Black holes and the double copy

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Abstract: Recently, a perturbative duality between gauge and gravity theories (the double copy) has been discovered, that is believed to hold to all loop orders. In this paper, we examine the relationship between classical solutions of non-Abelian gauge theory and gravity. We propose a general class of gauge theory solutions that double copy to gravity, namely those involving stationary Kerr-Schild metrics. The Schwarzschild and Kerr black holes (plus their higher-dimensional equivalents) emerge as special cases. We also discuss plane wave solutions. Furthermore, a recently examined double copy between the self-dual sectors of Yang-Mills theory and gravity can be reinterpreted using a momentum-space generalisation of the Kerr-Schild framework.

Contents

1 Introduction

There is a growing body of work examining the relationships between (non)-Abelian gauge theories and gravity, both with and without supersymmetry. Our primary interest in this article will be the *double copy* of ref. [1–3]. This postulates that scattering amplitudes in non-Abelian gauge theories can be expressed such that replacement of colour information by additional kinematic dependence, in a well-defined way, automatically leads to gravity amplitudes. For this to work, a certain duality between colour and kinematics — *BCJ duality* — has to be made manifest in the gauge theory.

BCJ duality, and the double copy, have been proven at tree level [2, 4–11]. The latter is then equivalent to the well-known KLT relations [12], whose origin is the relationship between open and closed string tree amplitudes. Remarkably, however, both BCJ duality and the double copy appear to be true at loop level, including in the absence of super-symmetry [3, 13–28]. Other investigations have verified their application in form factors rather than scattering amplitudes [29], and also in theories with fundamental matter [30]. All-order tests have been performed in the soft limit of fixed-angle scattering [16], and the Regge limit [31–34] (see also [35–47] for related studies). Despite this progress, however, it is perhaps fair to say that we do not fully understand the implications of BCJ duality, and the associated double copy. The main reason for this is that both ideas are currently defined in a purely perturbative context. Consequently, exploration of the deeper explanation of BCJ duality is hampered by the complexity of multiloop calculations. Motivated

by this, one may ask whether any features of the double copy manifest themselves in a classical context. If the answer to this question is yes, this potentially offers a significant insight into gauge and gravity theories.

In fact, a number of cases where classical solutions match up between gauge and gravity theories already exist in the literature. The case of linearised waves is quite obvious; see e.g. [48]. A more involved example is that of (non)-Abelian shockwaves, recently studied in [31] in the context of the high-energy (Regge) limit, where Feynman diagrams can be summed to all orders, exhibiting a double copy between the two theories. One attempt at a Lagrangian-level understanding of the double copy arises after projection to the self-dual sector of both theories, as examined in ref. [49]. There the action can be made manifestly cubic [50] in terms of an adjoint-valued scalar field. The 3-vertex for this field has the form of a product of two structure constants, one for the gauge group, and the other for a kinematic Lie algebra of area-preserving diffeomorphisms. Thus, BCJ duality is made manifest, and the double copy relationship to self-dual gravity is also straightforward.

Another framework for understanding the double copy at the Lagrangian level has been presented in ref. [2] (see also [38] and the more recent light-like gauge approach [51]). In that paper, additional terms are added to the Lagrangian of Yang-Mills theory order by order in perturbation theory, such that Feynman diagrams calculated with this effective Lagrangian are manifestly BCJ-dual. The double copy of this gives a gravitational Lagrangian as expected. However, it is not yet known how to generalise this construction to the nonperturbative level, in that a closed form for the all-order Lagrangian is not available. Beyond Lagrangian approaches, the recently studied scattering equations provide formulas for gauge theory and gravity amplitudes which exhibit an alternative form of double copy [52–54]. Those formulas can be directly obtained from the new ambitwistor string theories, where the distinction between gauge theory and gravity is expressed by a natural choice of worldsheet fields [55, 56]. Closer to our concerns here, the recent work [57] considers the relation between gauge symmetries and gravitational symmetries at the linearised level using a double copy prescription based on the convolution of fields.

The aim of this paper is to find more examples of classical solutions which match under the double copy. We observe that Kerr-Schild coordinates in gravity provide a natural relation to solutions in gauge theory. By examining this connection, we find an infinite class of solutions with the double-copy property. In particular, we will find single copies, or "square roots", of the Schwarzschild[1] and Kerr black holes (plus their higher dimensional generalisations). The proposed relation between the Schwarzschild and Coulomb solutions, for instance, is consistent with the expectation from the construction of these solutions in perturbation theory; see e.g. [59].

An interesting aspect of our work is the importance of a "zeroth" copy of the gauge theory result. This consists of replacing kinematic information in the gauge theory with additional colour information, and leads to a *biadjoint scalar theory* i.e. a cubic scalar

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[1]A previous proposal for a single copy for the Schwarzschild black hole has been made by relating the field strength and curvature tensors in the gauge / gravity theories. Our proposal differs from this as it is set up directly in terms of the gluon and graviton fields directly. We thank Nima Arkani-Hamed for discussions on this point, and Clifford Cheung for providing ref. [58].

theory, in which the scalar transforms in the adjoint of two Lie groups. Such theories have been shown to be increasingly important in studies of BCJ duality and the double copy [11, 53, 57, 73]. The biadjoint scalar theory will also provide a physical interpretation of the double copy procedure, that ties it to the usual formulation involving perturbative scattering amplitudes.

As we will review in what follows, the major ingredients in the double copy story (provided BCJ duality is satisfied) are:

- *Colour factors.* These are replaced by kinematic numerators, which has the effect of replacing Lorentz vectors by an outer product of two Lorentz vectors.

- *Scalar propagators.* Denominators of individual cubic diagrams are not replaced in the double copy procedure. These denominator factors can be interpreted as scalar propagators associated with the topology of the diagram.

In our examples, we will encounter clear analogues of these ingredients. Each example has a Lorentz vector in the gauge theory, such that the corresponding gravity solution consists of an outer product of this vector with itself. Scalar prefactors also arise, fixed by the zeroth copy, that are not squared when performing the double copy procedure. Furthermore, we will see explicitly that these scalar functions can be physically interpreted as scalar propagators integrated over source distributions of charge / mass. Whilst we remain far from understanding the general circumstances in which classical solutions must match up between Yang-Mills theory and gravity (including the appropriate gauges to be chosen on either side of the correspondence), we hope that our results provide a useful springboard for a more systematic and far-reaching investigation.

