Filomat 34:1 (2020), 99–110
https://doi.org/10.2298/FIL2001099S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Bitopological Approximation Space with Application to Data Reduction in Multi-valued Information Systems

A. S. Salama[a]

[a]*Department of Mathematics, Faculty of Science, Tanta University, Egypt*

Abstract. We presented bitopological approximation space as a generalization of classical approximation space. This generalization is based on a topological space that have a subbases generated by a family of binary relations defined on the universe of discourse. We studied some properties of rough sets on bitopological approximation spaces. Many new membership functions and inclusion functions are defined and are used for redefining the rough approximations. Finally, some real life application examples are given to illustrate the benefit of our approach.

1. Introduction

For a long time we dreamed that the general topological spaces can apply in life sciences. Since 2004, I have been working on topology with rough sets and with information systems and tried to reduct the information system by topology by using general binary relations instead of equivalence relations. In 2010, I solved the problem of finding the missing attribute values by using the topological base relation that defined in [8]. More generalizations of the topological approach of information systems can do using near open sets such as pre-open sets, e-open sets and so on [5].

Many approaches generalized rough sets and fuzzy sets using topological spaces [9, 11, 12, 17, 18, 19, 20]. Liu in [6] has studied a comparison of two types of rough sets induced by coverings and he in [7] has introduced the axiomatic systems for rough sets and fuzzy rough sets. The main purpose from these generalizations is to add new objects in the positive region of decision categories. Deleting some objects from negative region is equivalent target for the addition, objects in the positive region. A topological generalizations achievement many aspects such as reduction of large data sets and generate decision rules that help in data mining. Also, attribute reduction in ordering information systems based on evidence theory is one of the directions of study on this topic [10].

Rough sets and topology with their generalizations have developed the similarity measure at the base of granular computing methodology. Knowledge discovery, according to granular computing models based on rough sets and topological spaces is a different technique for data pre-processing, reduction, and data mining [21, 22, 23, 24].

In [1] Abu-Donia introduced multi knowledge bases using rough approximations and topology, they used these knowledge bases in applications. He also in [2], introduced a comparison between different

2010 *Mathematics Subject Classification*. 54A05; 54B05; 54D35; 03B70
Keywords. Rough sets; Rough Approximations; Right neighborhood, Accuracy Measure; Topological Spaces.
Received: 10 January 2019; Revised: 15 March 2019; Accepted: 14 April 2019
Communicated by Biljana Popović
Email address: dr_salama75@yahoo.com (A. S. Salama)

kinds of approximations using a family of binary relations. He with Salama in [3, 4] have generalized the classical rough approximation spaces using topological near open sets called δβ-open sets.

δβ-open sets have opened the door about using other topological approaches for information systems. In our approach we used any finite number of binary relations defined on attributes of an information system to generate two topological spaces and used them in generalizations. We organized our work at the following:

In Section 2 we defined the notion of bitopological approximation space with some important accuracy measures on it. We introduced in Section 3 some important properties of bitopological spaces and we defined the concept of rough set on it. In Section 4, we introduced an application approach for data reduction in multi-valued information systems. The conclusion of our work is given in Section 5.

2. Bitopological Approximation Space

The approximation space $A = (X, R)$ is the core of rough set theory, where X is the universe and R is an equivalence relation [8, 14]. The equivalence class $[x]_R$ is the basic tool for defining rough approximations, lower $\underline{R}(A) = \{x \in X : [x]_x \subseteq A\}$and upper $\overline{R}(A) = \{x \in X : [x]_x \bigcap A \neq \varphi\}$. In application this theory have a wide range when we use positive $POS_R(A) = \underline{R}(A)$, negative $NEG_R(A) = X - \overline{R}(A)$ and $BN_R(A) = \overline{R}(A) - \underline{R}(A)$ regions of some categories X of the universe.

To measure the quality of applications Pawlak define the accuracy measure as follows:

$$\alpha_R(A) = \frac{|\underline{R}(A)|}{|\overline{R}(A)|} \quad \text{where} \quad A \neq \varphi.$$

By accuracy measure we can capture the large of the boundary region of a given category. From the advantage of this theory is the ability to handle a category only using the given data.

In the case of equivalence relations the essential rough set approximations lower and upper coincide with the topological interior and the closure operation respectively. This open the door for using the theory of topological spaces for more generalizations of rough sets. A topological space (X, τ) consisting of a set X and family τ of subsets of X that satisfies the conditions:

1. The empty set and the universe are basic members in τ.
2. The arbitrary unions of members in τ are again a member in τ.
3. The finite intersections of members in τ are again a member in τ.

The topological interior and the topological closure of $A \subset X$ are defined as follows:

1. $int(A) = \bigcup\{G \subseteq X : G \in \tau, G \subseteq A\}$,
2. $cl(A) = \bigcap\{F \subseteq X : X - F \in \tau, A \subseteq F\}$.

We can generate a lower and upper approximation operator based on a topological bases generated by binary relations.

Suppose $R_i, i = 1, 2, 3, ..., n$ be any binary relations defined on a universe U, if $x, y \in U$ then the set $R_{i-R}(x) = \{y : xR_iy, (x, y) \in R_i, \forall i\}$is called the right blocks of x. The family of all right blocks$S_R = \{R_{i-R}(x) : x \in U\}$ is a subbase of a topology $\tau_R$on the universe. In the same way the left blocks $R_{i-L}(x) = \{y : yR_ix, (x, y) \in R_i, \forall i\}$ define a subbase $S_L = \{R_{i-L}(x) : x \in U\}$ of another topology $\tau_L$on the universe. The approximation space $BiT = (U, R_i, \tau_R, \tau_L)$is a generalization of the classical approximation space of Pawlak and we named it biotopological approximation space.

The subbase S_R defines lower and upper approximations of $A \subset U$as follows:

1. $\underline{S}_R(A) = \bigcup\{R_{i-R}(x) \in S_R : R_{i-R}(x) \subseteq A\}$,
2. $\overline{S}_R(A) = \bigcap\{R_{i-R}(x) \in S_R : R_{i-R}(x) \bigcap A \neq \varphi\}$,

Also, the subbase S_L defines lower and upper approximations of $A \subset U$as follows:

1. $\underline{S}_L(A) = \bigcup\{R_{i-L}(x) \in S_L : R_{i-L}(x) \subseteq A\}$,
2. $\overline{S}_L(A) = \bigcap\{R_{i-L}(x) \in S_L : R_{i-L}(x) \bigcap A \neq \varphi\}$,

The accuracy measures of a subset $A \subset U$of the above approximations defined as follows:

$$\alpha_{S_R}(A) = \frac{|A \bigcap \underline{S}_R(A)|}{|A \bigcup \overline{S}_R(A)|}, \alpha_{S_L}(A) = \frac{|A \bigcap \underline{S}_L(A)|}{|A \bigcup \overline{S}_L(A)|}.$$

A subset $A \subseteq U$ of a bitopological space $BiT = (U, R_i, \tau_R, \tau_L)$ is called $\beta_R$open if $A \subset cl_R(int_R(cl_R(A)))$and called$\beta_L$ if $A \subset cl_L(int_L(cl_L(A)))$. The subset $A \subseteq U$ is called βopen if $A \subset cl_R(int_R(cl_R(A))) \bigcap cl_L(int_L(cl_L(A)))$.

