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Bivariate-Schurer-Stancu Operators Based on (p, q)-Integers

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Abstract. The aim of this article is to introduce a bivariate extension of Schurer-Stancu operators based on (p, q)-integers. We prove uniform approximation by means of Bohman-Korovkin type theorem, rate of convergence using total modulus of smoothness and degree of approximation via second order modulus of smoothness, Peetre's K-functional, Lipschitz type class.

1. Introduction

In 1962, Schurer [13] introduced the following generalization of the classical Bernstein operators for all non-negative integer l and $n \in \mathbb{N}$

$$B_n^l(f;x) = \sum_{k=0}^{n+l} \binom{n+l}{k} x^k (1-x)^{n+l-k} f\Big(\frac{k}{n}\Big),$$

where $f \in C[0, l+1]$, $x \in [0,1]$ and $\mathbb{N}$ is the set of positive integers. Various modifications have introduced and studied their approximation properties in different functional spaces (see [16], [18], [19], [17] and references therein).

In recent past, the applications of q-calculus attracted the attention of mathematicians and has interesting impact in the research in approximation theory. It has been noticed that linear positive operators based on q-integers are quite effective as far as the rate of convergence is concerned. In 1987, Lupaş [7] first defined q-analogue of Bernstein operators. In 1997, Philips [12] studied other form of Bernstein-polynomials based on q-integers. Several extensions of q-linear positive operators have been studied by different researchers (see [2], [8] and references therein). Recently, Mursaleen et al [9] added an idea based on (p, q)-calculus in approximation theory and gave a (p, q) extension to the classical Bernstein operators. The motive of (p, q)-integers was to generalize various forms of q-oscillator algebras in physics [4]. Several generalization of Bernstein operators were studied using (p, q)-analogue and their approximation properties have been investigated. For instance, (p, q)-Bernstein-Stancu operators [10], (p, q)-Bernstein-Schurer operators [11], (p, q)-genuine Baskakov-Durrmeyer operators [5], (p, q)-Baskakov-Kantorovich operators [6], (p, q)-Schurer-Stancu operators [21], (p, q)-Baskakov-Durrmeyer-Stancu [3], (p, q)-Bivariate Berntein-Kantorovich

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operators [1], (p,q)-Bivariate Berntein-Chlodowsky operators [20] etc. were introduced and their approximation properties are studied. Motivated by the above generalizations, we present a bivariate extension of (p,q)-Schurer-Stancu operators in this paper.

Let $0 < q < p \leq 1$. Then, (p,q)-integers for non negative integers n, k are given by

$$[k]_{p,q} = \frac{p^k - q^k}{p - q} \qquad and \qquad [k]_{p,q} = 1 \qquad for \qquad k = 0.$$

(p,q)-binomial coefficient

$$\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!},$$

and (p,q)-binomial expansion

$$\begin{aligned}(ax+by)^n_{p,q} &= \sum_{k=0}^{n} \binom{n}{k}_{p,q} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} a^{n-k} b^k x^{n-k} y^k, \\ (x+y)^n_{p,q} &= (x+y)(px+py)(p^2x+q^2y)...(p^{n-1}x - q^{n-1}y).\end{aligned}$$

2. Construction of (p,q)-Bivariate-Schurer-Stancu operators

Let $I = [0, l+1]$ and $(x_1, x_2) \in I \times I = [0, l+1] \times [0, l+1]$. Then, for any function $f \in C(I \times I)$ and $(n_1, n_2) \in \mathbb{N} \times \mathbb{N}$, the operators $S^{\alpha_{12},\beta_{12}}_{n_{12},l} : C(I \times I) \to C([0,1] \times [0,1])$ is defined as follows

$$\begin{aligned} S^{\alpha_{12},\beta_{12}}_{n_{12},l}(f; p_{12}, q_{12}; x_1, x_2) = &\sum_{\nu_1=0}^{n_1+l} \sum_{\nu_2=0}^{n_2+l} s^{p_1,q_1}_{n_1,l,\nu_1}(x_1) s^{p_2,q_2}_{n_2,l,\nu_2}(x_2) \\ &\times f\left(\frac{p^{n_1-\nu_1}[\nu_1]_{p_1q_1} + \alpha_1}{[n_1]_{p_1q_1} + \beta_1}, \frac{p^{n_2-\nu_2}[\nu_2]_{p_2q_2} + \alpha_2}{[n_2]_{p_2q_2} + \beta_2} \right), \end{aligned} \tag{1}$$