The structure of the paper is as follows. In section 2, we briefly review aspects of BCJ duality and the double / zeroth copies. In section 3, we review properties of Kerr-Schild coordinates, and show how the self-dual double copy of ref. [49] can be cast in this language. In sections 4, we will examine familiar examples of GR solutions that emerge as special cases of the Kerr-Schild framework. In section 5, we consider some time-dependent solutions, which correspond to non-perturbative waves. Finally, we discuss our results and conclude in section 6.

2 The double copy

In this section, we will review the well-known double copy story for amplitudes [1–3]. We begin by noting that a general massless m-point L-loop gauge theory amplitude $\mathcal{A}_m^{(L)}$ in d space-time dimensions can be written as

$$\mathcal{A}_m^{(L)} = i^L g^{m-2+2L} \sum_{i \in \Gamma} \int \prod_{l=1}^{L} \frac{d^d p_l}{(2\pi)^d} \frac{1}{S_i} \frac{n_i \, c_i}{\prod_{\alpha_i} p_{\alpha_i}^2}, \tag{2.1}$$

where g is the coupling constant. Here the sum is over the complete set of graphs involving triple vertices, consistent with the given loop order and number of external particles, and S_i a symmetry factor for each graph i (the dimension of its automorphism group). The

denominator contains all relevant propagator momenta, and n_i is the kinematic numerator associated with each graph. Finally, c_i is the colour factor of each graph, obtained by dressing each triple vertex with structure constants. The restriction to cubic diagrams in eq. (2.1) uses the fact that four-gluon vertices can always be replaced with sums of products of three-gluon vertices. Furthermore, this representation is not unique - one may shuffle contributions between terms in the sum using (generalised) gauge transformations.

As explained in detail in [1], one may classify cubic diagrams into overlapping sets of three, whose colour factors are related by Jacobi identities, originating from the Lie algebra of the gauge group. BCJ duality is then the statement that, for each such Jacobi identity, the kinematic numerators (as functions of the set of independent external and loop momenta) can be chosen to obey the same relation. We take this as an indication that some kinematic symmetry algebra underlying the numerators exists. It is difficult to relate this kinematic algebra, in general, to a symmetry of the Lagrangian of the theory; a specific obstruction is the presence of the four-gluon operator in the Lagrangian. However, in some cases (such as the self-dual sector of Yang-Mills theory), BCJ duality can be made exact at the Lagrangian level, and the kinematic symmetry interpreted [49]. We return to this point in the following section. String theory amplitudes, in particular using the heterotic string [8] and the pure spinor formalism [9, 60], have also provided important insights into this algebra.

BCJ duality is not manifest for arbitrary choices of the numerators in eq. (2.1). Rather, the BCJ conjecture states that at least one choice of numerators exists such that the duality is manifest. Once this has been achieved, the *double copy* states that

$$\mathcal{M}_m^{(L)} = i^{L+1} \left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_{i \in \Gamma} \int \prod_{l=1}^{L} \frac{d^d p_l}{(2\pi)^d} \frac{1}{S_i} \frac{n_i \, \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2} \tag{2.2}$$

is an m-point, L-loop gravity amplitude, where the graviton has been defined via

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}. \tag{2.3}$$

Equation (2.2) has been obtained from eq. (2.1) by replacing the Yang-Mills coupling with its gravitational counterpart:

$$g \to \frac{\kappa}{2}, \quad \kappa^2 = 16\pi G_N, \tag{2.4}$$

where G_N is Newton's constant. Furthermore, the colour factors $\{c_i\}$ in the gauge theory have been replaced with a second set of kinematic numerators $\{\tilde{n}_i\}$, that may or may not come from the same gauge theory. The gravity theory thus obtained depends on the choice of the two gauge theories (e.g. on the amount of supersymmetry). If both gauge theories are the pure non-supersymmetric Yang-Mills theory, the gravity theory is General Relativity coupled to a 2-form field (equivalent to an axion in four dimensions) and a dilaton[2] [2, 3].

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[2]For recent work regarding the construction of pure gravity (no additional fields) from a double copy procedure, see ref. [30].

Similarly, one may start with eq. (2.1) and replace the kinematic numerators $\{n_i\}$ with a second set of colour factors $\tilde{c}_i$:

$$\mathcal{T}_m^{(L)} = i^L y^{m-2+2L} \sum_{i \in \Gamma} \int \prod_{l=1}^L \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{\tilde{c}_i \, c_i}{\prod_{\alpha_i} p_{\alpha_i}^2}, \tag{2.5}$$

where y is the appropriate coupling constant. The particle content of this theory is a set of scalar fields $\phi^{aa'}$, which transform in the adjoint representation of two Lie algebras. This is an example of a *biadjoint* scalar theory [11, 53, 73], as mentioned in the introduction. The equation of motion of such a theory is explicitly given by

$$\partial^2 \phi^{aa'} - y f^{abc} f^{a'b'c'} \phi^{bb'} \phi^{cc'} = 0, \tag{2.6}$$

where the second term arises from a cubic interaction involving both sets of structure constants.

As is clear in the above discussion, the definitions of BCJ duality and the double/zeroth copies are intrinsically perturbative. Nevertheless, we will see the ideas of this section (such as the replacement of coupling constants and colour information by kinematics) throughout the paper. We will also use the Minkowski metric $\mathrm{diag}(-, +, +, +, \ldots)$ throughout, unless otherwise stated.

3 Kerr-Schild coordinates and the double copy

In this section, we examine a particular choice of coordinates in gravity theories, namely *Kerr-Schild (KS) coordinates*, that will be crucial for what follows. These coordinates are applicable to a specific class of solutions of the Einstein equations, namely Kerr-Schild solutions (for a review, see e.g. [61]). These have the property that the spacetime metric $g_{\mu\nu}$ may be written in the form[3]

$$
\begin{aligned}
g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu} \\
&\equiv \eta_{\mu\nu} + k_\mu k_\nu \phi
\end{aligned} \tag{3.1}
$$

where ϕ is a scalar function, $\eta_{\mu\nu}$ is the Minkowski metric and the (co)vector field k_μ has the property that it is null with respect to both the Minkowski and full metric:

$$\eta^{\mu\nu} k_\mu k_\nu = 0 = g^{\mu\nu} k_\mu k_\nu. \tag{3.2}$$

Therefore, the inverse metric is simply

$$g^{\mu\nu} = \eta^{\mu\nu} - \phi k^\mu k^\nu, \tag{3.3}$$

where we raise the index on k using the Minkowski metric.

