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—预报误差模型的渐近性质

朱豫才

中国西安交通大学留学生

CIP-GEGEVENS KONINKLIJKE BIBLIOTHEEK, DEN HAAG

Zhu Yu-Cai

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BLACK BOX IDENTIFICATION OF MIMO TRANSFER FUNCTIONS - ASYMPTOTIC PROPERTIES OF PREDICTION ERROR MODELS

ZHU Yu-cai

Abstract: Identification of MIMO transfer functions is considered. The transfer function matrix is parametrized as black-box models, which have certain shift-properties; no structure or order is chosen a priori. In order to obtain a good transfer function estimate, we allow the order of the model to increase to infinity as the number of data tends to infinity. The expression of asymptotic covariance of the transfer function estimates is derived, which is asymptotic both in the number of data and in the model order. The result indicates that the joint covariance matrix of the transfer functions from inputs to outputs and from driving white noise sources to the additive output disturbances respectively is proportional to the Kronecker product of the inverse of the joint spectrum matrix for the inputs and driving noise and the spectrum matrix of the additive output noise. The factor of proportionality is the ratio of model order to number of data. The result is independent of the particular model structure used. This result is the MIMO extension of the theory of Ljung (1985). The application of this theory for defining the bounds of modelling errors is highlighted.

1 INTRODUCTION

Consider a discrete time system with m inputs and p outputs. A general linear time-invariant model for the relationship between inputs and outputs can be written

$$y(t) = \sum_{k=1}^{\infty} G_k \cdot u(t-k) + \nu(t) \qquad (1.1)$$

where: y(t) is a p-dimensional column output vector at time t; u(t) is an m-dimensional column input vector at time t;
G_k is a sequence of p x m matrices; and $\{\nu(t)\}$ is assumed to be a stochastic stationary process with zero mean values.
When the delay operator q^{-1} is introduced as

$$q^{-1}u(t) = u(t-1)$$

the model (1.1) can also be written

$$y(t) = G(q)u(t) + \nu(t) \qquad (1.2)$$

where

$$G(q) = \sum_{k=1}^{\infty} G_k \cdot q^{-k} \qquad (1.3)$$

The transfer function matrix for the model is given

$$G(e^{i\omega}) = \sum_{k=1}^{\infty} G_k \cdot e^{-i\omega k} \quad -\pi < \omega < \pi \qquad (1.4)$$

For the disturbance, the most common approach is to assume that $\nu(t)$ is the output vector of a stable filter driven by a white noise vector

$$\nu(t) = H(q)e(t) \qquad (1.5)$$

where

$$H(q) = \sum_{k=0}^{\infty} H_k q^{-k}$$

where $\{e(t)\}$ is white noise with covariance matrix R. Both H(q) and

$H^{-1}(q)$ are stable. Then the disturbance $\nu(t)$ will be a stationary process with spectral density

$$\Phi_{\nu}(\omega) = H(e^{i\omega}) \; R \; H^{T}(e^{-i\omega}) \tag{1.6}$$

where $H(e^{i\omega})$ is the p x p transfer function matrix of $H(q)$

$$H(e^{i\omega}) = \sum_{k=0}^{\infty} H_k \cdot e^{-i\omega k} \qquad -\pi \leq \omega \leq \pi \tag{1.7a}$$

and H_k is a sequence of pxp matrices, with

$$H_o = I_p \text{ (p x p identity matrix)} \tag{1.7b}$$

The problem of identification is to estimate an approximate estimation model of the system model above from observed input-output data. We denote the data sequence by Z^N:

$$Z^N \triangleq y(1),u(1),\ldots y(N),u(N) \tag{1.8}$$

where N is called sample number of the data sequence.
If we have parametrized the model in some way:

$$y(t) = G(q,\theta) \; u(t) + H \; (q,\theta) \; \varepsilon(t) \tag{1.9}$$

where θ is a (dx1) parameter vector, a commmon way for estimation is to compute the one-step ahead prediction according to (1.9)

$$\hat{y}(t|\theta) = (I_p - H^{-1}(q,\theta))y(t) + H^{-1}(q,\theta)G(q,\theta)u(t) \tag{1.10}$$

and then to determine the parameters by minimizing the squared prediction errors; that is determine $\hat{\theta}_N \in D_n \subset R^d$, such that

$$V = \frac{1}{N} \sum_{t=1}^{N} \hat{\varepsilon}^T(t,\theta)\hat{\varepsilon}(t,\theta) \tag{1.11}$$

is minimized, where

$$\hat{\varepsilon}(t,\theta) = y(t) - \hat{y}(t|\theta) = H^{-1}(q,\theta)\left[y(t) - G(q,\theta)u(t)\right] \tag{1.12}$$

Expression (1.11) can cover most of the time domain identification techniques in practice. It can be shown that specific methods, e.g. the least squares or maximum likelihood method or k-step ahead prediction

error method, can be obtained from (1.1) by taking a specific model structure.
After the parameter estimation, the transfer function estimate is taken as

$$\hat{G}_N(e^{i\omega}) = G(\hat{\theta}_N, e^{i\omega}) \tag{1.13}$$

Recently, Ljung and Yuan developed a theory to explain the properties of the transfer function estimate. In Ljung and Yuan (1985), it was shown that in SISO cases, for the Markov parameter model (impulse response model), the variance of the transfer function estimate is proportional to the noise to input signal ratio multiplied by the ratio of model order and number of samples. The extension of the result to MIMO Markov parameter models can be found in Yuan and Ljung (1984). In Ljung (1985), it has been shown that the same result holds for the polynomial-type of SISO models, e.g. ARMA model or ARMAX model. This work is to extend the result of Ljung (1985) to MIMO polynomial-type models.

In section 2 the Kronecker matrix product and some of its basic properties will be presented. This will prove useful in the derivation of the result. In section 3 the Box-Jenkins model will be introduced and the shift property of the polynomial-type models will be emphasized. The main result is in section 4. In section 5 an application of the theory is proposed. Section 6 gives conclusions.