where $S^{\alpha_{12},\beta_{12}}_{n_{12},l}(f; p_{12}, q_{12}, ; x_1, x_2) = S^{\alpha_1,\alpha_2,\beta_1,\beta_2}_{n_1,n_2,l_1,l_2}(f; p_1, q_1, p_2, q_2; x_1, x_2)$ and

$$s^{p_i,q_i}_{n_i,l,\nu_i}(x_i) = \frac{1}{p^{\frac{(n_i+l)(n_i+l-1)}{2}}} \binom{n_i+l}{\nu_i}_{p_i,q_i} p^{\frac{\nu_i(\nu_i-1)}{2}} x^{\nu_i} \prod_{j=0}^{n_i+l-\nu_i-1} (p_i^j - q_i^j x_i),$$

with the conditions
(i) for any positive real number p_i and q_i $(i = 1, 2)$ such that $0 < q_i < p_i \leq 1$,
(ii) for any non-negative real value of α_i and β_i $(i = 1, 2)$ such that $0 \leq \alpha_i \leq \beta_i$.

Remark 2.1. *One can find that*
(i) if $p_i = 1 (i = 1, 2)$*, then the operators defined by* (1) *reduce to q-Bivariate-Schurer-Stancu operators,*

$$S^{\alpha_{12},\beta_{12}}_{n_{12},l}(f; q_1, q_2; x_1, x_2) = \sum_{\nu_1=0}^{n_1+l} \sum_{\nu_2=0}^{n_2+l} s^{q_1}_{n_1,l,\nu_1}(x_1) s^{q_2}_{n_2,l,\nu_2}(x_2) f\left(\frac{[\nu_1]_{q_1} + \alpha_1}{[n_1]_{q_1} + \beta_1}, \frac{[\nu_2]_{q_2} + \alpha_2}{[n_2]_{q_2} + \beta_2} \right),$$

where

$$s^{q_i}_{n_i,l,\nu_i}(x_i) = \binom{n_i+l}{\nu_i}_{q_i} x^{\nu_i} \prod_{j=0}^{n_i+l-\nu_i-1} (1 - q_i^j x_i),$$

(ii) if $\alpha_i = \beta_i = 0, (i = 1, 2)$, then the operators defined by (1) *reduce to* (p, q)*-Bivariate-Bernstein-Schurer operators*

$$S_{n_{12},l}^{\alpha_{12},\beta_{12}}(f; p_{12}, q_{12}; x_1, x_2) = \sum_{\nu_1=0}^{n_1+l}\sum_{\nu_2=0}^{n_2+l} s_{n_1,l,\nu_1}^{p_1,q_1}(x_1) s_{n_2,l,\nu_2}^{p_2,q_2}(x_2) f\left(\frac{p^{n_1-\nu_1}[\nu_1]_{p_1q_1}}{[n_1]_{p_1q_1}}, \frac{p^{n_2-\nu_2}[\nu_2]_{p_2q_2}}{[n_2]_{p_2q_2}}\right),$$

and

(iii) if $l = 0$ and $\alpha_i = \beta_i = 0, (i = 1, 2)$, then the operators defined by (1) *reduce to* (p, q)*-Bivariate-Bernstein operators*

$$S_{n_{12}}^{\alpha_{12},\beta_{12}}(f; p_{12}, q_{12}; x_1, x_2) = \sum_{\nu_1=0}^{n_1}\sum_{\nu_2=0}^{n_2} s_{n_1,\nu_1}^{p_1,q_1}(x_1) s_{n_2,\nu_2}^{p_2,q_2}(x_2) f\left(\frac{p^{n_1-\nu_1}[\nu_1]_{p_1q_1}}{[n_1]_{p_1q_1}}, \frac{p^{n_2-\nu_2}[\nu_2]_{p_2q_2}}{[n_2]_{p_2q_2}}\right),$$

where

$$s_{n_i,\nu_i}^{p_i,q_i}(x_i) = \frac{1}{p^{\frac{(n_i)(n_i-1)}{2}}}\binom{n_i}{\nu_i}_{p_i,q_i} p^{\frac{\nu_i(\nu_i-1)}{2}} x^{\nu_i} \prod_{j=0}^{n_i-\nu_i-1}(p_i^j - q_i^j x_i).$$

