Black hole Area-Angular momentum inequality in non-vacuum spacetimes

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We show that the area-angular momentum inequality $A \geq 8\pi|J|$ holds for axially symmetric closed outermost stably marginally trapped surfaces. These are horizon sections (namely, apparent horizons) contained in otherwise generic black hole spacetimes, with non-negative cosmological constant and whose matter content satisfies the dominant energy condition.

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Introduction. Isolated stationary black holes cannot rotate arbitrarily fast. The total angular momentum J in Kerr solutions that are consistent with cosmic censorship is bounded from above by the square of the total mass M. The heuristic *standard picture of gravitational collapse* [1] suggests a more generic validity of this bound. The total mass-angular momentum inequality $J \leq M^2$ has been indeed extended to the dynamical case of vacuum axisymmetric black hole spacetimes [2–7]. Together with this inequality involving global quantities, a quasi-local version of it is desirable since it would offer a valuable insight into the gravitational collapse process in the presence of matter and/or multiple horizons. However, this attempt encounters immediately the ambiguities associated with the quasi-local definition of gravitational mass and angular momentum. In this context, an alternative (but related) bound on the angular momentum can be formulated in terms of a horizon area-angular momentum inequality $A \geq 8\pi|J|$. This inequality was conjectured for the non-vacuum axisymmetric stationary case (actually in the more general charged case) in [8] and then proved in [9, 10], whereas its validity in the vacuum axisymmetric dynamical case was conjectured and discussed in [11], partial results were given in [12, 13] and a complete proof in [14]. Equality holds in the extremal case. Here we reconciliate both results by extending the validity of the inequality to dynamical non-vacuum spacetimes, only requiring axisymmetry on the horizon.

The dynamical non-vacuum case. Proofs of $A \geq 8\pi|J|$ require some kind of geometric stability condition characterizing the surface $\mathcal{S}$ for which the inequality is proved. On the one hand, in the non-vacuum stationary case discussed in [9, 10] surfaces $\mathcal{S}$ are taken to be sections of black hole horizons modeled as outer trapping horizons [15]. This entails, first, the vanishing of the expansion $\theta^{(\ell)}$ associated with light rays emitted from $\mathcal{S}$ along the (outgoing) null normal ℓ^a [i.e. $\mathcal{S}$ is a marginally trapped surface] and, second, that when moving towards the interior of the black hole one finds fully trapped surfaces, so that the variation of $\theta^{(\ell)}$ along some future ingoing null normal k^a is negative (outer condition): $\delta_k\theta^{(\ell)} < 0$ (see [16] for a detailed discussion of this condition in the context of black hole extremality). The latter inequality acts as a stability condition on $\mathcal{S}$ and, actually, is closely related to the *stably outermost* condition imposed on marginally trapped surfaces contained in spatial 3-slices Σ when proving the existence of dynamical trapping horizons [17]. Such *stably outermost* condition means that the variation of $\theta^{(\ell)}$ along some outward deformation of $\mathcal{S}$ in the slice Σ is non-negative. That is, $\delta_v\theta^{(\ell)} \geq 0$ for some spacelike *outgoing* vector v tangent to Σ (see the generalization to spacetime normal vectors in [18]). Regarding now the vacuum dynamical case in [14], the inequality $A \geq 8\pi|J|$ is first proved for stable minimal surfaces $\mathcal{S}$ in a spatial maximal slice Σ, i.e. $\mathcal{S}$ is a local minimum of the area when considering arbitrary deformations of $\mathcal{S}$ in Σ, and then generalized for arbitrary surfaces, in particular horizon sections.

The present discussion of the inequality $A \geq 8\pi|J|$ closely follows the strategy and steps in [14], adapting them to the use of a stability condition in the spirit of those in [9, 10, 16–18], i.e. based on marginally trapped surfaces rather than on minimal surfaces. In the line of [17, 18] we will refer to a marginally trapped surface $\mathcal{S}$ as *(spacetime) stably outermost* (see Definition 1 below) if for some outgoing space-like vector or outgoing past null vector X^a it holds $\delta_X\theta^{(\ell)} \geq 0$. Then, it follows:

Theorem 1. *Given an axisymmetric closed marginally trapped surface $\mathcal{S}$ satisfying the (axisymmetry-compatible) spacetime stably outermost condition, in a spacetime with non-negative cosmological constant and fulfilling the dominant energy condition, it holds the inequality*

$$A \geq 8\pi|J| \ , \tag{1}$$

where A and J are the area and (Komar) angular momentum of $\mathcal{S}$. If equality holds, then $\mathcal{S}$ is a section of a non-expanding horizon with the geometry of extreme Kerr throat sphere.