2 KRONECKER PRODUCTS

The results here have been adapted from BREWER (1978) and Yuan and Ljung (1984).
Let

$$A = (a_{ij}) \ , \quad B = (b_{ij})$$

be m x n and p x r matrices, respectively. The Kronecker product of A and B is defined as an mp x nr matrix, denoted by $A \otimes B$

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \tag{2.1}$$

It is easy to show that

$$(A \otimes B)(C \otimes D) = AC \otimes BD \tag{2.2}$$

provided the dimensions are compatible. If A and B are square invertible matrices, then

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \tag{2.3}$$

and for any C and D

$$(C \otimes D)^* = C^* \otimes D^* \tag{2.4}$$

where * means conjugate transpose.

The column vector of matrix B(mxn) is defined as

$$\text{col } B \triangleq \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} \quad (mn \times 1) \tag{2.5}$$

where B_j is the j-th column of B.
If A is a p x m matrix and B is an m x r matrix, we have the following useful relationship by using Kronecker products

$$\text{col } AB = (I_r \otimes A) \text{ col } B = (B^T \otimes I_p) \text{ col } A \tag{2.6}$$

With the help of the Kronecker product, we can now present a matrix calculus and some of the properties.
Given A(m x n) and B(p x r), the matrix derivative is defined

$$\frac{\partial A}{\partial B} \triangleq \begin{pmatrix} \frac{\partial A}{\partial b_{11}} & \frac{\partial A}{\partial b_{12}} & & \frac{\partial A}{\partial b_{1r}} \\ \frac{\partial A}{\partial b_{21}} & & & \\ & & & \\ \frac{\partial A}{\partial b_{p1}} & & & \frac{\partial A}{\partial b_{pr}} \end{pmatrix} \tag{2.7}$$

Given A(dim m x n), F(dim s x t) and B(dim p x r), it can be shown that

$$\frac{\partial (AF)}{\partial B} = \frac{\partial A}{\partial B}(I_r \otimes F) + (I_p \otimes A)\frac{\partial F}{\partial B} \tag{2.8}$$

and it can also be shown that

$$\frac{\partial (A^{-1})}{\partial B} = -(I_p \otimes A^{-1})\frac{\partial A}{\partial B}(I_r \otimes A^{-1})$$

provided that A is a square and invertible matrix.

3 BLACK BOX MODELS AND SHIFT PROPERTY

In order to show the idea in a concrete way, we will take a special model structure, the so-called Box-Jenkins model. But the results holds for all the models which have the shift property.
The Box-Jenkins model is given as

$$\begin{aligned} G(q,\theta) &= A^{-1}(q,\theta)\, B(q,\theta) \\ H(q,\theta) &= C^{-1}(q,\theta)\, D(q,\theta) \end{aligned} \tag{3.1}$$

where $A(q,\theta)$, $B(q,\theta)$, $C(q,\theta)$ and $D(q,\theta)$ are polynomial matrices with dimension p x p, p x m, p x p and p x p respectively

$$\left.\begin{aligned} A(q,\theta) &= I_p + A_1 q^{-1} + \dots + A_n q^{-n} \\ B(q,\theta) &= \phantom{I_p + {}} B_1 q^{-1} + \dots + B_n q^{-n} \\ C(q,\theta) &= I_p + C_1 q^{-1} + \dots + C_n q^{-n} \\ D(q,\theta) &= I_p + D_1 q^{-1} + \dots + D_n q^{-n} \end{aligned}\right\} \tag{3.2}$$

Note that (3.2) is a special form of the Box-Jenkins model with

$$A_o = I_p, \quad B_o = 0, \quad C_o = I_p \quad \text{and} \quad D_o = I_p \tag{3.3}$$

Remark

When $A_o = I$, then $[A(q,\theta), B(q,\theta)]$ is called a monic ARMA model of $G(q,\theta)$. It can easily be shown that any ARMA model can be transferred into the monic ARMA model provided A_o is invertible. $B_o = 0$ means that $G(q,\theta)$ is strictly proper. This assumption is justified by the fact that most input-output systems are strictly proper. $C_o = D_o = I_p$ means that $H_o = I_p$ as in (1.7b). (3.2) has the order n.

Now we define the parameter vector as

$$\theta = \text{col}[A_1\ B_1\ C_1\ D_1 \quad A_2\ B_2\ C_2\ D_2 \quad \cdots \quad A_n\ B_n\ C_n\ D_n]$$

$$= \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{pmatrix} \ (d \times 1) \tag{3.4}$$

where

$$\theta_k = \text{col}[A_k\ B_k\ C_k\ D_k] \quad (s \times 1) \qquad \text{for } k = 1,\ldots,n \tag{3.5}$$

Here d is the number of parameters and $s = p(3p + m)$ for the Box-Jenkins model.

Now we shall show the shift property of model (3.1), which is a polynomial-type model.

Let

$$T(q,\theta) \triangleq \text{col}[G(\theta,q)\ H(\theta,q)]$$

$$= \begin{pmatrix} g_{11}(q,\theta) \\ \vdots \\ g_{pm}(q,\theta) \\ \vdots \\ h_{11}(q,\theta) \\ \vdots \\ h_{pp}(q,\theta) \end{pmatrix} \tag{3.6}$$

where $g_{ij}(q,\theta)$ and $h_{ij}(q,\theta)$ are the entries of rational matrices $G(q,\theta)$ and $H(q,\theta)$ respectively.
It is easy to verify that

$$\frac{\partial T^T(q,\theta)}{\partial \theta_k} = q^{-k} \; Z(q,\theta) \tag{3.7}$$

where $Z(q,\theta) \triangleq \frac{\partial}{\partial \theta_1} T^T(q,\theta).q$

Here $\frac{\partial T^T}{\partial \theta_k}(q,\theta)$ is a s x p(p+m) matrix. (3.7) holds because g_{ij} and h_{ij} are rational functions of q^{-1} and θ is specially decomposed as in (3.4). The reader can verify (3.7) by taking a SISO ARMA example.

Equation (3.7) is the so-called "shift property" of model set (3.1) and (3.2), which is one of the keys for deriving our result.

