the water-level in (9) becomes higher and as a consequence, more subchannels may be *turned on*, as long as (10) implies a higher value for M_r. Therefore, a higher power budget corresponds to a higher water-level, which generally results in the utilization of more subchannels and thus in a higher data rate.

Remark 2. From (8) and (9), we can write the optimum power allocation using the following expression:

$$P_i = \frac{P_{\text{budget}}}{M_r} + \left(\frac{1}{M_r}\sum_{m=1}^{M_r} g_m^{-1} - g_i^{-1}\right), \quad \forall i: 1 \leq i \leq M_r. \tag{13}$$

The first term is a constant power portion, while the second term is the distance between the mean of the inverse gain-to-noise ratios of all used subchannels and the g_i^{-1} of each subchannel. Figure 2 illustrates this remark, where the subchannels are sorted. Observe that for subchannel i the distance of g_i^{-1} to the mean value is positive, while for subchannel j, the distance is negative.

Remark 3. Figure 2 also illustrates a characteristic feature of the water-filling allocation strategy: water-filling allocates more power to the strongest subchannels.

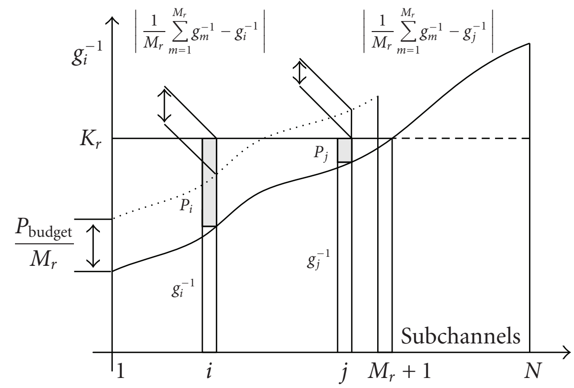

Figure 2: Graphical representation of (13).

3.2. *Margin maximization water-filling*

Using the Lagrange multipliers method and applying the Kuhn-Tucker conditions, we can derive the optimal solution to the MM problem in (7) as follows:

$$b_i = \begin{cases} K_m - \log_2(g_i^{-1}) & \text{for all used subchannels,} \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

where K_m is constant and depends on the target data rate.

Assuming that the subchannels are sorted in a descending order, K_m is given by

$$K_m = \frac{B_{\text{target}}}{M_m} + \frac{1}{M_m}\sum_{i=1}^{M_m} \log_2(g_i^{-1}), \tag{15}$$

where M_m is the total number of used subchannels determined according to the following criteria:

$$\frac{B_{\text{target}}}{M_m} + \frac{1}{M_m}\sum_{i=1}^{M_m} \log_2(g_i^{-1}) \geq \log_2(g_{M_m}^{-1}),$$
$$\frac{B_{\text{target}}}{M_m+1} + \frac{1}{M_m+1}\sum_{i=1}^{M_m+1} \log_2(g_i^{-1}) < \log_2(g_{M_m+1}^{-1}). \tag{16}$$

The analogy between (9) and (10) of the RM problem with (15) and (16) will be evident, as soon as we calculate the power of each subchannel allocated with b_i bits according to (14). Using (4), we get

$$P_i = \begin{cases} 2^{K_m} - g_i^{-1} & \forall i: 1 \leq i \leq M_m, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

In (17), we observe that the optimum bit solution to the MM problem results in a power distribution that follows a water-filling power allocation as in the RM problem. Therefore, a power distribution, similar to the one shown in Figure 1, holds also for the MM problem. In this case, the constant water-level is equal to

$$2^{K_m} = 2^{B_{\text{target}}/M_m} \prod_{i=1}^{M_m} (g_i^{-1})^{1/M_m} \tag{18}$$

while the total power is given by

$$P = \sum_{i=1}^{N} P_i = M_m 2^{K_m} - \sum_{i=1}^{M_m} g_i^{-1}. \quad (19)$$

Remark 4. We observe in (18), that the higher the target rate, the higher the water-level and consequently more subchannels may be used, as long as (16) implies a higher value for M_m. As a result, a higher target rate requires a higher total power consumption.

3.3. Duality conditions between RM and MM problems

The RM and MM problems admit a unique water-filling solution. The following proposition holds.

Proposition 1. *Let (b_i^w, P_i^w), for $1 \le i \le N$, be a water-filling bit and power allocation, where $\sum_{i=1}^{N} b_i^w = B_w$ and $\sum_{i=1}^{N} P_i^w = P_w$. Then,*

$$B_w = \max_{P_w} \sum_{i=1}^{N} b_i, \qquad P_w = \min_{B_w} \sum_{i=1}^{N} P_i. \quad (20)$$

Proof. We have shown in Section 3.1, that a water-filling power allocation provides the unique solution that maximizes the data rate subject to a total power constraint. In fact, the whole power budget is consumed (see Remark 1). Therefore, any other allocation (b_i, P_i) with $\sum_{i=1}^{N} P_i = P_w$ results in a total rate of $\sum_{i=1}^{N} b_i < B_w$. Therefore, the first part of (20) is true.

Moreover, we have shown in Section 3.2, that a water-filling bit allocation provides the unique solution that minimizes the total power subject to a target rate constraint. Therefore, any other allocation (b_i, P_i) with $\sum_{i=1}^{N} b_i = B_w$ results in a total power of $\sum_{i=1}^{N} P_i > P_w$. Consequently, the second part of (20) is also true. □

From the analytical expressions derived for the RM and MM problems, there exists a duality between the RM and MM problems under specific conditions. We are now in the position to define these conditions in the form of the following theorem.

Theorem 1. *Let (b_i^r, P_i^r), for $1 \le i \le N$, be the solution to the RM problem, where $\sum_{i=1}^{N} b_i^r = B_r$. Then, (b_i^r, P_i^r) is also the solution to the MM problem with $B_{target} = B_r$. Equivalently, let (b_i^m, P_i^m), for $1 \le i \le N$, be the solution to the MM problem, where $\sum_{i=1}^{N} P_i^m = P_m$. Then, (b_i^m, P_i^m) is also the solution to the RM problem with $P_{budget} = P_m$.*

Proof. Using Proposition 1, for the RM solution (b_i^r, P_i^r), we can write

$$P_r = \sum_{i=1}^{N} P_i^r = \min_{B_r} \sum_{i=1}^{N} P_i \quad (21)$$

which implies that (b_i^r, P_i^r) is also the solution to the MM problem, when $B_{\text{target}} = B_r$.

Similarly, for the MM solution (b_i^m, P_i^m), we can write

$$B_m = \sum_{i=1}^{N} b_i^m = \max_{P_m} \sum_{i=1}^{N} b_i \quad (22)$$

which implies that (b_i^m, P_i^m) is also the solution to the RM problem, when $P_{\text{budget}} = P_m$. □

4. TOTAL POWER AND PEAK-POWER CONSTRAINED LOADING

When introducing the peak-power constraint, the optimization problem becomes more complicated. Let $\overline{P}_i$, for $1 \le i \le N$, denote the maximum allowable power per subchannel. In multicarrier systems, a power spectral density (PSD) mask constraint is usually imposed by regulatory rules in order to control the level of interference into other communication systems operating in the neighborhood, for example, [18]. The RM and MM problems are formulated as follows.