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[3]Throughout the paper we will refer to $h_{\mu\nu}$ (the deviation from the Minkowski metric) as the graviton, despite the fact that we are working in the non-linear regime.

To understand the dynamics of Kerr-Schild metrics, we turn to the Einstein equations. Recall that the Einstein tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \tag{3.4}$$

using the conventional notation for the Ricci tensor $R_{\mu\nu}$ and scalar $R = R^\mu{}_\mu$. In terms of the function ϕ and vector k_μ introduced in eq. (3.1), one has

$$R^\mu{}_\nu = \frac{1}{2}\left(\partial^\mu\partial_\alpha\left(\phi k^\alpha k_\nu\right) + \partial_\nu\partial^\alpha\left(\phi k_\alpha k^\mu\right) - \partial^2\left(\phi k^\mu k_\nu\right)\right);$$
$$R = \partial_\mu\partial_\nu\left(\phi k^\mu k^\nu\right), \tag{3.5}$$

where, as usual, $R^\mu{}_\nu = g^{\mu\lambda}R_{\lambda\nu}$ but we have defined $\partial^\mu = \eta^{\mu\nu}\partial_\mu$. This mixed convention is useful, because it is only with this particular combination of indices that the Ricci tensor has the remarkable property that it is linear in ϕ (and indeed in $h_{\mu\nu}$).

A simplification occurs for the stationary case in which all time derivatives vanish. Without loss of generality, we may also set $k^0 = 1$, with all dynamics in the zeroth component contained in the function ϕ. The stationary nature of the solution means that the Ricci tensor can be simplified for each of the components, and one finds (using $\partial_i = \partial^i$, $k_i = k^i$)

$$R^0{}_0 = \frac{1}{2}\nabla^2\phi; \tag{3.6}$$

$$R^i{}_0 = -\frac{1}{2}\partial_j\left[\partial^i\left(\phi k^j\right) - \partial^j\left(\phi k^i\right)\right]; \tag{3.7}$$

$$R^i{}_j = \frac{1}{2}\partial_l\left[\partial^i\left(\phi k^l k_j\right) + \partial_j\left(\phi k^l k^i\right) - \partial^l\left(\phi k^i k_j\right)\right]; \tag{3.8}$$

$$R = \partial_i\partial_j\left(\phi k^i k^j\right), \tag{3.9}$$

where Latin indices run over the spacelike components.

Let us now interpret these equations in the spirit of the double copy. To that end, we define a vector field $A_\mu = \phi k_\mu$, with associated Abelian field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We will refer to this as a Kerr-Schild ansatz. More generally, one can consider a non-Abelian gauge field A_μ^a which, in this Kerr-Schild ansatz, can be written as $A_\mu^a = k_\mu\phi^a$. The vacuum Einstein equations $R_{\mu\nu} = 0$ imply, in the stationary case, that the gauge field satisfies the (Abelian) Maxwell equations

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu(\phi k^\nu) - \partial^\nu(\phi k^\mu)) = 0, \tag{3.10}$$

whose components are related to (3.6) and (3.7). It may seem surprising that the gauge field satisfies the Abelian equations. However, this reflects the linear structure of the Einstein equations in the Kerr-Schild coordinate system.

Going further, we may also interpret ϕ in the spirit of the zeroth copy. The Kerr-Schild ansatz for the gauge field A_μ is obtained by removing a factor of k_ν from the (non-perturbative) graviton $h_{\mu\nu}$. Repeating this, we find that the Kerr-Schild scalar function ϕ

is the field that survives upon taking the zeroth copy. This field then satisfies the equation of motion

$$\nabla^2 \phi = 0. \tag{3.11}$$

Thus, we see that eq. (3.11) is an Abelianised version of the biadjoint field equation of eq. (2.6). It is important to note that the scalar field ϕ plays a role analogous to the propagators in the amplitudes story. It is present, and unchanged, in the scalar, gauge and gravitational cases. Meanwhile, the gauge field A_μ is obtained by multiplying the scalar by a single copy of a vector field k_μ while the gravitational field is obtained by multiplying the scalar by two copies of the vector k_μ. Furthermore, eq. (3.11) gives us a direct physical interpretation of the scalar field ϕ. In the zeroth copy theory, considering the general case in which a source term is also present, the field ϕ will be the Green's function (scalar propagator) integrated over the source. This is the same idea as in the double copy of refs. [2, 3], reviewed here in section 2. In that case, one leaves propagator denominators intact. They correspond to the scalar propagators that one would obtain after taking a zeroth copy.

Let us make a few more remarks on the Kerr-Schild ansatz and the double copy. In general, the squaring map from gauge theory to gravity includes also a dilaton and a 2-form field. This follows from degree-of-freedom counting: two states in gauge theory squares into four states. Our double copy, using the Kerr-Schild ansatz for both gauge theory and gravity, can be expressed as $A_\mu = k_\mu \phi, h_{\mu\nu} = k_\mu k_\nu \phi$. This is manifestly symmetric, so the 2-form field is not present. In addition, since $k^2 = 0$ for the KS ansatz, the trace of the graviton is manifestly zero, removing the dilaton. Thus, the double copy map in this case is simply a map between gauge theory and Einstein gravity.

In this section, we have considered stationary Kerr-Schild solutions. We will see explicit examples in what follows. First, it is instructive to provide further motivation for relating this (Kerr-Schild) structure to the standard BCJ double copy of refs. [2, 3]. To this end, one may examine the relationship between scattering amplitudes in self-dual Yang-Mills theory and gravity [49], which admits a neat reinterpretation using the Kerr-Schild language, albeit in momentum space. This is the subject of the following section.