Lemma 2.2. *Let $e_{i,j} = x_1^i x_2^j, 0 \le i, j \le 2$ are the two dimensional test functions. Then, we have*

$$\begin{aligned}
S_{n_{12},l}^{\alpha_{12},\beta_{12}}(e_{0,0}; p_{12}, q_{12}; x_1, x_2) &= 1,\\
S_{n_{12},l}^{\alpha_{12},\beta_{12}}(e_{10}; p_{12}, q_{12}; x_1, x_2) &= \frac{[n_1 + l]x_1 + \alpha_1}{[n_1] + \beta_1},\\
S_{n_{12},l}^{\alpha_{12},\beta_{12}}(e_{01}; p_{12}, q_{12}; x_1, x_2) &= \frac{[n_2 + l]x_2 + \alpha_2}{[n_2] + \beta_2},\\
S_{n_{12},l}^{\alpha_{12},\beta_{12}}(e_{11}; p_{12}, q_{12}; x_1, x_2) &= \frac{[n_1 + l]x_1 + \alpha_1}{[n_1] + \beta_1}\frac{[n_2 + l]x_2 + \alpha_2}{[n_2] + \beta_2},\\
S_{n_{12},l}^{\alpha_{12},\beta_{12}}(e_{20}; p_{12}, q_{12}; x_1, x_2) &= \frac{[n_1 + l](p_1^{n_1+l-1} + 2\alpha_1)x_1}{([n_1] + \beta_1)^2} + \frac{q_1[n_1 + l][n_1 + l - 1]x_1^2}{([n_1] + \beta_1)^2} + \frac{\alpha_1^2}{([n_1] + \beta_1)^2},\\
S_{n_{12},l}^{\alpha_{12},\beta_{12}}(e_{02}; p_{12}, q_{12}; x_1, x_2) &= \frac{[n_2 + l](p_2^{n_2+l-1} + 2\alpha_2)x_2}{([n_2] + \beta_2)^2} + \frac{q_2[n_2 + l][n_2 + l - 1]x_2^2}{([n_2] + \beta_2)^2} + \frac{\alpha_2^2}{([n_2] + \beta_2)^2}.
\end{aligned}$$

Proof. From equation (1), we find that

$$\begin{aligned}
S_{n_{12},l}^{\alpha_{12}\beta_{12}}(e_{0,0}; p_{12}, q_{12}; x_1, x_2) &= S_{n_1,l_1}^{\alpha_1,\beta_1}(e_0; p_1, q_1; x_1) S_{n_2,l_2}^{\alpha_2,\beta_2}(e_0; p_2, q_2; x_2),\\
S_{n_{12},l}^{\alpha_{12}\beta_{12}}(e_{1,0}; p_{12}, q_{12}; x_1, x_2) &= S_{n_1,l_1}^{\alpha_1,\beta_1}(x_1; p_1, q_1; x_1) S_{n_2,l_2}^{\alpha_2,\beta_2}(e_0; p_2, q_2; x_2),\\
S_{n_{12},l}^{\alpha_{12}\beta_{12}}(e_{0,1}; p_{12}, q_{12}; x_1, x_2) &= S_{n_1,l_1}^{\alpha_1,\beta_1}(e_0; p_1, q_1; x_1) S_{n_2,l_2}^{\alpha_2,\beta_2}(x_2; p_2, q_2; x_2),\\
S_{n_{12},l}^{\alpha_{12}\beta_{12}}(e_{1,1}; p_{12}, q_{12}; x_1, x_2) &= S_{n_1,l_1}^{\alpha_1,\beta_1}(x_1; p_1, q_1; x_1) S_{n_2,l_2}^{\alpha_2,\beta_2}(x_2; p_2, q_2; x_2),\\
S_{n_{12},l}^{\alpha_{12}\beta_{12}}(e_{2,0}; p_{12}, q_{12}; x_1, x_2) &= S_{n_1,l_1}^{\alpha_1,\beta_1}(x_1^2; p_1, q_1; x_1) S_{n_2,l_2}^{\alpha_2,\beta_2}(e_0; p_2, q_2; x_2),\\
S_{n_{12},l}^{\alpha_{12}\beta_{12}}(e_{0,2}; p_{12}, q_{12}; x_1, x_2) &= S_{n_1,l_1}^{\alpha_1,\beta_1}(e_0; p_1, q_1; x_1) S_{n_2,l_2}^{\alpha_2,\beta_2}(x_2^2; p_2, q_2; x_2),
\end{aligned}$$

using these equalities, we can easily prove Lemma 2.2.