RM loading problem:

$$\begin{aligned} &\max \sum_{i=1}^{N} \log_2\left(1 + P_i \cdot g_i\right), \\ &\text{subject to} \sum_{i=1}^{N} P_i \le P_{\text{budget}}, \\ &0 \le P_i \le \overline{P}_i, \qquad \forall i : 1 \le i \le N. \end{aligned} \quad (23)$$

MM loading problem:

$$\begin{aligned} &\min \sum_{i=1}^{N} \frac{2^{b_i} - 1}{g_i}, \\ &\text{subject to} \sum_{i=1}^{N} b_i = B_{\text{target}}, \\ &0 \le b_i \le \overline{b}_i, \qquad \forall i : 1 \le i \le N. \end{aligned} \quad (24)$$

In the RM problem, we observe that the peak-power constraint upper bounds the possible power allocation in each subchannel. In the MM problem, the peak-power constraint is transformed into a maximum bit allocation constraint, denoted as $\overline{b}_i$ for $1 \le i \le N$, which upper bounds the possible bit allocation in each subchannel and is defined by

$$\overline{b}_i = \log_2\left(1 + \overline{P}_i \cdot g_i\right). \quad (25)$$

The RM and MM problems in (23) and (24) also belong to the class of convex optimization problems with convex constraint sets, and therefore a unique global solution exists.

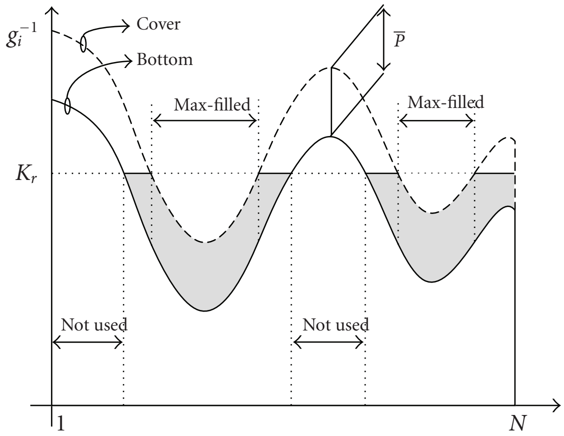

Figure 3: The concept of "cap-limited" water-filling.

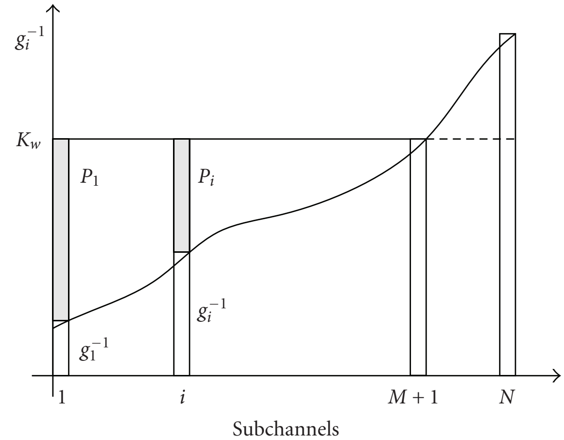

Figure 4: The optimal water-filling.

(1) sort g_is in descending order
(2) set $j = 1$
(3) apply WATER-FILLING (8)–(10), over subchannels $j, \dots, N$
(4) if $P_j > \overline{P}_j$ then
(5) set $P_j = \overline{P}_j$
(6) reduce the available power $P_{\text{budget}} = P_{\text{budget}} - \overline{P}_j$
(7) update $j = j + 1$
(8) go to step (3)
(9) end if

Algorithm 1: Cap-limited water-filling.

4.1. *Rate maximization water-filling*

By using the Lagrange multipliers approach and applying the Kuhn-Tucker conditions, we can derive the optimal solution to the RM problem in (23) as follows[2]:

$$P_i = \begin{cases} \overline{P}_i & \text{if } \sum_{i=1}^{N} \overline{P}_i \le P_{\text{budget}}, \\ [K_r - g_i^{-1}]_0^{\overline{P}_i} & \text{if } \sum_{i=1}^{N} \overline{P}_i > P_{\text{budget}}, \end{cases} \quad (26)$$

where K_r is a constant and is determined by the solution to the following nonlinear equation:

$$\sum_{i=1}^{N} [K_r - g_i^{-1}]_0^{\overline{P}_i} = P_{\text{budget}}. \quad (27)$$

The RM solution in (26) is again water-filling, however in this case, the spectrum vessel has a limited depth of $\overline{P}_i$ and is covered by a cap. When $\overline{P}_i = \overline{P}$ for all subchannels, then the shape of the cap is identical to the vessel's bottom, that is, the inverse of the g_is. The concept of the "cap-limited" water-filling is illustrated in Figure 3 subject to a common PSD mask for all subchannels.

In order to obtain the solution in (26), we need to determine the constant K_r. In Section 3.1, an iterative algorithm for the calculation of the water-level of the total power constrained RM was presented. When the peak-power constraint is introduced, the RM problem in (23) can be treated using an iterative water-filling process [21], which is described using the pseudocode of Algorithm 1.

Algorithm 1 is optimal, however its direct implementation is not efficient and presents $\mathcal{O}(N^2)$ complexity. In order to overcome such a high computational load, an iterative algorithm of reduced complexity can be constructed by exploiting the fact that in every new iteration of Algorithm 1, the participating subchannels are allocated more power with respect to the previous iteration.

First, consider the optimal water-filling solution in Section 3.1, which is described in Figure 4, where the subchannels are sorted. Denoting as $\mathbf{P}_N = [P_1, P_2, \dots, P_N]$ the optimum N-point water-filling power vector, $\mathbf{P}_N$ satisfies the following set of equations:

$$\begin{gathered} P_1 + \cdots + P_M = P_{\text{budget}}, \\ P_i > 0, \quad \forall i : 1 \le i \le M, \\ P_i + g_i^{-1} = K_w, \quad \forall i : 1 \le i \le M, \end{gathered} \quad (28)$$

where K_w is the water-level and subchannels from $M+1$ to N are turned off, that is, they are loaded with zero power, and the following proposition holds.

Proposition 2. *Given the sorted water-filling power allocation vector $\mathbf{P}_N$, if one removes subchannels $1, \dots, L$ and reduces P_{budget} by $\sum_{i=1}^{L} P_i$, then the new optimal water-filling solution is the $(N-L)$-point power vector $\mathbf{P}_{N-L} = \mathbf{P}_N - \{P_1, \dots, P_L\} = [P_{L+1}, \dots, P_N]$.*

