

Black holes in $f(R)$ theoriesA. de la Cruz-Dombriz,^{*} A. Dobado,[†] and A. L. Maroto[‡]*Departamento de Física Teórica I, Universidad Complutense de Madrid, 28040 Madrid, Spain*

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In the context of $f(R)$ theories of gravity, we address the problem of finding static and spherically symmetric black hole solutions. Several aspects of constant curvature solutions with and without electric charge are discussed. We also study the general case (without imposing constant curvature). Following a perturbative approach around the Einstein-Hilbert action, it is found that only solutions of the Schwarzschild-(anti) de Sitter type are present up to second order in perturbations. Explicit expressions for the effective cosmological constant are obtained in terms of the $f(R)$ function. Finally, we have considered the thermodynamics of black holes in anti-de Sitter space-time and found that this kind of solution can only exist provided the theory satisfies $R_0 + f(R_0) < 0$. Interestingly, this expression is related to the condition which guarantees the positivity of the effective Newton's constant in this type of theories. In addition, it also ensures that the thermodynamical properties in $f(R)$ gravities are qualitatively similar to those of standard general relativity.

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I. INTRODUCTION

In the last years, increasing attention has been paid to modified theories of gravity in order to understand several open cosmological questions such as the accelerated expansion of the Universe [1] and the dark matter origin [2]. Some of those theories modify general relativity by adding higher powers of the scalar curvature R , the Riemann and Ricci tensors, or their derivatives [3]. Lovelock and $f(R)$ theories are some examples of these attempts. It is therefore quite natural to ask about black hole (BH) features in those gravitational theories since, on the one hand, some BH signatures may be peculiar to Einstein's gravity and others may be robust features of all generally covariant theories of gravity. On the other hand, the results obtained may lead to rule out some models which will be in disagreement with expected physical results. For those purposes, research on thermodynamical quantities of BHs is of particular interest.

In this work we will restrict ourselves to the so-called $f(R)$ gravity theories (see [4]) in metric formalism in Jordan's frame. In this frame, the gravitational Lagrangian is given by $R + f(R)$ where $f(R)$ is an arbitrary function of R and Einstein's equations are usually fourth order in the metric [see [5] for several proposed $f(R)$ functions compatible with local gravity tests and other cosmological constraints]. An alternative approach would be to use the Einstein's frame, where ordinary Einstein's gravity coupled to a scalar plus a massive spin-2 field is recovered. Even if a mathematical correspondence could be established between those two frames, in the last years

some controversy has remained about their physical equivalence.

Previous literature on $f(R)$ theories [6] proved in Einstein's frame that the Schwarzschild solution is the only static spherically symmetric solution for an action of the form $R + aR^2$ in $D = 4$. In [7] uniqueness theorems of spherically symmetric solutions for general polynomial actions in arbitrary dimensions using Einstein's frame were proposed (see also [8] for additional results). See also [9] for spherical solutions with sources.

Using the Euclidean action method (see for instance [10,11]) in order to determine different thermodynamical quantities, anti-de Sitter (AdS) BHs in $f(R)$ models have been studied [12]. In [13] the entropy of the Schwarzschild-de Sitter BH was calculated for some particular cosmologically viable models in vacuum and their cosmological stability was discussed.

BH properties have also been widely studied in other modified gravity theories. For instance, [14,15] studied BHs in Einstein's theory with a Gauss-Bonnet term and cosmological constant. Different results were found depending on the dimension D and the sign of the constant horizon curvature k . For $k = 0, -1$, the Gauss-Bonnet term does not modify AdS BH thermodynamics at all [only the horizon position is modified with respect to the Einstein-Hilbert (EH) theory] and BHs are not only locally thermodynamically stable but also globally preferred. Nevertheless for $k = +1$ and $D = 5$ (for $D \geq 6$ thermodynamics is again essentially that for the AdS BH) there exist some features not present in the absence of the Gauss-Bonnet term. Gauss-Bonnet and/or Riemann squared interaction terms were studied in [16] concluding that in this case phase transitions may occur with $k = -1$.