3.1 Kerr-Schild-like approach to self-dual solutions

In the previous section, we reviewed the properties of Kerr-Schild solutions in General Relativity. These are a special class of solutions whose general form is that of eq. (3.1), where k^μ is defined as a function of the spacetime coordinates i.e. in position space. From the point of view of particle scattering, it is natural to apply a similar ansatz in momentum space, which corresponds to the vector k^μ becoming a differential operator in position space, $k^\mu \to \hat{k}^\mu$. To this end, let us suppose that we can write the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{3.12}$$

$$= \eta_{\mu\nu} + \kappa \hat{k}_\mu \hat{k}_\nu(\phi) \tag{3.13}$$

where to be general we take $\hat{k}_\mu$ to be an arbitrary linear differential operator. We assume that this operator $\hat{k}$ commutes with itself, $[\hat{k}_\mu, \hat{k}_\nu] = 0$, in order that the metric be symmet-

ric. For example, we may take $\hat{k}_\mu = \alpha_\mu^\nu \partial_\nu$ where α is a constant matrix. We also restrict our attention to double copies with no dilaton field. We therefore wish to restrict to trace-free $h_{\mu\nu}$. Thus, we assume that $\hat{k}^2 = 0$ in the sense that $\eta^{\mu\nu}\hat{k}_\mu\hat{k}_\nu(\phi) = 0$. Furthermore, we take $\hat{k}_\mu(\phi)\eta^{\mu\nu}\hat{k}_\nu(\psi) = 0$. Hence, the inverse metric is

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa\hat{k}^\mu\hat{k}^\nu(\phi). \tag{3.14}$$

The solutions that underlie the double-copy in the self-dual sector (we are now considering four space-time dimensions) are precisely of this form. Using light-cone coordinates

$$u = t - z, \quad v = t + z, \quad w = x + iy \quad \bar{w} = x - iy, \tag{3.15}$$

such that the Minkowski line element becomes

$$ds^2 = -du\,dv + dw\,d\bar{w}, \tag{3.16}$$

we can express the relevant $\hat{k}^\mu$ for self-dual theory as

$$\hat{k}_u = 0, \quad \hat{k}_v = \frac{1}{4}\partial_w, \quad \hat{k}_w = 0, \quad \hat{k}_{\bar{w}} = \frac{1}{4}\partial_u. \tag{3.17}$$

Note that $\hat{k} \cdot \partial \equiv 0$. The Christoffel symbols are

$$\Gamma^\rho{}_{\mu\nu} = \frac{\kappa}{2}\left(\partial_\mu\hat{k}^\rho\hat{k}_\nu\phi + \partial_\nu\hat{k}^\rho\hat{k}_\mu\phi - \partial^\rho\hat{k}_\mu\hat{k}_\nu\phi + \kappa(\hat{k}^\rho\hat{k}^\sigma\phi)(\partial_\sigma\hat{k}_\mu\hat{k}_\nu\phi)\right), \tag{3.18}$$

where we have raised the index on the partial derivatives using the Minkowski metric. Using eq. (3.18), we find that the vacuum Einstein equations are

$$R_{\mu\nu} = \frac{\kappa}{2}\left[-\hat{k}_\mu\hat{k}_\nu\partial^2\phi + \kappa(\hat{k}_\mu\hat{k}_\nu\partial_\rho\partial_\sigma\phi)(\hat{k}^\rho\hat{k}^\sigma\phi) - \kappa(\hat{k}_\mu\hat{k}_\rho\partial^\sigma\phi)(\hat{k}_\nu\hat{k}_\sigma\partial^\rho\phi)\right] = 0. \tag{3.19}$$

This differs from the usual Kerr-Schild case because the Ricci tensor is no longer linear in ϕ; we recover linearity if we allow $\hat{k}_\mu$ to be an ordinary vector (rather than a linear operator).

The Einstein equation (3.19) is equivalent to a single scalar equation

$$\partial^2\phi - \frac{\kappa}{2}(\hat{k}^\mu\hat{k}^\nu\phi)(\partial_\mu\partial_\nu\phi) = 0. \tag{3.20}$$

Expanding this using eq. (3.17), one finds that this is none other than the Plebanski equation for self-dual gravity. The Plebanski equation is known to provide an explicit example of a classical double copy, where the gauge theory side is given by a particular form of the self-dual Yang-Mills equation [49]. The next order of business is therefore to examine the Yang-Mills side of the correspondence.

We can perform a similar manipulation for self-dual Yang-Mills theory. In this case, we assume that the gauge field has the "Kerr-Schild" form

$$A_\mu^a = \hat{k}_\mu\phi^a, \tag{3.21}$$

where ϕ^a are Lie-algebra-valued scalars, and the 4-vector $\hat{k}$ is the linear differential operator appearing in eq. (3.17). It is straightforward to calculate the Yang-Mills equations in this gauge; the result is

$$\hat{k}_\nu \partial^2 \phi^a + 2g f^{abc}(\hat{k}^\mu \phi^b)(\hat{k}_\nu \partial_\mu \phi^c) = 0. \tag{3.22}$$

Multiplying by T^a and expanding using the metric in eq. (3.16) and eq. (3.17) yields

$$\hat{k}_\nu \left(\partial^2 \Phi + ig \left[\partial_w \Phi, \partial_u \Phi \right] \right) = 0, \tag{3.23}$$

where $\Phi = \phi^a T^a$. This equation is equivalent to the standard self-dual YM equation, in the form which makes the double copy manifest (see ref. [49] for a detailed review). We therefore see that this self-dual double copy arises from a momentum-space Kerr-Schild description. This is appealing in that the Kerr-Schild ansätze for the gauge field and graviton, eqs. (3.13) and (3.21), are manifest double copies of each other.

In this section, we have seen that the self-dual scattering amplitudes of ref. [49] can be expressed using a momentum-space generalisation of Kerr-Schild coordinates. We are now further encouraged in our identification of the stationary Kerr-Schild double copy structure of section 3 with the standard BCJ double copy. Let us move on to study some examples.

4 Stationary Kerr-Schild solutions

We have already seen in section 3 that there is a special class of gravitational Kerr-Schild solutions, which map to solutions of the Abelian Yang-Mills equations. Given the comments in the previous section, we can make this more formal as follows: let eq. (3.1) be a stationary solution of the Einstein equations ($\partial_0 \phi = \partial_0 k^\mu = 0$), and choose $k^0 = 1$. Then

$$A_a^\mu = c_a \, \phi \, k^\mu \tag{4.1}$$

is a solution of the Yang-Mills equations, for an arbitrary choice of constants c_a (since this ansatz linearises the Yang-Mills equations). This constitutes a large general class of solutions that can be identified between gauge and gravity theories. We can then refer to the gauge theory solution as a *single copy*, or "square root", of the gravity solution. Let us begin with a familiar example.

