Bivariate cubic periodic spline interpolation on a three direction mesh

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BIVARIATE CUBIC PERIODIC SPLINE INTERPOLATION ON A THREE DIRECTION MESH

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1. Introduction

The problem of periodic spline interpolation on the real line is extensively studied and an abundance of results is available nowadays. In case the nodes are equally spaced, the theory of periodic spline interpolation may be imbedded in a natural way in the beautiful theory of univariate cardinal spline interpolation, initiated by I.J. Schoenberg in his monograph [6]. A part of the univariate theory of cardinal spline functions has been extended to $\mathbb{R}^2$ by using box-splines (cf. [1]). For instance, a space of spline functions spanned by translates of a fixed bivariate box spline on a three-direction mesh has been studied with respect to the problem of cardinal spline interpolation.

In this report we consider the problem of (double) periodic bivariate cubic spline interpolation in $\mathbb{R}^2$, identified here with the x_1-x_2 plane, supplied with the three-direction mesh consisting of the mesh-lines

$$x_1 = i,\ x_2 = j,\ x_2 - x_1 = k \quad ((i,j,k) \in \mathbb{Z}^3,\ (x_1,x_2) \in \mathbb{R}^2) .$$

Note that the three direction mesh defines a uniform triangulation of $\mathbb{R}^2$, which is also called a type-1 triangulation.

In order to formulate our problem of periodic spline interpolation the following definitions are needed.

Definition 1.1. Let $m,n \in \mathbb{N}$. A complex-valued function f defined on $\mathbb{R}^2$ is called m,n periodic if and only if

$$f(x_1 + m, x_2) = f(x_1, x_2) = f(x_1, x_2 + n) \quad ((x_1,x_2) \in \mathbb{R}^2) .$$

Definition 1.2. A function s, defined on $\mathbb{R}^2$, is called a cubic spline function if and only if

i) $s \in C^1(\mathbb{R}^2)$, i.e., the function s is differentiable and the first order partial derivatives denoted by s_{x_1} and s_{x_2} are continuous on $\mathbb{R}^2$,

ii) on each triangle of the type-1 triangulation s coincides with a bivariate polynomial of degree at most 3.

The space of cubic spline functions will be denoted by capital S. We go on by defining the m,n periodic spline space $S_{m,n}$.

<u>Definition 1.3.</u> Let $m,n \in \mathbb{N}$. A function s belongs to $S_{m,n}$ iff

i) $s \in S$,

ii) s is an m,n periodic function.

Finally, we define the space $Y_{m,n}$ of m,n periodic sequences as follows.

<u>Definition 1.4.</u> A (complex-valued) sequence (y_μ) $(\mu \in \mathbb{Z}^2)$ belongs to $Y_{m,n}$ iff

$$y_{\mu_1+m,\mu_2} = y_{\mu_1,\mu_2} = y_{\mu_1,\mu_2+n} \qquad (\mu = (\mu_1,\mu_2) \in \mathbb{Z}^2) .$$

Now we are in a position to formulate our interpolation problem.
Let $m,n \in \mathbb{N}$, $(y_\mu) \in Y_{m,n}$, and $\xi \in [0,1)^2 \subset \mathbb{R}^2$. Find a function $s \in S_{m,n}$ such that

$$(1.1) \qquad s(\xi + \mu) = y_\mu \qquad (\mu \in \mathbb{Z}^2) .$$

Observe that the interpolation points are located at $\xi + \mathbb{Z}^2$, i.e. at a shift of the lattice points $(i,j) \in \mathbb{Z}^2$. Mostly, one has $\xi = (0,0)$ or $\xi = (\frac{1}{2},\frac{1}{2})$. It is obvious that

$$(1.2) \qquad \dim Y_{m,n} = mn .$$

In Section 2 we will show that $\dim S_{m,n} = 2mn + 2$. So, we cannot expect that each $(y_\mu) \in Y_{m,n}$ corresponds with a unique $s \in S_{m,n}$ satisfying (1.1). In other words the interpolation problem is not <u>unisolvent</u>. Since we wish to have an interpolation scheme that guarentees existence and uniqueness of interpolation, further conditions for the interpolating spline functions are needed. These conditions may be imposed upon the interpolating spline function in several ways.
In this report we require the interpolating spline function stems from an mn dimensional subspace of $S_{m,n}$ spanned by translates of a finitely supported spline function $s \in S$; for instance a box spline or a spline function in S having minimal support. To be more precise, such a subspace is of the type

$$(1.3) \qquad T_{m,n}(\varphi) = \{ \sum_{\nu \in \mathbb{Z}^2} a_\nu \varphi(x - \nu) \mid (a_\nu) \in Y_{m,n} \} ,$$

where φ is a finitely supported function.
In Section 3 we will derive a necessary and sufficient condition for the unisolvence of our interpolation problem in case an interpolating function must be chosen in $T_{m,n}(\varphi)$. In Section 4 this condition will be applied to

a specific situation, where $\xi \in \{(0,0),(\frac{1}{2},\frac{1}{2})\}$ and where φ is a specific finitely supported spline function in S; namely a cubic spline function having the smallest compact support. Section 5 deals with the computation of an interpolating periodic spline function with the help of Fast Fourier Transforms.

The final section is devoted to the order of approximation in case a sufficiently smooth 1,1 periodic function f has been interpolated at the points $(h_1(\xi_1 + i), h_2(\xi_2 + j))$ $((i,j) \in \mathbb{Z}^2)$ with $h_1 = \frac{1}{m}$, $h_2 = \frac{1}{n}$ and $\xi \in [0,1)^2$ by a scaled function $s(mx_1, nx_2)$ with $s \in S_{m,n}$. It seems that the approximation is of order $|h|^2 = h_1^2 + h_2^2$.

2. <u>The dimension of the space $S_{m,n}$</u>

Let D denote the rectangular $[0,m] \otimes [0,n]$ in $\mathbb{R}^2$, and let S(D) be the space of cubic spline functions consisting of the restrictions of the cubic spline functions $s \in S$ to the rectangular D.

In [2], Chui and Wang constructed various bases of the space S(D), starting with a finitely supported cubic spline function B^1 introduced by P.O. Fredricson [3]. The support of B^1 together with the triangles that divide B^1 into polynomial pieces is given in Fig. 2.1 below.

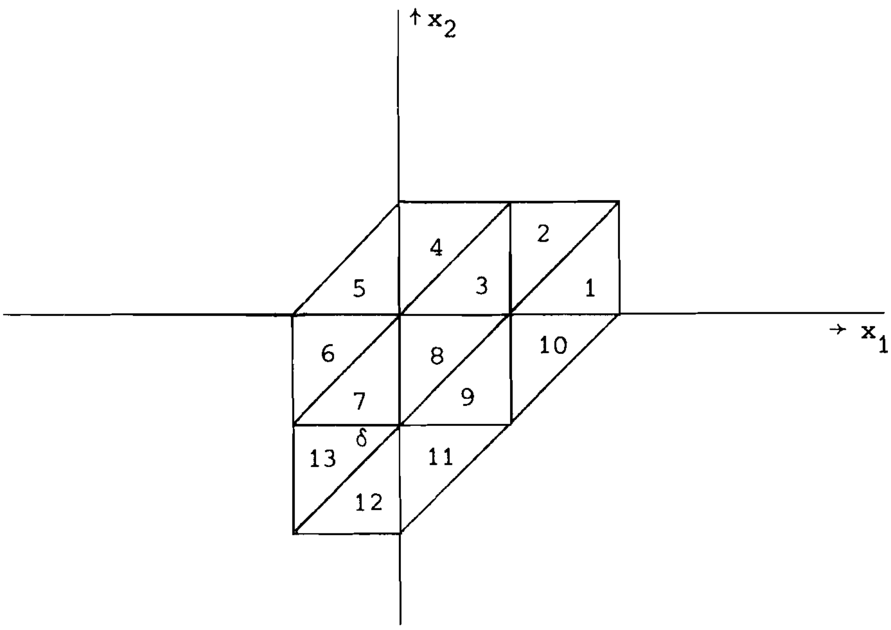


Fig. 2.1

For completeness, we list here the polynomial pieces of B^1 that correspond to the triangles in the support of B^1.