Note that axisymmetry is only required on the horizon surface (this includes the intrinsic geometry of $\mathcal{S}$ and a certain component of its extrinsic geometry, see below), so that J accounts solely for the angular momentum of the black hole (horizon) in an otherwise generically non-axisymmetric spacetime. Actually, no other geometric requirement is imposed outside $\mathcal{S}$. Regarding the topology of the marginally trapped surface $\mathcal{S}$, this is always a topological sphere (for

$J \neq 0$) as a consequence of the stability condition combined with the dominant energy condition. Therefore, we can assume in the following that $\mathcal{S}$ is a sphere S^2 without loss of generality.

Main steps. The proof in [14] has two parts. First, a geometric part providing a lower bound on the area A. And second, a part making use of variational principles to relate that lower area bound to an upper bound on the angular momentum J and, in a subsequent step, to prove rigidity. Here we recast the first geometric part in the new setting and recover exactly the functional needed in the second variational part, so that results in [12, 14] can be directly applied.

Let us first introduce some notation and consider a closed orientable 2-surface $\mathcal{S}$ embedded in a spacetime M with metric g_{ab} and Levi-Civita connection ∇_a, satisfying the dominant energy condition and with non-negative cosmological constant $\Lambda \geq 0$. We denote the induced metric on $\mathcal{S}$ as q_{ab}, with Levi-Civita connection D_a, Ricci scalar ${}^2R$ and volume element ϵ_{ab} (we will denote by dS the area measure on $\mathcal{S}$). Let us consider null vectors ℓ^a and k^a spanning the normal plane to $\mathcal{S}$ and normalized as $\ell^a k_a = -1$, leaving a (boost) rescaling freedom $\ell'^a = f\ell^a$, $k'^a = f^{-1}k^a$. The expansion $\theta^{(\ell)}$ and the shear $\sigma^{(\ell)}_{ab}$ associated with the null normal ℓ^a are given by

$$\theta^{(\ell)} = q^{ab}\nabla_a \ell_b \ , \ \ \sigma^{(\ell)}_{ab} = q^c{}_a q^e{}_b \nabla_c v_d - \frac{1}{2}\theta^{(\ell)} q_{ab} \ , \quad (2)$$

whereas the normal fundamental form $\Omega^{(\ell)}_a$ is

$$\Omega^{(\ell)}_a = -k^c q^d{}_a \nabla_d \ell_c \ . \quad (3)$$

Transformation rules under a null normal rescaling are

$$\theta^{(\ell')} = f\theta^{(\ell)} \ , \ \sigma^{(\ell')}_{ab} = f\sigma^{(\ell)}_{ab} \ , \ \Omega^{(\ell')}_a = \Omega^{(\ell)}_a + D_a(\mathrm{ln} f). \quad (4)$$

We characterize now the surfaces $\mathcal{S}$ for which the result in Theorem 1 holds. First, we impose $\mathcal{S}$ to be axisymmetric, with axial Killing vector η^a, i.e. $\mathcal{L}_\eta q_{ab} = 0$. The associated (Komar) angular momentum is expressed in terms of $\Omega^{(\ell)}_a$ as

$$J = \frac{1}{8\pi}\int_{\mathcal{S}} \Omega^{(\ell)}_a \eta^a dS \ , \quad (5)$$

where the divergence-free character of η^a together with the transformations properties of $\Omega^{(\ell)}_a$ in (4) guarantee the invariance of J under rescaling of the null normals. We also assume a tetrad $(\xi^a, \eta^a, \ell^a, k^a)$ on $\mathcal{S}$, adapted to axisymmetry in the sense that $\mathcal{L}_\eta \ell^a = \mathcal{L}_\eta k^a = 0$, with ξ^a is a unit vector tangent to $\mathcal{S}$ and orthogonal to η^a, i.e. $\xi^a\eta_a = \xi^a\ell_a = \xi^a k_a = 0$, $\xi^a\xi_a = 1$. We can then write the induced metric on $\mathcal{S}$ as $q_{ab} = \frac{1}{\eta}\eta_a\eta_b + \xi_a\xi_b$, with $\eta = \eta^a\eta_a$, so that