4.1 The Schwarzschild black hole

The Schwarzschild black hole is the most general spherically symmetric solution of the vacuum Einstein equations. It must be static and asymptotically flat, by Birkhoff's theorem (see e.g. [62]). One can source this with a pointlike mass M, via the energy-momentum tensor

$$T^{\mu\nu} = M v^\mu v^\nu \delta^{(3)}(\mathbf{x}), \tag{4.2}$$

where $v^\mu = (1, 0, 0, 0)$ is a vector pointing purely in the time direction. The solution of the Einstein equations

$$G_{\mu\nu} = \frac{\kappa^2}{2} T_{\mu\nu} \tag{4.3}$$

for the metric $g_{\mu\nu}$ depends on the choice of gauge - in this case a choice of coordinate system. It is well-known, however, that a Kerr-Schild form exists, in which the exterior Schwarzschild metric has the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2GM}{r}k_\mu k_\nu, \tag{4.4}$$

where G is Newton's constant, and

$$k^\mu = \left(1, \frac{x^i}{r}\right), \quad r^2 = x^i x_i, \quad i = 1 \ldots 3. \tag{4.5}$$

Comparing eq. (4.4) with eq. (2.3) and using the relation

$$\kappa^2 = 16\pi G, \tag{4.6}$$

one finds, using a convenient normalisation, that the (non-perturbative) graviton for the exterior Schwarzschild solution is given by

$$h_{\mu\nu} = \frac{\kappa}{2}\phi k_\mu k_\nu, \quad \phi = \frac{M}{4\pi r}. \tag{4.7}$$

According to the results of section 3, one may take a single copy of this solution:

$$A^\mu = \frac{g c_a T^a}{4\pi r}\left(1, \frac{\mathbf{x}}{r}\right) \equiv \frac{g c_a T^a}{4\pi r}k_\mu, \tag{4.8}$$

where k_μ is defined as in eq. (4.5). We have obtained this from eq. (4.7) via the replacements

$$\frac{\kappa}{2} \to g, \quad M \to c_a T^a, \quad k_\mu k_\nu \to k_\mu, \quad \frac{1}{4\pi r} \to \frac{1}{4\pi r}. \tag{4.9}$$

This all makes very good sense from a double copy point of view. The first replacement is the usual identification of coupling constants in the two theories. The second replacement replaces a charge in the gravity theory (a mass) with a corresponding colour charge. Indeed, substituting eq. (4.8) into the Abelian Maxwell equations, one finds

$$\partial_\mu F^{\mu\nu} = j^\nu, \tag{4.10}$$

where the current

$$j^\nu = -g(c_a T^a)v^\nu \delta^{(3)}(\mathbf{x}) \tag{4.11}$$

corresponds to a static colour source located at the origin, with 4-velocity $v^\mu = (1, \mathbf{0})$. It is interesting that this happens: the double copy is a statement about the gauge field and graviton. Here we have inserted extra degrees of freedom into the theory, for the purpose of sourcing the gauge fields. The single copy, applied to the fields, has correctly identified the required source in the gauge theory. The final replacement in eq. (4.9) corresponds to a scalar propagator that remains unchanged on the gravity side, in line with our previous comments.

– 10 –

Let us now physically interpret the gauge solution of eq. (4.8). Given that this is a solution of the Abelian Maxwell equations, we can perform a gauge transformation according to

$$A_\mu^a \to A_\mu^a + \partial_\mu \chi^a(x). \tag{4.12}$$

Let us choose

$$\chi^a = -\frac{gc_a}{4\pi} \log\left(\frac{r}{r_0}\right), \tag{4.13}$$

where r_0 is an arbitrary length scale to make the logarithm dimensionless. In the new gauge, one has

$$A_\mu = \left(\frac{gc_a T^a}{4\pi r}, 0, 0, 0\right). \tag{4.14}$$

This is recognisable as the Coulomb solution for the superposition of static colour charge that we have located at the origin, which is not surprising. It is well-known in Maxwell electromagnetism, for example, that the most general solution consistent with spherical symmetry is the Coulomb solution plus a radiation field. A static solution rules out the latter. Our results are also consistent with ref. [63], which addresses the solution of the Yang-Mills equations with an arbitrary static source. The authors argue that one may gauge away the non-Abelian nature of the Yang-Mills field, leaving the Abelian-like Coulomb solution.

We stress that the double copy between the Coulomb and Schwarzschild solutions only manifests itself with a particular gauge choice. The role of Kerr-Schild coordinates on the gravity side was crucial here. Before moving on, it is worth pointing out that the Schwarzschild double copy works also in higher dimensions, as it must do if our double copy interpretation is correct. The d-dimensional generalisation of the Schwarzschild black hole was first found by Tangherlini [64], and in Kerr-Schild coordinates the metric is given by

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{\mu}{r^{d-3}} k_\mu k_\nu, \tag{4.15}$$

where k^μ is a simple generalisation of eq. (4.5), such that $i = 1 \ldots (d-1)$, and the parameter μ is related to the mass M via[4]

$$M = \frac{\Omega_{d-2}}{8\pi G} \mu, \tag{4.16}$$

with

$$\Omega_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} \tag{4.17}$$

the area of a unit $(d-2)$-sphere. The Newtonian potential obtained from the metric of eq. (4.15) is

$$\phi = \frac{4\pi}{\Omega_{d-2}} \frac{GM}{r}. \tag{4.18}$$

Taking the single copy of this result, one obtains

$$A^\mu = \frac{gT_a}{\Omega_{d-2}} k^\mu. \tag{4.19}$$

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[4]Note that in eq. (4.16) we have absorbed a factor of $2/(d-2)$ into Newton's constant. This ensures that the relation $\kappa^2 = 16\pi G$ holds for arbitrary values of d.

This can be obtained from the d-dimensional Coulomb solution

$$A^\mu = \left(\frac{gT_a}{\Omega_{d-2}r^{d-3}}, \mathbf{0} \right) \tag{4.20}$$

via a gauge transformation according to eq. (4.12), choosing

$$\chi = \int^r dr' A^0(r') = \frac{gT_a}{(4-d)\Omega_{d-2}} r^{4-d}, \tag{4.21}$$

valid for $d > 4$. The uniqueness of the Coulomb-like solution can again be understood from an electromagnetic analogue of Birkhoff's theorem, which fixes the Tangherlini solution in GR.