[2] The following notation is used, where x, a, and c are real numbers with $a > c$: $[x]_c^a = \begin{cases} a, & x \ge a, \\ x, & c < x < a, \\ c, & x \le c. \end{cases}$

```
(1)  sort g_i s in descending order
(2)  set j = 1
(3)  apply WATER-FILLING (8)–(10),
     over subchannels j,...,N
(4)  set M̄ = j
(5)  if P_j > P̄_j then
(6)      update M̄ = M̄ + 1
(7)      if P_M̄ > P̄_M̄ then
(8)          go to step (6)
(9)      else
(10)         set P_i = P̄_i, for i = j,...,M̄ − 1
(11)         reduce the available power
             P_budget = P_budget − Σ_{i=j}^{M̄−1} P̄_i
(12)         set j = M̄
(13)         go to step (3)
(14)     end if
(15) end if
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Algorithm 2: Low-complexity cap-limited water-filling.

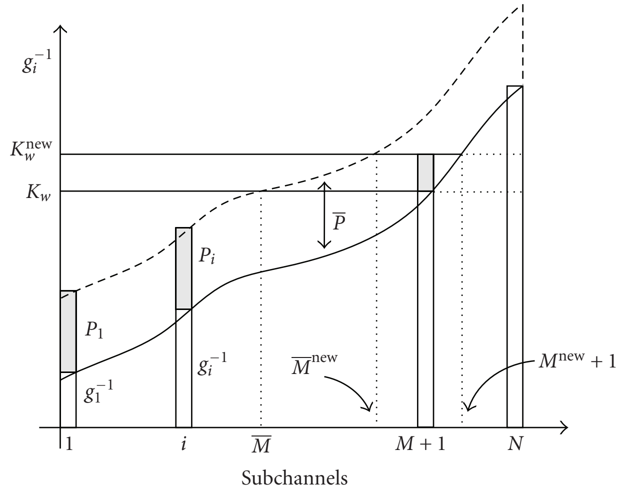


Figure 5: Graphical representation of iterative water-filling.

Proof. We prove for $L = 1$. Let $\mathbf{P}'_{N-1} = [P'_2, \ldots, P'_N]$ be the optimum vector. Then, $\mathbf{P}'_{N-1}$ should satisfy (28):

$$P'_2 + \cdots + P'_{M'} = P'_{\text{budget}},$$
$$P'_i > 0, \quad \forall i : 2 \le i \le M', \tag{29}$$
$$P'_i + g_i^{-1} = K'_w, \quad \forall i : 2 \le i \le M',$$

where $P'_{\text{budget}} = P_{\text{budget}} - P_1$, $K'_w = (P'_{budget} + \sum_{i=2}^{M'} g_i^{-1})/(M'-1)$, and subchannels from $M'+1$ to N are turned off.

For $M' = M$, the constant K'_w becomes

$$\begin{aligned} K'_w &= \frac{1}{M-1}\left(P_{\text{budget}} - P_1 + \sum_{i=2}^{M} g_i^{-1}\right) \\ &= \frac{1}{M-1}\left[\left(P_{\text{budget}} + \sum_{i=1}^{M} g_i^{-1}\right) - \left(g_1^{-1} + P_1\right)\right] \\ &= \frac{1}{M-1}(M \cdot K_w - K_w) = K_w. \end{aligned} \tag{30}$$

From (28) and (30), we derive that the power vector $\mathbf{P}_{N-1} = [P_2, \ldots, P_N]$ satisfies (29) and therefore $\mathbf{P}_{N-1}$ is the optimal vector. The proof for $L > 1$ is similar. □

As suggested by Algorithm 1, if the power allocated to subchannel i (staring from the one with the highest g_i) exceeds $\overline{P}_i$, then we set $P_i = \overline{P}_i$, reduce P_{budget} by $\overline{P}_i$, exclude subchannel i from the optimization problem, and perform water-filling to the remaining subchannels. Since P_{budget} is reduced by an amount of power less than the optimal power assigned by the previous water-filling, then according to Proposition 2 and Remark 1, the new solution has higher K_w and additional subchannels may be turned on. As a result, all subchannels participating in the next water-filling will be assigned additional power. Based on this remark, the new optimal algorithm is described by the pseudocode in Algorithm 2.

Algorithm 2 is explained using Figure 5 subject to common PSD mask for all subchannels. Given the initial water-filling solution with cut-off subchannel index M and water-level K_w, the algorithm determines the first subchannel, denoted as $\overline{M}$, where the power assignment does not violate $\overline{P}$. Then, it upper bounds all subchannels from 1 to $\overline{M}-1$ with $\overline{P}$ and reduces the power budget by the total power assigned so far. At the next step, the algorithm proceeds to successive water-filling over the subchannels ranging from $\overline{M}$ to N. The new water-filling solution determines a new higher water-level, K_w^{new}, corresponding to new subchannel indexes $\overline{M}^{\text{new}} > \overline{M}$ and $M^{\text{new}} \ge M$. This procedure is repeated until the water-filling allocation does not violate $\overline{P}$ in any of the subchannels involved in the new iteration.

The complexity improvement of the iterative water-filling scheme described by Algorithm 2 compared with Algorithm 1 depends on the total number of iterations that water-filling has executed. If $L \le N$ is the number of iterations, then the computational complexity of Algorithm 2 is $\mathcal{O}(N(L + \log_2 N))$. The lower is L compared to N, the higher is the computational complexity improvement. In [21], a suboptimum algorithm for the RM problem in (23) is described that uses an iterative search-secant method to determine the root of (27), by noting that (27) admits a root when $\sum_{i=1}^{N} \overline{P}_i > P_{\text{budget}}$. The search-secant process is subject to a tolerance variable that affects the speed of convergence, as well as the accuracy of the final result. Generally, there is a tradeoff between the speed of convergence and accuracy. The method presents a computational complexity that grows linearly with $L'N$, where L' is the number of the search-secant iterations.

Remark 5. Similar to Remark 1, we observe from (26) and (27) that the optimal "cap-limited" water-filling solution consumes the total available power budget.

4.2. Margin maximization water-filling

In order to obtain the optimal solution to the peak-power constrained MM problem, we will use the duality between the RM and MM problems developed in Section 3.3. We have shown in Section 3.2 that the optimal bit solution for the MM problem under a total power constraint results in a power allocation, which follows a water-filling distribution. Given a peak-power constraint, this power allocation should also follow the "cap-limited" water-filling concept of Figure 3. The optimal bit solution to (24) is therefore given by

$$b_i = \left[K_m - \log_2\left(g_i^{-1}\right)\right]_0^{\overline{b}_i}, \quad \forall i : 1 \le i \le N, \qquad (31)$$

where K_m is the solution to the following nonlinear equation:

$$\sum_{i=1}^{N}\left[K_m - \log_2\left(g_i^{-1}\right)\right]_0^{\overline{b}_i} = B_{\text{target}}. \qquad (32)$$

It can be easily verified that Theorem 1 applies also for the total and peak-power constrained RM and MM problems.

5. INTEGER-BIT CONSTRAINED LOADING

The Lagrangian methods described in the previous sections provide the optimal loading solutions, where generally the bit assignment in each subchannel takes real values. However, due to implementation constraints, only integer bit values are of practical interest, that is, design of realistic constellation encoders and decoders. As a consequence, the proposed Lagrangian algorithms in the literature include a final suboptimal bit-rounding step with appropriate power scaling to preserve the power budget and target error rate constraints.

The integer-bit constrained loading problem, also referred to as discrete loading, belongs to the class of combinatorial optimization problems. The RM and MM formulations of the previous sections apply here, along with the additional integer-bit constraint: $b_i \in \mathbb{Z}_+$ for $1 \le i \le N$.