(2.2) Table of polynomial pieces

$$1: \frac{1}{3}(2 - x_1)^3 \ ,$$

$$2: \frac{1}{3}(1 - x_2)^2(4 + 2x_2 - 3x_1) \ ,$$

$$3: \frac{1}{3}(1 - x_2)^2(1 + 2x_2) + (1 - x_2)(x_1 - x_2)(1 - x_1) \ ,$$

$$4: \frac{1}{3}(1 - x_2)^2(1 - x_2 + 3x_1) \ ,$$

$$5: \frac{1}{3}(1 + x_1 - x_2)^3 \ ,$$

$6\colon\ \frac{1}{3}(1 + x_1)^2(1 + x_1 - 3x_2)$,

$7\colon\ \frac{1}{3}(1 + x_1)^2(1 - 2x_1) + (1 + x_1)(x_1 - x_2)(1 + x_2)$,

$8\colon\ \frac{4}{3} + x_2 - x_1^2 - (1 + x_2 - x_1)^2 - x_1(1 + x_2 - x_1)(1 + x_2)$,

$9\colon\ (2 - x_1 + x_2)(2 - x_1 + x_2 - \frac{2}{3}(2 - x_1 + x_2)^2 + (1 - x_1)(1 + x_2))$,

$10\colon\ \frac{1}{3}(2 - x_1 + x_2)^2(2 - 2x_2 - x_1)$,

$11\colon\ \frac{1}{3}(2 - x_1 + x_2)^2(2 + 2x_1 + x_2)$,

$12\colon\ \frac{1}{3}(2 + x_2)^3$,

$13\colon\ \frac{1}{3}(1 + x_1)^2(4 - 2x_1 + 3x_2)$.

The Fourier-transform $\check{B}^1$ of the function B^1 is given by

$$(2.3)\qquad \check{B}^1(\omega_1,\omega_2) = 2\int_{\mathbb{R}^2} B^1(x_1,x_2)e^{-j\omega_1x_1-j\omega_2x_2}dx_1dx_2 =$$

$$= 2\,\frac{1-e^{-j\omega_1}}{j\omega_1}\cdot\frac{1-e^{-j\omega_2}}{j\omega_2}\cdot\frac{1-e^{-j(\omega_1+\omega_2)}}{j(\omega_1+\omega_2)}\,\check{X}_\delta(\omega_1,\omega_2)\ ,$$

where X_δ is the characteristic function of the triangle δ (cf. Fig. 2.1), i.e., $X(x) = 1$ if $x \in \delta$, otherwise $X(x) = 0$, and $j^2 = -1$.

From B^1, the spline function B^2 is defined by

$$(2.4)\qquad B^2(x) = B^1(-x)\qquad (x \in \mathbb{R}^2)\ .$$

In order to obtain various bases of $S(D)$ one considers the translates of B^1 and B^2.

Let $p \in \{1,2\}$ and let Ω_p be defined by

$$\Omega_p := \{\nu \in \mathbb{Z}^2 \mid \operatorname{supp} B^p(x - \nu) \cap \operatorname{int}(D) \neq \emptyset\}\ .$$

Here int(D) denotes the interior of D.

It is known (cf. [2]) that the collection of spline functions

$$\mathcal{B} = \{B^1(x - \nu),\ B^2(x - \mu) \mid \nu \in \Omega_1,\ \mu \in \Omega_2\}$$

spans $S(D)$ and that $\dim S(D) = 2mn + 4m + 4n + 3$.

The collection $\mathcal{B}$ has cardinality 2mn + 4m + 4n + 6 which equals dim S(D) +3. Now, Chui and Wang have given criteria to determine which three elements may be deleted from $\mathcal{B}$ to give a basis of S(D). We state one of these criteria as a lemma.

Lemma 2.1. (Chui and Wang [2]). If $\nu^1 \in \Omega_1$ and $\nu^2 \in \Omega_1$ are distinct but lie on the same mesh-line, than for any $\mu^1 \in \Omega_2$ the collection

$$\hat{\mathbf{B}} = \{B^1(x-\nu),\ B^2(x-\mu)\ \mid\ \nu \in \Omega_1\backslash\{\nu^1,\nu^2\},\ \mu \in \Omega_2\backslash\{\mu^1\}\}$$

is a basis of S(D).

Our next goal is to identify by means of a linear system of point evaluations those spline functions $s \in S(D)$ which can be regarded as restrictions to D of m,n periodic spline functions in $S_{m,n}$. To do so, we need the second order partial derivative $s_{x_1x_2}$ of a spline function s in S(D). Since the mixed second order derivatives of the basis functions B^1 and B^2 are discontinuous only on the diagonals of the three-direction mesh, we have to distinguish at a point $\nu \in \mathbb{Z}^2$ the two limit values

$$s^r_{x_1x_2}(\nu),\ s^\ell_{x_1x_2}(\nu)\ , \tag{2.5}$$

where $s^r_{x_1x_2}(\nu)$ is the limit value of $s_{x_1x_2}(x)$ when $x = (x_1,x_2)$ approaches $\nu = (\nu_1,\nu_2)$ from that side of the diagonal $x_1 - x_2 = \nu_1 - \nu_2$ for which $x_1 - x_2 > \nu_1 - \nu_2$, and where $s^\ell_{x_1x_2}(\nu)$ is the limit value of $s_{x_1x_2}(x)$ when x approaches ν from the other side of the diagonal.
The announced linear system of point evaluations will be given in the following lemma.

Lemma 2.2. A function $s \in S(D)$ belongs to $S_{m,n}(D)$ iff

$$A_1 \begin{cases} \left.\begin{aligned} & s(i,n) = s(i,0) \\ & s_{x_1}(i,n) = s_{x_1}(i,0), \\ & s_{x_2}(i,n) = s_{x_2}(i,0), \end{aligned}\right\} (i = 0,1,\dots,m) \\ s^r_{x_1x_2}(i,n) = s^r_{x_1x_2}(i,0), \qquad (i = 0,1,\dots,m-1)\ , \end{cases}$$

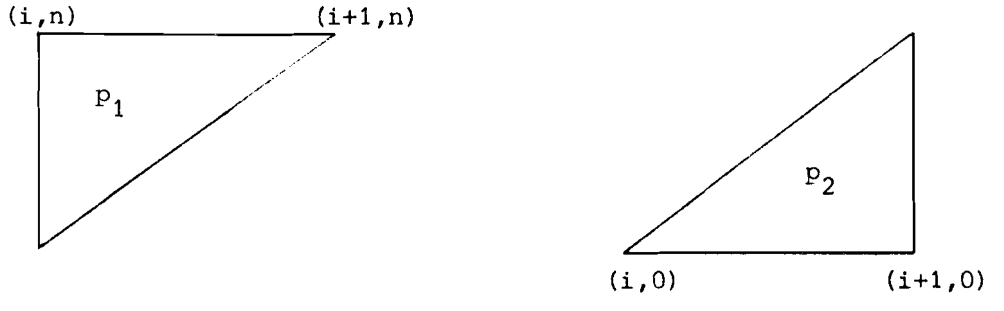
$$A_2 \left\{ \begin{aligned} & s(m,j) = s(0,j), \\ & s_{x_1}(m,j) = s_{x_1}(0,j), \\ & s_{x_2}(m,j) = s_{x_2}(0,j), \\ & s^{\ell}_{x_1x_2}(m,j) = s^{\ell}_{x_1x_2}(0,j) \end{aligned} \right\} (j = 0,1,\dots,n-1) \ .$$