Lemma 2.3. *Let $S^{\alpha_{12},\beta_{12}}_{n_{12},l}(f;p_{12},q_{12},;x_1,x_2)$ be defined by* (1)*. Then, we have*

$$\begin{aligned} S^{\alpha_{12}\beta_{12}}_{n_{12},l}(((t_1-x_1)^2);p_{12},q_{12};x_1,x_2) &= \frac{x_1^2}{([n_1]+\beta_1)^2}\Big([n_1+l][n_1+l-1]q_1-2[n_1+l]([n_1]+\beta_1)+([n_1]+\beta_1)^2\Big)\\ &+ \frac{[n_1+l](p_1^{n_1+l-1}+2\alpha_1)-2\alpha_1([n_1]+\beta_1)}{([n_1]+\beta_1)^2}x_1+\frac{\alpha_1^2}{(n+\beta_1)^2}.\\ S^{\alpha_{12}\beta_{12}}_{n_{12},l}(((t_2-x_2)^2);p_{12},q_{12};x_1,x_2) &= \frac{x_2^2}{([n_2]+\beta_2)^2}\Big([n_2+l][n_2+l-1]q_2-2[n_2+l]([n_2]+\beta_2)+([n_2]+\beta_2)^2\Big)\\ &+ \frac{[n_2+l](p_2^{n_2+l-1}+2\alpha_2)-2\alpha_2([n_2]+\beta_2)}{([n_2]+\beta_2)^2}x_2+\frac{\alpha_2^2}{(n+\beta_2)^2}. \end{aligned}$$

Proof. In view of Lemma 2.2 and linearity property, it is easy to prove Lemma 2.3.

3. Main Results

Definition 3.1. *Let $X, Y \subset \mathbb{R}$ be any two given intervals and the set $B(X\times Y) = \{f : X\times Y \to \mathbb{R} | f$ is bounded on $X\times Y\}$. For $f \in B(X\times Y)$, let the function $\omega_{total}(f;\cdot,*) : [0,\infty)\times[0,\infty) \to \mathbb{R}$, defined for any $(\delta_1,\delta_2) \in [0,\infty)\times[0,\infty)$ by*

$$\omega_{total}(f;\delta_1,\delta_2) = \sup_{|x-x'|\le\delta_1,|y-y'|\le\delta_2}\{|f(x,y)-f(x',y')| : (x,y),(x',y') \in [0,\infty)\times[0,\infty)\},$$

is called the first order modulus of smoothness of the function f or the total modulus of continuity of the function f.

In order to get the rate of convergence and degree of approximation for the operators $S^{\alpha_{12}\beta_{12}}_{n_{12},l}$, we consider $p_i = p_{n_i}$ and $q_i = q_{n_i}$ for $i = 1,2$ such that $0 < q_{n_i} < p_{n_i} \le 1$ satisfying

$$\lim_{n_i\to\infty} p_{n_i}^{n_i} \to a_i, \lim_{n_i\to\infty} q_{n_i}^{n_i} \to b_i \text{ where } 0 \le a_i < b_i < 1 \tag{2}$$

and

$$\lim_{n_i\to\infty} p_{n_i} \to 1, \lim_{n_i\to\infty} q_{n_i} \to 1 (i = 1,2). \tag{3}$$

Here, we recall the following result due to Volkov [15]:

Theorem 3.2. *Let I and J be compact intervals of the real line. Let $L_{n_1,n_2} : C(I\times J) \to C(I\times J), (n_1,n_2) \in \mathbb{N}\times\mathbb{N}$ be linear positive operators. If*

$$\begin{aligned} \lim_{n_1,n_2\to\infty} L_{n_1,n_2}(e_{ij}) &= e_{ij}, (i,j) \in \{(0,0),(1,0),(0,1)\},\\ \lim_{n_1,n_2\to\infty} L_{n_1,n_2}(e_{20}+e_{02}) &= e_{20}+e_{02}, \end{aligned}$$

uniformly on $I\times J$, then the sequence $(L_{n_1,n_2}f)$ converges to f uniformly on $I\times J$ for any $f \in C(I\times J)$.

Theorem 3.3. *Let $e_{ij}(x_1,x_2) = x_1^i x_2^j (0 \le i+j \le 2, i,j \in \mathbb{N})$ be the test functions defined on $I\times I$ and $(p_{n_i}), (q_{n_i}), i = 1,2$ be the sequences defined by* (2) *and* (3)*. If*

$$\lim_{n_1,n_2\to\infty} (S^{\alpha_{12}\beta_{12}}_{n_{12},l} e_{ij})(x_1,x_2) = e_{ij}(x_1,x_2),$$

uniformly on $I\times I$, then

$$\lim_{n_1,n_2\to\infty} (S^{\alpha_{12}\beta_{12}}_{n_{12},l} f)(x_1,x_2) = f(x_1,x_2),$$

uniformly for any $f \in C(I\times I)$.