Another approach is given by Lovelock gravities, which are free of ghosts and where field equations contain no more than second derivatives of the metric. These theories

^{*}dombriz@fis.ucm.es[†]dobado@fis.ucm.es[‡]maroto@fis.ucm.es

were studied in [17] and the corresponding entropy was evaluated.

The paper is organized as follows: in Sec. II we present some general results for $f(R)$ gravities for interesting physical situations in metric formalism. In Secs. III and IV, BHs in $f(R)$ gravities are studied and explicit Einstein's field equations are presented for static and spherically symmetric metrics. Section V is devoted to find perturbative solutions for static and spherically symmetric background metrics: general metric coefficients are found depending on $f(R)$ derivatives evaluated at background scalar curvature. Sections VI and VII are widely devoted to study thermodynamical quantities and their consequences in local and global stability for some particular $f(R)$ models. Finally, we include some conclusions.

II. GENERAL RESULTS

In order to study the basics of the solutions of general $f(R)$ gravity theories, let us start from the action

$$S = S_g + S_m \quad (1)$$

where S_g is the D -dimensional gravitational action:

$$S_g = \frac{1}{16\pi G_D} \int d^D x \sqrt{|g|} (R + f(R)) \quad (2)$$

with $G_D \equiv M_D^{2-D}$ being the D -dimensional Newton's constant, M_D the corresponding Planck mass, g the determinant of the metric g_{AB} , ($A, B = 0, 1, \dots, D-1$), R the scalar curvature, and $R + f(R)$ is the function defining the theory under consideration. As the simplest example, the EH action with cosmological constant Λ_D is given by $f(R) = -(D-2)\Lambda_D$. The matter action S_m defines the energy-momentum tensor as

$$T^{AB} = -\frac{2}{\sqrt{|g|}} \frac{\delta S_m}{\delta g_{AB}}. \quad (3)$$

From the above action, the equations of motion in the metric formalism are just

$$R_{AB}(1 + f'(R)) - \frac{1}{2}(R + f(R))g_{AB} + (\nabla_A \nabla_B - g_{AB} \square) f'(R) + 8\pi G_D T_{AB} = 0 \quad (4)$$

where R_{AB} is as usual the Ricci tensor and $\square = \nabla_A \nabla^A$ with ∇ the usual covariant derivative. Thus for the vacuum EH action with cosmological constant we have

$$R_{AB} - \frac{1}{2}Rg_{AB} + \frac{D-2}{2}\Lambda_D g_{AB} = 0 \quad (5)$$

which means $R_{AB} = \Lambda_D g_{AB}$ and $R = D\Lambda_D$. Coming back to the general case, the required condition to get constant scalar curvature solutions $R = R_0$ (from now R_0 will denote a constant curvature value) in vacuum implies:

$$R_{AB}(1 + f'(R)) - \frac{1}{2}g_{AB}(R + f(R)) = 0. \quad (6)$$

Taking the trace in the previous equation, R_0 must be a root

of the equation:

$$2(1 + f'(R_0))R_0 - D(R_0 + f(R_0)) = 0. \quad (7)$$

For this kind of solution, an effective cosmological constant may be defined as $\Lambda_D^{\text{eff}} \equiv R_0/D$. Thus any constant curvature solution $R = R_0$ with $1 + f'(R_0) \neq 0$ fulfills

$$R_{AB} = \frac{R_0 + f(R_0)}{2(1 + f'(R_0))} g_{AB}. \quad (8)$$

On the other hand, one can consider

$$2R(1 + f'(R)) - D(R + f(R)) = 0 \quad (9)$$

as a differential equation for the $f(R)$ function so that the corresponding solution would admit any curvature R value. The solution of this differential equation is just

$$f(R) = aR^{D/2} - R \quad (10)$$

where a is an arbitrary constant. Thus the gravitational Lagrangian becomes proportional to $aR^{D/2}$ which will have solutions of constant curvature for arbitrary R . The reason is that this action is scale invariant since a/G_D is a nondimensional constant.