A subset $A \subseteq U$ that discernible the objects using the decision attribute in the bitopological space $BiT = (U, R_i, \tau_R, \tau_L)$ is called δ_R open if $A = \overline{S}_{R\delta}(A)$, where $\overline{S}_{R\delta}(A) = \bigcap\{G \in \tau_R : A \subset G\}$and it called δ_L open if $A = \overline{S}_{L\delta}(A)$, where $\overline{S}_{L\delta}(A) = \bigcap\{G \in \tau_L : A \subset G\}$. The subset $A \subseteq U$ is called δopen if $A = \overline{S}_{R\delta}(A) \bigcap \overline{S}_{L\delta}(A)$.

A subset $A \subseteq U$ that discernible the objects using the decision attribute in the bitopological space $BiT = (U, R_i, \tau_R, \tau_L)$ is called δ_R^{-o}open if $A \subseteq cl_R(int_R(\overline{S}_{R\delta}(A)))$and it is called δ_L^{-o}open if $A \subseteq cl_L(int_L(\overline{S}_{L\delta}(A)))$. The subset $A \subseteq U$ is called $(\delta\beta)$open if $A \subseteq cl_R(int_R(\overline{S}_{R\delta}(A))) \bigcap cl_L(int_L(\overline{S}_{L\delta}(A)))$.

The family of all $\delta\beta$open (resp. $R$and Lopen) sets of $A \subseteq U$ is denoted by $(\delta\beta)O(U)$ (resp. $RO(U)$and $LO(U)$). The complement of $(\delta\beta)$ open (resp. $R$and Lopen) set is $(\delta\beta)$ closed (resp. $R$and Lclosed) set. We denote the set of all $(\delta\beta)$closed (resp. $R$and Lopen closed) sets by $(\delta\beta)C(U)$(resp. $RC(U)$and $LC(U)$).

Let $BiT = (U, R_i, \tau_R, \tau_L)$ be a bitopological space. Rlower approximation and Rupper approximation of the subset $A \subseteq U$ is defined as follows:

1. $\underline{BiT}_R(A) = \bigcup\{G \in RO(U) : G \subseteq A\}$,
2. $\overline{BiT}_R(A) = \bigcap\{F \in RC(U) : F \supseteq A\}$.

Let $BiT = (U, R_i, \tau_R, \tau_L)$ be a bitopological space. Llower approximation and Lupper approximation of the subset $A \subseteq U$ is defined as follows:

1. $\underline{BiT}_L(A) = \bigcup\{G \in LO(U) : G \subseteq A\}$,
2. $\overline{BiT}_L(A) = \bigcap\{F \in LC(U) : F \supseteq A\}$.

. Let $BiT = (U, R_i, \tau_R, \tau_L)$ be a bitopological space. $(\delta\beta)$lower approximation and $(\delta\beta)$upper approximation of the subset $A \subseteq U$ is defined as follows:

1. $\underline{BiT}_{(\delta\beta)}(A) = \bigcup\{G \in (\delta\beta)O(A) : G \subseteq A\}$,
2. $\overline{BiT}_{(\delta\beta)}(A) = \bigcap\{F \in (\delta\beta)C(U) : F \supseteq A\}$.

The accuracy measures of a subset $A \subset U$of the above approximations defined as follows:

$$\alpha_R(A) = \frac{|A \bigcap \underline{BiT}_R(A)|}{|A \bigcup \overline{BiT}_R(A)|}, \alpha_L(A) = \frac{|A \bigcap \underline{BiT}_L(A)|}{|A \bigcup \overline{BiT}_L(A)|}, \alpha_{(\delta\beta)}(A) = \frac{|A \bigcap \underline{BiT}_{(\delta\beta)}(A)|}{|A \bigcup \overline{BiT}_{(\delta\beta)}(A)|}$$

Example 2.1. *Let $U = \{a, b, c, d, e\}$ be a universe we define three relations on U by:*

$$R_1 = \{(a, a), (a, e), (b, c), (b, d), (c, e), (d, a), (d, e), (e, e)\},$$

$$R_2 = \{(a, c), (a, a), (a, e), (b, c), (b, d), (c, e), (d, a), (d, e), (e, e), (b, e)\},$$

$$R_3 = \{(c, c), (a, a), (a, e), (b, c), (b, d), (c, e), (d, a), (d, e), (e, e), (d, d)\}$$

Then we have:

$$S_R = \{\{a, e\}, \{c, d\}, \{e\}\},$$

$S_L = \{\{a,d\},\{b\},\{a,c,d,e\}\}$.

Then the topologies associated with these relations are:

$\tau_R = \{U,\varphi,\{e\},\{c,d\},\{a,e\},\{c,d,e\},\{a,c,d,e\}\}$,

$\tau_L = \{U,\varphi,\{b\},\{a,d\},\{a,b,d\},\{a,c,d,e\}\}$.

Table 1 below showing the degree of accuracy measure $\alpha_R(A)$,$\alpha_L(A)$ and $\alpha_{(\delta\beta)}(A)$ for some subsets of the universe.

$A \subseteq U$	$\alpha_R(A)\%$	$\alpha_L(A)\%$	$\alpha_{(\delta\beta)}(A)\%$
$\{a,c\}$	0	50	100
$\{b,e\}$	33	33	100
$\{a,b,e\}$	66	100	100
$\{a,c,d\}$	50	66	100
$\{b,c,e\}$	20	75	100
$\{c,d,e\}$	60	75	100
$\{a,c,d,e\}$	80	80	80
$\{a,b,d,e\}$	40	100	100
$\{b,c,d,e\}$	60	80	100

Table 1: Accuracy measure $\alpha_R(A)$,$\alpha_L(A)$ and $\alpha_{(\delta\beta)}(A)$ of some subsets

We see that the degree of exactness of the set $\{c,d,e\}$ by using R accuracy measure equal to 60%, by using Laccuracy measure equal to 75% and by using $(\delta\beta)$accuracy measure equal to100%. Consequently, $(\delta\beta)$accuracy measure is the best accuracy measure.