At the end of this section, a gradient of the prediction is introduced which will be important for the asymptotic distribution. We will give an expression of the gradient which is convenient for our purpose.

$$\psi(t,\theta) = \frac{d\hat{y}^T(t|\theta)}{d\theta} \quad (d \times p) \tag{3.8}$$

From (1.10) we get

$$H(q,\theta)\hat{y}(t|\theta) = H(q,\theta)y(t) - y(t) + G(q,\theta)u(t) \tag{3.9}$$

According to the relation (2.6) we have

$$H(q,\theta)\hat{y}(t|\theta) = (u^T(t)\otimes I_p) \text{ col } G(q,\theta) - y(t) + (y^T(t)\otimes I_p)\text{col}H(q,\theta) \tag{3.10}$$

$$\hat{y}^T(t|\theta)H^T(q,\theta) = (\text{col } G(q,\theta))^T(u(t)\otimes I_p)-y^T(t)+(\text{col}H(q,\theta))^T(y(t)\otimes I_p) \tag{3.11}$$

Using (2.8) we obtain the relation

$$\frac{d}{d\theta}\hat{y}^T(t|\theta)H^T(q,\theta) + (I_d \otimes \hat{y}^T(t|\theta))\frac{dH^T(q,\theta)}{d\theta} =$$

$$\frac{d}{d\theta}(\mathrm{col}G(q,\theta))^T(u\otimes I_p) + \frac{d}{d\theta}(\mathrm{col}H(q,\theta))^T(y\otimes I_p) \tag{3.12}$$

It can be shown that (using the properties of the Kronecker product)

$$(I_d \otimes \hat{y}^T)\frac{dH^T(q,\theta)}{d\theta} = \frac{d}{d\theta}(\mathrm{col}H(q,\theta))^T(\hat{y}\otimes I_p) \tag{3.13}$$

Substituting (3.13) into (3.12) leads to

$$\frac{d\hat{y}^T}{d\theta} = \frac{d}{d\theta}T^T(q,\theta).(\zeta(t,\theta)\otimes I_p).(H^T)^{-1} \tag{3.14}$$

where

$$\zeta(t,\theta) = \begin{bmatrix} u(t) \\ \hat{\varepsilon}(t,\theta) \end{bmatrix} \quad \text{and } \hat{\varepsilon}(t,\theta) = y(t) - \hat{y}(t|\theta)$$

It is also easy to show that

$$(\zeta(t,\theta)\otimes I_p).(H^T)^{-1} = (I_{m+p}\otimes(H^T)^{-1}).(\zeta(t,\theta)\otimes I_p) \tag{3.15}$$

Then (3.14) becomes

$$\psi(t,\theta) = \frac{d}{d\theta}(T^T(q,\theta))(I_{m+p}\otimes(H^T(q,\theta))^{-1})(\zeta(t,\theta)\otimes I_p) \tag{3.16}$$

4 ASYMPTOTIC PROPERTIES OF THE MODEL

In this section the main result of the paper will be developed. First some formal assumptions will be given. Then several lemmas will be proved. Finally, we will end up with Theorem 4.1 which gives the expression of the covariance matrix of the transfer function estimates.

To estimate a transfer function matrix is basically a non-parametric problem. Since the system is viewed as a black box, the internal parametrization via θ is merely a vehicle to arrive at this estimate. Then, it is natural to let the model order n depend on the number of observed data

$$n = n(N) \tag{4.1}$$

in order to get the best transfer function estimates. Typically, we allow n(N) tends to infinity when N tends to infinity:

$$n(N) \to \infty \quad \text{as } N \to \infty \tag{4.2}$$

When the model order n increases, the model may lose "parameter identifiability", but it will retain "system identifiability" under weak conditions on the experiment design. See Gustavsson et al. (1977) for a discussion of this point. To deal with this problem, we introduce a regularization procedure in the following way. Let

$$\theta^*(n) = \arg\min_{\theta \in D_n} \bar{E}\hat{\varepsilon}^T(t,\theta)\hat{\varepsilon}(t,\theta) \tag{4.3}$$

where

$$\bar{E}\hat{\varepsilon}^T(t,\theta)\hat{\varepsilon}(t,\theta) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E\,\hat{\varepsilon}^T(t,\theta)\hat{\varepsilon}(t,\theta)$$

(If the minimum is not unique, let $\theta^*(n)$ denote any of the parameter vectors leading to such a minimum).
Here n emphasises that the minimum is carried out over n-th order models.

Now define the estimate $\hat{\theta}_N(n,\delta)$ by

$$\hat{\theta}_N(n,\delta) = \arg\min_{\theta \in D_n} V_N(\theta,\delta,n) \tag{4.4}$$

where

$$V_N(\theta,\delta,n) = \frac{1}{2}\left[\frac{1}{N}\sum_{t=1}^{N} \hat{\varepsilon}^T(t,\theta)\,\hat{\varepsilon}(t,\theta) + \delta(\theta-\theta^*(n))^T(\theta-\theta^*(n))\right] \tag{4.5}$$

Here δ is a regularization parameter, helping us to select a unique minimizing element in (4.4) in cases where $\delta = 0$ leads to non-unique minima.

The procedure here is a technical way of dealing with the unique estimate

$$\hat{G}_N(e^{i\omega}) = G(e^{i\omega},\hat{\theta}_N) = \lim_{\delta\to 0} G(e^{i\omega}, \hat{\theta}_N(n,\delta)) \tag{4.6}$$

by a sequence of unique parameter estimates $\{\hat{\theta}_N(m,\delta)\}$ rather than by the possibly non-unique (but realizable) estimate θ_N.