$$\begin{aligned} \Omega^{(\ell)}_a &= \Omega^{(\eta)}_a + \Omega^{(\xi)}_a \\ \Omega^{(\ell)}_a \Omega^{(\ell)a} &= \Omega^{(\eta)}_a \Omega^{(\eta)a} + \Omega^{(\xi)}_a \Omega^{(\xi)a} \ , \end{aligned} \quad (6)$$

with $\Omega^{(\eta)}_a = \eta^b \Omega^{(\ell)}_b \eta_a/\eta$ and $\Omega^{(\xi)}_a = \xi^b \Omega^{(\ell)}_b \xi_a$. In addition, we demand $\Omega^{(\ell)}$ to be also axisymmetric, $\mathcal{L}_\eta \Omega^{(\ell)}_a = 0$. Second, $\mathcal{S}$ is taken to be a marginal trapped surface: $\theta^{(\ell)} = 0$. We will refer to ℓ^a as the *outgoing* null vector. Third, a stability condition must be imposed on $\mathcal{S}$, namely we demand the marginally trapped surface to be a *spacetime stably outermost* in the following sense:

Definition 1. *Given a closed marginally trapped surface $\mathcal{S}$ we will refer to it as spacetime stably outermost if there exists an outgoing ($-k^a$-oriented) vector $X^a = \gamma\ell^a - \psi k^a$, with $\gamma \geq 0$ and $\psi > 0$, such that the variation of $\theta^{(\ell)}$ with respect to X^a fulfills the condition*

$$\delta_X \theta^{(\ell)} \geq 0. \quad (7)$$

If, in addition, X^a (in particular γ, ψ) and $\Omega^{(\ell)}_a$ are axisymmetric, we will refer to $\delta_X\theta^{(\ell)} \geq 0$ as an (axisymmetry-compatible) spacetime stably outermost condition.

Here δ denotes a variation operator associated with a deformation of the surface $\mathcal{S}$ (c.f. for example [17, 19]). Two remarks are in order. First, note that the characterization of a marginally trapped surface as spacetime stably outermost is independent of the choice of future-oriented null normals ℓ^a and k^a. Indeed, given $f > 0$, for $\ell'^a = f\ell^a$ and $k'^a = f^{-1}k^a$ we can write $X^a = \gamma\ell^a - \psi k^a = \gamma'\ell'^a - \psi' k'^a$ (with $\gamma' = f^{-1}\gamma \geq 0$ and $\psi' = f\psi > 0$), and it holds $\delta_X\theta^{(\ell')} = f \cdot \delta_X\theta^{(\ell)} > 0$. Second, the proof of inequality (1) would only require the vector X^a in the stability condition to be outgoing past null. We have, however, kept a more generic characterization in Definition 1 that directly extends the stably outermost condition in [17] (in particular, $\mathcal{S}$ is spacetime stably outermost if there exists a ($-k^a$-oriented) vector for which the surface is stably outermost in the sense of [18]).

We can now establish the lower bound on the horizon area by following analogous steps to those in [14]. First, we derive a generic inequality on $\mathcal{S}$, provided by the following lemma.

Lemma 1. *Given a closed marginally trapped surface $\mathcal{S}$ satisfying the spacetime stably outermost condition for an axisymmetric X^a, then for all axisymmetric $\alpha > 0$ it holds*

$$\begin{aligned} &\int_{\mathcal{S}} \left[D_a\alpha D^a\alpha + \frac{1}{2}\alpha^2\, {}^2R \right] dS \geq \\ &\int_{\mathcal{S}} \left[\alpha^2 \Omega^{(\eta)}_a \Omega^{(\eta)a} + \alpha\beta\sigma^{(\ell)}_{ab}\sigma^{(\ell)ab} + G_{ab}\alpha\ell^a(\alpha k^b + \beta\ell^b) \right] dS \ , \end{aligned} \quad (8)$$

where $\beta = \alpha\gamma/\psi \geq 0$.