According to the above approximations any subset $A \subseteq U$has the following regions:

1. The Rinternal edg of A, $\underline{Edg}_R(A) = A - \underline{BiT}_R(A)$.
2. The Linternal edg of A, $\underline{Edg}_L(A) = A - \underline{BiT}_L(A)$.
3. The $(\delta\beta)$internal edg of A, $\underline{Edg}_{(\delta\beta)}(A) = A - \underline{BiT}_{(\delta\beta)}(A)$.
4. The Rexternal edg of A, $\overline{Edg}_R(A) = \overline{BiT}_R(A) - A$.
5. The Lexternal edg of A, $\overline{Edg}_L(A) = \overline{BiT}_L(A) - A$.
6. The $(\delta\beta)$ external edg of A, $\overline{Edg}_{(\delta\beta)}(A) = \overline{BiT}_{(\delta\beta)}(A) - A$.
7. The Rboundary of A, $BON_R(A) = \overline{BiT}_R(A) - \underline{BiT}_R(A)$.
8. The Lboundary of A, $BON_L(A) = \overline{BiT}_L(A) - \underline{BiT}_L(A)$.
9. The $(\delta\beta)$ boundary of A, $BON_{(\delta\beta)}(A) = \overline{BiT}_{(\delta\beta)}(A) - \underline{BiT}_{(\delta\beta)}(A)$.
10. The R negative of A, $NEG_R(A) = U - \overline{BiT}_R(A)$.
11. The L negative of A, $NEG_L(A) = U - \overline{BiT}_L(A)$.
12. The $(\delta\beta)$ negative of A, $NEG_{(\delta\beta)}(A) = U - \overline{BiT}_{(\delta\beta)}(A)$.

Proposition 2.1. *For any bitopological approximation space $BiT = (U,R_i,\tau_R,\tau_L)$, and for any $A \subseteq X$we have:*

1. $BON_R(A) = \underline{Edg}_R(A) \bigcup \overline{Edg}_R(A)$
2. $BON_L(A) = \underline{Edg}_L(A) \bigcup \overline{Edg}_L(A)$
3. $BON_{(\delta\beta)}(A) = \underline{Edg}_{(\delta\beta)}(A) \bigcup \overline{Edg}_{(\delta\beta)}(A)$

Proof. all three parts can proved as follows:

$BON_R(A) = \overline{BiT}_R(A) - \underline{BiT}_R(A) = (\overline{BiT}_R(A) - A) \bigcup (A - \underline{BiT}_R(A))$ but $\overline{Edg}_R(A) = \overline{BiT}_R(A) - A$ and $\underline{Edg}_R(A) = A - \underline{BiT}_R(A)$, then we have $BON_R(A) = \underline{Edg}_R(A) \bigcup \overline{Edg}_R(A)$.

The next two propositions give the connection between the classical lower and apper edge and that of our approach. Here $\underline{R}(A)$and $\overline{R}(A)$are the classical lower and upper approximations of rough sets. □

Proposition 2.2. *For any bitopological approximation space $BiT = (U, R_i, \tau_R, \tau_L)$, and for any $A \subseteq X$we have:*

1. $\overline{R}(A) - \underline{BiT}_R(A) = \overline{Edg}(A) \bigcup \underline{Edg}_R(A)$
2. $\overline{R}(A) - \underline{BiT}_L(A) = \overline{Edg}(A) \bigcup \underline{Edg}_L(A)$
3. $\overline{R}(A) - \underline{BiT}_{(\delta\beta)}(A) = \overline{Edg}(A) \bigcup \underline{Edg}_{(\delta\beta)}(A)$
4. $\overline{BiT}_R(A) - \underline{R}(A) = \overline{Edg}_R(A) \bigcup \underline{Edg}(A)$
5. $\overline{BiT}_L(A) - \underline{R}(A) = \overline{Edg}_L(A) \bigcup \underline{Edg}(A)$
6. $\overline{BiT}_{(\delta\beta)}(A) - \underline{R}(A) = \overline{Edg}_{(\delta\beta)}(A) \bigcup \underline{Edg}(A)$

Proof. Obvious. □

Proposition 2.3. *For any bitopological approximation space $BiT = (U, R_i, \tau_R, \tau_L)$, and for any $A \subseteq X$we have:*

1. $\underline{Edg}(A) = \underline{Edg}_R(A) \bigcup (\underline{BiT}_R(A) - \underline{R}(A))$
2. $\underline{Edg}(A) = \underline{Edg}_L(A) \bigcup (\underline{BiT}_L(A) - \underline{R}(A))$
3. $\underline{Edg}(A) = \underline{Edg}_{(\delta\beta)}(A) \bigcup (\underline{BiT}_{(\delta\beta)}(A) - \underline{R}(A))$
4. $\overline{Edg}(A) = \overline{Edg}_R(A) \bigcup (\overline{R}(A) - \overline{BiT}_R(A))$
5. $\overline{Edg}(A) = \overline{Edg}_L(A) \bigcup (\overline{R}(A) - \overline{BiT}_L(A))$
6. $\overline{Edg}(A) = \overline{Edg}_{(\delta\beta)}(A) \bigcup (\overline{R}(A) - \overline{BiT}_{(\delta\beta)}(A))$

Proof. Obvious. □

Let $BiT = (U, R_i, \tau_R, \tau_L)$be a bitopological approximation space and $A \subseteq U$. Then we define the following membership functions:

1. $x \underline{\in}_R A$ iff $x \in \underline{BiT}_R(A)$, that called R-strong membership function.
2. $x \underline{\in}_L A$ iff $x \in \underline{BiT}_L(A)$, that called L-strong membership function.
3. $x \underline{\in}_{(\delta\beta)} A$ iff $x \in \underline{BiT}_{(\delta\beta)}(A)$, that called $(\delta\beta)$-strong membership function.
4. $x \overline{\in}_R A$ iff $x \in \overline{BiT}_R(A)$, that called R-weak membership function.
5. $x \overline{\in}_L A$ iff $x \in \overline{BiT}_L(A)$, that called L-weak membership function.
6. $x \overline{\in}_{(\delta\beta)} A$ iff $x \in \overline{BiT}_{(\delta\beta)}(A)$, that called $(\delta\beta)$-weak membership function.