Now we will address the issue of finding some general criteria to relate solutions of the EH action with solutions of more general $f(R)$ gravities, not necessarily of constant curvature R . Let g_{AB} , a solution of EH gravity with cosmological constant, i.e.,

$$R_{AB} - \frac{1}{2}Rg_{AB} + \frac{D-2}{2}\Lambda_D g_{AB} + 8\pi G_D T_{AB} = 0. \quad (11)$$

Then g_{AB} is also a solution of any $f(R)$ gravity, provided the following compatibility equation:

$$f'(R)R_{AB} - \frac{1}{2}g_{AB}[f(R) + (D-2)\Lambda_D] + (\nabla_A \nabla_B - g_{AB} \square) f'(R) = 0 \quad (12)$$

obtained from (4) is fulfilled. In the following we will consider some particularly interesting cases. The simplest possibility is obviously vacuum ($T_{AB} = 0$) with vanishing cosmological constant $\Lambda_D = 0$. Then the above Eq. (11) becomes

$$R_{AB} = \frac{1}{2}Rg_{AB}, \quad (13)$$

which implies $R = 0$ and $R_{AB} = 0$. Consequently, g_{AB} is also a solution of any $f(R)$ gravity provided $f(0) = 0$, which is for instance the case when $f(R)$ is analytical around $R = 0$. When the cosmological constant is different from zero ($\Lambda_D \neq 0$), but still $T_{AB} = 0$, we have also constant curvature with $R_0 = D\Lambda_D$ and $R_{AB} = \Lambda_D g_{AB}$. Then the compatibility equation (12) reduces to (7). In other words, g_{AB} is also a solution of $f(R)$ provided $f(D\Lambda_D) = \Lambda_D(2 - D + 2f'(D\Lambda_D))$. Notice that it would also be a solution for any R_0 in the particular case $f(R) = aR^{D/2} - R$.

Next we can consider the case with $\Lambda_D = 0$ and conformal matter ($T = T_A^A = 0$). For a perfect fluid, this means having the equation of state $\rho = (D - 1)p$ where p is the pressure and ρ the energy density. In this case (11) implies

$$R = 0; \quad R_{AB} = 8\pi G_D T_{AB}. \quad (14)$$

Then, provided $f(0) = f'(0) = 0$, g_{AB} is also a solution of any $f(R)$ gravity. This result could have particular interest in cosmological calculations for ultrarelativistic matter (i.e., conformal) dominated universes. For the case of conformal matter with nonvanishing Λ_D , we have again constant $R = R_0$ with $R_0 = D\Lambda_D$ and g_{AB} is a solution of $f(R)$ provided that once again $f(D\Lambda_D) = \Lambda_D(2 - D + 2f'(D\Lambda_D))$.

III. BLACK HOLES IN $f(R)$ GRAVITIES

Now we consider the external metric for the gravitational field produced by a nonrotating object in $f(R)$ gravity theories. The most general static and spherically symmetric $D \geq 4$ dimensional metric can be written as (see [18]):

$$ds^2 = e^{-2\Phi(r)} A(r) dt^2 - A^{-1}(r) dr^2 - r^2 d\Omega_{D-2}^2 \quad (15)$$

or alternatively

$$ds^2 = \lambda(r) dt^2 - \mu^{-1}(r) dr^2 - r^2 d\Omega_{D-2}^2 \quad (16)$$

where $d\Omega_{D-2}^2$ is the metric on the S^{D-2} sphere and identification $\lambda(r) = e^{-2\Phi(r)} A(r)$ and $\mu(r) = A(r)$ can be straightforwardly established.