To prove it we essentially follow the discussion in section 3.3. of [20]. First, we evaluate $\delta_X\theta^{(\ell)}/\psi$ for the vector $X^a = \gamma\ell^a - \psi k^a$ provided by Definition 1, with axisymmetric γ and ψ (use e.g. Eqs. (2.23) and (2.24) in [19]) and impose $\theta^{(\ell)} = 0$. We can write

$$\begin{aligned} \frac{1}{\psi}\delta_X\theta^{(\ell)} \ = \ & -\frac{\gamma}{\psi}\left[\sigma^{(\ell)}_{ab}\sigma^{(\ell)ab} + G_{ab}\ell^a\ell^b\right] \\ & - {}^2\Delta \mathrm{ln}\psi - D_a\mathrm{ln}\psi D^a \mathrm{ln}\psi + 2\Omega^{(\ell)}_a D^a\mathrm{ln}\psi \quad (9) \\ & - \left[-D^a\Omega^{(\ell)}_a + \Omega^{(\ell)}_c\Omega^{(\ell)c} - \frac{1}{2}{}^2R + G_{ab}k^a\ell^b \right] . \end{aligned}$$

We multiply now the expression by α^2 and integrate on $\mathcal{S}$. Using $\int_{\mathcal{S}} \frac{\alpha^2}{\psi}\delta_X\theta^{(\ell)} dS \geq 0$, integrating by parts to remove boundary terms, we can write

$$\begin{aligned}
0 \le \quad & \int_{\mathcal{S}} \alpha\beta \left[-\sigma^{(\ell)}_{ab}\sigma^{(\ell)\,ab} - G_{ab}\ell^a\ell^b\right] dS \\
+ & \int_{\mathcal{S}} \alpha^2 \left[-\Omega^{(\ell)}_a\Omega^{(\ell)\,a} + \frac{1}{2}{}^2R - G_{ab}k^a\ell^b\right] dS \\
+ & \int_{\mathcal{S}} \left[2\alpha D_a\alpha D^a\ln\psi - \alpha^2 D_a\ln\psi D^a\ln\psi\right] dS \\
+ & \int_{\mathcal{S}} \left[2\alpha^2\Omega^{(\ell)}_a D^a\ln\psi - 2\alpha\Omega^{(\ell)}_a D^a\alpha\right] dS \,. \qquad (10)
\end{aligned}$$

From the axisymmetry of α and ψ, $\Omega^{(\eta)\,a} D_a\alpha = \Omega^{(\eta)\,a} D_a\psi = 0$, and using (6) we can write

$$\begin{aligned}
0 \le \quad & \int_{\mathcal{S}} \alpha\beta \left[-\sigma^{(\ell)}_{ab}\sigma^{(\ell)\,ab} - G_{ab}\ell^a\ell^b\right] dS \\
+ & \int_{\mathcal{S}} \alpha^2 \left[-\Omega^{(\eta)}_a\Omega^{(\eta)\,a} + \frac{1}{2}{}^2R - G_{ab}k^a\ell^b\right] dS \\
+ & \int_{\mathcal{S}} \Big[2(D^a\alpha)(\alpha D_a\ln\psi - \alpha\Omega^{(\xi)}_a) \qquad (11) \\
& -(\alpha D_a\ln\psi - \alpha\Omega^{(\xi)}_a)(\alpha D^a\ln\psi - \alpha\Omega^{(\xi)\,a})\Big] dS \,.
\end{aligned}$$

Making use of the Young's inequality in the last integral

$$D^a\alpha D_a\alpha \ge 2D^a\alpha(\alpha D_a\ln\psi - \alpha\Omega^{(\xi)}_a) - |\alpha D\ln\psi - \alpha\Omega^{(\xi)}|^2$$

inequality (8) follows for all axisymmetric $\alpha > 0$.

Inequality (8) constitutes the first key ingredient in the present discussion and the counterpart of inequality (15) in [14] [inserting their Eqs. (30) and (31)]. In this spacetime version, the geometric meaning of each term in inequality (8) is apparent. For our present purposes, we first disregard the positive-definite gravitational radiation shear squared term. Imposing Einstein equations, we also disregard the cosmological constant and matter terms [25], under the assumption of non-negative cosmological constant $\Lambda \ge 0$ and the dominant energy condition (note that $\alpha k^b + \beta\ell^b$ is a non-spacelike vector). Therefore

$$\int_{\mathcal{S}} \left[D_a\alpha D^a\alpha + \frac{1}{2}\alpha^2\,{}^2R\right] dS \ge \int_{\mathcal{S}} \alpha^2\Omega^{(\eta)}_a\Omega^{(\eta)\,a} dS. \quad (12)$$

This geometric inequality completes the first stage towards the lower bound on A.