Proof. First, we observe that if $s \in S_{m,n}(D)$ then s as well as s_{x_1} and s_{x_2} must satisfy the periodicity conditions:

$$B_1 \left\{ \begin{aligned} & s(x_1,n) = s(x_1,0), \\ & s_{x_2}(x_1,n) = s_{x_2}(x_1,0), \quad (0 \le x_1 \le m) \ , \end{aligned} \right.$$

$$B_2 \left\{ \begin{aligned} & s(m,x_2) = s(0,x_2), \\ & s_{x_1}(m,x_2) = s_{x_1}(0,x_2), \quad (0 \le x_2 \le n) \ . \end{aligned} \right.$$

So, we have to prove the equivalence of the two systems (A_1,A_2) and (B_1,B_2). It suffices to prove that A_1 and B_1 are equivalent, since the proof of the equivalence of A_2 and B_2 runs along similar lines. Therefore, let us assume that a function $s \in S(D)$ satisfies B_1. Then the first three equations of A_1 are trivially satisfied. In order to establish the fourth equation of A_1 we consider the two line segments $[(i,n),(i+1,n)]$ and $[(i,0),(i+1,0)]$ and the two adjoining polynomial pieces p_1,p_2 of $s \in S(D)$ as shown by the figures below



From B_1 it follows that the polynomial p, defined by

$$p(x_1,x_2) = p_2(x_1,x_2) - p_1(x_1,x_2+n) \ ,$$

and its partial derivative p_{x_2} vanishes on the line segment $[(i,0),(i+1,0)]$.

Hence

$$p(x_1,x_2) = x_2^2(a + bx_1 + cx_2)$$

for some constants a, b and c. Consequently, $p_{x_1x_2}(i,0) = 0$, which implies $s^r_{x_1x_2}(i,n) = s^r_{x_1x_2}(i,0)$.

Now, we assume that A_1 holds. Then the polynomial p introduced above has the properties

$$p(i,0) = p(i + 1,0) = p_{x_1}(i,0) = p_{x_1}(i+1,0) = p_{x_2}(i,0) =$$
$$= p_{x_2}(i+1,0) = 0 ,$$
$$p_{x_1x_2}(i,0) = 0 .$$

It is easy to verify that these properties have the consequence that x_2^2 is a divisor of $p(x_1,x_2)$; this in turn implies B_1. □

The next step is to select a basis of the space S(D) and to convert the system of point evaluations A_1,A_2 into a linear system of equations for the coefficients in the expansion of a function $s \in S(D)$ with respect to that basis. Due to Lemmma 2.2 the linear system then describes the subspace $S_{m,n}(D)$ of S(D). So, let $\nu^1 = (-1,0)$, $\nu^2 = (-1,1)$ and $\mu^1 = (0,-1)$. Then, according to Lemma 2.1, a function $s \in S(D)$ can uniquely be represented by

$$s(x) = \sum_{\nu\in\Omega_1} c_\nu B^1(x - \nu) + \sum_{\mu\in\Omega_2} d_\mu B^2(x - \mu) , \tag{2.6}$$

with $c_{\nu^1} = c_{\nu^2} = d_{\mu^1} = 0$.

By using this representation, the function values of B^1, B^2, and their derivatives at the lattice points, we may deduce that the linear system (A_1,A_2) leads to a linear system of equations for the coefficients c_ν and d_μ which can be described as follows.

Let A, A', B and B' be the matrices

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix},$$

$$A' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}, \quad B' = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Then from A_1 the following linear system may be computed

$$M\begin{cases} A\begin{bmatrix} c_{i,n} \\ c_{i,n+1} \\ d_{i+1,n} \\ d_{i+1,n-1} \end{bmatrix} + B\begin{bmatrix} c_{i-1,n} \\ c_{i-1,n+1} \\ d_{i,n} \\ d_{i,n-1} \end{bmatrix} = A\begin{bmatrix} c_{i,0} \\ c_{i,1} \\ d_{i+1,0} \\ d_{i+1,-1} \end{bmatrix} + B\begin{bmatrix} c_{i-1,0} \\ c_{i-1,1} \\ d_{i,0} \\ d_{i,-1} \end{bmatrix} \\ (i = 0,1,\dots,m-1) , \\ A'\begin{bmatrix} c_{m,n} \\ c_{m,n+1} \\ d_{m+1,n} \end{bmatrix} + B'\begin{bmatrix} c_{m-1,n} \\ c_{m-1,n+1} \\ d_{m,n} \\ d_{m,n-1} \end{bmatrix} = A'\begin{bmatrix} c_{m,0} \\ c_{m,1} \\ d_{m+1,0} \end{bmatrix} + B'\begin{bmatrix} c_{m-1,0} \\ c_{m-1,1} \\ d_{m,0} \\ d_{m,-1} \end{bmatrix} . \end{cases}$$

Similarly, we may compute from A_2 the linear system

$$N\begin{cases} A\begin{bmatrix} d_{m,j} \\ d_{m+1,j} \\ c_{m,j+1} \\ c_{m-1,j+1} \end{bmatrix} + B\begin{bmatrix} d_{m,j-1} \\ d_{m+1,j-1} \\ c_{m,j} \\ c_{m-1,j} \end{bmatrix} = A\begin{bmatrix} d_{0,j} \\ d_{1,j} \\ c_{0,j+1} \\ c_{-1,j+1} \end{bmatrix} + B\begin{bmatrix} d_{0,j-1} \\ d_{1,j-1} \\ c_{0,j} \\ c_{-1,j} \end{bmatrix} \\ (j = 0,1,\dots,n-1) . \end{cases}$$

We observe that in case the coefficients c_ν, d_μ satisfy the relations

$$\begin{bmatrix} c_{i,n} \\ c_{i,n+1} \end{bmatrix} = \begin{bmatrix} c_{i,0} \\ c_{i,n} \end{bmatrix} \qquad (i = -1,\dots,m) ,$$

$$\begin{bmatrix} d_{i,n} \\ d_{i,n-1} \end{bmatrix} = \begin{bmatrix} d_{i,0} \\ d_{i,-1} \end{bmatrix} \qquad (i = 0,\dots,m+1) ,$$

$$\begin{bmatrix} c_{m,j} \\ c_{m-1,j} \end{bmatrix} = \begin{bmatrix} c_{0,j} \\ c_{-1,j} \end{bmatrix} \qquad (j = 0,1,\dots,n) ,$$

$$\begin{bmatrix} d_{m,j} \\ d_{m+1,j} \end{bmatrix} = \begin{bmatrix} d_{0,j} \\ d_{1,j} \end{bmatrix} \qquad (j = -1,\dots,n-1) ,$$

then the equations M and N are trivially satisfied. In fact this means that we have found the subspace $S^{(1)}_{m,n}(D)$ of $S_{m,n}(D)$ consisting of restrictions to

D of functions in the space $S^{(1)}_{m,n} \subset S_{m,n}$ given by

$$S^{(1)}_{m,n} = \{\sum_{\nu} (c_{\nu} B^1(x-\nu) + d_{\nu} B^2(x-\nu)) \mid (c_{\nu}),(d_{\nu}) \in Y_{m,n}\ c_{\nu^1} = c_{\nu^2} = d_{\mu^1} = 0\} \ .$$

First, we assume that $n \geq 2$; the case $n = 1$ will be treated separately. As a consequence of Lemma 2.1 and the periodicity of the sequences (c_{ν}) and (d_{ν}) with the restrictions $c_{\nu^1} = c_{\nu^2} = d_{\mu^1} = 0$ one has