According to the definitions of membership functions above we can redefine the lower and the upper approximations of a set $A \subseteq X$ as follows:

(1)$\underline{BiT}_R(A) = \{x \in A : x \underline{\in}_R A\}$,

(2)$\underline{BiT}_L(A) = \{x \in A : x \underline{\in}_L A\}$,

(3)$\underline{BiT}_{(\delta\beta)}(A) = \{x \in A : x \underline{\in}_{(\delta\beta)} A\}$,

(4)$\overline{BiT}_R(A) = \{x \in A : x \overline{\in}_R A\}$,

(5)$\overline{BiT}_L(A) = \{x \in A : x \overline{\in}_L A\}$,

(6)$\overline{BiT}_{(\delta\beta)}(A) = \{x \in A : x \overline{\in}_{(\delta\beta)} A\}$

Let $BiT = (U, R_i, \tau_R, \tau_L)$be a bitopological approximation space and $A, B \subseteq U$. Then we have the following cases with respect to A and B:

1. R-roughly bottom equal ($A \eqsim_R B$) if $\underline{BiT}_R(A) = \underline{BiT}_R(B)$,
2. L-roughly bottom equal ($A \eqsim_L B$) if $\underline{BiT}_L(A) = \underline{BiT}_L(B)$,
3. $(\delta\beta)$-roughly bottom equal ($A \eqsim_{(\delta\beta)} B$) if $\underline{BiT}_{(\delta\beta)}(A) = \underline{BiT}_{(\delta\beta)}(B)$,
4. R-roughly top equal ($A \simeq_R B$) if $\overline{BiT}_R(A) = \overline{BiT}_R(B)$,
5. L-roughly top equal ($A \simeq_L B$) if $\overline{BiT}_L(A) = \overline{BiT}_L(B)$,
6. $(\delta\beta)$-roughly top equal ($A \simeq_{(\delta\beta)} B$) if $\overline{BiT}_{(\delta\beta)}(A) = \overline{BiT}_{(\delta\beta)}(B)$,
7. R-roughly equal ($A \approx_R B$) if ($A \eqsim_R B$) and ($A \simeq_R B$),
8. L-roughly equal ($A \approx_L B$) if ($A \eqsim_L B$) and ($A \simeq_L B$),
9. $(\delta\beta)$-roughly equal ($A \approx_{(\delta\beta)} B$) if ($A \eqsim_{(\delta\beta)} B$) and ($A \simeq_{(\delta\beta)} B$).

Let $BiT = (U, R_i, \tau_R, \tau_L)$be a bitopological approximation space and $A, B \subseteq U$. Then we have the following cases with respect to A and B:

1. A is R roughly bottom included in B ($A \subset^R_\sim B$) if $\underline{BiT}_R(A) \subseteq \underline{BiT}_R(B)$
2. A is L roughly bottom included in B ($A \subset^L_\sim B$) if $\underline{BiT}_L(A) \subseteq \underline{BiT}_L(B)$
3. A is $(\delta\beta)$ roughly bottom included in B ($A \subset^{(\delta\beta)}_\sim B$) if $\underline{BiT}_{(\delta\beta)}(A) \subseteq \underline{BiT}_{(\delta\beta)}(B)$
4. A is R roughly top included in B ($A \subset^\sim_R B$) if $\overline{BiT}_R(A) \subseteq \overline{BiT}_R(B)$
5. A is L roughly top included in B ($A \subset^\sim_L B$) if $\overline{BiT}_L(A) \subseteq \overline{BiT}_L(B)$
6. A is $(\delta\beta)$ roughly top included in B ($A \subset^\sim_{(\delta\beta)} B$) if $\overline{BiT}_{(\delta\beta)}(A) \subseteq \overline{BiT}_{(\delta\beta)}(B)$
7. A is R roughly included in B($A \subset^\sim_{\sim R} B$) if ($A \subset^R_\sim B$) and ($A \subset^\sim_R B$)
8. A is L roughly included in B($A \subset^\sim_{\sim L} B$) if ($A \subset^L_\sim B$) and ($A \subset^\sim_L B$)
9. A is $(\delta\beta)$ roughly included in B($A \subset^\sim_{\sim(\delta\beta)} B$) if ($A \subset^{(\delta\beta)}_\sim B$) and ($A \subset^\sim_{(\delta\beta)} B$)

3. Properties of bitopological rough approximations

In this section, we introduced some important properties of bitopological spaces and we will define the concept of rough set in $BiT = (U, R_i, \tau_R, \tau_L)$. For any bitopological approximation space $BiT = (U, R_i, \tau_R, \tau_L)$a subset A of U is called:

1. R definable set if $\overline{BiT}_R(A) = \underline{BiT}_R(A)$ or $BON_R(A) = \varphi$
2. L definable set if $\overline{BiT}_L(A) = \underline{BiT}_L(A)$ or $BON_L(A) = \varphi$
3. $(\delta\beta)$ definable set if $\overline{BiT}_{(\delta\beta)}(A) = \underline{BiT}_{(\delta\beta)}(A)$ or $BON_{(\delta\beta)}(A) = \varphi$
4. R rough if $\overline{BiT}_R(A) \neq \underline{BiT}_R(A)$ or $BON_R(A) \neq \varphi$
5. L rough set if $\overline{BiT}_L(A) \neq \underline{BiT}_L(A)$ or $BON_L(A) \neq \varphi$
6. $(\delta\beta)$ rough set if $\overline{BiT}_{(\delta\beta)}(A) \neq \underline{BiT}_{(\delta\beta)}(A)$ or $BON_{(\delta\beta)}(A) \neq \varphi$

For any bitopological approximation space $BiT = (U, R_i, \tau_R, \tau_L)$a subset A of U is called:

1. Roughly R definable, if $\underline{BiT}_R(A) \neq \varphi$ and $\overline{BiT}_R(A) \neq U$.
2. Roughly L definable, if $\underline{BiT}_L(A) \neq \varphi$ and $\overline{BiT}_L(A) \neq U$.
3. Roughly $(\delta\beta)$ definable, if $\underline{BiT}_{(\delta\beta)}(A) \neq \varphi$ and $\overline{BiT}_{(\delta\beta)}(A) \neq U$.
4. Internally R undefinable, if $\underline{BiT}_R(A) = \varphi$ and $\overline{BiT}_R(A) \neq U$.
5. Internally L undefinable, if $\underline{BiT}_L(A) = \varphi$ and $\overline{BiT}_L(A) \neq U$.
6. Internally $(\delta\beta)$ undefinable, if $\underline{BiT}_{(\delta\beta)}(A) = \varphi$ and $\overline{BiT}_{(\delta\beta)}(A) \neq U$.

7. Externally R undefinable, if $\underline{BiT}_R(A) \neq \varphi$ and $\overline{BiT}_R(A) = U$.
8. Externally L undefinable, if $\underline{BiT}_L(A) \neq \varphi$ and $\overline{BiT}_L(A) = U$.
9. Externally $(\delta\beta)$ undefinable, if $\underline{BiT}_{(\delta\beta)}(A) \neq \varphi$ and $\overline{BiT}_{(\delta\beta)}(A) = U$.
10. Totally R undefinable, if $\underline{BiT}_R(A) = \varphi$ and $\overline{BiT}_R(A) = U$.
11. Totally L undefinable, if $\underline{BiT}_L(A) = \varphi$ and $\overline{BiT}_L(A) = U$.
12. Totally $(\delta\beta)$ undefinable, if $\underline{BiT}_{(\delta\beta)}(A) = \varphi$ and $\overline{BiT}_{(\delta\beta)}(A) = U$.