Remark 6. The monotonicity and concave nature of the rate function in (1), along with the monotonicity and convex nature of the power function in (4), as well as of the corresponding discrete incremental (33) and decremental (34) power cost functions defined below, guarantee the existence of a unique optimum solution for each of the RM and MM discrete loading problems based on appropriate greedy algorithms. The optimality is addressed in [26, 31, 32], using the matroid theory.

5.1. Optimum greedy algorithms

The solution to the integer-bit loading problem is provided using a greedy algorithm, which defines an appropriate bit allocation cost function and iteratively assigns one bit at a time to the least cost-expensive subchannel. In general, a greedy algorithm is characterized by the following two properties [33]. First, at each step, the algorithm always moves its operating point along the direction that guarantees the largest increment (decrement) to the assigned objective function to be maximized (minimized). Second, a greedy algorithm proceeds only in a forward way, that is, it never tracks back. Two greedy loading methods are used: the bit-filling [23, 31] and the bit-removal [25, 32].

Considering that subchannel i carries b_i bits, the power needed to transmit one more bit in this subchannel is given by

$$\Delta P_i^{+}(b_i) = \frac{2^{b_i}}{g_i}, \quad \forall b_i : 0 \le b_i < \overline{b}_i \qquad (33)$$

while the power saved by removing one bit from this subchannel is given by

$$\Delta P_i^{-}(b_i) = \frac{2^{(b_i-1)}}{g_i}, \quad \forall b_i : 0 < b_i \le \overline{b}_i \qquad (34)$$

and the maximum[3] number of bits that can be assigned to each subchannel is

$$\overline{b}_i = \left\lfloor \log_2\left(1 + \overline{P}_i \cdot g_i\right)\right\rfloor. \qquad (35)$$

The incremental power in (33) constitutes the cost function of the bit-filling process, while the decremental power in (34) constitutes the cost function of the bit-removal process. In particular, the bit-filling algorithm starts from an initial all-zero bit allocation, $b_i = 0$ for $1 \le i \le N$, and then adds one bit at a time to the subchannel that requires the minimum additional power until the total power budget is consumed (RM case) or the target rate is achieved (MM case). On the other hand, the bit-removal algorithm starts from an initial maximum bit allocation, $b_i = \overline{b}_i$ for $1 \le i \le N$, and then removes one bit at a time from the subchannel that saves the maximum power until the target rate is achieved (MM case). Note that if $\sum_{i=1}^{N}\overline{P}_i \le P_{\text{budget}}$, the maximum bit allocation $b_i = \overline{b}_i$ used initially by the bit-removal algorithm, is the direct solution to the RM case. In Appendix A, the following theorem is proved.

Theorem 2. *Given a target rate B_{target}, the bit-filling and bit-removal algorithms result in the same optimum bit and power allocation.*

The following remarks are also in order.

Remark 7. For the nondiscrete RM problems formulated in the previous sections, we have noted that the optimal solution results in the consumption of the total available power budget. In the discrete RM problem, however, the optimum integer-bit solution results in total power that is generally less or equal to the power budget.

Remark 8. Although bit-filling and bit-removal provide the same solution, the computational load associated with each

[3] In practice, a maximum size in the embedded constellation is also imposed [7]. Let $b_{\max}$ denote the maximum number of bits that can be allocated in a subchannel. Then, (35) is written as $\overline{b}_i = \min(b_{\max}, \lfloor\log_2(1 + \overline{P}_i \cdot g_i)\rfloor)$.

method mainly depends on the target data rate. The complexity of the bit-filling is $\mathcal{O}(B_{\text{target}}N)$, while the complexity of the bit-removal is $\mathcal{O}((B_{\max} - B_{\text{target}})N)$, where $B_{\max}$ is the data rate corresponding to the $\overline{b}_i$ bit-profile. If B_{target} is close to $B_{\max}$, then bit-removal converges faster.

Remark 9. If bit-filling is left free to proceed above B_{target}, by adding one bit at a time to the least power cost-expensive subchannel, then it will terminate at the $\overline{b}_i$ allocation.

Also for the discrete loading RM and MM problems, there exist exact conditions for their equivalence, as in the water-filling case. Theorem 1 developed in Section 3.3 holds also for the case of discrete loading, where the only difference is that due to the integer-bit allocation, the MM solution under the duality conditions is also the RM solution with $P_{\text{budget}} \geq P_m$ (see Theorem 1 for details). The proof is given in Appendix B. Another approach is provided in [34].

5.2. Efficient integer-bit allocation profiles

The high computational load of the greedy bit-filling and bit-removal algorithms is an important disadvantage for practical systems with large number of subchannels and high data rate demands. In [27], an efficient discrete bit allocation profile was developed by recognizing that the order of the subchannels, which participate in the single-bit incremental process of bit-filling, is specific and includes a characteristic circular repetition.

The characteristic bit allocation profile is calculated as follows:

$$b_i' = \begin{cases} \lfloor \log_2(k_{i_{\min}}) \rfloor + 1, & i = i_{\max}, \\ \lfloor \log_2(k_{i_{\min}}) \rfloor - \lfloor \log_2(k_i) \rfloor & \text{otherwise,} \end{cases} \tag{36}$$

where $i_{\max} = \arg\max\{g_i\}$, $i_{\min} = \arg\min\{g_i\}$, and $k_i = g_{i_{\max}}/g_i$ for $1 \leq i \leq N$.

The allocation b_i' depends only on the system g_i values and presents an optimum bit allocation profile of a continuous greedy bit-filling process, where any total power or peak-power constraints are at the moment ignored. In other words, if bit-filling is continuously applied, then it will reach the allocation b_i' after $\sum_{i=1}^{N} b_i'$ steps, where $\sum_{i=1}^{N} b_i'$ is the data rate that corresponds to the b_i' profile. From (36), we observe that $b_i' = 0$ for $i = i_{\min}$. Depending on the value of k_i, b_i' may be zero for other subchannels as well. Assuming that the subchannels are sorted, that is, $i_{\max} = 1$ and $i_{\min} = N$, the following remarks are in order.

Remark 10. Given allocation (36) and assuming that bit-filling is applied, then one bit has to be added to all the subchannels $i : 2 \leq i \leq N$, before we can further increase the bits in subchannel $i = 1$ by one. The order, in which the subchannels are assigned by one more bit, depends on the power cost function (33) of each subchannel and generally it does not coincide with the descending order of the g_i values.