<u>Further assumptions</u>

Assume that the true system can be described by

$$y(t) = G_o(q)\ u(t) + H_o(q)\ e(t) \tag{4.7}$$

where $\{e(t)\}$ is a white noise vector with covariance matrix R and bounded fourth moments. Moreover, G_o and H_o are stable filters. The output noise spectrum is then

$$\Phi_v(\omega) = H_o(e^{i\omega})R\ H_o^T(e^{-i\omega}) \tag{4.8}$$

Assume the predictor filters $H^{-1}(q,\theta)$ and $H^{-1}G(q,\theta)$ in (1.10) along with their first-, second-, and third-order derivatives with repsect to θ are uniformly stable filters in $\theta \in D_n$ for each given n. Let

$$T_n^*(e^{i\omega}) = T(e^{i\omega},\theta^*(n))$$

$$\hat{T}(e^{i\omega},n,\delta) = T(e^{i\omega},\hat{\theta}_N(n,\delta))$$

$$T_o(e^{i\omega}) = \mathrm{col}\left[G_o(e^{i\omega})\ H_o(e^{i\omega})\right] \tag{4.9}$$

Assume that

$$\lim_{n\to\infty}\ n^2\ E\left[\hat{\varepsilon}(t,\theta^*(n))-e(t)\right]^T\left[\hat{\varepsilon}(t,\theta^*(n))-e(t)\right] = 0 \tag{4.10}$$

which implies that $T_n^*(e^{i\omega})$ tends to $T_o(e^{i\omega})$ as n tends to infinity, i.e. the transfer functions estimates are consistent.

In the same way that $Z(q,\theta)$ defined in (3.7), we denote $Z_o(q)$ as

$$Z_o(q) = \frac{\partial T_o^T(q)}{\partial \theta_1} \cdot q \tag{4.11}$$

and

$$Z_o(e^{i\omega}) = \frac{\partial T_o^T(e^{i\omega})}{\partial \theta_1} \cdot e^{i\omega} \tag{4.12}$$

Assume that

$$Z_o^T(e^{i\omega})\ Z_o(e^{-i\omega}) \text{ is invertible} \tag{4.13}$$

Further, assume that

$$r_u(\tau) = \bar{E}\ u(t)u^T(t-\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E\left[u(t)u^T(t-\tau)\right] \tag{4.14a}$$

$$r_{ue}(\tau) = \bar{E}\ u(t)\ e^T(t-\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E\left[u(t)e^T(t-\tau)\right] \tag{4.14b}$$

$$r_{eu}(\tau) = \bar{E}\ e(t)\ u^T(t-\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E\left[e(t)\ u(t-\tau)\right] \tag{4.14c}$$

exist and that

$$\begin{aligned} r_{ue}(\tau) &= 0 \qquad \text{for } \tau < 0 \\ r_{eu}(\tau) &= 0 \qquad \text{for } \tau > 0 \end{aligned} \tag{4.15}$$

Let the spectrum $\Phi_u(\omega)$ be defined as

$$\Phi_u(\omega) = \sum_{\tau=-\infty}^{\infty} r_u(\tau)\ e^{-i\tau\omega} \tag{4.16}$$

Let $\Phi_{ue}(\omega)$ and $\Phi_{eu}(\omega)$ be defined similarly. Finally, assume that

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{t=1}^{N} E\left[\frac{d}{d\theta} \hat{\varepsilon}^T(t,\theta(n))\hat{\varepsilon}(t,\theta(n))\Big|_{\theta=\theta^*} \right] = 0 \quad (n \text{ fixed}) \tag{4.17}$$

Let us now start the derivation of the expression of the covariance matrix. Denote $\chi(t-1,\theta)$ as the s x p dimensional process

$$\chi(t-1,\theta) = \frac{\partial \hat{y}^T(t|\theta)}{\partial \theta_1} = Z(q,\theta)(I_{m+p} \otimes (H^T)^{-1}).(\zeta(t-1,\theta) \otimes I_p) \quad (4.18)$$

where

$$Z(q,\theta) = \frac{\partial T^T(q,\theta)}{\partial \theta_1} .q \quad (4.19)$$

Then from (3.16) we have

$$\psi(t,\theta) = \begin{pmatrix} \chi(t-1,\theta) \\ \chi(t-2,\theta) \\ \vdots \\ \chi(t-n,\theta) \end{pmatrix} \quad (4.20)$$

Denote the d x d matrix

$$\bar{E}\, \psi(t,\theta)\, \psi^T(t,\theta) \triangleq M_n(\theta) \quad (4.21)$$

It consists of n x n blocks each of dimension s x s, and the k-j block is

$$\bar{E}\, \chi(t-k,\theta)\, \chi^T(t-j,\theta) \triangleq r_\chi(j-k,\theta) \quad (4.22)$$

$M_n(\theta)$ is called block Toeplitz covariance of the s x p dimensional process $\chi(\theta,t)$.

Introduce the s x d matrix

$$W_n(\omega) = \left[e^{+i\omega} I_s \quad e^{+2i\omega} I_s \quad \dots \quad e^{+ni\omega} I_s\right] \quad (4.23)$$

It is well-known that the spectrum of $\chi(t,\theta)$ is

$$\Phi_\chi(\omega,\theta) = \lim_{n\to\infty} \frac{1}{n} W_n(\omega)\, M_n(\theta)\, W_n^T(-\omega)$$

$$= Z(e^{i\omega},\theta)(I_{m+p} \otimes (H^T(e^{i\omega},\theta))^{-1})(\Phi_\zeta(\omega) \otimes I_p)(I_{m+p} \otimes (H^{-1}(e^{-i\omega},\theta)).Z^T(e^{-i\omega},\theta)$$

$$\Phi_\chi(\omega,\theta) = Z(e^{i\omega},\theta)(\Phi_\zeta(\omega)\otimes[(H^T(e^{i\omega},\theta))^{-1}(H^{-1}(e^{-i\omega},\theta))])\ Z^T(e^{-i\omega},\theta) \tag{4.24}$$

Now we have the following result:

Lemma 4.1

Assume that (4.14)-(4.17) hold. Suppose also that

$$C > \|\Phi_u(\omega)\| \qquad \lambda_{min}\,(\Phi_u(\omega) > \Delta > 0 \qquad \forall\omega \tag{4.25}$$

where λ_{min} denotes the minimum singular value of the matrix, and

$$\frac{1}{\sqrt{n(N)}} \sum_{\tau=-n(N)}^{n(N)} |\tau|\ \|r_u(\tau)\| \to 0 \quad \text{as } N \to \infty,\ n(N) \to \infty \tag{4.26}$$