For obvious reasons the $\Phi(r)$ function is called the anomalous redshift. Notice that a photon emitted at r with proper frequency ω_0 is measured at infinity with frequency $\omega_\infty = e^{-\Phi(r)} \sqrt{A(r)} \omega_0$. As the metric is static, the scalar curvature R in D dimensions depends only on r and it is given, for the metric parametrization (15), by

$$R(r) = \frac{1}{r^2} [D^2 - 5D + 6 + rA'(r)(-2D + 3r\Phi'(r) + 4) - r^2 A''(r) - A(r)(D^2 - 5D + 2r^2 \Phi'(r)^2 - 2(D - 2)r\Phi'(r) - 2r^2 \Phi''(r) + 6)], \quad (17)$$

where the prime denotes derivative with respect to r . At this stage, it is interesting to ask which are the most general static and spherically symmetric metrics with constant scalar curvature R_0 . This curvature can be found solving the equation $R = R_0$. Then it is immediate to see that for a constant $\Phi(r) = \Phi_0$ the general solution is

$$A(r) = 1 + a_1 r^{3-D} + a_2 r^{2-D} - \frac{R_0}{D(D-1)} r^2 \quad (18)$$

with a_1 and a_2 being arbitrary integration constants. In fact, for the particular cases $D = 4$, $R_0 = 0$, and $\Phi_0 = 0$, the metric can be written exclusively in terms of the function

$$A(r) = 1 + \frac{a_1}{r} + \frac{a_2}{r^2}. \quad (19)$$

By establishing the identifications $a_1 = -2G_N M$ and $a_2 = Q^2$, this solution corresponds to a Reissner-Nordström solution, i.e., a charged massive BH solution with mass M and charge Q . Further comments about this result will be made below.

IV. CONSTANT CURVATURE BLACK HOLE SOLUTIONS

By inserting the metric (15) into the general $f(R)$ gravitational action S_g in (2), and making variations with respect to the $A(r)$ and $\Phi(r)$ functions, we find the equations of motion:

$$(2 - D)(1 + f'(R))\Phi'(r) - r[f'''(R)R'(r)^2 + f''(R)(\Phi'(r)R'(r) + R''(r))] = 0 \quad (20)$$

and

$$2rA(r)f'''(R)R'(r)^2 + f''(R)[2DA(r)R'(r) - 4A(r)R'(r) + 2rA(r)R''(r) + A'(r)rR'(r)] + g'(R)[-2rA(r)\Phi'(r)^2 + 2DA(r)\Phi'(r) - 4A(r)\Phi'(r) - rA''(r) + 2rA(r)\Phi''(r) + A'(r)(2 - D + 3r\Phi'(r))] - r(R + f(R)) = 0, \quad (21)$$

where f' , f'' , and f''' denote derivatives of $f(R)$ with respect to the curvature R .

The above equations look in principle quite difficult to solve. For this reason we will first consider the case of constant scalar curvature $R = R_0$ solutions. Then the equations of motion reduce to

$$(2 - D)(1 + f'(R))\Phi'(r) = 0 \quad (22)$$

and

$$R + f(R) + (1 + f'(R)) \left[A''(r) + (D - 2) \frac{A'(r)}{r} - (2D - 4) \frac{A(r)\Phi'(r)}{r} - 3A'(r)\Phi'(r) + 2A(r)\Phi'^2(r) - 2A(r)\Phi''(r) \right] = 0. \quad (23)$$

As commented in the previous sections, the constant curvature solutions of $f(R)$ gravities are given by

$$R_0 = \frac{Df(R_0)}{2(1 + f'(R_0)) - D} \quad (24)$$

whenever $2(1 + f'(R_0)) \neq D$. Thus from (22) $\Phi'(r) = 0$ and then (23) becomes

$$R_0 + f(R_0) + (1 + f'(R_0)) \left[A''(r) + (D - 2) \frac{A'(r)}{r} \right] = 0. \quad (25)$$