4.2 The Kerr black hole

In the previous subsection, we have seen how the Kerr-Schild story of section 3 can be used to obtain a single copy of the Schwarzschild solution, namely a Coulomb solution. We saw also that the sources for the two solutions had a double copy structure, with a static colour charge on the gauge side becoming a static mass in the gravity theory. In this section we study another example, namely the Kerr (uncharged, rotating) black hole. In Kerr-Schild coordinates, the graviton field is

$$h_{\mu\nu} = \phi(r)k_\mu k_\nu, \tag{4.22}$$

where

$$\phi(r) = \frac{2MGr^3}{r^4 + a^2z^2} \tag{4.23}$$

and

$$k_\mu = \left(1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r} \right). \tag{4.24}$$

Note that r is no longer simply the modulus of the vector (x, y, z). It is instead defined implicitly via the equation

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1 \tag{4.25}$$

except for the region $\{x^2 + y^2 \leq a^2, z = 0\}$ (i.e. a disc of radius a about the origin in the (x, y) plane), where $r = 0$.

Following the Kerr-Schild single copy procedure as we did for the Schwarzschild solution, one may construct the gauge field

$$A_\mu^a = \frac{g}{4\pi}\phi(r)c_a k_\mu, \tag{4.26}$$

where again this is a solution to the Abelian Maxwell equations in the vacuum region described above. The rotation introduces a magnetic component to the Maxwell field.

We may interpret the gauge and gravity solutions further by determining the sources that create them. Given that we are only concerned with the vacuum solution for the Kerr metric, this means that we should look for the minimal possible source that will generate this. The source is a disk whose mass distribution exhibits a ring singularity at $x^2 + y^2 = a^2$

(which generates the well-known curvature singularity of the metric there). This was first explored by Israel [65]; we have also found ref. [66] useful.

One may introduce the spheroidal coordinates

$$x = \sqrt{r^2 + a^2} \sin\theta \cos\phi, \quad y = \sqrt{r^2 + a^2} \sin\theta \sin\phi, \quad z = r\cos\theta, \qquad (4.27)$$

where r, θ and ϕ play the role of a radial, polar angular and azimuthal angular coordinate respectively. Surfaces of constant $r = R$ are ellipsoids, such that $R \to 0$ converges to the disk of radius a in the (x, y) plane. In these coordinates, the energy-momentum tensor one finds for the Kerr metric is [65][5]

$$T^{\mu\nu} = \sigma \left(w^\mu w^\nu + \zeta^\mu \zeta^\nu \right), \quad \sigma = -\frac{M}{8\pi^2 a \cos\theta} \delta(z) \Theta(a - \rho), \qquad (4.28)$$

where we have introduced the radial and spacelike 4-vectors (in the spheroidal coordinate system (t, r, ϕ, θ))

$$w^\mu = \tan\theta \left(1, 0, 1/(a\sin^2\theta), 0 \right), \quad \zeta^\mu = \left(0, 1/(a\cos\theta), 0, 0 \right), \qquad (4.29)$$

and also defined

$$\rho = a\sin\theta. \qquad (4.30)$$

As discussed in [65], this has the form of a negative proper surface density rotating about the z-axis with superluminal velocity, and balanced by a radial pressure. The former and latter effects are the first and second terms in eq. (4.28) respectively.

We may repeat the above exercise for the square root of the Kerr solution. Substituting the gauge field into the Abelian Maxwell equations, one finds a source current

$$j^\mu = -\delta(z) \Theta(a - \rho) \frac{g(c_a T^a)}{4\pi} \frac{1}{a^2 \cos\theta} \left(\sec^2\theta, 0, \frac{\sec^2\theta}{a}, 0 \right). \qquad (4.31)$$

Here we have performed the coupling constant and mass replacements of eq. (4.9). Introducing the vector [65]

$$\xi^\mu = \left(1, 0, \frac{1}{a}, 0 \right), \qquad (4.32)$$

one may rewrite eq. (4.31) as

$$j^\mu = q\xi^\mu, \quad q = -\delta(z) \Theta(a - \rho) \frac{g(c_a T^a)}{4\pi a^2} \sec^3\theta. \qquad (4.33)$$

This has the form of a distribution of colour charge rotating about the z-axis.

Some additional comments are in order regarding the source terms we have found above. Firstly, it is not obviously the case that the energy-momentum tensor of eq. (4.28) is a "double copy" of the current of eq. (4.33). Perhaps the correct way to view this, however, is as follows. One may write eq. (4.28) as

$$T^{\mu\nu} = \delta(z) \Theta(a - \rho) \left(-\frac{M\sec^3\theta}{8\pi a^2} \right) \left[\xi^\mu \xi^\nu - \cos^2\theta \, \tilde\eta^{\mu\nu} \right] \qquad (4.34)$$

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[5]Note that our definition of σ in eq. (4.28) differs from that of [65] by a factor of two, due to us having normalized our surface integral as being over both the upper and lower surfaces of the disk.

where, in the Cartesian coordinates (t, x, y, z), $\tilde{\eta}^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 0)$. The first term of eq. (4.34) consists of a pure double copy of the gauge theory solution, where the rotating charge has been replaced with a similar rotating mass distribution. However, this would not be stable in the gravity theory, and thus would not lead to a static solution for the metric. The additional terms in eq. (4.34) thus supply the radial pressure that is needed to stabilise the disk.

The Kerr black hole has a higher-dimensional extension, known as the Myers-Perry black hole [67]. A major difference is that, in d space-time dimensions, there are $(d-1)/2$ independent rotation planes if d is odd, and $(d-2)/2$ if d is even; this is the dimension of the Cartan subgroup of $\mathrm{SO}(d-1)$. So multiple angular momenta are allowed, one per rotation plane, which makes the problem challenging. This challenge is alleviated by the fact that Myers-Perry black holes are also Kerr-Schild solutions. We can take the graviton field (4.22), and write the general solution as

$$\phi(r) = \begin{cases} \dfrac{\mu r^2}{\Pi F} & \text{if } d \text{ is odd,} \\[2ex] \dfrac{\mu r}{\Pi F} & \text{if } d \text{ is even,} \end{cases} \tag{4.35}$$

and

$$k_\mu dx^\mu = \begin{cases} dt + \displaystyle\sum_{i=1}^{(d-1)/2} \dfrac{r(x^i dx^i + y^i dy^i) + a_i(x^i dy^i - y^i dx^i)}{r^2 + a_i^2} & \text{if } d \text{ is odd,} \\[3ex] dt + \dfrac{z dz}{r} + \displaystyle\sum_{i=1}^{(d-2)/2} \dfrac{r(x^i dx^i + y^i dy^i) + a_i(x^i dy^i - y^i dx^i)}{r^2 + a_i^2} & \text{if } d \text{ is even.} \end{cases} \tag{4.36}$$