Let $A_d = (a_{ij})$ be an arbitrary d x d matrix whose elements depend on n(N) such that

$$\frac{1}{n(N)} W_n(\omega)\ A_d\ W_n^T(-\omega) \to A(\omega) \quad (s \times s) \qquad \text{as } n(N) \to \infty \tag{4.27}$$

and

$$\lim_{n(N)\to\infty} \sup \|A_d\| < C$$

Here $\|.\|$ is the matrix norm. Then if $n(N) \to \infty$ as $N \to \infty$

$$\lim_{n(N)\to\infty} \frac{1}{n(N)} W_n(\omega)\ (M_n(\theta) + \delta I_d)^{-1}\ A_d\ W_n^T(-\omega)$$

$$= (\Phi_\chi(\omega) + \delta I_s)^{-1} A(\omega) \tag{4.28}$$

Proof: The matrix $M_n(\theta) + \delta I_d$ is the block Toeplitz covariance matrix of the s x p dimensional process

$$\chi(t,\theta) + \sqrt{\delta}\ w(t)$$

where w(t) is an s x p dimensional white noise process with

$R_w = Ew(t)\, w^T(t) = I_s$. The spectrum of this process is given by $(\Phi_\chi(\omega) + \delta I_s)$. The result follows from the corollary to Yuan and Ljung (1984) Lemma 4.3. (Take $\hat{W}_d^*(\omega) = W_n(\omega)$, $\bar{R}_d = M_n(\theta) + \delta I_d$).

□

Similarly we have

Corollary 4.1

$$\lim_{n(N)\to\infty} \frac{1}{n(N)} W_n(\omega)\ A_d(M_n(\theta) + \delta I_d)^{-1}\ W_n^T(-\omega)$$

$$= A(\omega)\ (\Phi_\chi(\omega) + \delta I_s)^{-1} \tag{4.29}$$

Let us now consider the parameter estimate (4.4). First, from (4.3) and (4.4) we have as in Ljung (1978)

$$\hat{\theta}_n(n,\delta) \to \theta^*(n) \quad \text{w.p.1} \quad \text{as } N \to \infty \tag{4.30}$$

From the definition (4.4) and Taylor's expansion, we have

$$0 = V_N'(\hat{\theta}_N(n,\delta),\delta)$$

$$= V_N'(\theta^*(n),\delta) + V_N''(\xi_N^n,\delta)(\hat{\theta}_N(n,\delta) - \theta^*(n)) \tag{4.31}$$

where ξ_N^n belongs to a neighbourhood of $\theta^*(n)$ and from (4.30)

$$\lim_{N\to\infty} |\xi_N^n - \theta^*(n)| = 0 \qquad \text{w.p.1} \tag{4.32}$$

Hence

$$\left[\hat{\theta}_N(n,\delta) - \theta^*(n)\right] = -\left[V_N''(\xi_N^n,\delta)\right]^{-1} V_N'(\theta^*(n),\delta) \tag{4.33}$$

We shall consider each of the factors of the right-hand side of (4.33) in the following lemmas.

For the proof of the following lemma, we introduce

$$\hat{\varepsilon}(t,\theta^*(n)) \overset{\Delta}{=} e(t) + r(t,\theta^*(n)) \tag{4.34}$$

From (4.10) we have

$$E\left[r^T(t,\theta^*(n))r(t,\theta^*(n))\right] \leq C_n^2 / n^2 \quad , \quad \lim_{n\to\infty} C_n = 0 \tag{4.35}$$

<u>Lemma 4.2</u> Under previous assumptions and (4.35)

$$\lim_{n\to\infty} \frac{1}{n(N)} W_n(\omega) V_N''(\xi_N^n,\delta) W_n^T(-\omega)$$

$$= \Phi_\chi(\omega) + \delta I_s \quad \text{w.p. 1} \quad \text{as } N \to \infty \tag{4.36}$$

The proof is given in Appendix 1.

<u>Lemma 4.3</u>: Under condition (4.35) and previous assumptions we have

$$\sqrt{N}(V_N'(\theta^*(n),\delta)) \in \text{As } N(0,Q(n)) \quad \text{as } N \to \infty \tag{4.37}$$

where

$$\lim_{n\to\infty} \frac{1}{n(N)} W_n(\omega) Q(n) W_n^T(-\omega) = \Phi_{\chi R}(\omega)$$

$$= Z(e^{i\omega},\theta^*(n))(\Phi_\zeta(\omega) \left[(H^T(e^{i\omega},\theta^*))^{-1}R(H^{-1}(e^{-i\omega},\theta^*(n))\right]Z^T(e^{-i\omega},\theta^*(n)) \tag{4.38}$$

The proof is given in Appendix 2.

Combining these two lemmas we obtain the following result:

<u>Lemma 4.4</u>

Under the conditions of Lemma 4.3 we have

$$\sqrt{N}\left[\hat{\theta}_N(n,\delta) - \theta^*(n)\right] \in \text{As } N(0,R(n,\delta)) \tag{4.39}$$

where

$$\lim_{n\to\infty} \frac{1}{n(N)} \cdot W_n(\omega)) R(n,\delta) W_n^T(-\omega)$$

$$= \left[\Phi_\chi(\omega) + \delta I_s\right]^{-1} \Phi_{\chi R}(\omega) \left[\Phi_\chi(\omega) + \delta I_s\right]^{-1} \tag{4.40}$$

Proof: (4.33) and lemma 4.3 imply that (4.39) holds with

$$R(n,\delta) = \left[V_N''(\xi_N^n,\delta)\right]^{-1} Q(n) \left[V_N''(\xi_N^n,\delta)\right]^{-1} \tag{4.41}$$

Applying Lemma 4.2, Lemma 4.1 and Corollary 4.1 successively, we obtain then (4.40). ▱

Now we consider

$$\sqrt{N}\left[\hat{T}_N(e^{i\omega},n,\delta) - T^*(e^{i\omega},n)\right] \tag{4.42}$$

By Taylor's expansion

$$\hat{T}_N(e^{i\omega},n,\delta) - T^*(e^{i\omega},n) = T(e^{i\omega},\hat{\theta}_N(n,\delta)) - T(e^{i\omega},\theta^*(n))$$

$$= T'_{\theta^T}(e^{i\omega},\theta^*(n))(\hat{\theta}_N-\theta^*(n)) + O(|\hat{\theta}_N-\theta^*(n)|) \tag{4.43}$$