(2.7) $$\dim S^{(1)}_{m,n}(D) = 2mn - 3 \ .$$

Now, we will show that the space $S_{m,n}(D)$ can be written as a direct sum of two spaces, one of them equals $S^{(1)}_{m,n}(D)$. To this end, let $s \in S_{m,n}(D)$. Since $S_{m,n}(D) \subset S(D)$, s can be represented by formula (2.6). The coefficients c_{ν} and d_{ν} satisfy the equations M and N, where in addition $c_{\nu^1} = c_{\nu^2} = d_{\mu^1} = 0$. By substracting an appropriate function $s_1 \in S^{(1)}_{m,n}(D)$ (uniquely determined) from s, we may assume that $s - s_1$ belongs to the space $S^{(2)}_{m,n}(D)$ given by

$$S^{(2)}_{m,n}(D) = \{ \sum_{\nu\in\Omega_1} c_{\nu} B^1(x-\nu) + \sum_{\nu\in\Omega_2} d_{\nu} B^2(x-\nu) \mid$$

$$c_{i,j} = 0 \ (i = -1,\dots,m-2;\ j = 0,1,\dots,n-1) \ ,$$

$$d_{i,j} = 0 \ (i = 0,1,\dots,m-1;\ j = -1,0,\dots,n-2)$$

$$(c_{\nu}),\ (d_{\nu}) \text{ satisfy } M \text{ and } N\} \ .$$

It is clear that

$$S_{m,n}(D) = S^{(1)}_{m,n}(D) \oplus S^{(2)}_{m,n}(D) \ ,$$

For functions $s \in S^{(2)}_{m,n}(D)$, the equations M and N can be replaced by the equations

$$M_1: \begin{bmatrix} c_{i,n} \\ c_{i,n+1} \\ d_{i+1,n} \\ d_{i+1,n-1} \end{bmatrix} = C \begin{bmatrix} c_{i-1,n} \\ c_{i-1,n+1} \\ d_{i,n} \\ d_{i,n-1} \end{bmatrix} \qquad (i = 0,1,\dots,m-2) \ .$$

$$M_2: \begin{bmatrix} c_{m-1,n} \\ c_{m-1,n+1} \\ d_{m,n} \\ d_{m,n-1} \end{bmatrix} = C \begin{bmatrix} c_{m-2,n} \\ c_{m-2,n+1} \\ d_{m-1,n} \\ d_{m-1,n-1} \end{bmatrix} + \begin{bmatrix} c_{m-1,0} \\ c_{m-1,1} \\ d_{m,0} \\ d_{m,-1} \end{bmatrix} .$$

$$M_3: \begin{bmatrix} c_{m,n} \\ c_{m,n+1} \\ d_{m+1,n} \end{bmatrix} = C' \left(\begin{bmatrix} c_{m-1,n} \\ c_{m-1,n+1} \\ d_{m,n} \\ d_{m,n-1} \end{bmatrix} - \begin{bmatrix} c_{m-1,0} \\ c_{m-1,1} \\ d_{m,0} \\ d_{m,-1} \end{bmatrix} \right) + \begin{bmatrix} c_{m,0} \\ c_{m,1} \\ d_{m+1,0} \end{bmatrix} .$$

$$N_1: \begin{bmatrix} d_{m,j} \\ d_{m+1,j} \\ c_{m,j+1} \\ c_{m-1,j+1} \end{bmatrix} = C \begin{bmatrix} d_{m,j-1} \\ d_{m+1,j-1} \\ c_{m,j} \\ c_{m-1,j} \end{bmatrix} \qquad (j = 0,1,\ldots,n-2) .$$

$$N_2: \begin{bmatrix} d_{m,n-1} \\ d_{m+1,n-1} \\ c_{m,n} \\ c_{m-1,n} \end{bmatrix} = C \begin{bmatrix} d_{m,n-2} \\ d_{m+1,n-2} \\ c_{m,n-1} \\ c_{m-1,n-1} \end{bmatrix} + \begin{bmatrix} d_{0,n-1} \\ d_{1,n-1} \\ c_{0,n} \\ c_{-1,n} \end{bmatrix} .$$

Here $C = -A^{-1}B$, $C' = -A'^{-1}B'$,

$$C = \begin{bmatrix} -2 & 0 & -1 & -2 \\ -2 & 0 & -2 & -1 \\ 3 & 0 & 2 & 2 \\ 3 & 0 & 1 & 3 \end{bmatrix}, \quad C' = \begin{bmatrix} -2 & 0 & -1 & -2 \\ -2 & 0 & -2 & -1 \\ 3 & 0 & 2 & 2 \end{bmatrix} .$$

The characteristic polynomial of C is simple; it is given by $\lambda(\lambda - 1)^3$. The eigenspace E_1 corresponding to the eigenvalue 1 is two dimensional:

$$E_1 = \langle (3,0,-1,-4),\ (1,1,-1,-1) \rangle .$$

With help of Cayley's theorem, C^k (k is a positive integer) may easily be computed. One has

$$C^k = \begin{bmatrix} 1-3k & 0 & -k & -2k \\ 1-3k & 0 & -k-1 & 1-2k \\ 3k & 0 & k+1 & 2k \\ 3k & 0 & k & 2k+1 \end{bmatrix} .$$

In the equations M_1, the number $c_{-1,n+1}$ occurs. Note that $c_{-1,n+1} \notin \Omega_1$. However, the value of $c_{-1,n+1}$ is not of importance, since the second column of C consists only of zeros. The same will be true for the number $d_{m+1,-1}$ in the equations N_1. Therefore, we take $c_{-1,n+1} = d_{m+1,-1} = 0$. The total number of unknowns in the equations is equal to 4n + 4m + 6, i.e., dim S(D) + - dim $S^{(1)}_{m,n}(D)$, while the total number of equations is equal to 4n + 4m + 3. In order to solve these equations we first observe that a solution is completely determined by the values of the six unknowns $c_{-1,n}$, $d_{0,n}$, $d_{0,n-1}$, $d_{m,-1}$, $c_{m,0}$, and $c_{m-1,0}$, since then the other unknowns easily follow as given below

$$
\begin{aligned}
c_{i,n} &= -(3i+2)c_{-1,n} - (i+1)d_{0,n} - 2(i+1)d_{0,n-1} , \\
c_{i,n+1} &= -(3i+2)c_{-1,n} - id_{0,n} - (2i+1)d_{0,n-1} , \\
d_{i+1,n} &= (3i+3)c_{-1,n} + (i+2)d_{0,n} + (2i+2)d_{0,n-1} , \\
d_{i+1,n-1} &= (3i+3)c_{-1,n} + (i+1)d_{0,n} + (2i+3)d_{0,n-1} , \\
&(i = 0,1,\dots,m-2) ,
\end{aligned}
$$

$$
\begin{aligned}
c_{m-1,n} &= -(3m-1)c_{-1,n} - md_{0,n} - 2md_{0,n-1} + c_{m-1,0} , \\
c_{m-1,n+1} &= -(3m-1)c_{-1,n} - (m-1)d_{0,n} - (2m-1)d_{0,n-1} + 3d_{m,-1} + \\
&\qquad + 3c_{m,0} + 3c_{m-1,0} , \\
d_{m,n} &= 3mc_{-1,n} + (m+1)d_{0,n} + 2md_{0,n-1} - 2d_{m,-1} - c_{m,0} - 2c_{m-1,0} , \\
d_{m,n-1} &= 3mc_{-1,n} + md_{0,n} + (2m+1)d_{0,n-1} + d_{m,-1} , \\
c_{m,n} &= -(3m+2)c_{-1,n} - (m+1)d_{0,n} - 2(m+1)d_{0,n-1} + c_{m,0} , \\
c_{m,n+1} &= -(3m+2)c_{-1,n} - md_{0,n} - (2m+1)d_{0,n-1} + 3d_{m,-1} + 4c_{m,0} + 2c_{m-1,0} , \\
d_{m+1,n} &= (3m+3)c_{-1,n} + (m+2)d_{0,n} + (2m+2)d_{0,n-1} - 2d_{m,-1} - c_{m-1,0} , \\
d_{m,j} &= -(3j+2)d_{m,-1} - (j+1)c_{m,0} - 2(j+1)c_{m-1,0} , \\
d_{m+1,j} &= -(3j+2)d_{m,-1} - jc_{m,0} - (2j+1)c_{m-1,0} , \\
c_{m,j+1} &= (3j+3)d_{m,-1} + (j+2)c_{m,0} + (2j+2)c_{m-1,0} , \\
c_{m-1,j+1} &= (3j+3)d_{m,-1} + (j+1)c_{m,0} + (2j+3)c_{m-1,0} , \\
&(j = 0,1,\dots,n-2) ,
\end{aligned}
$$