For each rotation plane, there is a rotation parameter a_i, and a pair of coordinates (x^i, y^i). We have used the functions

$$\Pi = \prod_{i=1}^{(d-2)/2} (r^2 + a_i^2), \qquad F = 1 - \sum_{i=1}^{(d-1)/2} \frac{a_i^2(x^{i2} + y^{i2})}{(r^2 + a_i^2)^2}. \tag{4.37}$$

Finally, the radial variable r is defined via

$$\sum_{i=1}^{(d-1)/2} \frac{(x^{i2} + y^{i2})}{(r^2 + a_i^2)^2} = 0 \quad \text{if } d \text{ is odd,} \qquad \frac{z^2}{r^2} + \sum_{i=1}^{(d-2)/2} \frac{(x^{i2} + y^{i2})}{(r^2 + a_i^2)^2} = 0 \quad \text{if } d \text{ is even.} \tag{4.38}$$

The Myers-Perry black holes provide a straightforward extension of our discussion on the Kerr black hole. They allow for solutions to Maxwell's equations in higher dimensions based on the Kerr-Schild ansatz, $A_\mu = \phi \, k_\mu$.

We point out that not all vacuum asymptotically flat black holes in higher dimensions are of Kerr-Schild type. See [68] for a review. Such solutions may have a variety of horizon topologies and are much harder to construct, the simplest example being the black ring (horizon topology $S^1 \times S^2$ in five dimensions). It would be interesting to analyse their gauge theory counterparts. One way to proceed would be the following. In the case of five

dimensions, there is a solution-generating technique for vacuum black holes because, after symmetry considerations, the Einstein equations reduce to a two-dimensional integrable system. If an extension of this technique to gauge theory exists, a map between the two types of solutions may be constructed. This would provide non-Kerr-Schild examples of the double copy.

4.3 Black branes

One other type of vacuum solution worth mentioning is that of black branes. These are black holes with horizons that extend in extra spatial dimensions. Since the direct product of two Ricci-flat manifolds is also a Ricci-flat manifold, solutions with extended horizons are trivially constructed from lower-dimensional black holes. Consider the metric of a $(d + m)$-dimensional black brane which uniformly extends a d-dimensional black hole with metric $g_{\mu\nu}$ along $\mathbb{R}^m$ ($\hat{\alpha} = 1 \ldots m$),

$$ds^2_{\text{brane}} = g_{\mu\nu}dx^\mu dx^\nu + dz_{\hat{\alpha}}dz^{\hat{\alpha}}. \tag{4.39}$$

The simplest example is the Schwarzschild string, $\text{Schwarz}_d \times \mathbb{R}$. The solutions discussed in the previous sections can therefore be trivially uplifted to new Kerr-Schild solutions,

$$g_{MN} = \eta_{MN} + \phi k_M k_N, \tag{4.40}$$

where $M, N = 1 \ldots d + m$, and we have the extensions $\eta_{MN} = \text{diag}(-1, 1, 1, \ldots)$ and $k_M = \{k_\mu, 0, 0, \ldots\}$. The corresponding gauge theory solutions are then simply $A_M = \phi k_M$.

The reason why there is no such black string solution in four dimensions is that there is no vacuum asymptotically flat black hole in three dimensions. This case deserves some comments. Consider a "cosmic string" which is a distribution of mass along the z axis in four dimensions,

$$T^{MN} = \sigma v^M v^N \delta(x)\delta(y), \qquad v^M = (1, 0, 0, 1). \tag{4.41}$$

This is an extension of the source (4.2) along an extra dimension with coordinate z, where we have $d = 3$ and $m = 1$. Naively, the Kerr-Schild solution would be

$$\phi(r) = \sigma \log \frac{r}{r_0}, \qquad k^M = \left(1, \frac{x}{r}, \frac{y}{r}, 0\right), \tag{4.42}$$

with $r = \sqrt{x^2 + y^2}$, where $(\partial_x^2 + \partial_y^2)\phi = 0$. However, this solution is valid only on the gauge theory side of the double copy, where one obtains the standard electromagnetic field generated by a charged line. The Kerr-Schild gravity solution constructed in this way does not satisfy the vacuum Einstein equations, which is consistent with that fact that, from eq. (3.8), we see that the Einstein equations are stronger than their gauge theory counterparts. In fact, it is well known that the space-time solution corresponding to the source (4.41) is flat, except for a conical singularity on the z axis (which has observational consequences). Despite the similarity with previous cases, what happens here is that the Einstein equations have no propagating degrees of freedom in three dimensions (the z direction is a mere spectator), i.e. there is no graviton to be obtained as the double copy of gluons.

5 Time-dependent Kerr-Schild solutions

We have now seen two specific examples in which stationary Kerr-Schild solutions in gravity have well-defined single copies in Yang-Mills theory. In this section, we briefly discuss well-known time-dependent solutions.

5.1 Plane wave solutions

Plane wave (pp-wave) solutions are arguably the simplest time-dependent vacuum solutions in either gauge or gravity theories. Unsurprisingly, gauge theory plane waves are related to gravitational pp-waves by a double copy prescription; this is manifestly the case for linearised waves (see e.g. [48][6]), and it extends to the non-perturbative case. Since pp-waves are Kerr-Schild solutions, they can be written as in equation (3.1). Using a pair of lightcone coordinates $x^\mu = (u, v, x^i)$, with $i = 1 \ldots d - 2$, we can express a pp-wave using

$$k_\mu dx^\mu = du = dz - dt, \qquad \phi = \phi(u, x^i). \tag{5.1}$$

Then the Einstein equations are simply

$$\partial_i \partial^i \phi = 0. \tag{5.2}$$

Non-Abelian plane wave solutions also have the form [70]

$$A_\mu^a = k_\mu \phi^a(u, x^i), \tag{5.3}$$

where ϕ^a fulfills the "propagator" equation (5.2). As in the stationary cases considered previously, the Kerr-Schild language makes the double copy explicit. Furthermore, the non-Abelian solution is also a solution of the Abelian Yang-Mills equations. This is, of course, what happens on the gravity side: a coordinate system has been chosen such that the expression for the metric, expanded in κ, terminates at first order. The graviton field obtained is truly non-perturbative, but still admits a simple double copy interpretation.