Remark 11. If bit-removal is applied in (36), then one bit is first removed from subchannel $i = 1$ and then, one bit has to be removed from all the subchannels $i : 2 \leq i \leq N^*$, before we can further decrease the bits in subchannel $i = 1$ by one, where N^* corresponds to the first nonzero bit-loaded subchannel.

The importance of allocation (36) in providing low complexity bit loading follows from Remarks 10 and 11 along with the next theorem.

Theorem 3. *The integer bit allocation $b_i = [b_i' + z]_0^{\overline{b}_i}$, for $1 \leq i \leq N$ and $z \in \mathbb{Z}$, is efficient [35]:*

$$\Delta P_i^+(b_i) \geq \Delta P_j^+(b_j - 1), \quad \forall i, j : 1 \leq i, j \leq N. \tag{37}$$

Proof. This theorem is proved by substituting $b_i = [b_i' + z]_0^{\overline{b}_i}$ in (33) and showing that (37) is true, $\forall i, j : 1 \leq i, j \leq N$. □

Theorem 3 states that every up- or downshift of (36) corresponds to an optimum discrete bit allocation under the power minimization goal, taking also into account the low (all-zeros) and the upper ($\overline{b}_i$) bounds of the valid bit vectors. In [27], the following theorem was proven.

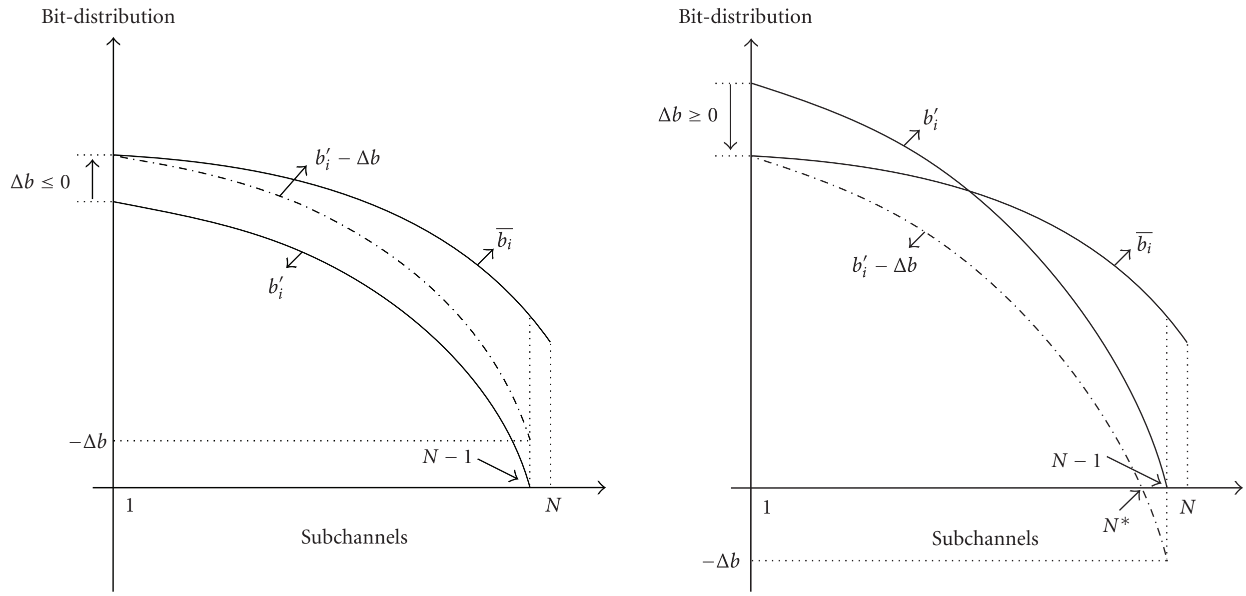
Theorem 4. $b_{i_{\max}}' - \overline{b}_{i_{\max}} = \max\{b_i' - \overline{b}_i\}$, $\forall i : 1 \leq i \leq N$.

According to Theorem 4, if (36) is shifted by $\Delta b = b_{i_{\max}}' - \overline{b}_{i_{\max}}$, then $\overline{b}_i \geq b_i' - \Delta b$, $\forall i : 1 \leq i \leq N$, where the sign of Δb determines the up- or downshift. This is illustrated in Figure 6 for the two possible cases, where the subchannels are sorted. In Figure 6(b), the b_i' profile violates the maximum allowable allocation $\overline{b}_i$ in some of the subchannels. This is due to the fact that profile (36) does not include any power or PSD restrictions.

Using Theorem 4, along with Remarks 10 and 11, we can use the bit-profile b_i' as an initial optimum allocation and then perform a multiple-bit addition or removal process, that converges to the optimum bit solution with no more than a single bit difference per subchannel. In the following section efficient loading algorithms for the discrete RM and MM problems are presented.

6. LOW-COMPLEXITY INTEGER-BIT LOADING

In the previous section, it was made clear that in each allocation step, the greedy algorithm updates the bit-profile according to a power cost function until the system constraints are met, that is, the total power budget is consumed for the RM case or the target data rate is achieved for the MM case. At the end of the greedy process, the respective objective is satisfied, that is, rate maximization or margin maximization. The system constraints define a pair of low and maximum bit allocation limits. The greedy loading process can be described as a continuous bit-by-bit allocation procedure, since at each step it updates the bit-profile by moving on *efficient* bit allocations, see (37), within the set of all possible bit-profiles. For the RM problem, the upper bit allocation limit is determined by the total power budget constraint, while for the MM problem the upper bound is directly calculated by (35). In fact, the upper bound forthe RM problem coincides



(a) (b)

FIGURE 6: Examples of the b_i' bit-profile with respect to the $\overline{b}_i$ upper bound.

with the rate maximization solution. In the rest of this section, we present efficient discrete loading algorithms by exploiting the characteristic bit-profile b_i' defined in (36). These algorithms are based on a multiple-bit loading process that moves the b_i' profile towards the optimum solution.

6.1. Discrete rate maximization

First, we address the total power constrained problem. In contrast to the case of a PSD mask, where the maximum allowable bit allocation is directly determined by (35), the bit upper limit in the total power constrained problem is not straightforward. However, we know from Theorem 3 that every shift of the b_i' bit-profile corresponds to an efficient allocation. Thus, if the available power budget is not exceeded, the new bit allocation is valid within the system constraints.

Since there is no explicit bit upper limit defined, we will use notation $[x]_c^a$ with $a = \infty$, that is, $[x]_0^\infty = \max(x, 0)$. The data rate and total power of bit allocation $b_i' + \alpha$, where $\alpha \in \mathbb{Z}$, are, respectively,

$$B(\alpha) = \sum_{i=1}^{N} [b_i' + \alpha]_0^\infty, \qquad P(\alpha) = \sum_{i=1}^{N} \frac{2^{[b_i'+\alpha]_0^\infty} - 1}{g_i}. \quad (38)$$

Let $P(0) < P_{\text{budget}}$. In order to obtain the maximum possible data rate, we want to upshift profile b_i' by $\alpha \geq 0$, so that

$$P(\alpha) \leq P_{\text{budget}} < P(\alpha + 1). \quad (39)$$

From (38), we can write

$$P(\alpha) = 2^\alpha \left(P(0) + \sum_{i=1}^{N} g_i^{-1} \right) - \sum_{i=1}^{N} g_i^{-1}. \quad (40)$$

Using (40), we derive the integer solution of (39) as

$$\alpha = \left\lfloor \log_2 \left(\frac{P_{\text{budget}} + \sum_{i=1}^{N} g_i^{-1}}{P(0) + \sum_{i=1}^{N} g_i^{-1}} \right) \right\rfloor. \quad (41)$$

The difference between the total power budget and the power corresponding to the b_i' profile upshifted by (41) may allow the allocation of a limited number of additional bits, less than N. We can use the greedy bit-filling process to allocate these bits.

If $P(0) > P_{\text{budget}}$, then we want to downshift profile b_i' by $\alpha \leq 0$, so that (39) also holds. Note that $|\alpha + 1| < |\alpha|$, when $\alpha \leq 0$. It turns out that α is given by (41). Since the value of $|\alpha|$ may be greater than the smallest nonzero value of b_i' for $1 \leq i \leq N$, the total power of the downshifted bit-profile $[b_i' + \alpha]_0^\infty$ may be higher than expected and therefore additional downshifting may be necessary. In this case, the new value of α is calculated using (41), where the upper limit of the summations is replaced by N^*, which denotes the total number of nonzero bit-loaded subchannels, and $P(0)$ is replaced by the total power $P(\alpha)$, which corresponds to the downshifted bit profile of the previous step. At the end of the downshift process, we use greedy bit-filling to add any additional bits less than N^* if there is available power. The pseudocode in Algorithm 3 describes the low-complexity discrete loading for the total power constrained RM problem.

In the case of a peak-power constraint, the optimum RM solution can be calculated by directly allocating $\overline{b}_i$ bits and then, if necessary, we perform bit-removal in order to discard the most power-expensive bits until the total power constraint is met.