Thus (4.39) implies that the variable in (4.42) will have an asymptotic normal distribution with covariance matrix

$$\bar{P}(\omega,n,\delta) = \frac{d}{d\theta^T} T(e^{i\omega},\theta)\ R(n,\delta)\ \frac{d}{d\theta} T^T(e^{-i\omega},\theta) \tag{4.44}$$

Now consider the p(p+m) x d matrix

$$\frac{d}{d\theta^T} T(e^{i\omega},\theta) = \left[\frac{\partial}{\partial\theta_1^T} T(e^{i\omega},\theta)\ \ \frac{\partial}{\partial\theta_2^T} T(e^{i\omega},\theta) \ \dots\ \frac{\partial}{\partial\theta_n^T} T(e^{i\omega},\theta)\right]$$

Using the shift property (3.7) we have

$$\frac{\partial}{\partial\theta_k^T} T(e^{i\omega},\theta) = e^{-ik\omega} \frac{\partial}{\partial\theta_1^T} T(e^{i\omega},\theta).e^{i\omega}$$

$$= e^{-ik\omega} Z^T(e^{i\omega},\theta)$$

And using (4.23) we have

$$\frac{d}{d\theta^T} T(e^{i\omega},\theta) = Z^T(e^{i\omega},\theta)\, W_n(-\omega) \tag{4.45}$$

Notice the minus sign in $W_n(-\omega)$.

Combining (4.44) and (4.45) gives

$$\bar{P}(\omega,n,\delta) = Z^T(e^{i\omega},\theta)\, W_n(-\omega)\, R(n,\delta)\, W_n^T(+\omega)\, Z(e^{-i\omega},\theta) \tag{4.46}$$

According to Lemma 4.4 and the fact that $\hat{\varepsilon}(t,\theta^*(n)) \to e(t)$

$$\lim_{n\to\infty} \frac{1}{n(N)} \bar{P}(\omega,n,\delta)$$

$$= Z_o^T(e^{i\omega})\left[Z_o(e^{-i\omega})\, S(-\omega)\, Z_o^T(e^{i\omega}) + \delta I_s\right]^{-1}.$$

$$\cdot Z_o(e^{-i\omega})\, S_R(-\omega)\, Z_o^T(e^{i\omega}) \cdot \left[Z_o(e^{-i\omega}) S(-\omega) Z_o^T(e^{i\omega}) + \delta I_s\right]^{-1} \cdot Z_o(e^{-i\omega}) \tag{4.47}$$

where

$$S = \Phi_\zeta \otimes (H_o^T)^{-1} H_o^{-1} \quad , \quad S_R = \Phi_\zeta \otimes (H_o^T)^{-1} R . H_o^{-1} \tag{4.48}$$

Consider the limit of (4.47) as $\delta \to 0$. Apply the matrix inversion lemma

$$[A + BCD]^{-1} = A^{-1}B[C^{-1}+DA^{-1}B]^{-1}DA^{-1}$$

to $[Z_o(e^{-i\omega})S(-\omega)Z_o^T(e^{i\omega}) + \delta I_s]^{-1} Z_o(e^{-i\omega})$, suppressing arguments and indices

$$[Z\ SZ^T + \delta I]^{-1} Z$$

$$= \left[\frac{1}{\delta} I - \frac{1}{\delta} Z[\delta S^{-1} + Z^T Z]^{-1} Z^T \right] Z$$

$$= \frac{1}{\delta} Z - \frac{1}{\delta} Z\left[\delta(Z^TZ)^{-1}S^{-1} + I\right]^{-1}(Z^TZ)^{-1}(Z^TZ)$$

$$\approx \left[\text{small } \delta\right] \approx \frac{1}{\delta} Z - \frac{1}{\delta} Z\left[I - \delta(Z^TZ)^{-1} S^{-1}\right]$$

$$= Z(Z^TZ)^{-1}S^{-1} \qquad (4.49)$$

Hence the limit of (4.47) as $\delta \to 0$ is

$$S^{-1}(-\omega)\ S_R(-\omega)\ S^{-1}(-\omega)$$

$$= \Phi_\zeta^{-1}(-\omega) \otimes \left[H(e^{+i\omega})\ R\ H^T(e^{-i\omega})\right]$$

$$= \Phi_\zeta^{-1}(-\omega) \otimes \Phi_v(\omega) = \left[\Phi_\zeta^{-1}(\omega)\right]^T \otimes \Phi_v(\omega)$$

Now it is time to state the main result.

Theorem 4.1: Consider the estimate $\hat{T}_N(e^{i\omega},n,\delta)$ under the assumptions (4.3) - (4.17), (4.25) and (4.26).

Then

$$\sqrt{N}\left[\hat{T}_N(e^{i\omega},n,\delta) - T_n^*(e^{i\omega})\right] \in \text{As } N(0,P(\omega,n,\delta)) \qquad (4.50)$$
$$\text{as } N \to \infty \text{ for fixed } n,\delta$$

where

$$\lim_{\delta=0}\ \lim_{n\to\infty}\ \frac{1}{n} P(\omega,n,\delta) = \left[\Phi_\zeta^{-1}(\omega)\right]^T \otimes \Phi_v(\omega)$$

$$= \left(\begin{bmatrix} \Phi_u(\omega) & \Phi_{ue}(\omega) \\ \Phi_{eu}(\omega) & R \end{bmatrix}^{-1}\right)^T \otimes \Phi_v(\omega) \qquad (4.51)$$

This result is very general.

In order to understand what sort of result we have obtained in (4.51), let us make one more assumption. Assume that the system operates in an open loop. Then we have

$$\Phi_{ue}(\omega) = \Phi_{eu}(\omega) = 0$$

and

$$\mathrm{cov}\left[\mathrm{col}\ \hat{G}_N(e^{i\omega},n)\right] \approx \frac{n}{N}\left[\Phi_u^{-1}(\omega)\right]^T \otimes \Phi_v(\omega) \tag{4.52}$$

$$\mathrm{cov}\left[\mathrm{col}\ \hat{H}_N(e^{i\omega},n)\right] \approx \frac{n}{N} R^{-1} \otimes \Phi_v(\omega) \tag{4.53}$$

We see that $\hat{G}$ and $\hat{H}$ are asymptotically uncorrelated. The expression (4.52) says that the covariance of $\hat{G}$ at a given frequency is proportional to the (generalized) noise-to-signal ratio at that frequency. The covariance increases with the order n, not with the number of parameters d. The result in Theorem 4.1 brings us new theoretical insight into identification, together with physical feelings, such as "noise-to-signal ratio".