Proof. Using the Theorem 3.2 and Lemma 2.2, Theorem 3.3 can easily be proved.

Theorem 3.4. *[14] Let* $L : C([0,\infty)\times[0,\infty)) \to B([0,\infty)\times[0,\infty))$ *be a linear positive operator. For any* $f \in C(X\times Y)$, *any* $(x,y) \in X \times Y$ *and any* $\delta_1, \delta_2 > 0$, *the following inequality*

$$\begin{aligned}|(Lf)(x,y) - f(x,y)| &\leq |Le_{0,0}(x,y) - 1||f(x,y)| \\ &+ \Big[Le_{0,0}(x,y) + \delta_1^{-1}\sqrt{Le_{0,0}(x,y)(L(\cdot - x^2))^2(x,y)} + \delta_2^{-1}\sqrt{Le_{0,0}(x,y)(L(* - y^2))^2(x,y)} \\ &+ \delta_1^{-1}\delta_2^{-1}\sqrt{(Le_{0,0})^2(x,y)(L(\cdot - x^2))^2(x,y)(L(* - y^2))^2(x,y)}\Big]\omega_{total}(f;\delta_1,\delta_2),\end{aligned}$$

holds.

Theorem 3.5. *Let* $f \in C(I \times I)$ *and* $(x_1, x_2) \in I \times I$. *Then, for* $(n_1, n_2) \in \mathbb{N}$ *and for any* $\delta_1, \delta_2 > 0$, *we have*

$$|(S_{n_{12},l}^{\alpha_{12}\beta_{12}} f)(x_1,x_2) - f(x_1,x_2)| \leq 4\omega_{total}(f;\delta_1,\delta_2),$$

where $\delta_{n_{12},l}^{\alpha_{12}\beta_{12}}(x_i) = \sqrt{S_{n_{12},l}^{\alpha_{12}\beta_{12}}(((t_i - x_i)^2); p_{12}, q_{12}; x_1, x_2)}$.

Proof Using Theorem 3.4 and Lemma 2.3 , we can arrive at the proof of the Theorem 3.5.

4. Local approximations

Let $C_B^2(I) = \{f \in C_B(I) : f^{(i,j)} \in C_B(I), 1 \leq i,j \leq 2\}$, where $C_B(I)$ is the space of all bounded and uniformly continuous functions on I and $f^{i,j}$ is $(i,j)^{th}$-order of partial derivative with respect to x, y of f, endowed with the norm

$$\| f \|_{C_B^2(I)} = \|f\|_{C_B(I)} + \sum_{i=1}^{2}\Big\|\frac{\partial^i f}{\partial x_1^i}\Big\| + \sum_{i=1}^{2}\Big\|\frac{\partial^i f}{\partial x_2^i}\Big\|.$$

The Peetre's K-functional of the function $f \in C_B(I)$ is given by

$$K(f;\delta) = \inf_{g\in C_B(I)^2}\{\|f - g\|_{C_B(I)} + \delta\|g\|_{C_B(I)^2}, \delta > 0\}. \tag{4}$$

The following inequality

$$K(f;\delta) \leq M_1\{\omega_2(f;\sqrt{\delta}) + min(1,\delta)\|f\|_{C_B(I)}\},$$

holds for all $\delta > 0$ where M_1 is a constant independent of δ and f and $\omega_2(f;\sqrt{\delta})$ is the second order modulus of continuity which is defined in a similar manner as the second order modulus of continuity for one variable case

$$\omega(f;\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta},}\ \sup_{x+2h\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|.$$

Theorem 4.1. *Let* (q_{n_i}) *and* (p_{n_i}) *for* $i = 1,2$ *are the real sequences defined in* (2) *and* (3). *Then, for* $f \in C_B^2(I \times I)$, *we have the following*

$$\begin{aligned}|S_{n_{12},l}^{\alpha_{12}\beta_{12}}(g;x_1,x_2) - f(x_1,x_2)| &\leq 4K(f;M_{n_1,n_2}(x_1,x_2)) + \omega\left(f;\sqrt{\Big(\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}\Big)^2 + \Big(\frac{[n_2+l]x_2+\alpha_2}{[n_2]+\beta_2}\Big)^2}\right) \\ &\leq M\left\{\omega_2\Big(f:\sqrt{M_{n_1,n_2}(x_1,x_2)}\Big) + min\{1, M_{n_1,n_2}(x_1,x_2)\}\|f\|_{C_B^2(I)}\right\} \\ &\quad +\omega\left(f;\sqrt{\Big(\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}\Big)^2 + \Big(\frac{[n_2+l]x_2+\alpha_2}{[n_2]+\beta_2}\Big)^2}\right),\end{aligned}$$