$$d_{m,n-1} = -(3n-1)d_{m,-1} - nc_{m,0} - 2nc_{m-1,0} + d_{0,n-1} ,$$

$$d_{m+1,n-1} = -(3n-1)d_{m,-1} - (n-1)c_{m,0} - (2n-1)c_{m-1,0} + 3c_{-1,n} + \\ + 3d_{0,n} + 3d_{0,n-1} ,$$

$$c_{m,n} = 3nd_{m,-1} + (n+1)c_{m,0} + 2nc_{m-1,0} - 2c_{-1,n} - d_{0,n} - 2d_{0,n-1} ,$$

$$c_{m-1,n} = 3nd_{m,-1} + nc_{m,0} + (2n+1)c_{m-1,0} + c_{-1,n} .$$

The following observation is important. We cannot choose the starting numbers $c_{-1,n}$, $d_{0,n}$, $d_{0,n-1}$, $d_{m,-1}$, $c_{m,0}$, $c_{m-1,0}$ arbitrarely, since at the left members of the foregoing equations the variables $c_{m-1,n}$, $d_{m,n-1}$ and $c_{m,n}$ occur twice. This leads to the following three equations for the starting numbers

$$-(3m-1)c_{-1,n} - md_{0,n} - 2md_{0,n-1} + c_{m-1,0} = 3nd_{m,-1} + nc_{m,0} + \\ + (2n+1)c_{m-1,0} + c_{-1,n} ,$$

$$3mc_{-1,n} + md_{0,n} + (2m+1)d_{0,n-1} + d_{m,-1} = -(3n-1)d_{m,-1} - nc_{m,0} + \\ - 2nc_{m-1,0} + d_{0,n-1} ,$$

$$-(3m+2)c_{-1,n} - (m+1)d_{0,n} - 2(m+1)d_{0,n-1} + c_{m,0} = 3nd_{m,-1} + \\ + (n+1)c_{m,0} + 2nc_{m-1,0} - 2c_{-1,n} - d_{0,n} - 2d_{0,n-1} .$$

Surprisingly, these three equations are identical to the equation

$$3mc_{-1,n} + md_{0,n} + 2md_{0,n-1} + 3nd_{m,-1} + nc_{m,0} + 2nc_{m-1,0} = 0 .$$

We conclude that the six starting numbers are restricted to only one linear equation. Hence $\dim S^{(2)}_{m,n}(D) = 5$, and therefore

$$\dim S_{m,n} = \dim S_{m,n}(D) = \dim S^{(1)}_{m,n}(D) + \dim S^{(2)}_{m,n}(D) = \\ = 2mn - 3 + 5 = 2mn + 2 .$$

We now turn to the case where $n = 1$. Due to our choice of $\nu^2 = (-1,1)$, one has $c_{-1,n} = c_{\nu^2} = 0$. Our conclusion is that $\dim S^{(2)}_{m,1} = 4$. Moreover, $\dim S^{(1)}_{m,1} = 2m - 2$. Hence, again one has that $\dim S_{m,n} = 2mn + 2$. So, we may finish this section with the following theorem.

<u>Theorem 2.3</u>. Let $m,n \in \mathbb{N}$. Then

$$\dim S_{m,n} = 2mn + 2 .$$

3. Translates of a fixed finitely supported function

In this section we collect some results with respect to the space

$$(3.1) \qquad T(\varphi) := \{\sum_\nu a_\nu \varphi(x - \nu) \mid \nu \in \mathbb{Z}^2,\ a_\nu \in \mathbb{C}\} ,$$

where φ is a function defined on $\mathbb{R}^2$ having a finite support. Of special interest for us is the space $T_{m,n}(\varphi)$ as defined in (1.3), which of course is a linear subspace of $T(\varphi)$. We will derive now a fundamental relation for functions $f \in T(\varphi)$, which can be utilized to investigate problems of interpolation in $T(\varphi)$.

Lemma 3.1. Let $s \in T(\varphi)$. Then for all $x^1, x^2 \in \mathbb{R}^2$ the following relation holds

$$(3.2) \qquad \sum_\nu \varphi(x^2 - \nu) s(x^1 + \nu) = \sum_\nu \varphi(x^1 - \nu) s(x^2 + \nu) .$$

Proof. Since $s \in T(\varphi)$, the function s may be written in the form

$$s(x^1) = \sum_\mu a_\mu \varphi(x^1 - \mu) .$$

Hence, by changing the order of summation and replacing ν by $\mu - \nu$, one gets

$$\sum_\nu \varphi(x^2 - \nu) s(x^1 + \nu) = \sum_\nu \varphi(x^2 - \nu) \sum_\mu a_\mu \varphi(x^1 + \nu - \mu) =$$

$$= \sum_\mu a_\mu \sum_\nu \varphi(x^2 - \nu)\varphi(x^1 + \nu - \mu) = \sum_\mu a_\mu \sum_\nu \varphi(x^2 + \nu - \mu)\varphi(x^1 - \nu)$$

$$= \sum_\nu \varphi(x^1 - \nu) \sum_\mu a_\mu \varphi(x^2 + \nu - \mu) = \sum_\nu \varphi(x^1 - \nu) s(x^2 + \nu) . \qquad \square$$

Note that the unicity of the coefficients a_μ in the proof of the previous lemma is not required. If the function φ is such that $\sum_\nu a_\nu \varphi(x - \nu) = 0$ $(x \in \mathbb{R}^2)$ implies $a_\nu = 0$ $(\nu \in \mathbb{Z}^2)$, then we call the set of translates $\{\varphi(x - \nu) \mid \nu \in \mathbb{Z}^2\}$ independent on $\mathbb{R}^2$. In this case, one evidently has that for each $s \in T(\varphi)$, the coefficients a_ν are uniquely determined.
The independence of the set of translates is needed in the following lemma.