Proposition 3.1. *For any bitopological approximation space $BiT = (U, R_i, \tau_R, \tau_L)$and for all $x, y \in U$, we have:*

1. *if $x \in \overline{BiT}_R(\{y\})$ and $y \in \overline{BiT}_R(\{x\})$ then $\overline{BiT}_R(\{x\}) = \overline{BiT}_R(\{y\})$.*
2. *if $x \in \overline{BiT}_L(\{y\})$ and $y \in \overline{BiT}_L(\{x\})$ then $\overline{BiT}_L(\{x\}) = \overline{BiT}_L(\{y\})$.*
3. *if $x \in \overline{BiT}_{(\delta\beta)}(\{y\})$ and $y \in \overline{BiT}_{(\delta\beta)}(\{x\})$ then $\overline{BiT}_{(\delta\beta)}(\{x\}) = \overline{BiT}_{(\delta\beta)}(\{y\})$.*

Proof. (1) By definition of R upper approximation of a set is the τ_R closure of this set, and since $cl_{\tau_R}(\{y\})$ is R closed set containing x while $\overline{BiT}_R(\{x\})$ is the smallest Rclosed set containing x, thus $\overline{BiT}_R(\{x\}) \subseteq \overline{BiT}_R(\{y\})$. The opposite inclusion follows by symmetry $\overline{BiT}_R(\{y\}) \subseteq \overline{BiT}_R(\{x\})$. Hence $\overline{BiT}_R(\{x\}) = \overline{BiT}_R(\{y\})$. The proof of Parts (2) and (3) are by the same way. □

4. Data deduction in information systems using bitopological spaces

Decision tables (information systems) are widely used in applications and it have many different types. Some of them, its rows are represented objects, while its columns are labeled by attributes. In this system, independent attributes named condition attributes and dependent attributes called decision attributes. For the decision table $T = (U, C, D)$, $U$is the set of objects, $C$is the set of independent attributes and D is the dependent attribute. On any decision table we define an information function f that maps the direct product of $U$and C into the set of all attribute values V_C. A decision table is called incomplete when some values of specified attributes are missing.

Reduction of attributes and derivation of decision rules from decision tables are important for applications and this will done using hybird approach between rough set theory and topology. By reduction of the information table we mean smaller subsets $B \subseteq C$of attributes that preserve the quality of approximations.

With respect to our approach the subset $B' \subset B \subseteq C$ is a reduct of subset $B$with respect to the bitopological approximation space $BiT = (U, C, \tau_R, \tau_L)$ if it is a minimal subset of Bwhich keeps the quality of classification unchanged. Another important topic in rough set theory on decision table is called decision rules. A decision rule r generated by a reduct $B \subseteq C$ is represented in the following form :

$r = \bigwedge_{a \in B}(f(a(x), v)) \rightarrow (D(x), w)$ where $v \in V_a$and $w \in V_D$.

We denote by $s = (a(x), v)$and by $t = (D(x), w)$ to the condition and decision parts of a decision rule, respectively. Let $r_s$and r_t be the set of objects satisfying condition and decision parts of the decision rule r. Objects satisfying both condition and decision parts of the decision rule are called the support of this rule. Decision rules are certain if $r_s \subseteq r_t$.

Another important issue in information systems is the discover of dependency among attributes. the set of attributes $A \subseteq C$ depends totally on the set of attributes $B \subseteq C$, denoted $B \Rightarrow A$ if the set of all values of attributes from A are contained in the values of attributes from B. In other words, A depends totally on B, if there exists a functional dependency between values of A and B.

When we need to measure the dependency of attributes we define three important regions of each approximation type of a subset $A \subset C$ as follows:

1. The R positive region $POS_R(A) = \underline{BiT}_R(A)$.
2. The L positive region $POS_L(A) = \underline{BiT}_L(A)$.
3. The $(\delta\beta)$ positive region $POS_{(\delta\beta)}(A) = \underline{BiT}_{(\delta\beta)}(A)$.

4. The R negative region $NEG_R(A) = U - \overline{BiT}_R(A)$
5. The L negative region $NEG_L(A) = U - \overline{BiT}_L(A)$
6. The $(\delta\beta)$ negative region $NEG_{(\delta\beta)}(A) = U - \overline{BiT}_{(\delta\beta)}(A)$,
7. The R boundary region $BON_R(A) = \overline{BiT}_R(A) - \underline{BiT}_R(A)$
8. The L boundary region $BON_L(A) = \overline{BiT}_L(A) - \underline{BiT}_L(A)$
9. The $(\delta\beta)$ boundary region $BON_{(\delta\beta)}(A) = \overline{BiT}_{(\delta\beta)}(A) - \underline{BiT}_{(\delta\beta)}(A)$,

Formally dependency can be defined in the following way. Let A and B be subsets of C. We will say that A depends on B with respect to τ_R, denoted $B \underset{R}{\Rightarrow} A$, if $\gamma_R(A, B) = \frac{|POS_R(A)|}{|U|}$, and with respect to τ_L,$B \underset{L}{\Rightarrow} A$, if

$$\gamma_L(A, B) = \frac{|POS_L(A)|}{|U|}.$$

Example 4.1. *Consider the multi-valued information system given in Table 2 below:*

In this table the set $U = \{u1, u2, u3, u4, u5, u6, u7\}$is a set of objects and $C = \{A_1, A_2, A_3\}$is the set of condition attributes and Decision = $\{D\}$is the decision attribute. In this example we need to determine the condition attributes that support the decision attribute.

D	A3	A2	A1	U
Yes	{3}	{1,2,3}	{0}	u_1
No	{3,4}	{1,2}	{0, 1}	u_2
Yes	{3}	{1,3}	{2}	u_3
No	{4}	{1,2,4}	{1}	u_4
Maybe	{3,4}	{4}	{1}	u_5
Yes	{3}	{1,2}	{1,2}	u_6
Maybe	{3,4}	{1,2,3}	{0,2}	u_7

Table 2: Multi-valued Information System

The power set of condition attributes is given as follows:

$$P(C) = \{C, \varphi, \{A_1\}, \{A_2\}, \{A_3\}, \{A_1, A_2\}, \{A_1, A_3\}, \{A_2, A_3\}\}$$

Now we define the following binary relations on Uas follows:

$$R_{B\subseteq C} = \{(x, y) : f_{B\subseteq C}(x) \subseteq f_{B\subseteq C}(y), \forall B \subseteq C, B \neq \varphi, \forall x, y \in U\}$$