where $M_{n_1,n_2}(x_1,x_2) = \Big(\delta_{n_{12},l}^{\alpha_{12}\beta_{12}}(x_1)\Big)^2 + \Big(\delta_{n_{12},l}^{\alpha_{12}\beta_{12}}(x_2)\Big)^2$.

Proof. Consider the auxiliary operators

$$\widehat{S_{n_{12},l}^{\alpha_{12}\beta_{12}}}(f;x_1,x_2) = S_{n_{12},l}^{\alpha_{12}\beta_{12}}(f;x_1,x_2) - f\left(\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}+x_1, \frac{[n_2+l]x_2+\alpha_2}{[n_2]+\beta_2}+x_2\right) + f(x_1,x_2). \tag{5}$$

Then

$$\widehat{S_{n_{12},l}^{\alpha_{12}\beta_{12}}}(f;x_1,x_2) \leq 3\|f\|_{C_B(I)}, \ \widehat{S_{n_{12},l}^{\alpha_{12}\beta_{12}}}(t_1-x_1;x_1,x_2)=0 \quad \text{and} \quad \widehat{S_{n_{12},l}^{\alpha_{12}\beta_{12}}}(t_2-x_2;x_1,x_2)=0. \tag{6}$$

Let $g \in C_B^2(I)$ and $(x_1,x_2) \in I$. By the Taylor's theorem, we have

$$\begin{aligned} g(u_1,u_2)-g(x_1,x_2) &= \frac{\partial g(x_1,x_2)}{\partial x_1}(u_1-x_1) + \int_{x_1}^{u_1}(u_1-\alpha_1)\frac{\partial^2 g(\alpha_1,x_1)}{\partial \alpha_1^2}d\alpha_1 \\ &\quad + \frac{\partial g(x_1,x_2)}{\partial x_2}(u_2-x_2) + \int_{x_2}^{u_2}(u_2-\alpha_2)\frac{\partial^2 g(\alpha_2,x_2)}{\partial \alpha_2^2}d\alpha_2. \end{aligned} \tag{7}$$

Applying the auxiliary operator defined by (5) on both sides of (7), we find

$$\begin{aligned} &\widehat{S_{n_{12},l}^{\alpha_{12}\beta_{12}}}(g;x_1,x_2)-g(x_1,x_2) \\ &= \widehat{S_{n_{12},l}^{\alpha_{12}\beta_{12}}}\left(\int_{x_1}^{u_1}(u_1-\alpha_1)\frac{\partial^2 g(\alpha_1,x_1)}{\partial\alpha_1^2}d\alpha_1;x_1,x_2\right) + \widehat{S_{n_{12},l}^{\alpha_{12}\beta_{12}}}\left(\int_{x_2}^{u_2}(u_2-\alpha_2)\frac{\partial^2 g(\alpha_2,x_2)}{\partial\alpha_2^2}d\alpha_2;x_1,x_2\right) \\ &= S_{n_{12},l}^{\alpha_{12}\beta_{12}}\left(\int_{x_2}^{u_2}(u_2-\alpha_2)\frac{\partial^2 g(\alpha_2,x_2)}{\partial\alpha_2^2}d\alpha_2;x_1,x_2\right) - \int_{x_1}^{\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}+x_1}\left(\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}+x_1-\alpha_1\right)\frac{\partial^2 g(\alpha_1,x_1)}{\partial\alpha_1^2}d\alpha_1 \\ &+ S_{n_{12},l}^{\alpha_{12}\beta_{12}}\left(\int_{x_2}^{u_2}(u_2-\alpha_2)\frac{\partial^2 g(\alpha_2,x_2)}{\partial\alpha_2^2}d\alpha_2;x_1,x_2\right) - \int_{x_2}^{\frac{[n_2+l]x_2+\alpha_2}{[n_2]+\beta_2}+x_2}\left(\frac{[n_2+l]x_2+\alpha_2}{[n_2]+\beta_2}+x_2-\alpha_2\right)\frac{\partial^2 g(\alpha_2,x_2)}{\partial\alpha_2^2}d\alpha_2. \end{aligned}$$