Lemma 3.2. Let φ be a finitely supported function such that the translates $\{\varphi(x - \nu) \mid \nu \in \mathbb{Z}^2\}$ are independent on $\mathbb{R}^2$, then

$$\dim T_{m,n}(\varphi) = mn .$$

Proof. If $s \in T_{m,n}(\varphi) \subset T(\varphi)$, then it follows from the unicity of the coefficients a_ν in the representation $s(x) = \sum_\nu a_\nu \varphi(x - \nu)$ and the m,n periodicity of s that $(a_\nu) \in Y_{m,n}$. Take mn independent sequences $(a_\nu)^{(i)}$ $(i = 1,2,\ldots$ $\ldots,mn)$ in $Y_{m,n}$. It is obvious that the corresponding functions

$$s_i(x) = \sum_\nu (a_\nu)^{(i)} \varphi(x - \nu)$$

form a basis of $T_{m,n}(\varphi)$. □

In $T_{m,n}(\varphi)$ we consider the interpolation problem stated as follows. Let $(y_\mu) \in Y_{m,n}$ and $\xi \in [0,1)^2 \subset \mathbb{R}^2$. Find a function $s \in T_{m,n}(\varphi)$ such that

$$(3.3) \qquad s(\xi + \mu) = y_\mu \quad \text{for all } \mu \in \mathbb{Z}^2 .$$

Since $\dim Y_{m,n} = \dim T_{m,n}(\varphi)$ the natural question arises to determine the points $\xi \in [0,1)^2$ for which the corresponding interpolation problem is unisolvent. The following theorem gives a useful criterium for this question. In order to formulate this criterium, we need the so-called characteristic function Φ defined by

$$(3.4) \qquad \Phi(z) = \sum_\nu \varphi(\xi - \nu) z^\nu .$$

$$(z = (z_1,z_2) \in \mathbb{C}^2, \ \nu = (\nu_1,\nu_2) \in \mathbb{Z}^2, \ z^\nu := z_1^{\nu_1} z_2^{\nu_2}).$$

Theorem 3.3. Let φ be a finitely supported function such that the translates $\{\varphi(x - \nu) \mid \nu \in \mathbb{Z}^2\}$ are independent on $\mathbb{R}^2$. Then the interpolation problem (3.3) is unisolvent if and only if the associated characteristic function Φ has no zeros at the points $z = (z_1,z_2) \in \mathbb{R}^2$ for which

$$(3.5) \qquad z_1^m = 1, \ z_2^n = 1 .$$

Proof. If Φ has a zero (w_1,w_2) for which $w_1^m = w_2^n = 1$ then the function s given by

$$s(x) = \sum_\nu w^\nu \varphi(x - \nu) ,$$

evidently belongs to $T_{m,n}(\varphi)$, it does not vanish identically, and

$$s(\xi + \mu) = \sum_{\nu} w^{\nu}\varphi(\xi + \mu - \nu) = \sum_{\nu} w^{\nu+\mu}\varphi(\xi - \nu) =$$

$$= w^{\mu} \sum_{\nu} w^{\nu}\varphi(\xi - \nu) = w^{\mu}\Phi(w) = 0 \qquad (\mu \in \mathbb{Z}^2) .$$

Hence the interpolation problem is not unisolvent.
If Φ has no zeros for which (3.5) holds, then the three polynomials $z^{\nu^0}\Phi(z)$, $z_1^m - 1$, and $z_2^n - 1$ have no common zeros. Here ν^0 is chosen in such a way that $z^{\nu^0}\Phi(z)$ is a polynomial; since φ is finitely supported such a choice can always be made. It follows from the well-known Hilbert's Nullstellen Satz (cf. Van der Waerden [7]) that polynomials $p_1(z)$, $p_2(z)$, and $p_3(z)$ exist such that

$$(3.6) \qquad 1 = p_1(z)z^{\nu^0}\Phi(z) + p_2(z)(z_1^m - 1) + p_3(z)(z_2^n - 1) \qquad (z \in \mathbb{C}^2) .$$

Now let E_1, E_2 be the shift operators (defined on all functions f) given by

$$(3.7) \qquad \begin{cases} (E_1 f)(x) = f(x + e^1) , \\ (E_2 f)(x) = f(x + e^2) \end{cases}$$

for all $x \in \mathbb{R}^2$, where $e^1 = (1,0)$, and $e^2 = (0,1)$.
Furthermore, we define for any $\mu \in \mathbb{Z}^2$ the operator E^{μ} as follows

$$(3.8) \qquad E^{\mu} := E_1^{\mu_1} E_2^{\mu_2} \qquad (\mu = (\mu_1, \mu_2)) .$$

With the help of the shift operators, the interpolation conditions (3.3), and the substitutions $\xi = x^2$, $x = x^1$ in (3.2), we obtain

$$\sum_{\nu} \varphi(\xi - \nu)E^{\nu}s(x) = \sum_{\nu} \varphi(x - \nu)y_{\nu} ,$$

which may also be written as

$$(3.9) \qquad \Phi(E)s(x) = \sum_{\nu} \varphi(x - \nu)y_{\nu} .$$

Since s is an m,n periodic function, one has

$$E_1^m s(x) = s(x), \quad E_2^n s(x) = s(x) .$$

Using these relations and (3.8), we conclude from (3.6) that

$$s(x) = p_1(E)E^{\nu_0}\Phi(E)s(x) = p_1(E)E^{\nu_0}\sum_{\nu}\varphi(x-\nu)y_{\nu}\ .$$

This function uniquely solves the interpolation problem. $\square$

4. Translates of the spline function B^1

As announced in the introductionary section, we will now apply Theorem 3.3 to a special case, where φ is the Fredricson spline B^1, as introduced in Section 2 (see Fig. 2.1). With respect to the point $\xi \in [0,1)^2$, we restrict ourselves to the two cases:

$$\xi^0 := (0,0), \ \xi^{\frac{1}{2}} := (\tfrac{1}{2},\tfrac{1}{2}) .$$

This means, that our interpolation problem is considered at the lattice points $\mathbb{Z}^2$ or at the mid-points $\xi^{\frac{1}{2}} + \mathbb{Z}^2$. In order to apply Theorem 3.3 we have to verify the condition that the collection $\{B^1(x - \nu) \mid \nu \in \mathbb{Z}^2\}$ is independent on $\mathbb{R}^2$. To do so, let $\sum_\nu a_\nu B^1(x - \nu) = 0$ $(x \in \mathbb{R}^2)$. Then, by taking the function value, and the first order partial derivatives at an arbitrary point $x = (i,j)$, we get the three equations

$$a_{i,j} + a_{i-1,j} + a_{i,j+1} = 0$$

$$a_{i,j} - a_{i-1,j} = 0$$

$$a_{i,j} - a_{i,j+1} = 0 .$$

Hence $a_{i,j} = 0$ for all $(i,j) \in \mathbb{Z}^2$. This proves the independence on $\mathbb{R}^2$ of the collection $\{B^1(x - \nu) \mid \nu \in \mathbb{Z}^2\}$.

The next step is to compute the associated characteristic functions given by

$$\Phi_0(z) := \sum_\nu B^1(\xi^0 - \nu)z^\nu, \ \Phi_{\frac{1}{2}}(z) := \sum_\nu B^1(\xi^{\frac{1}{2}} - \nu)z^\nu .$$

It turns out that

$$(4.1) \quad \begin{cases} \Phi_0(z) = \frac{1}{3}(z_1 z_2)^{-1}(z_1 + z_2 + 1) , \\ \Phi_{\frac{1}{2}}(z) = \frac{1}{24} z_1^{-1}(1 + 4z_1 + 14z_1 z_2 + 4z_1^2 z_2 + z_1^2 z_2^2) . \end{cases}$$

Due to the dominant term $14z_1z_2$ the function $\Phi_{\frac{1}{2}}$ has no zeros on the polydisk $|z_1| = |z_2| = 1$. The zeros of Φ_0 on the given polydisk must satisfy $z_1 + z_2 + 1 = 0$. Hence $(z_1,z_2) = (-\frac{1}{2} \pm \frac{1}{2}i\sqrt{3}, -\frac{1}{2} \pm \frac{1}{2}i\sqrt{3})$. Note that $z_1^3 = z_2^3 = 1$. Having examined the zeros of Φ_0 and $\Phi_{\frac{1}{2}}$ on the polydisk, Theorem 3.3 now leads to the following

Theorem 4.1. Let $(y_\mu) \in Y_{m,n}$. Then there exists a unique spline function $s \in T_{n,m}(B^1)$ such that $s(\xi + \mu) = y_\mu$ in the following two cases:

i) $\xi = \xi_0$, n,m arbitrary,

ii) $\xi = \xi_{\frac{1}{2}}$, 3 is not a divisor of n and m.