```
(1) calculate initial bit-profile b'_i
(2) calculate total power P(0) from (38)
(3) calculate α from (41)
(4) if α ≠ 0 then
(5)     set b'_i = max(0, b'_i + α) for 1 ≤ i ≤ N
(6)     calculate P(α) from (38)
(7)     go to step (3)
(8) else if power available then
(9)     do GREEDY BIT-FILLING
(10) end if
```

Algorithm 3: Discrete total power constrained rate maximization.

```
(1) calculate initial bit-profile b'_i
(2) calculate total rate B(0) from (38)
(3) calculate α from (42)
(4) if α ≠ 0 then
(5)     set b'_i = max(0, b'_i + α) for 1 ≤ i ≤ N
(6)     calculate B(α) from (38)
(7)     go to step (3)
(8) else if target rate not met then
(9)     do GREEDY BIT-FILLING
(10) end if
```

Algorithm 4: Discrete total power-constrained margin maximization.

6.2. Discrete margin maximization

In the MM problem, we assume that the total power budget is sufficient in order to support the desired target rate. As in the previous section, we first address the total power-constrained loading. Given the initial bit-profile b_i' with a total data rate of $B(0)$, see (38), we can perform multiple-bits loading by directly calculating the allocation $b_i = [b_i' + \alpha]_0^\infty$, where

$$\alpha = \left\lfloor \frac{B_{\text{target}} - B(0)}{N^*} \right\rceil. \tag{42}$$

In (42), N^* is the total number of nonzero bit-loaded subchannels, as defined in Remark 11, and the sign of α depends on whether data rate increase (if $B_{\text{target}} > B(0)$) or decrease (if $B_{\text{target}} < B(0)$) is required. The new bit allocation $b_i = [b_i' + \alpha]_0^\infty$ is efficient according to Theorem 3 and optimum under the power minimization goal. However, when downshift is performed, the resulting data rate might be greater than expected due to the low (zero) bit-limit. Therefore successive, but limited, number of multiple-bits loading steps may be necessary until α becomes zero. Then, according to Remarks 10 and 11, the bit-profile allocated so far differs from the target rate solution at most in a single bit per subchannel. The remaining bits can be allocated to the appropriate subchannels based on the respective cost function (33) or (34). The pseudocode in Algorithm 4 describes the low-complexity MM loading.

The above results also hold for the peak-power constrained loading. However, in this case, two important points should be noted. First, an explicit bit upper limit exists and the data-rate expression in (38) becomes

$$B(\alpha) = \sum_{i=1}^{N} [b_i' + \alpha]_0^{\overline{b}_i}. \tag{43}$$

Second, in (42), the value of α is calculated by the difference between the desired rate and the rate of the bounded b_i' profile. Since the difference between b_i' and $[b_i']_0^{\overline{b}_i}$ may be large, a result of $a = 0$ may not indicate the maximum of one bit difference convergence. In order to overcome such a situation, if $b'_{i_{\max}}$ violates the upper limit $\overline{b}_{i_{\max}}$, we apply Theorem 4 and move the b_i' bit-profile within the bit-limits.

6.3. Numerical example

Figure 7 shows an example of bit and power allocation using the CSA(6) standard ADSL loop in Table 47 of ANSI T1.413-1995 [36]. The system parameters are: $N = 256$ subchannels, subcarrier spacing 4.3125 kHz, -140 dBm/Hz additive white Gaussian noise (AWGN) plus near-end crosstalk (NEXT) generated by 20 high-rate DSL (HDSL) neighboring lines, and a 40-kHz lower band edge. The loading constraint values are: -40 dBm/Hz PSD mask, 100 mWatt total power budget, and $b_{\max} = 15$. We also consider 6 dB margin and 3 dB coding gain. Assuming a maximum bit-error rate of 10^{-7}, the corresponding SNR gap equals 12.8 dB.

Figure 7 shows the maximum bit allocation, $\overline{b}_i$, the initial bit allocation, b_i', and the target bit allocation that corresponds to 80% of the maximum possible data rate, that is, $B_{\text{target}} = 0.80 \cdot \sum_{i=1}^{N} \overline{b}_i$. Figure 7 also shows the transmit PSD that corresponds to the maximum and the target rate allocation. The sawtooth shape of the PSD is common to all discrete bit-loading algorithms and is the result of the stepwise power distribution due to the integer bit constraint. Since there is a 40-kHz lower band edge, subchannels 1–9 are not used. Also, note that for the requested target rate, the loading algorithm does not utilize subchannels 28–52. The remaining power, that is, the difference between the total power of the target rate allocation and the power budget, can be used in order to increase the system margin in all subchannels.

In [27], numerical results show that exploiting the efficient bit allocation profile described in Section 5.2, a computational complexity improvement of up to 6 times compared with the greedy bit-filling and the bit-removal methods is achieved. Although the results correspond to the case of total and peak-power constraints, the complexity improvement for the total power constraint is only the same or higher. Indeed, in the latter case, the algorithm experiences less differences between the actual and the expected power or rate (step 6 of Algorithms 3 and 4), thus the optimum bit-allocation is reached with less shifting operations of the b_i' profile. It has to be noted that according to Remarks 10 and 11, the shifting of the b_i' profile converges to the target rate with only one bit difference per subchannel. Therefore, the final greedy bit-filling or bit-removal steps in Algorithms 3 and 4 require only the calculation of the cost function (33) or (34) and the determination of the least or most power-expensive subchannels,

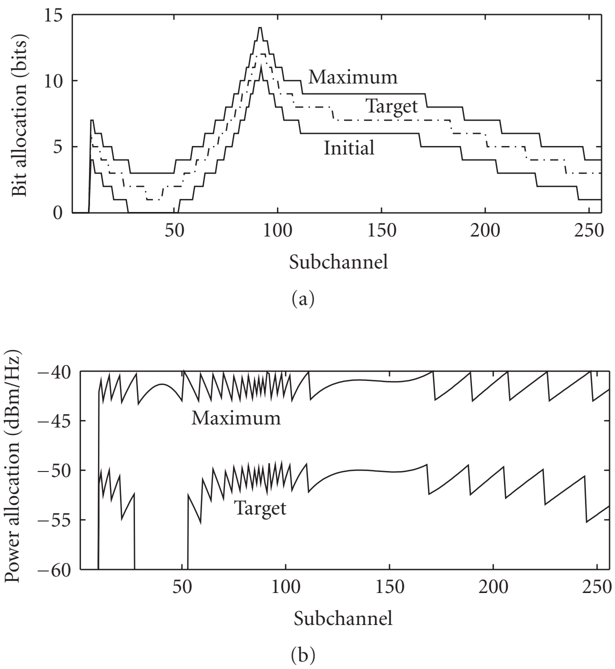


Figure 7: Example of bit and power allocation using the CSA(6) standard ADSL loop.

respectively. As a result of Remarks 10 and 11, after each subchannel selection, there is no need to update the corresponding cost function, thus the total complexity of the final greedy process is reduced.

6.4. *When perfect CSI is not known*

The bit and power loading algorithms described in the previous sections presume that an estimation of the instantaneous CSI, that is, the subchannel gain-to-noise ratio values, is available. When the channel is constant or slow time varying, this is not a complex task. For "always on" links, such as DSL, CSI is obtained during the modems' training. For burst transmission, such as in wireless LANs, CSI can be estimated using a suitable preamble structure or inbound training information. However, in order to account for the limitations imposed by the time-varying behavior of the wireless channels, such as noisy or outdated CSI, alternative adaptive MCM schemes have gained research attention, for example, statistical adaptive MCM and adaptive MCM with partial CSI. References [37–39] can motivate the interested reader on this topic.