Hence,

$$\begin{aligned} &|\widehat{S_{n_{12},l}^{\alpha_{12}\beta_{12}}}(g;x_1,x_2)-g(x_1,x_2)| \\ &\leq S_{n_{12},l}^{\alpha_{12}\beta_{12}}\left(\left|\int_{x_2}^{u_2}|u_2-\alpha_2|\left|\frac{\partial^2 g(\alpha_2,x_2)}{\partial\alpha_2^2}\right|d\alpha_2\right|;x_1,x_2\right) + \int_{x_1}^{\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}+x_1}\left|\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}+x_1-\alpha_1\right|\left|\frac{\partial^2 g(\alpha_1,x_1)}{\partial\alpha_1^2}\right|d\alpha_1\Bigg| \\ &+S_{n_{12},l}^{\alpha_{12}\beta_{12}}\left|\int_{x_2}^{u_2}|u_2-\alpha_2|\left|\frac{\partial^2 g(\alpha_2,x_2)}{\partial\alpha_2^2}\right|d\alpha_2\Big|;x_1,x_2\right| + \Bigg|\int_{x_2}^{\frac{[n_2+l]x_2+\alpha_2}{[n_2]+\beta_2}+x_2}\left|\frac{[n_2+l]x_2+\alpha_2}{[n_2]+\beta_2}+x_2-\alpha_2\right|\left|\frac{\partial^2 g(\alpha_2,x_2)}{\partial\alpha_2^2}\right|d\alpha_2\Bigg| \\ &\leq \left\{S_{n_{12},l}^{\alpha_{12}\beta_{12}}((u_1-x_1)^2:x_1,x_2) + \left(\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}\right)^2\right\}\|g\|_{C_B^2(I)} \\ &\quad + \left\{S_{n_{12},l}^{\alpha_{12}\beta_{12}}((u_2-x_2)^2:x_1,x_2) + \left(\frac{[n_2+l]x_2+\alpha_2}{[n_2]+\beta_2}\right)^2\right\}\|g\|_{C_B^2(I)} \\ &= \{\left(\delta_{n_{12},l}^{\alpha_{12}\beta_{12}}(x_1)\right)^2 + \left(\delta_{n_{12},l}^{\alpha_{12}\beta_{12}}(x_2)\right)^2\}\|g\|_{C_B^2(I)}, \end{aligned}$$

and

$$\begin{aligned}
|S_{n_{12},l}^{\alpha_{12}\beta_{12}}(g;x_1,x_2)-f(x_1,x_2)| &\leq |\widehat{S_{n_{12},l}^{\alpha_{12}\beta_{12}}}(f-g;x_1,x_2)|+|\widehat{S_{n_{12},l}^{\alpha_{12}\beta_{12}}}(g;x_1,x_2)-g(x_1,x_2)|+|g(x,y)-f(x,y)| \\
&\quad +\left|f\left(\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}+x_1,\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}+x_1\right)-f(x_1,x_2)\right| \\
&\leq 3||f-g||_{C_B(I)}+||f-g||_{C_B(I)}+|\widehat{S_{n_{12},l}^{\alpha_{12}\beta_{12}}}(g;x_1,x_2)-g(x_1,x_2)| \\
&\quad +\left|f\left(\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}+x_1,\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}+x_1\right)-f(x_1,x_2)\right| \\
&\leq 4||f-g||_{C_B(I)}+\{\left(\delta_{n_{12},l}^{\alpha_{12}\beta_{12}}(x_1)\right)^2+\left(\delta_{n_{12},l}^{\alpha_{12}\beta_{12}}(x_2)\right)^2\}||g||_{C_B^2(I)} \\
&\quad +\left|f\left(\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}+x_1,\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}+x_1\right)-f(x_1,x_2)\right| \\
&\leq 4||f-g||_{C_B(I)}+2M_{n_1,n_2}(x_1,x_2)||_{C_B^2(I)} \\
&\quad +\omega\left(f;\sqrt{\left(\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}+x_1\right)^2+\left(\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}+x_1\right)^2}\right).
\end{aligned}$$

Next, using the equation (4), we get

$$\begin{aligned}
|S_{n_{12},l}^{\alpha_{12}\beta_{12}}(g;x_1,x_2) &- f(x_1,x_2)|\leq 4K(f;M_{n_1,n_2}(x_1,x_2)) \\
&+ \omega\left(f;\sqrt{\left(\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}+x_1\right)^2+\left(\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}+x_1\right)^2}\right) \\
&\leq M\left\{\omega_2\left(f:\sqrt{(}M_{n_1,n_2}(x_1,x_2))\right)+min\{1,M_{n_1,n_2}(x_1,x_2)\}||f||_{C_B^2(I)}\right\} \\
&+ \omega\left(f;\sqrt{\left(\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}+x_1\right)^2+\left(\frac{[n_1+l]x_1+\alpha_1}{[n_1]+\beta_1}+x_1\right)^2}\right).
\end{aligned}$$