5. The computation of the interpolating periodic spline function

It is known that the numerical computation of a univariate interpolating spline function on a uniform mesh does not involve matrix inversions (cf. Meinardus, Merz [4]). We will show that with respect to our interpolation problem matrix inversions are not needed either. So let the conditions in Theorem 3.3 be satisfied, and let $s \in T_{n,m}(\varphi)$ be the unique function interpolating the m,n periodic sequence (y_μ) of data at the points $\xi + \mu$. Since $s \in T_{n,m}(\varphi) \subset T(\varphi)$, the interpolation conditions imply that

$$\sum_\nu a_\nu \varphi(\xi + \mu - \nu) = y_\mu \qquad (\mu \in \mathbb{Z}^2)$$

(cf. (3.1)) with $(a_\mu) \in Y_{m,n}$. This may also be written as

$$\sum_\nu a_{\nu+\mu} \varphi(\xi - \nu) = y_\mu \qquad (\mu \in \mathbb{Z}^2) \ . \tag{5.1}$$

Our next purpose is to write (5.1) as a matrix equation by mapping an m,n periodic sequence (a_μ) to the matrix

$$A = \begin{pmatrix} a_{0,0} & \cdots & a_{0,n-1} \\ \vdots & & \vdots \\ a_{m-1,0} & \cdots & a_{m-1,n-1} \end{pmatrix} . \tag{5.2}$$

Let $\underset{\sim}{Q}_k$ denote the $k \times k$ permutation matrix

$$\underset{\sim}{Q}_k = \begin{pmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ \vdots & & & & & & & \vdots \\ 0 & & & & & & & 0 \\ 0 & & & & & & & 1 \\ 1 & 0 & . & . & & & & 0 \end{pmatrix} . \tag{5.3}$$

Then the sequences $(a_{\mu+e^1})$, $(a_{\mu+e^2})$ correspond to the matrices $\underset{\sim}{Q}_m A$ and $A\underset{\sim}{Q}_n^{-1}$, respectively. In general, if $\nu = (\nu_1,\nu_2) \in \mathbb{Z}^2$, then the sequence $(a_{\mu+\nu})$ is connected with the matrix $\underset{\sim}{Q}_m^{\nu_1} A \underset{\sim}{Q}_n^{-\nu_2}$. Consequently, Relation 5.1 can be represented by the matrix equation

$$\sum_\nu \varphi(\xi - \nu) \underset{\sim}{Q}_m^{\nu_1} A \underset{\sim}{Q}_n^{-\nu_2} = Y \ . \tag{5.4}$$

The matrix Ω_k is orthogonal and its eigenvalues $\lambda_0,\lambda_1,\ldots,\lambda_{k-1}$ with corresponding eigenvectors $v^0,v^1,\ldots,v^{k-1}$ are given by

$$\lambda_i = \zeta_k^i ,$$

(5.5) $$v^i = (1,\lambda_i,\lambda_i^2,\ldots,\lambda_i^{n-1}) ,$$

where $\zeta_k = e^{2\pi j/k}$, i.e., the k-th root of unity ($j^2 = -1$).

The matrix $T_k = \frac{1}{\sqrt{k}}(v^0,v^1,\ldots,v^{k-1})$ consisting of eigenvectors of Ω_k has the property that $\Omega_k = T_k\Lambda_k T_k^*$, where Λ_k is the diagonal matrix $\Lambda_k = \mathrm{diag}(1,\zeta_k,\zeta_k^2,\ldots,\zeta_k^{k-1})$, and T_k^* the adjoint of T_k. Note that $T_k^* = T_k^{-1}$. Substituting the relation $\Omega_k = T_k\Lambda_k T_k^*$ in (5.4), we get

(5.6) $$\sum_\nu \varphi(\xi - \nu)\Lambda_m^{\nu_1} T_m^* A T_n \Lambda_n^{-\nu_2} = T_m^* Y T_n .$$

Setting $\tilde{A} = T_m^* A T_n$, $\tilde{Y} = T_m^* Y T_n$, and using the definition of the characteristic function Φ (cf. (3.4)), we obtain

(5.7) $$\tilde{A}_{p,q} = \frac{\tilde{Y}_{p,q}}{\Phi(\zeta_m^p,\zeta_n^{-q})} \qquad (p = 0,1,\ldots,m-1;\ q = 0,1,\ldots,n-1) .$$

Here $\tilde{A}_{p,q}$ and $\tilde{Y}_{p,q}$ denote the p-q-th entry of the matrices $\tilde{A}$ and $\tilde{Y}$ respectively.

For numerical computations the matrix $\tilde{Y}$ may be calculated by means of Fast Fourier Transforms (cf. Merz [5]). Subsequently, the matrix $\tilde{A}$ follows from (5.7), and, finally A may be computed again by means of Fast Fourier Transforms.

Note that $\Phi(\zeta_m^p,\zeta_n^{-q}) \neq 0$, since we have assumed that $\Phi(z)$ has no zeros (z_1,z_2) for which $z_1^m = z_2^n = 1$.

6. The order of convergence

In Lemma 3.1 a fundamental relation has been given, which shall now be used to obtain a qualitative result for the order of convergence in case an 1-1 periodic sufficiently snooth function has been interpolated by a cubic 1-1 periodic spline function from the scaled space $T_h(B^1)$ defined below. Here $h = (h_1,h_2)$ with $h_1 = 1/n$, $h_2 = 1/m$ and $n,m \in \mathbb{N}$. For ease of notation we write $xy = (x_1y_1,x_2y_2)$, $x/y = (x_1/y_1,x_2/y_2)$. Now the space $T_h(B^1)$ is defined as

$$(6.1) \qquad T_h(B') = \{x \mapsto s(x/h) \mid s \in T_{n,m}(B')\} .$$

Our interpolation problem may be formulated as follows. Let $\xi \in [0,1)^2 \subset \mathbb{R}^2$, and let f be an 1-1 periodic function. Find a function $s_h \in T_h(B^1)$ such that

$$(6.2) \qquad S_h(\xi h + \mu h) = f(\xi h + \mu h) \qquad (\mu \in \mathbb{Z}^2) .$$

By setting $s_h(x) = s(x/h)$ with $s \in T_{n,m}(B^1)$, condition (6.2) may be read as $s(\xi + \mu) = f(\xi h + \mu h)$. In order to guarantee existence and uniqueness for each $n,m \in \mathbb{N}$ we assume that the associated characteristic function $\Phi(z)$ has no zeros on the polydisk $|z_1| = |z_2| = 1$ (cf. Theorem 3.3).
Our purpose is to examine the order of convergence of the quantity $|f(x^0) - s_h(x^0)|$ at an arbitrary point $x^0 \in [0,1)^2$ for h_1 and h_2 tending to zero.
So, let $s_h \in T_h(B^1)$ be the unique interpolant of f, and let $s_h(x) = s(x/h)$ with $s \in T_{n,m}(B^1)$. Then, due to Lemma 3.1 one has

$$\sum_\nu B^1(\xi - \nu)s(x + \nu) = \sum_\nu B^1(x - \nu)f(\xi h + \nu h) .$$

Hence,

$$(6.3) \qquad \sum_\nu B^1(\xi - \nu)(s(x + \nu) - f(xh + \nu h)) = g(x)$$

with

$$g(x) = \sum_\nu B^1(x - \nu)f(\xi h + \nu h) - \sum_\nu B^1(\xi - \nu)f(xh + \nu h) .$$

Since $\Phi(z) = \sum_\nu B^1(\xi - \nu)z^\nu$ has no zeros on the polydisk $|z_1| = |z_2| = 1$, coefficients A_ν exist such that

$$(6.4) \qquad \frac{1}{\Phi(z)} = \sum_\nu A_\nu z^\nu$$

which converges absolutely on the polydisk $|z_1| = |z_2| = 1$.
Then by setting $x^0 = xh$, we conclude from (6.3) and (6.4) that

$$(6.5) \qquad s_h(x^0) - f(x^0) = s(x) - f(hx) = \sum_{\mu} A_\mu g(x+\mu) \ .$$