For this done, we constract the following relations on condition attributes:

$$R_{\{A_1\}} = \{(u1, u1), (u1, u2), (u1, u7), (u2, u2), (u3, u3), (u3, u7), (u4, u4), (u4, u2), (u4, u5), (u4, u6), (u5, u5), (u5, u2), (u5, u4), (u5, u6), (u6, u6), (u7, u7)\},$$

$$R_{\{A_2\}} = \{(u1, u1), (u1, u7), (u2, u2), (u2, u1), (u2, u4), (u2, u6), (u2, u7), (u3, u1), (u3, u3), (u3, u7), (u4, u4), (u5, u4), (u5, u5), (u6, u6), (u6, u1), (u6, u2), (u6, u4), (u6, u7), (u7, u1), (u7, u7)\},$$

$$R_{\{A_3\}} = \{(u1, u1), (u1, u2), (u1, u3), (u1, u5), (u1, u6), (u1, u7), (u2, u2), (u2, u5), (u2, u7), (u4, u2), (u4, u5), (u4, u7), (4, 4), (u3, u1), (u3, u2), (u3, u3), (u3, u5), (u3, u6), (u3, u7), (u6, u1), (u6, u2), (u6, u3), (u6, u5), (u6, u6), (u6, u7), (u5, u2), (u5, u5), (u5, u7), (u7, u2), (u7, u5), (u7, u7)\}$$

$$R_{\{A_1,A_2\}} = \{(u1,u1),(u2,u2),(u3,u3),(u3,u7),(u4,u4),\\ (u5,u4),(u5,u5),(u6,u6),(u7,u7)\}$$

$$R_{\{A_1,A_3\}} = \{(u1,u1),(u1,u2),(u1,u7),(u2,u2),(u3,u3),(u3,u6),(u3,u7),\\ (u4,u4),(u4,u2),(u4,u5),(u5,u5),(u5,u2),(u6,u6),(u7,u7)\} \quad ,$$

$$R_{\{A_2,A_3\}} = \{(u1,u1),(u1,u7),(u2,u2),(u2,u7),(u3,u3),(u3,u1),\\ (u3,u7),(u4,u4),(u5,u5),(u6,u6),(u6,u1),(u6,u2),(u6,u7),(u7,u7)\} \ ,$$

$$R_C = \{(u1,u1),(u1,u7),(u2,u2),(u3,u3),(u3,u7),(u4,u4),\\ (u5,u5),(u4,u6),(u6,u6),(u7,u7)\} \quad ,$$

The right and left blocks of $R_{\{A_1\}}$are a subbase as follows:

$$S_{\{A_1\}-R} = \{\{u1,u2,u7\},\{u2\},\{u3,u7\},\{u2,u4,u5,u6\},\{u6\},\{u7\}\}$$

$$S_{\{A_1\}-L} = \{\{u1\},\{u1,u2\},\{u3\},\{u4,u5\},\{u4,u5,u6\},\{u1,u3,u7\}\}$$

The right and left blocks of $R_{\{A2\}}$ are a subbase as follows:

$$S_{\{A2\}-R} = \{\{u1,u7\},\{u1,u2,u4,u6,u7\},\{u1,u3,u7\},\{u4\},\{u4,u5\}\}$$

$$S_{\{A_2\}-L} = \{\{u1,u2,u3,u6,u7\},\{u2,u6\},\{u3\},\{u2,u4,u5,u6\},\{u5\}\}$$

The right and left blocks of $R_{\{A3\}}$ are a subbase as follows:

$$S_{\{A3\}-R} = \{\{u1,u2,u3,u5,u6,u7\},\{u2,u5,u7\},\{u2,u4,u5,u7\}\}$$

$$S_{\{A3\}-L} = \{\{u1,u3,u6\},\{u4\},U\}$$

The right and left blocks of $R_{\{A1,A2\}}$ are a subbase as follows:

$$S_{\{A1,A2\}-R} = \{\{u1\},\{u2\},\{u3,u7\},\{u4,u5\},\{u5\},\{u6\},\{u3,u7\}\}$$

$$S_{\{A1,A2\}-L} = \{\{u1\},\{u2\},\{u3\},\{u4\},\{u4,u5\},\{u6\},\{u7\}\}$$

The right and left blocks of $R_{\{A1,A3\}}$ are a subbase as follows:

$$R_{\{A1,A3\}-R} = \{\{u1,u2,u7\},\{u2\},\{u3,u6,u7\},\{u2,u4,u5\},\{u2,u5\},\{u6\},\{u7\}\}$$

$$R_{\{A1,A3\}-L} = \{\{u1\},\{u1,u2,u4,u5\},\{u3\},\{u4\},\{u3,u6\},\{u1,u3,u7\}\}$$

The right and left blocks of $R_{\{A2,A3\}}$ are a subbase as follows:

$$S_{\{A2,A3\}-R} = \{\{u1,u7\},\{u2,u7\},\{u1,u3,u7\},\{u4\},\{u5\},\{u1,u2,u6,u7\},\{u7\}\}$$

$$S_{\{A2,A3\}-L} = \{\{u1,u3,u6\},\{u2,u6\},\{u3\},\{u4\},\{u5\},\{u6\},\{u1,u2,u3,u6,u7\}\}$$

The right and left blocks of R_C are a subbase as follows:

$$S_{C-R} = \{\{u1,u7\},\{u2\},\{u3,u7\},\{u4\},\{u5\},\{u6\},\{u7\}\}$$

$$S_{C-L} = \{\{u1\},\{u2\},\{u3\},\{u4\},\{u5\},\{u6\},\{u1,u3,u7\}\}$$

The two topologies we need in the reduction process is given by:

$$\tau_R = \{U,\varphi,\{u2\},\{u4\},\{u5\},\{u6\},\{u7\},\{u1,u7\},\{u3,u7\},\\ \{u2,u4\},\{u4,u6\},\{u4,u7\},\{u2,u5\},\{u2,u6\},\{u2,u7\},\\ \{u4,u5\},\{u5,u6\},\{u5,u7\},\{u6,u7\},\{u3,u4,u7\},\\ \{u1,u4,u7\},\{u2,u3,u7\},\{u1,u2,u7\},\{u3,u5,u7\},\\ \{u1,u5,u7\},\{u3,u6,u7\},\{u1,u6,u7\},\{u1,u3,u7\},\\ \{u2,u4,u5\},\{u2,u4,u6\},\{u2,u4,u7\},\{u4,u5,u6\},\\ \{u4,u5,u7\},\{u2,u3,u4,u7\},\{u1,u2,u4,u7\},\\ \{u3,u4,u5,u7\},\{u1,u4,u5,u7\},\{u2,u4,u5,u6,u7\},\\ \{u2,u3,u4,u5,u6,u7\},\{u1,u2,u4,u5,u6,u7\}\}$$