In a second stage, under the assumption of axisymmetry we evaluate inequality (12) along the lines in [14]. First, we note that the sphericity of $\mathcal{S}$ follows from Lemma 1 under the outermost stably and dominant energy conditions together with $\Lambda \ge 0$ since, upon the choice of a constant α in Eq. (8), it implies (for non-vanishing angular momentum) a positive value for the Euler characteristic of $\mathcal{S}$. Then, the following form for the axisymmetric line element on $\mathcal{S}$ is adopted

$$ds^2 = q_{ab}dx^a dx^b = e^\sigma\left(e^{2q}d\theta^2 + \sin^2\theta d\varphi^2\right) \,, \qquad (13)$$

with σ and q functions on θ satisfying $\sigma + q = c$, where c is a constant. This coordinate system can always be found in axisymmetry [26]. We can then write $dS = e^c dS_0$, with $dS_0 = \sin\theta d\theta d\varphi$. In addition, the squared norm η of the axial Killing vector $\eta^a = (\partial_\varphi)^a$ is given by

$$\eta = e^\sigma \sin^2\theta \,. \qquad (14)$$

Regarding the left hand side in (12), we proceed exactly as in [14]. In particular, choosing $\alpha = e^{c-\sigma/2}$, the evaluation of the left-hand-side in inequality (12) results in (see [14])

$$\begin{aligned}
& \int_{\mathcal{S}} \left[D_a\alpha D^a\alpha + \frac{1}{2}\alpha^2\,{}^2R\right] dS \qquad (15) \\
& = e^c\left[4\pi(c+1) - \int_{\mathcal{S}}\left(\sigma + \frac{1}{4}\left(\frac{d\sigma}{d\theta}\right)^2\right) dS_0\right].
\end{aligned}$$

The second key ingredient in the present discussion concerns the evaluation of the right hand side in (12), in particular the possibility of making contact with the variational functional $\mathcal{M}$ employed in [12, 14].

Due to the S^2 topology of $\mathcal{S}$, we can always write $\Omega^{(\ell)}_a$ in terms of a divergence-free and an exact form

$$\Omega^{(\ell)}_a = \frac{1}{2\eta}\epsilon_{ab}D^b\bar{\omega} + D_a\lambda \,, \qquad (16)$$

where $\bar{\omega}$ and λ are fixed up to a constant. From the axisymmetry of q_{ab} and $\Omega^{(\ell)}_a$ functions $\bar{\omega}$ and λ are axially symmetric, so that $\Omega^{(\eta)}_a = \frac{1}{2\eta}\epsilon_{ab}D^b\bar{\omega}$ is the divergence-free part whereas $\Omega^{(\xi)}_a$ is the exact (gauge) part. In particular, $\eta^a\Omega^{(\ell)}_a = \frac{1}{2\eta}\epsilon_{ab}\eta^a D^b\bar{\omega}$ and expressing ξ^a as $\xi_b = \eta^{-1/2}\epsilon_{ab}\eta^a$, we have

$$\Omega^{(\ell)}_a\eta^a = \frac{1}{2\eta^{1/2}}\xi^a D_a\bar{\omega} \,. \qquad (17)$$

Plugging this expression into Eq. (5) and using (13) we find

$$J = \frac{1}{8}\int_0^\pi \partial_\theta\bar{\omega}\, d\theta = \frac{1}{8}\left(\bar{\omega}(\pi) - \bar{\omega}(0)\right) \qquad (18)$$

which is identical to the relation between J and the *twist potential* ω in Eq. (12) of [12]. As a remark, we note that if the axial vector η^a on $\mathcal{S}$ extends to a spacetime neighbourhood of $\mathcal{S}$ (something not needed in the present discussion), we can define the *twist* vector of η^a as $\omega_a = \epsilon_{abcd}\eta^b\nabla^c\eta^d$ and the relation $\xi^a\omega_a = \xi^a D_a\bar{\omega}$ holds. In the vacuum case, a twist potential ω satisfying $\omega_a = \nabla_a\omega$ can be defined, so that $\bar{\omega}$ and ω coincide on $\mathcal{S}$ up to a constant. Note however that $\bar{\omega}$ on $\mathcal{S}$ can be defined always.