Coming back to (25), and using (24), we get

$$A''(r) + (D-2)\frac{A'(r)}{r} = -\frac{2}{D}R_0. \quad (26)$$

This is a $f(R)$ -independent linear second order inhomogeneous differential equation which can be easily integrated to give the general solution

$$A(r) = C_1 + C_2 r^{3-D} - \frac{R_0}{D(D-1)} r^2 \quad (27)$$

which depends on two arbitrary constants C_1 and C_2 . However this solution has no constant curvature in the general case since, as we found above, the constant curvature requirement demands $C_1 = 1$. Then, for negative R_0 , this solution is basically the D -dimensional generalization obtained by Witten [11] of the BH in AdS space-time solution considered by Hawking and Page [10]. With the natural choice $\Phi_0 = 0$ the solution can be written as

$$A(r) = 1 - \frac{R_S^{D-3}}{r^{D-3}} + \frac{r^2}{l^2}, \quad (28)$$

where

$$R_S^{D-3} = \frac{16\pi G_D M}{(D-2)\mu_{D-2}} \quad (29)$$

with

$$\mu_{D-2} = \frac{2\pi^{(D-1)/(2)}}{\Gamma(\frac{D-1}{2})} \quad (30)$$

being the area of the $D-2$ sphere, $l^2 \equiv -D(D-1)/R_0$ is the asymptotic AdS space scale squared, and M is the mass parameter usually found in the literature.

Thus we have concluded that the only static and spherically symmetric vacuum solutions with constant (negative) curvature of any $f(R)$ gravity is just the Hawking-Page BH in AdS space. However this kind of solution is not the most general static and spherically symmetric metric with constant curvature as can be seen by comparison with the solutions found in (18). Therefore we have to conclude that there are constant curvature BH solutions that cannot be obtained as vacuum solutions of any $f(R)$ theory. As we show below, in the $D=4$ case, we see that the most general case can be described as a charged BH solution in $f(R)$ -Maxwell theory.

Indeed, let us consider now the case of charged black holes in $f(R)$ theories. We will limit ourselves to the $D=4$ case, since in other dimensions the curvature is not necessarily constant. The action of the theory is now the generalization of the Einstein-Maxwell action:

$$S_g = \frac{1}{16\pi G_4} \int d^4x \sqrt{|g|} (R + f(R) - F_{\mu\nu} F^{\mu\nu}) \quad (31)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Considering an electromagnetic potential of the form $A_\mu = (V(r), \vec{0})$ and the static

spherically symmetric metric (15), we find that the solution with constant curvature R_0 reads

$$V(r) = \frac{Q}{r} \\ \lambda(r) = \mu(r) = 1 - \frac{2G_4 M}{r} + \frac{(1 + f'(R_0))Q^2}{r^2} - \frac{R_0}{12} r^2. \quad (32)$$

Notice that unlike the EH case, the contribution of the black hole charge to the metric tensor is corrected by a $(1 + f'(R_0))$ factor.

V. PERTURBATIVE RESULTS

In the previous section, we have considered static spherically symmetric solutions with constant curvature. In EH theory, this would provide the most general solution with spherical symmetry. However, it is not guaranteed to be the case also in $f(R)$ theories. The problem of finding the general static spherically symmetric solution in arbitrary $f(R)$ theories without imposing the constant curvature condition is in principle too complicated. For that reason, in this section we will present a perturbative analysis of the problem, assuming that the modified action is a small perturbation around EH theory.

Therefore let us consider a $f(R)$ function of the form

$$f(R) = -(D-2)\Lambda_D + \alpha g(R) \quad (33)$$

where $\alpha \ll 1$ is a dimensionless parameter and $g(R)$ is assumed to be analytic in α . By using the metric parametrization given by (16) the equations of motion become

$$\begin{aligned} & \lambda(r)(1 + f'(R))\{2\mu(r)[(D-2)\lambda'(r) + r\lambda''(r)] \\ & + r\lambda'(r)\mu'(r)\} - 2\lambda(r)^2\{2\mu(r)[(D-2)R'(r)f''(R) \\ & + rf^{(3)}(R)R'(r)^2 + rR''(r)f''(R)] + rR'(r)\mu'(r)f''(R)\} \\ & - r\mu(r)\lambda'(r)^2(1 + f'(R)) + 2r\lambda(r)^2(R + f(R)) = 0 \end{aligned} \quad (34)$$