Now, we discuss the degree of approximation for the (p,q)-Bivariate-Schurer-Stancu operators in the Lipschitz class. We define the Lipschitz class $Lip_M^*(\gamma_1,\gamma_2)$ by means of two variables as follows:

$$|f(t_1,t_2)-f(x_1,x_2)|\leq M|t_1-x_1|^{\gamma_1}|t_2-x_2|^{\gamma_2},$$

where $0<\gamma_1,\gamma_2\leq 1$ and for any $(t_1,t_2),(x_1,x_2)\in I\times I$.

Theorem 4.2. *Let $f\in Lip_M^*(\gamma_1,\gamma_2)$ and $(q_{n_i}),(p_{n_i})$, $i=1,2$ are defined in* (2) *and* (3)*. Then for all $(x_1,x_2)\in I\times I$, we have*

$$|S_{n_{12},l}^{\alpha_{12}\beta_{12}}(g;x_1,x_2)-f(x_1,x_2)|\leq M\delta_{n_1}^{\gamma_1/2}(x_1)\delta_{n_2}^{\gamma_2/2}(x_2)$$

where $\delta_{n_i}(x_i)=S_{n_{12},l}^{\alpha_{12}\beta_{12}}(((t_i-x_i)^2);p_{12},q_{12};x_1,x_2)$.

Proof Since $f\in Lip_M^*(\gamma_1,\gamma_2)$, we can write

$$\begin{aligned}
|S_{n_{12},l}^{\alpha_{12}\beta_{12}}(f;q_{n_{12}},p_{n_{12}};x_1,x_2) &- f(x_1,x_2)| \\
&\leq S_{n_{12},l}^{\alpha_{12}\beta_{12}}(|f(t_1,t_2)-f(x_1,x_2)|;q_{n_{12}},p_{n_{12}};x_1,x_2) \\
&\leq MS_{n_{12},l}^{\alpha_{12}\beta_{12}}(|t_1-x_1|^{\gamma_1}|t_2-x_2|^{\gamma_2};q_{n_{12}},p_{n_{12}};x_1,x_2) \\
&= MS_{n_{12},l}^{\alpha_{12}\beta_{12}}(|t_1-x_1|^{\gamma_1};q_{n_{12}},p_{n_{12}};x_1,x_2)S_{n_{12},l}^{\alpha_{12}\beta_{12}}(|t_2-x_2|^{\gamma_2};q_{n_{12}},p_{n_{12}};x_1,x_2).
\end{aligned}$$

Next, we use the Hölder inequality with $p = \frac{2}{\gamma_1}$, $q = \frac{2}{2-\gamma_1}$ and $p = \frac{2}{\gamma_2}$, $q = \frac{2}{2-\gamma_2}$, respectively, we have

$$\begin{aligned} |S_{n_{12},l}^{\alpha_{12}\beta_{12}}(f;q_{n_{12}},p_{n_{12}};x_1,x_2) &- f(x_1,x_2)| \\ &\leq \left\{S_{n_{12},l}^{\alpha_{12}\beta_{12}}((t_1-x_1)^2;q_{n_{12}},p_{n_{12}};x_1,x_2)\right\}^{\frac{\gamma_1}{2}} \\ &\times \left\{S_{n_{12},l}^{\alpha_{12}\beta_{12}}((1;q_{n_{12}},p_{n_{12}};x_1,x_2)\right\}^{\frac{2}{2-\gamma_1}} \\ &\times \left\{S_{n_{12},l}^{\alpha_{12}\beta_{12}}((t_2-x_2)^2;q_{n_{12}},p_{n_{12}};x_1,x_2)\right\}^{\frac{\gamma_2}{2}} \\ &\times \left\{S_{n_{12},l}^{\alpha_{12}\beta_{12}}((1;q_{n_{12}},p_{n_{12}};x_1,x_2)\right\}^{\frac{2}{2-\gamma_2}} \\ &= M\delta_{n_1}^{\gamma_1/2}(x_1)\delta_{n_2}^{\gamma_2/2}(x_2). \end{aligned}$$

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