From Eqs. (16) and (13) and the choice of α, we have

$$\alpha^2\Omega^{(\eta)}_a\Omega^{(\eta)\,a} = \frac{\alpha^2}{4\eta^2}D_a\bar{\omega}D^a\bar{\omega} = \frac{1}{4\eta^2}\left(\frac{d\bar{\omega}}{d\theta}\right)^2 \,. \qquad (19)$$

Using this and (15) in (12) we recover *exactly* the bound

$$A \ge 4\pi e^{\frac{\mathcal{M}-8}{8}} \,, \qquad (20)$$

with the action functional

$$\mathcal{M} = \frac{1}{2\pi}\int_{\mathcal{S}}\left[\left(\frac{d\sigma}{d\theta}\right)^2 + 4\sigma + \frac{1}{\eta^2}\left(\frac{d\bar{\omega}}{d\theta}\right)^2\right] dS_0 \,, \quad (21)$$

in Ref. [14], so that the rest of the proof reduces to that in this reference. Namely, the upper bound in [12] for J

$$e^{\frac{\mathcal{M}-8}{8}} \geq 2|J| \,, \tag{22}$$

together with inequality (20) lead to the area-angular momentum inequality (1) and, in addition, a rigidity result follows: if equality in (1) holds, first, the intrinsic geometry of $\mathcal{S}$ is that of an extreme Kerr throat sphere [11] and, second, the vanishing of the positive-definite terms in (8), implies in particular the vanishing of the shear $\sigma^{(\ell)}_{ab}$, so that $\mathcal{S}$ is an *instantaneous* (non-expanding) isolated horizon [21].

Discussion. We have shown that axisymmetric stable marginally trapped surfaces (in particular, apparent horizons) satisfy the inequality $A \geq 8\pi|J|$ in generically dynamical, non-necessarily axisymmetric, spacetimes with matter. There are two key ingredients enabling the shift from the initial data discussion of inequality (1) in [14] to a spacetime context. First, the derivation of the geometric inequality (8) where the spacetime interpretation of each term in the right hand side is transparent and, more importantly, the global sign is controlled by standard physical assumptions on the matter energy content. This relaxes the counterpart maximal slicing hypothesis in [14]. Second, using the spherical topology of $\mathcal{S}$ we express the quadratic term controlling the angular momentum in the inequality in terms of a potential $\bar{\omega}$ living solely on the sphere and leading to an exact match with the key variational functional in [14]. This permits to avoid any further assumption on the spacetime geometry. A critical ingredient in the present derivation is the stability assumption (7), essentially the *stably outermost* condition for marginally trapped surfaces in [18] that naturally extends the stability condition used in [17] (in the context of spatial 3-slices) to general spacetime embeddings of marginally trapped surfaces. This stability condition, very closely related to the *outer* horizon condition in [15], essentially implies that outer trapping horizons (non-necessarily *future*) satisfy the area-angular momentum inequality (1). If in addition a $\theta^{(k)} < 0$ condition is imposed, then inequality (1) implies that the surface gravity κ of dynamical and isolated horizons [21] is non-negative under $\Lambda \geq 0$ and the dominant energy condition, with $\kappa = 0$ corresponding to the extremal case. Finally, in Ref. [14] the following question is posed: *how small a black hole can be?* Though, according to inequality (1) rotating classical black holes cannot be arbitrarily small, under the light of Eq. (8) it seems also reasonable to expect violations of $A \geq 8\pi|J|$ in near extremal semi-classical collapse due to corrections violating the dominant energy condition, in particular relevant when the black hole is small. This is also consistent with the violations of inequality (1) found in Ref. [22], in the context of black holes accreting matter that violates the null (and therefore the dominant) energy condition. Equation (8) provides a tool to estimate such possible violations.

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[25] In the study of the charged case, the stress-energy tensor of the electromagnetic field is kept. In this case, expression (5) must be completed with an angular momentum electromagnetic contribution [23]. This will be addressed in a forthcoming work.
[26] Note that this is essentially the coordinate system employed in the definition of isolated horizon mass multipoles in [24], where the form of the metric determinant permits to keep the standard orthogonality relations in a spherical harmonic decomposition.