5.2 Shockwave solutions

Perhaps the simplest case of a time-dependent field solution with a source term is a shockwave. This corresponds to an infinitely boosted particle, whose field (in either the gauge or gravity context) is Lorentz contracted so that it lies in a flat plane transverse to the particle direction. In gravity, shockwaves are described by the Aichelburg-Sexl metric [71]. In four dimensions, the metric takes the pp-wave form (5.1) with

$$\phi(u, x^i) = C \, \delta(u) \log |\mathbf{x}|. \tag{5.4}$$

In their original paper, they showed explicitly that one may obtain the shockwave metric by a singular coordinate transformation of the Schwarzschild black hole - namely, the coordinate transformation corresponding to an infinite boost of the latter.

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[6]Linearised gravitational waves have recently been examined in the language of Maxwell's equations, in [69].

The perturbative relationship between the shockwave solutions in QCD and gravity was recently discussed extensively in [31], which used Feynman diagram arguments in the Regge limit (corresponding to the scattering of two highly boosted particles) to perturbatively construct the shockwave solution in both Yang-Mills theory and gravity, making clear the double copy relationship.[7] Furthermore, they found that the double copy was insensitive to whether an abelian or non-abelian shockwave was used on the gauge theory side. This is entirely consistent with our results regarding the Schwarzschild metric. We have shown that this metric is a double copy of a point charge; boosting this in the gauge and gravity cases leads to the appropriate shockwaves in both theories.

6 Discussion

In this paper, we have examined how certain classical gauge and gravity solutions can be related by a double copy procedure, analogous to that postulated to exist in perturbative quantum field theory [2, 3]. In particular, we considered Kerr-Schild metrics in the gravity side of the correspondence, and an associated Kerr-Schild ansatz for the gauge field. Kerr-Schild metrics have the property that the graviton factorises into an outer product of two Lorentz vectors, dressed by an overall scalar function. For the self-dual solutions, we used a slight modification of the Kerr-Schild ansatz, and the results motivated us to look for position-space examples.

We found an infinite class of Kerr-Schild double copies, namely those for which the graviton and gauge field are stationary. This allowed us to find single copies, or square roots, of well-known black hole solutions (including the Schwarzschild and Kerr solutions, as well as their higher-dimensional generalisations). These single copies were found to be solutions of the Abelian Yang-Mills (Maxwell) equations, trivially embedded in the non-Abelian theory. The sources needed to generate these solutions also had an interesting double copy interpretation, involving the replacement of colour charge by mass.

Our work leaves several questions unanswered. First, why should the single copies of these stationary gravity solutions be Abelian? Perhaps we should not be surprised that the single copy gauge field is Abelian, in view of the remarkable fact that the Einstein equations for Kerr-Schild spacetimes are linear in $h_{\mu\nu}$. It has been argued before for static sources (applicable to the Schwarzschild case) that any non-Abelian component of the solution can be gauged away [63]. It may be simply that the Abelian nature is forced upon the gauge theory by the use of Kerr-Schild coordinates: for a double copy to a Kerr-Schild solution to exist, it must be true that the gauge theory terminates at first order in the coupling constant, forbidding the presence of (non-Abelian) higher order corrections. Indeed, in the Schwarzschild-Tangherlini examples, where the source is pointlike, it is easy to see that a stationary, spherically symmetric solution of the YM equations using the KS ansatz, which has the property that the solution can be expanded perturbatively in the coupling, must be Abelian. Another possibility is that there may exist non-Abelian solutions that double copy to the same gravitational solution as an Abelian-like gauge theory object. This is possible due to the fact that information is lost when performing the double copy. An explicit example of this is the analysis of ref. [16], which showed that the infrared singularities of

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[7]See [32–34] for other recent work looking at the Regge limit from a double copy point of view.

QED and QCD both double copy to the infrared singularities of GR. A related observation was made recently in ref. [72]. It was found that gravitational Compton scattering could be obtained as a double copy of Compton scattering in either QCD or QED, depending on the choice of the colour-ordered amplitudes involved in the double copy. Finally, we point out that, at the level of the pure gauge theory amplitude, the kinematic numerators (or the colour-ordered amplitudes) are the same for photons and gluons. It is the colour dependence that leads to the vanishing of the colour-dressed amplitude in the Abelian case.

A second question is whether or not the Kerr-Schild double copies are genuine manifestations of the double copy as usually defined [2, 3]. There are a number of arguments for why this is the case. Firstly, the Kerr-Schild picture is analogous to the self-dual double copy, as discussed in section 3.1, as well as to other known cases of classical double copies. Secondly, the point source for the Schwarzschild solution has an obvious single copy which sources the Coulomb solution in gauge theory. This property is reminiscent of the perturbative double copy of infrared singularities discussed in ref. [16]. In fact, the Schwarzschild/Coulomb case has been studied recently in ref. [59], where the field configurations were constructed from a perturbative approach. Finally, we interpreted the Kerr-Schild function ϕ by taking a zeroth copy to a biadjoint scalar theory. This, by definition, fixes that part of the gauge theory solution that should not be squared upon performing the double copy to gravity. In all cases we examined, the function ϕ was a (scalar) propagator integrated over the distribution of charge, as expected from the conventional double copy, in which denominators of propagators are left intact.

In addition to stationary Kerr-Schild double copies, we mentioned the case of pp-waves (shockwaves are a particular example), in which time-dependent solutions match under the double copy. However, these solutions have a simple time-dependence, that disappears in an infinite momentum frame. We also found that the single copies of the gravity solutions were Abelian-like in the gauge theory, which may also be a consequence of the Kerr-Schild coordinates, as discussed above.

We hope that this preliminary study provides a useful precursor for further work. There are many avenues to be explored. It is fair to say that we still lack a general understanding of which gauges should be chosen in the gauge and gravity theories for the double copy to be manifest. The recent proposal of ref. [57] has related the gauge symmetries on both sides using a convolution procedure for the double copy, but only at a linearised level. The Kerr-Schild solutions that we analysed here are a guide to further progress, in that more involved double copy procedures for non-perturbative solutions should explain the Kerr-Schild cases. It would be interesting then to analyse the consequences of a general double copy procedure. We expect that the theory of perturbations on backgrounds related by the double copy will itself exhibit double copy properties. It would also be very interesting to examine quantum-corrected solutions in the two theories, rather than purely classical examples, perhaps following the lines of ref. [72]. Work in this regard is ongoing.

Acknowledgments

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