Since

$$\sum_{\nu} B^1(x+\mu-\nu) f(\xi h + \nu h) = \sum_{\nu} B^1(x-\nu) f(\xi h + \nu h + \mu h) \ ,$$

one has

$$(6.6) \qquad g(x+\mu) = \sum_{\nu} B^1(x-\nu) f(\xi h + \nu h + \mu h) - \sum_{\nu} B^1(\xi-\nu) f(x^0 + \nu h + \mu h) \ .$$

Using Taylors formula, we get

$$(6.7) \qquad f(x^0 + \nu h + \mu h) = f(x^0 + \mu h) + (D_{\nu h} f)(x^0 + \mu h) + \\ + \tfrac{1}{2}(D^2_{\nu h} f)(x^0 + \mu h) + \mathcal{O}(|h|^3) \ .$$

Here $D_{\nu h}$ denotes the directional derivative in the direction $\nu h = (\nu_1 h_1, \nu_2 h_2)$, and $|h| = \sqrt{h_1^2 + h_2^2}$. Note that the term $f(x^0 + \nu h + \mu h)$ occurs in the right-hand member of (6.6), and that only those values of ν are of importance for which $\xi - \nu$ belongs to the support of B^1. We conclude that the term $\mathcal{O}(|h|^3)$ can be considered as uniform with respect to ν, x^0, and μ.
Again using Taylors formula for the term $f(\xi h + \nu h + \mu h)$, we get

$$(6.8) \qquad f(\xi h + \nu h + \mu h) = f(x^0 + \mu h) + (D_{\xi h - xh + \nu h} f)(x^0 + \mu h) + \\ + \tfrac{1}{2}(D^2_{\xi h - xh + \nu h} f)(x^0 + \mu h) + \mathcal{O}(|h|^3) \ .$$

Since in this case only those values of ν will be needed for which $x - \nu$ belongs to the support of B^1; we may conclude again that the term $\mathcal{O}(|h|^3)$ is uniform with respect to ν, x^0, and μ. We go on by substituting (6.7) and (6.8) in (6.6), making use of the nice property that the translates of B^1 constitute a partition of unity, i.e.,

$$(6.9) \qquad \sum_{\nu} B^1(x-\nu) = 1 \ , \\ B^1(x) \geq 0 \quad (x \in \mathbb{R}^2) \ .$$

This property may be shown by using the Fourier transform $\check{B}^1$ (of B^1) (cf. (2.3)). The substitution leads to

$$(6.10)\qquad g(x+\mu) = (D_{\xi h - xh}f)(x^0+\mu h) + \sum_\nu B^1(x-\nu)(D_{\nu h}f)(x^0+\mu h) +$$
$$+ \tfrac{1}{2}\sum_\nu B^1(x-\nu)(D^2_{\xi h - xh+\nu h}f)(x^0+\mu h) +$$
$$- \sum_\nu B^1(\xi-\nu)(D_{\nu h}f)(x^0+\mu h) - \tfrac{1}{2}\sum_\nu B^1(\xi-\nu)(D^2_{\nu h}f)(x^0+\mu h)$$
$$+ \mathcal{O}(|h|^3)\ .$$

In order to simplify the right-hand member of (6.10) we need Poisson's formula, which in our context may be stated as follows. Let φ be a smooth finitely supported function having Fourier transform $\check{\varphi}$. Then for all $x \in \mathbb{R}^2$, one has

$$(6.11)\qquad \sum_\nu \varphi(x-\nu) = \sum_\nu \check{\varphi}(2\pi\nu)e^{2\pi i(x,\nu)}\ .$$

Here (x,ν) denotes the inner product of x and ν.
Now, we apply (6.11) to the function $(x,z)B^1(x)$ for an arbitrary $z \in \mathbb{R}^2$. Since by (2.3) $\check{B}^1(2\pi\nu) = \check{B}^1_{\omega_1}(2\pi\nu) = \check{B}^1_{\omega_2}(2\pi\nu) = 0$ for all $\nu \neq 0$, $\check{B}(0) = 1$, $\check{B}^1_{\omega_1}(0) = -j/3$, $\check{B}^1_{\omega_2}(0) = j/3$, one has

$$(6.12)\qquad \sum_\nu (x-\nu,z)B^1(x-\nu) = \frac{1}{3}z_1 - \frac{1}{3}z_2 \qquad (x \in \mathbb{R}^2,\ z=(z_1,z_2) \in \mathbb{R}^2)\ .$$

It follows from (6.9) and (6.12) that

$$\sum_\nu B^1(x-\nu)(D_{\nu h}f)(x^0+\mu h) =$$
$$= \sum_\nu(\nu_1h_1B^1(x-\nu)f_{x_1}(x^0+\mu h) + \nu_2h_2B^1(x-\nu)f_{x_2}(x^0+\mu h)) =$$
$$= (x_1-\tfrac{1}{3})h_1f_{x_1}(x^0+\mu h) + (x_2+\tfrac{1}{3})h_2f_{x_2}(x^0+\mu h) =$$
$$= D_{xh+\beta h}f(x^0+\mu h)\ ,$$

with $\beta = (-\frac{1}{3},\frac{1}{3})$. Hence, (cf. (6.11))

$$(6.13)\qquad g(x+\mu) = \tfrac{1}{2}\sum_\nu B^1(x-\nu)(D^2_{\xi h - xh+\nu h}f)(x^0+\mu h) +$$
$$- \tfrac{1}{2}\sum_\nu B^1(\xi-\nu))D^2_{\nu h}f)(x^0+\mu h) + \mathcal{O}(|h|^3) =$$
$$= \tfrac{1}{2}\sum_\nu B^1(x-\nu)(D^2_{\xi h - xh+\nu h}f)(x^0) - \tfrac{1}{2}\sum_\nu B^1(\xi-\nu)(D^2_{\nu h}f)(x^0) + \mathcal{O}(|\mu||h|^3)\ .$$

The series $\sum_\mu A_\mu z^\mu$ is the Laurent expansion of $\frac{1}{\Phi(z)}$ converging on the polydisk $|z_1| = |z_2| = 1$. Therefore the sequence (A_μ) decays exponentially when $|\mu| \to \infty$. Therefore $\sum_\mu |\mu A_\mu| < \infty$. Note that $\sum_\mu A_\mu = (\sum_\mu B^1(\xi - \mu))^{-1} = 1$. By virtue of (6.5) and (6.13) we finally conclude that

$$(6.14) \qquad s_h(x^0) - f(x^0) = \tfrac{1}{2} \sum_\nu B^1(x - \nu)(D^2_{\xi h - xh + \nu h} f)(x^0) + \\ - \tfrac{1}{2} \sum_\nu B^1(\xi - \nu)(D^2_{\nu h} f)(x^0) + \mathcal{O}(|h|^3) \ .$$

The foregoing formula may be represented in a nicer form by introducing the scaled Fredricson spline B^1_h:

$$B^1_h(x) := B^1(x/h) \ .$$

Moreover we set $\xi_\nu := (\xi + \nu)h$. So, formula (6.19) may be read as follows

$$(6.15) \qquad s_h(x^0) - f(x^0) = \tfrac{1}{2} \sum_\nu B'_h(x^0 - \nu h) D^2_{\xi_\nu - x^0} f)(x^0) + \\ - \tfrac{1}{2} \sum_\nu B'_h(\xi_\nu) D^2_{\nu h} f(x^0) + \mathcal{O}(|h|^3) \ .$$

It is clear that (6.15) shows that $|s_h(x_0) - f(x_0)| = \mathcal{O}(|h|^2)$ $(|h| \to 0)$.

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