In the development of Theorem 4.1, we have used the shift property of the model structure, and the prediction error criterion. Therefore, it should be clear that the result holds for all the polynomial-type models which have the shift property. The Box-Jenkins model can cover many special parametrizations of this class, but not all of them. (4.52) is consistent with the result of Yuan and Ljung (1984), taking note of different definitions of $r_u(\tau)$.

The author would like to point out that the right-hand side of (3.22) of Ljung (1985) should be complex conjugated (or transposed).

Following the same argument in the proof of Corollary 3.3 of Ljung (1985), we have

<u>Corollary 4.2</u>

Consider the same situation as in Theorem 4.1, but assume that $H(q,\theta)$ is fixed, and independent of θ.

Assume that the system operates in open loop, i.e. $\Phi_{ue} \equiv 0$, and

$$n^2 \| G_n^*(e^{i\omega}) - G_o(e^{i\omega}) \| \to 0 \quad \text{as } n \to \infty \tag{4.54}$$

Then

$$\sqrt{N}\left[\text{col } \hat{G}_N(e^{i\omega},n,\delta) - \text{col } G_n^*(e^{i\omega})\right] \in \text{As } N(0,P(\omega,n,\delta)) \tag{4.55}$$

$$\lim_{\delta\to 0} \lim_{n\to\infty} \frac{1}{n} P(\omega,n,\delta) = \left[\Phi_u^{-1}(\omega)\right]^T \otimes \Phi_v(\omega) \tag{4.56}$$

A special case of Corollary 4.2 is to let $G(q,\theta)$ as given in (3.1) and

$$H(q,\theta) = I \tag{4.57}$$

This is called the output error method.

Because the expressions of the result are remarkably simple, they are very useful in applications. Ljung and Yuan have used the (other version of the) result for input design and order selection. Here, another application of the result will be proposed.

5 UPPER BOUND OF IDENTIFICATION ERRORS

We know that every model is subject to errors. In the field of system identification, most of the attention is focused on how to describe the model and how to obtain the parameters of the model; less attention has been paid to the study of the errors of the model. In principle, in order to use a black-box model of a system, one needs to model the error and to estimate the error as well. Theorem 4.1 describes the errors of MIMO black-box models in a stochastical way. Recently, robust control theory has been developed (see Vidyasagar (1985)), which is more suitable for industrial process control than the state space method. For the application of this new theory, one needs not only a model of the process, but also an upper bound of the model uncertainty (modelling errors) in the frequency domain. We will show how to derive an upper bound of the model uncertainty of black-box models (or identification errors), based on Theorem 4.1.

Assume that open-loop identification has been performed.
Denote $\Delta G(e^{i\omega})$ as the error of the model

$$\Delta G(e^{i\omega}) = \hat{G}_N(e^{i\omega},n) - G_o(e^{i\omega}) \qquad (5.1)$$

$$\Delta g_{ij}(e^{i\omega}) = \hat{g}^N_{ij}(e^{i\omega},n) - g^0_{ij}(e^{i\omega})$$

Then, from Theorem 4.1 and (4.52) we know that $\Delta g_{ij}(e^{i\omega})$ follows, asymptotically, the normal distribution and

$$\mathrm{var}[\Delta g_{ij}(e^{i\omega})] \approx \frac{n}{N}\,[\Phi_u^{-1}(\omega)]_{jj}\cdot\Phi_{v_i}(\omega) \qquad \forall i,j \qquad (5.2)$$

where $[\Phi_u^{-1}(\omega)]_{jj}$ is the (j,j) entry of the matrix $\Phi_u^{-1}(\omega)$, and $\Phi_{v_i}(\omega)$ is the spectrum of $V_i(t)$, and equals the (i,i) entry of the matrix $\Phi_v(\omega)$. Therefore, asymptotically, we can define the 3σ bound for the error

$$ub_{ij}(\omega) = 3\sqrt{\frac{n}{N}\,[\Phi_u^{-1}(\omega)]_{jj}\cdot\Phi_{v_i}(\omega)} \qquad (5.3)$$

with

$$|\Delta g_{ij}(e^{i\omega})| < ub_{ij}(\omega) \qquad \text{w.p. } 99.7\% \qquad (5.4)$$

Finally, we get an upper bound matrix

$$UB(\omega) = \{ub_{ij}(\omega)\} \qquad (5.5)$$

We can compute $UB(\omega)$ by (5.3), using the estimates of $\Phi_u(\omega)$ and $\Phi_v(\omega)$ and this quantity can be used for robust controller design of the feedback system.

Details on the estimation of the upper bound can be found in Zhu (1987a, 1987b).

6 CONCLUSIONS AND REMARKS

In this work, the asymptotic theory of the prediction error identification of Ljung (1985) has been extended to the MIMO case. The result has the same form as the SISO case. We would like to mention that the result is not only valid for the prediction error method family. The open-loop version of the result holds also for the spectral analysis, see Zhu (1987a, 1987b). Therefore, we can say that the result holds for (almost) all the identification methods which are based on the stochastic estimation theory for linear time-invariant systems. The key to arriving at this result is to let the order of the model go to infinity. One need not worry too much about "infinitely high order" models. Some numerical tests have shown that the asymptotic variance expression is a reasonable approximation for the true variance of the low order model; see Ljung and Yuan (1985), and Ljung (1985). For industrial process identification, we may have a large amount of I/O data, and we have to use very high order models (25-50-th order, for example) to fit the highly complex dynamics of the process (see Backx 1987). Hence, the asymptotic covariance will be a very good approximation of the true covariance.

The derivation of the upper bound of the identification errors from this theory completes the contribution of identification to robust control.