7. CONCLUSIONS

In this work, we surveyed the area of bit and power loading in constrained multicarrier communication systems in the single-user context. We discussed the optimal solutions to the main classes of loading problems, namely, rate maximization and margin maximization, under a set of specification and implementation constraints. We presented the water-filling power allocation policy under a total power constraint and the cap-limited water-filling concept was introduced when the peak-power constraint is included. Moreover, the loading problem was addressed subject to the integer-bit restriction and the optimal discrete solution was examined using combinatorial optimization methods. We reviewed existing loading algorithms and highlighted some ideas for low-complexity solutions.

APPENDICES

A. PROOF OF THEOREM 2: EQUIVALENCE OF BIT-FILLING AND BIT-REMOVAL

We denote as $\mathbf{b} \in \mathbb{Z}_+^N$ the N-point bit vector calculated in each allocation step by the bit-filling method. Then for all components of $\mathbf{b} = [b_1, \dots, b_N]$, the following relation holds:

$$\Delta P_i^+(b_i) > \Delta P_j^+(b_j - 1), \quad \forall i, j : 1 \le i, j \le N \tag{A.1}$$

and the bit-distribution vector $\mathbf{b}$ is said to be *BF-efficient*. The above definition was first used by Campello [35]. On the other hand, if $\mathbf{b} \in \mathbb{Z}_+^N$ is the N-point bit vector calculated in each allocation step by the bit-removal method, then the following relation holds:

$$\Delta P_m^-(b_m) < \Delta P_n^-(b_n + 1), \quad \forall m, n : 1 \le m, n \le N. \tag{A.2}$$

Similarly, we define that bit vector $\mathbf{b} \in \mathbb{Z}_+^N$ is *BR-efficient* if equation (A.2) holds.

In order to show that the bit-filling and bit-removal methods are equivalent, we have to prove that for a given target rate $B_{\text{target}} = \sum_{i=1}^N b_i$, there exists only one *BF-efficient* solution and only one *BR-efficient* solution and that an N-point *BF-efficient* vector is also *BR-efficient*.

We assume that there exist two different *BF-efficient* vectors $\mathbf{b} = [b_1, \dots, b_N]$ and $\mathbf{r} = [r_1, \dots, r_N]$, so that $B_{\text{target}} = \sum_{i=1}^N b_i = \sum_{i=1}^N r_i$. Consequently, there exist at least two points, named i_0 and j_0, where $b_{i_0} > r_{i_0}$ and $b_{j_0} < r_{j_0}$, otherwise the above bit-distributions are equal.

Since $\mathbf{r}$ is *BF-efficient*, then

$$\Delta P_{i_0}^+(r_{i_0}) > \Delta P_{j_0}^+(r_{j_0} - 1). \tag{A.3}$$

From the inequality $b_{j_0} < r_{j_0}$, we have

$$\Delta P_{j_0}^+(r_{j_0} - 1) \ge \Delta P_{j_0}^+(b_{j_0}). \tag{A.4}$$

Combining (A.3) and (A.4), we get

$$\Delta P_{i_0}^+(r_{i_0}) > \Delta P_{j_0}^+(b_{j_0}). \tag{A.5}$$

From the inequality $b_{i_0} > r_{i_0}$, we have

$$\Delta P_{i_0}^+(b_{i_0} - 1) \ge \Delta P_{i_0}^+(r_{i_0}). \tag{A.6}$$

Combining (A.5) and (A.6) we get

$$\Delta P_{i_0}^+(b_{i_0} - 1) > \Delta P_{j_0}^+(b_{j_0}), \tag{A.7}$$

which means that vector $\mathbf{b}$ is not *BF-efficient* and this is in contradiction to our initial assumption. Therefore, there is only one bit-profile that is *BF-efficient* for a given target rate. A similar proof can be derived for the uniqueness of the *BR-efficient* bit solution.

Next, we consider a bit distribution vector **b**, which is *BF-efficient*, that is, $\forall i, j : 1 \leq i, j \leq N$:

$$\begin{aligned}\Delta P_i^+(b_i) > \Delta P_j^+(b_j - 1) &\Longrightarrow \frac{2^{b_i}}{g_i} > \frac{2^{b_j - 1}}{g_j}\\ &\Longrightarrow \frac{2^{(b_i+1)-1}}{g_i} > \frac{2^{b_j-1}}{g_j}\\ &\Longrightarrow \Delta P_i^-(b_i+1) > \Delta P_j^-(b_j),\end{aligned} \tag{A.8}$$

therefore **b** is also *BR-efficient*.

B. PROOF OF THEOREM 1: DUALITY CONDITIONS FOR THE DISCRETE RM AND MM PROBLEMS

Let (b_i^G, P_i^G), for $1 \leq i \leq N$, be *efficient* greedy bit and power allocation profiles that satisfy (37) and let $\sum_{i=1}^N b_i^G = B^G$ and $\sum_{i=1}^N P_i^G = P^G$. From the definition of the greedy bit allocation process in Section 5.1, we have

$$P_i^G = \min_{b_i^G} P_i, \quad \forall i : 1 \leq i \leq N, \tag{B.1}$$

$$P^G = \min_{B^G} \sum_{i=1}^N P_i. \tag{B.2}$$

For a given subchannel g_i, the data rate function $b_i = \log_2(1 + P_i \cdot g_i)$ is strictly increasing with respect to P_i, while the integer function $\lfloor b_i \rfloor$ is increasing with respect to P_i. Therefore,

$$P_i^G = \min_{b_i^G} P_i, \quad \forall i \Longrightarrow b_i^G = \max_{P_i^G} b_i, \ \forall i \Longrightarrow B^G = \max_{P^G} \sum_{i=1}^N b_i. \tag{B.3}$$

Let (b_i^R, P_i^R), for $1 \leq i \leq N$, be the optimum greedy solution of the integer RM problem, where $\sum_{i=1}^N b_i^R = B^R$ and $\sum_{i=1}^N P_i^R = P^R$. Then according to (B.2), we can write

$$P^R = \min_{B^R} \sum_{i=1}^N P_i \tag{B.4}$$

which means that (b_i^R, P_i^R) is also the solution to the integer MM problem subject to a target rate of B^R.

Similarly, let (b_i^M, P_i^M), for $1 \leq i \leq N$, be the optimum greedy solution of the integer MM problem, where $\sum_{i=1}^N b_i^R = B^R$ and $\sum_{i=1}^N P_i^R = P^R$. Then according to (B.3), we can write

$$B^M = \max_{P^M} \sum_{i=1}^N b_i \tag{B.5}$$

which means that (b_i^M, P_i^M) is also the solution to the integer RM problem subject to a total power of P^M.

REFERENCES

[1] J. A. C. Bingham, "Multicarrier modulation for data transmission: an idea whose time has come," *IEEE Communications Magazine*, vol. 28, no. 5, pp. 5–14, 1990.

[2] Z. Wang and G. B. Giannakis, "Wireless multicarrier communications," *IEEE Signal Processing Magazine*, vol. 17, no. 3, pp. 29–48, 2000.

[3] K. Taura, M. Tsujishita, M. Takeda, H. Kato, M. Ishida, and Y. Ishida, "A digital audio broadcasting (DAB) receiver," *IEEE Transactions on Consumer Electronics*, vol. 42, no. 3, pp. 322–327, 1996.

[4] H. Sari, G. Karam, and I. Jeanclaude, "Transmission techniques for digital terrestrial TV broadcasting," *IEEE Communications Magazine*, vol. 33, no. 2, pp. 100–109, 1995.