$\tau_L = \{U, \varphi, \{u1\}, \{u2\}, \{u3\}, \{u4\}, \{u5\}, \{u6\}, \{u1, u3, u7\},$
$\{u1, u2\}, \{u1, u3\}, \{u1, u4\}, \{u1, u5\}, \{u1, u6\}, \{u2, u3\}, \{u2, u4\},$
$\{u2, u5\}, \{u1, u6\}, \{u3, u4\}, \{u3, u5\}, \{u3, u6\}, \{u1, u2, u3, u7\},$
$\{u4, u5\}, \{u4, u6\}, \{u1, u3, u4, u7\}, \{u5, u6\}, \{u1, u3, u5, u7\},$
$\{u1, u3, u6, u7\}, \{u1, u2, u3\}, \{u1, u2, u4\}, \{u1, u2, u5\}, \{u1, u2, u6\},$
$\{u1, u3, u4\}, \{u1, u3, u5\}, \{u1, u3, u6\}, \{u1, u4, u5\}, \{u1, u4, u6\},$
$\{u1, u5, u6\}, \{u2, u3, u4\}, \{u2, u3, u5\}, \{u2, u3, u6\}, \{u2, u4, u5\},$
$\{u2, u4, u6\}, \{u1, u2, u3, u4, u7\}, \{u2, u5, u6\}, \{u3, u4, u5\},$
$\{u1, u2, u3, u4, u7\}, \{u1, u2, u3, u5, u7\}, \{u1, u2, u3, u6, u7\},$
$\{u3, u4, u6\}, \{u4, u5, u6\}, \{u1, u3, u4, u5, u7\}, \{u1, u3, u5, u6, u7\},$
$\{u1, u2, u3, u4\}, \{u1, u2, u3, u5\}, \{u1, u2, u3, u6\}, \{u2, u3, u4, u5\},$
$\{u2, u3, u4, u6\}, \{u1, u2, u3, u4, u7\}, \{u3, u4, u5, u6\},$
$\{u1, u2, u3, u4, u5\}, \{u2, u3, u4, u5, u6\}, \{u1, u2, u3, u4, u6\},$
$\{u1, u2, u3, u4, u5, u6\}\}$

Now these topologies are considered as the basic knowledge base for our system that we can generate decision rules from Table 2.
The discernible subsets of the decision attribute are:

$$D1 = Decision(Yes) = \{u1, u3, u6\},$$

$$D2 = Decision(No) = \{u2, u4\},$$

$$D1 = Decision(Maybe) = \{u5, u7\},$$

Now we need to calculate $\overline{S}_{R\delta}(D_i), i = 1, 2, 3$ *and* $\overline{S}_{L\delta}(D_i),\ i = 1, 2, 3,$ *and then we know if the above categories about decision are* δ_R*open* $(D_i = \overline{S}_{R\delta}(D_i))$ *or* δ_L*open* $(D_i = \overline{S}_{L\delta}(D_i))$.
$\delta_R(D_1) = \overline{S}_{R\delta}(D_1) = \bigcap\{G \in \tau_R : D_1 \subset G\} = \{u1, u3, u6, u7\}$, *then* D_1*is not* δ_R*open category.*
$\delta_L(D_1) = \overline{S}_{L\delta}(D_1) = \bigcap\{G \in \tau_L : D_1 \subset G\} = \{u1, u3, u6\}$, *then* D_1*is* δ_L*open category.*
$\delta_R(D_2) = \overline{S}_{R\delta}(D_2) = \bigcap\{G \in \tau_R : D_2 \subset G\} = \{u2, u4\}$, *then* D_2*is* δ_R*open category.*
$\delta_L(D_2) = \overline{S}_{L\delta}(D_2) = \bigcap\{G \in \tau_L : D_2 \subset G\} = \{u2, u4\}$, *then* D_2*is* δ_L*open category.*
$\delta_R(D_3) = \overline{S}_{R\delta}(D_3) = \bigcap\{G \in \tau_R : D_3 \subset G\} = \{u5, u6, u7\}$, *then* D_3*is not* δ_R*open category.*
$\delta_L(D_3) = \overline{S}_{L\delta}(D_3) = \bigcap\{G \in \tau_L : D_3 \subset G\} = \{u1, u3, u5, u7\}$, *then* D_3*is not* δ_L*open category.*

We notice that $\delta_R(D_1) \cap \delta_L(D_1) = D_1$, $\delta_R(D_2) \cap \delta_L(D_2) = D_2$*and* $\delta_R(D_3) \cap \delta_L(D_3) = D_3$, *hence all decision catogries are* δ *open.*

If we tried to separate among these decision categories using $(\delta\beta)$*open approach, then we need to calculate* $cl_R(int_R(\delta_R(D_i))),\ i = 1, 2, 3$ *and* $cl_L(int_L(\delta_L(D_i))),\ i = 1, 2, 3.$

$$\delta_R^{-o}(D_1) = cl_R(int_R(\delta_R(D_1))) = U, \delta_L^{-o}(D_1) = cl_L(int_L(\delta_L(D_1))) = \{u1, u3, u5, u6, u7\},$$

$$\delta_R^{-o}(D_2) = cl_R(int_R(\delta_R(D_2))) = \{u2, u4\}, \delta_L^{-o}(D_2) = cl_L(int_L(\delta_L(D_2))) = \{u2, u4, u6\},$$

$$\delta_R^{-o}(D_3) = cl_R(int_R(\delta_R(D_3))) = \{u3, u5, u6, u7\}, \delta_L^{-o}(D_3) = cl_L(int_L(\delta_L(D_3))) = \{u1, u3, u5, u6, u7\},$$

Now we notice that $D_i \subseteq \delta_R^{-o}(D_i) \cap \delta_L^{-o}(D_i)$, $\forall i = 1, 2, 3$*then* $D_i, \forall i = 1, 2, 3$ *are* $(\delta\beta)$ *open. After many calculations using the topologies* τ_R*and* τ_L *the Rlower approximation and Rupper approximation of the subset* $D_i, i = 1, 2, 3$ *are given by:*
$\underline{BiT}_R(D_1) = \{u1, u6\}$, $\overline{BiT}_R(D_1) = \{u1, u3, u6, u7\}$, *then* $BON_R(D_1) = \{u3, u7\}$
$\underline{BiT}_R(D_2) = \{u2, u4\}$, $\overline{BiT}_R(D_2) = \{u2, u4\}$, *then* $BON_R(D_2) = \varphi$
$\underline{BiT}_R(D_3) = \{u5, u7\}$, $\overline{BiT}_R(D_3) = \{u1, u3, u5, u6, u7\}$, *then* $BON_R(D_3) = \{u1, u3, u6\}$
Also, the Llower approximation and Lupper approximation of the subset $D_i, i = 1, 2, 3$ *are given by:*
$\underline{BiT}_L(D_1) = \{u1, u3, u6\}$, $\overline{BiT}_L(D_1) = \{u1, u3, u6, u7\}$, *then* $BON_L(D_1) = \{u7\}$,
$\underline{BiT}_L(D_2) = \{u2, u4\}$, $\overline{BiT}_L(D_2) = \{u2, u4\}$, *then* $BON_L(D_2) = \varphi$,

$\underline{BiT}_L(D_3) = \{u5, u7\}$, $\overline{BiT}_L(D_3) = \{u1, u3, u5, u7\}$, *then* $BON_L(D_3) = \{u1, u3\}$.