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Appendix 1: The Proof of Lemma 4.2

By standard arguments and a law of large numbers (see e.g. Ljung, 1978) we have

$$V_N''(\theta,\delta) \to \lim_{N\to\infty} E\, V_N''(\theta,\delta) \qquad \text{as } N \to \infty \tag{A1.1}$$

w.p. 1 and uniformly in $\theta \in D_n$.

After some calculation, we have

$$V_N''(\theta,\delta) = \delta I_d + \frac{1}{N}\sum_{t=1}^{N} \psi\cdot\psi^T - \frac{1}{N}\sum_{t=1}^{N} \psi'\cdot(I_d \otimes r(t,\theta)) \tag{A1.2}$$

where $\psi'(t,\theta)$ is the d x dp second derivative matrix of $\hat{y}(t|\theta)$. Combining (A1.2), (4.32) and (4.34) gives

$$V_N''(\xi_N^n,\delta) \to \delta I + \bar{E}\psi(t,\theta^*(n))\psi^T(t,\theta^*(n))$$

$$+ \bar{E}\psi'(t,\theta^*(n))\ (I_d \otimes r(t,\theta^*(n))) \tag{A1.3}$$

using the fact that e(t) and $\psi'(t,\theta)$ are independent.

Remark

The reason why $\psi(t,\theta)$ and $\psi'(t,\theta)$ are independent of e(t) is due to the fact that the "prediction error" criterion is used: $\hat{y}(t|\theta)$ is dependent only on the previous y and u, i.e. $\hat{y}(t|\theta)$ is only dependent on the previous e, and e(t) is an independent variable, therefore $\hat{y}(t|\theta)$ and e(t) are independent, so that $\psi(t,\theta)$, $\psi'(t,\theta)$ are also independent of e(t).

It remains to be shown that the operator norm of the last term of (A1.3) tends to zero as n tends to infinity. We note that $\bar{E}\psi'(t,\theta^*(n))r(t,\theta^*(n))$ is a symmetric matrix and for the (k,j) element of the matrix, using (4.35)

$$|\bar{E}\,\hat{y}^T_{\theta_k\theta_j}(t\ \theta).r(t,\theta^*(n))| <$$

$$\sqrt{E\{r^T(t,\theta^*(n)).r(t,\theta^*(n))\}E\{y^T_{\theta_k\theta_j}(t|\theta)\ y_{\theta_k\theta_j}(t|\theta)\}}$$

$$< C.C_n/n \quad , \quad (C \text{ is a constant})$$

since the predictor filter and their second derivatives are stable.

Hence

$$\sum_{k=1}^{d} |\bar{E}\,\hat{y}^T_{\theta_k\theta_j}(t|\theta)\ r(t,\theta^*(n)| < s.C.C_n \tag{A1.4}$$

Now, the operator norm of any symmetric matrix is bounded by its absolute row sums. Hence we have

$$\|\bar{E}\ \psi'(t,\theta^*(n))(I_d \otimes r(t,\theta^*(n)))\| \to 0 \quad \text{as } n \to \infty \tag{A1.5}$$

In view of the definitions (4.21) and (A1.5) we obtain (4.36).

Appendix 2: The proof of Lemma 4.3

We have from (4.5)

$$V'_N(\theta^*(n),\delta) = \frac{-1}{N}\sum_{t=1}^{N} \psi(t,\theta^*(n))\ \hat{\varepsilon}(t,\theta^*(n)) \tag{A2.1}$$

According to (4.17b) the expected value of (A2.1) tends to zero faster than $1/\sqrt{N}$. From Ljung et al. (1979) it follows that

$$\sqrt{N}\ V'_N(\theta^*(n,\delta),\delta)\ \in \text{As } N(0,\ Q(n)) \tag{A2.2}$$

where

$$Q(n) = \lim_{N\to\infty} E\ N\ V'_N(\theta^*(n),\delta)\left[V'_N(\theta^*(n),\delta)\right]^T$$

$$= \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N}\sum_{s=1}^{N} E\left[\psi(t,\theta^*(n))(e(t)+r(t,\theta^*(n)).\right.$$

$$(e^T(s)+r^T(s,\theta^*(n))\psi^T(s,\theta^*(n))\big]$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E\big[\psi(t,\theta^*(n)\ e(t).e^T(t)\ \psi^T(t,\theta^*(n))\big]$$

$$+ \lim_{N\to\infty} \frac{2}{N} \sum_{t=1}^{N} \sum_{s=1}^{N} E\ \psi(t,\theta^*(n))r(t,\theta^*(n))\ e^T(s)\psi^T(s,\theta^*(n))$$

$$+ \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} \sum_{s=1}^{N} E\ \psi(t,\theta^*(n)).r(t,\theta^*(n))r^T(s,\theta^*(n))\ \psi^T(s,\theta^*(n)) \tag{A2.3}$$

The second and the third sums are obtained as filtered white noise and filtered deterministic input. According to Ljung (1985) the values of the entries of these limits are bounded by

$$C\sqrt{\big[Er^T(t,\theta^*(n))\ r(t,\theta^*(n))\big]} = C.C_n/n$$

which shows that their matrix norm is bounded by $C.C_n$. The first term of (A2.3) is

$$\bar{E}\big[\psi(t,\theta^*(n))e(t).e^T(t)\psi^T(t,\theta^*(n))\big] \overset{\Delta}{=} M_e(\theta^*(n)) \tag{A2.4}$$

This is a d x d block Toeplitz covariance matrix of the s x 1 process

$$\chi(t-1,\theta^*(n))e(t) = Z\big(q,\theta^*(n)\ (I_{m+p}\otimes(H^T)^{-1}\big)\big(\zeta(t-1,\theta^*(n)\big)\ e(t) \tag{A2.5}$$

Hence

$$\lim_{n\to\infty} \frac{1}{n(N)}\ W_n(\omega)\ M_e(\theta^*(n))\ W_n^T(-\omega) =$$

$$= Z\big(e^{i\omega},\theta^*(n)\big)\big\{\Phi_\zeta(\omega)\ \big[H^T\big(e^{i\omega},\theta^*(n)\big)^{-1}RH^{-1}\big(e^{-i\omega},\theta^*(n)\big)\big]\big\}Z^T\big(e^{-i\omega},\theta^*(n)\big) \tag{A2.6}$$

and the lemma is proved.

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