[5] M. Russel and G. L. Stüber, "Terrestrial digital video broadcasting for mobile reception using OFDM," *Wireless Personal Communications*, vol. 2, no. 1-2, pp. 45–66, 1995.

[6] L. Thibault and M. T. Le, "Performance evaluation of COFDM for digital audio broadcasting—part I: parametric study," *IEEE Transactions on Broadcasting*, vol. 43, no. 1, pp. 64–75, 1997.

[7] "Asymmetrical Digital Subscriber Line (ADSL) Transceivers," July 1999, ITU-T Recommendation G.992.1.

[8] "Very-High-Bit-Rate Digital Subscriber Line (VDSL) Metallic Interface—part 1: functional requirements and common specification—part 3: technical specification of a multi-carrier modulation transceiver," ANSI Contribution T1E1.4/2000-013R1, 2000-009R3.

[9] J. A. C. Bingham, *ADSL, VDSL, and Multicarrier Modulation*, John Wiley & Sons, New York, NY, USA, 2000.

[10] E. Biglieri, "Coding and modulation for a horrible channel," *IEEE Communications Magazine*, vol. 41, no. 5, pp. 92–98, 2003.

[11] S. Baig and N. D. Gohar, "A discrete multitone transceiver at the heart of the PHY layer of an in-home power line communication local area network," *IEEE Communications Magazine*, vol. 41, no. 4, pp. 48–53, 2003.

[12] R. van Nee and R. Prasad, *OFDM for Wireless Multimedia Communications*, Artech House, Boston, Mass, USA, 2000.

[13] R. van Nee, G. Awater, M. Morikura, H. Takanashi, M. Webster, and K. W. Halford, "New high-rate wireless LAN standards," *IEEE Communications Magazine*, vol. 37, no. 12, pp. 82–88, 1999.

[14] J. Heiskala and J. Terry, *OFDM Wireless LANs: A Theoretical and Practical Guide*, Sams, Indianapolis, Indiana, 2002.

[15] J. M. Cioffi, "A multicarrier primer," November 1991, ANSI Contribution T1E1.4/91-157, Clearfield, Fla, USA.

[16] A. Peled and A. Ruiz, "Frequency domain data transmission using reduced computational complexity algorithms," in *Proceedings of IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP '80)*, vol. 5, pp. 964–967, Denver, Colo, USA, April 1980.

[17] T. Starr, J. M. Cioffi, and P. J. Silverman, *Understanding Digital Subscriber Line Technology*, Prentice-Hall, Upper Saddle River, NJ, USA, 1999.

[18] "Spectrum Management for Loop Transmission Systems," ANSI Standard T1.417-2001.

[19] P. S. Chow, J. M. Cioffi, and J. A. C. Bingham, "Practical discrete multitone transceiver loading algorithm for data transmission over spectrally shaped channels," *IEEE Transactions on Communications*, vol. 43, no. 2–4, pp. 773–775, 1995.

[20] A. Leke and J. M. Cioffi, "A Maximum rate loading algorithm for discrete multitone modulation systems," in *Proceedings of the IEEE Global Telecommunications Conference (GLOBECOM '97)*, vol. 3, pp. 1514–1518, Phoenix, Ariz, USA, November 1997.

[21] E. Baccarelli, A. Fasano, and M. Biagi, "Novel efficient bit-loading algorithms for peak-energy-limited ADSL-type multicarrier systems," *IEEE Transactions on Signal Processing*, vol. 50, no. 5, pp. 1237–1247, 2002.

[22] B. S. Krongold, K. Ramchandran, and D. L. Jones, "Computationally efficient optimal power allocation algorithms for multicarrier communication systems," *IEEE Transactions on Communications*, vol. 48, no. 1, pp. 23–27, 2000.

[23] D. Hughes-Hartogs, "Ensemble modem structure for imperfect transmission media," 4 679 227 (July 1987), 4 731 816 (March 1988) and 4 833 796 (May 1989).

[24] J. Campello, "Practical bit loading for DMT," in *Proceedings of IEEE International Conference on Communications (ICC '99)*, vol. 2, pp. 801–805, Vancouver, BC, Canada, June 1999.

[25] R. V. Sonalkar and R. R. Shively, "Efficient bit-loading algorithm for DMT applications," *IEEE Communications Letters*, vol. 4, no. 3, pp. 80–82, 2000.

[26] A. Fasano, G. Di Blasio, E. Baccarelli, and M. Biagi, "Optimal discrete bit loading for DMT based constrained multicarrier systems," in *Proceedings of IEEE International Symposium on Information Theory (ISIT '02)*, p. 243, Lausanne, Switzerland, June-July 2002.

[27] N. Papandreou and T. Antonakopoulos, "A new computationally efficient discrete bit-loading algorithm for DMT applications," *IEEE Transactions on Communications*, vol. 53, no. 5, pp. 785–789, 2005.

[28] J. M. Cioffi, G. P. Dudevoir, M. V. Eyuboglu, and G. D. Forney, "MMSE decision-feedback equalizers and coding. II. Coding results," *IEEE Transactions on Communications*, vol. 43, no. 10, pp. 2595–2604, 1995.

[29] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, John Wiley & Sons, New York, NY, USA, 1991.

[30] R. K. Sundaram, *A First Course in Optimization Theory*, Cambridge University Press, Cambridge, UK, 1996.

[31] E. Baccarelli and M. Biagi, "Optimal integer bit-loading for multicarrier ADSL systems subject to spectral-compatibility limits," *Signal Processing*, vol. 84, no. 4, pp. 729–741, 2004.

[32] A. Fasano, "On the optimal discrete bit loading for multicarrier systems with constraints," in *Proccedings of the 57th IEEE Semiannual Vehicular Technology Conference (VTC '03)*, vol. 2, pp. 915–919, Jeju, Korea, April 2003.

[33] G. L. Nemhauser and L. A. Wolsey, *Integer and Combinatorial Optimization*, John Wiley & Sons, New York, NY, USA, 1988.

[34] A. Fasano and G. Di Blasio, "The duality between margin maximization and rate maximization discrete loading problems," in *Proceedings of the 5th IEEE Workshop on Signal Processing Advances in Wireless Communications (SPAWC '04)*, pp. 621–625, Lisbon, Portugal, July 2004.

[35] J. Campello, "Optimal discrete bit loading for multicarrier modulation systems," in *Proceedings of IEEE International Symposium on Information Theory (ISIT '98)*, p. 193, Cambridge, Mass, USA, August 1998.

[36] "Asymmetric Digital Subscriber Line (ADSL) Metallic Interface," ANSI Standard T1.413-1995.

[37] Z. Song, K. Zhang, and Y. L. Guan, "Statistical adaptive modulation for QAM-OFDM systems," in *Proceedings of IEEE Global Telecommunications Conference (GLOBECOM '02)*, vol. 1, pp. 706–710, Taipei, Taiwan, November 2002.

[38] Y. Yao and G. B. Giannakis, "Rate-maximizing power allocation in OFDM based on partial channel knowledge," *IEEE Transactions on Wireless Communications*, vol. 4, no. 3, pp. 1073–1083, 2005.

[39] S. Ye, R. S. Blum, and L. J. Cimini Jr., "Adaptive OFDM systems with imperfect channel state information," *IEEE Transactions on Wireless Communications*, vol. 5, no. 11, pp. 3255–3265, 2006.