Also, the $(\delta\beta)$*lower approximation and* $(\delta\beta)$*upper approximation of the subset* $D_i, i = 1, 2, 3$ *are given by:*

$\underline{BiT}_{(\delta\beta)}(D_1) = \{u1, u3, u6\}$,$\overline{BiT}_{(\delta\beta)}(D_1) = \{u1, u3, u6, u7\}$, *then* $BON_{(\delta\beta)}(D_1) = \{u7\}$

$\underline{BiT}_{(\delta\beta)}(D_2) = \{u2, u4\}$,$\overline{BiT}_{(\delta\beta)}(D_2) = \{u2, u4\}$, *then* $BON_{(\delta\beta)}(D_2) = \varphi$

$\underline{BiT}_{(\delta\beta)}(D_3) = \{u5, u7\}$,$\overline{BiT}_{(\delta\beta)}(D_3) = \{u3, u5, u7\}$. *then* $BON_{(\delta\beta)}(D_3) = \{u3\}$.

The conclusions about the accurate approach (exactly$(\delta\beta)$ *approach) with respect to the information given in Table 2 are given as the following:*

1. *The decision value "Yes" is not exactly such that* $\{u7\}$*is in the boundary region.*
2. *The decision value "No" is exactly 100% such that its boundary region is empty.*
3. *The decision value "Maybe" is not exactly such that* $\{u7\}$*is in its boundary region.*

5. Conclusion

We generalized Pawlak approximation space by family of binary relations to bitopological approximation space. Using Rblocks and Lblocks of these relations we generated two topological spaces and using them we defined the lower and upper approximations of any subset in the universe. We generalized the rough approximations of any subset to three regions namely, R region, Lregion and $(\delta\beta)$-region and used these regions for reduction of information systems.

The same category in a descision system can distinguishable by three accuracy measures $\alpha_R(D)$,$\alpha_L(D)$ and $\alpha_{(\delta\beta)}(D)$. The applications are proved that the measure $\alpha_{(\delta\beta)}(D)$is the best than other measures.

References

[1] H. M. Abu-Donia, Multi knowledge based rough approximations and applications, Knowledge-Based Systems, 26 (2012) 20-29.
[2] H. M. Abu-Donia, Comparison between different kinds of approximations by using a family of binary relations, Knowledge-Based Systems, 21 (2008) 911-919.
[3] H. M. Abu-Donia, A. S. Salama, Generalization of Pawlak's rough approximation spaces by using δβ-open sets, International Journal of Approximate Reasoning, 53 (2012) 1094-1105.
[4] H. M. Abu-Donia, A. S. Salama, Approximation operators by using finite family of reflexive relations, International Journal of Applied Mathematical Research, 4 (**??**) (2015) 376-392.
[5] Won Keun Min, Remarks on è-open sets in generalized topological spaces, Applied Mathematics Letters, 24 (2011) 165–168.
[6] G. Liu and Y. Sai, A comparison of two types of rough sets induced by coverings, International Journal of Approximate Reasoning, 50 (2009) 521-528.
[7] G.L. Liu, Axiomatic systems for rough sets and fuzzy rough sets, International Journal of Approximate Reasoning, 48 (2008) 857-867.
[8] A. S. Salama ; Topological Solution for missing attribute values in incomplete information tables, Information Sciences, 180 (2010) 631-639.
[9] Y. Ouyang, Z. Wang and H. Zhang, On fuzzy rough sets based on tolerance relations Information Sciences, 180 (2010) 532-542.
[10] Wei-hua Xu, Xiao-yan Zhang, Jian-min Zhong and Wen-xiu Zhang, Attribute reduction in ordered information systems based on evidence theory , Knowledge and Information Systems, Volume 25, Number 1, Pages 169-184, (2010).
[11] Yingjie Yang , Chris Hinde; A new extension of fuzzy sets using rough sets: R-fuzzy sets, Information Sciences, 180, 354–365 (2010).
[12] Lingyum Yang, Luoshan Xu, Topological properties of generalized approximation spaces, Information Sciences, 181 (2011) 3570-3580.
[13] Pawlak, Z., Some Issues on Rough Sets, Transactions on Rough Sets, I, Journal Subline. Lectures Notes in Computer Science, 3100 (2004) 1-58.
[14] W. Zhu, Generalized rough sets based on relations, Information Sciences, 177 (2007) 4997-5011.
[15] W. Zhu, Relationship between generalized rough sets based on binary relation and covering, Information Sciences, 179 (2009) 210-225.
[16] W. Zhu, Relationship among basic concepts in covering-based rough sets, Information Sciences, 179 (2009) 2478-2486.
[17] William Zhu; Topological approaches to covering rough sets, Information Sciences 177, 1499–1508 (2007).
[18] Suyun Zhao , Eric C.C. Tsang; On fuzzy approximation operators in attribute reduction with fuzzy rough sets, Information Sciences 178, 3163–3176 (2008).
[19] Pei Zhi, Daowu Pei, Li Zheng, Topology vs generalized rough sets, International Journal of Approximate Reasoning 52 (2011) 231-239.

[20] Pei Zhi, Daowu Pei, Li Zheng, Covering rough sets based on neighborhoods an approach without using neighborhoods, International Journal of Approximate Reasoning 52 (2011) 461-472.

[21] O. G. El Barbary, A. S. Salama, El Sayed Atlam, Granular information retrieval using neighborhood systems, Math Meth Appl Sci. (2017);1–17.

[22] S. Butenkov, A. Zhukov, A. Nagorov, N. Krivsha, Granular computing models and methods based on the spatial granulation, Procedia Computer Science, 103 (2017) 295 – 302.

[23] Silvia Calegari, Davide Ciucci, Granular computing applied to ontologies, International Journal of Approximate Reasoning, 51 (2010) 391–409.

[24] Mohammed M. Eissa, Mohammed Elmogy, Mohammed Hashem, Rough – Granular Computing knowledge discovery models for medical classification, Egyptian Informatics Journal, (2016) 17, 265–272.