$$\begin{aligned} & - \lambda(r)\mu'(r)[2(D-2)\lambda(r) + r\lambda'(r)](1 + f'(R)) \\ & + \mu(r)\{2\lambda(r)R'(r)[2(D-2)\lambda(r) + r\lambda'(r)]f''(R) \\ & + r(1 + f'(R))(\lambda'(r)^2 - 2\lambda(r)\lambda''(r))\} \\ & - 2r\lambda(r)^2(R + f(R)) = 0 \end{aligned} \quad (35)$$

where prime denotes the derivative with respect to the corresponding argument and $R \equiv R(r)$ is given by (17). Now, assuming that the $\lambda(r)$ and $\mu(r)$ functions appearing in the metric (16) are also analytical in α , they can be written as follows:

$$\begin{aligned}\lambda(r) &= \lambda_0(r) + \sum_{i=1}^{\infty} \alpha^i \lambda_i(r) \\ \mu(r) &= \mu_0(r) + \sum_{i=1}^{\infty} \alpha^i \mu_i(r)\end{aligned}\quad (36)$$

where $\{\lambda_0(r), \mu_0(r)\}$ are the unperturbed solutions for the EH action with the cosmological constant given by

$$\begin{aligned}\mu_0(r) &= 1 + \frac{C_1}{r^{D-3}} - \frac{\Lambda_D}{(D-1)} r^2 \\ \lambda_0(r) &= -C_2(D-2)(D-1)\mu_0(r)\end{aligned}\quad (37)$$

which are the standard BH solutions in a D -dimensional AdS space-time. Note that the factor C_2 can be chosen by performing a coordinate t reparametrization so that both functions could be identified. For the moment, we will keep the background solutions as given in (37) and we will discuss the possibility of getting $\lambda(r) = \mu(r)$ in the perturbative expansion later on.

By inserting (33) and (36) in (34) and (35) we obtain the following first order equations:

$$(D-3)\mu_1(r) + r\mu_1'(r) + \frac{2\Lambda_D g'(R_0) - g(R_0)}{D-2} r^2 = 0 \quad (38)$$

$$\begin{aligned}C_2[C_1(D-1)r^{3-D} - \Lambda_D r^2 + D-1]g(R_0)r^2 \\ + \left[C_1(D-3)r^{3-D} + \frac{2\Lambda_D}{D-1} r^2 \right] \lambda_1(r) \\ + C_2(D-2)(D-1)(\Lambda_D r^2 - D+3)\mu_1(r) \\ + \left(1 + C_1 r^{3-D} - \frac{\Lambda_D r^2}{D-1} \right) [2C_2(1-D)r^2 \Lambda_D g'(R_0) \\ + r\lambda_1'(r)] = 0\end{aligned}\quad (39)$$

whose solutions are

$$\begin{aligned}\lambda_1(r) &= C_4(D-1)(D-2) \\ &+ \frac{(C_1 C_4 - C_2 C_3)(D-2)(D-1)}{r^{D-3}} \\ &- [C_4(D-2)\Lambda_D + C_2(g(R_0) - 2\Lambda_D g'(R_0))]r^2\end{aligned}\quad (40)$$

$$\mu_1(r) = \frac{C_3}{r^{D-3}} + \frac{(g(R_0) - 2\Lambda_D g'(R_0))}{(D-2)(D-1)} r^2. \quad (41)$$

Up to second order in α the equations are

$$(D-3)\mu_2(r) + r\mu_2'(r) + \frac{(g(R_0) - 2\Lambda_D g'(R_0))}{D-2} \left(g'(R_0) - \frac{2D}{D-2} \Lambda_D g''(R_0) \right) r^2 = 0 \quad (42)$$

$$\begin{aligned}\left[-C_1(D-3)r^{3-D} - \frac{2\Lambda_D r^2}{D-1} \right] \lambda_2(r) + C_2(D-2)(D-1)(-\Lambda_D r^2 + D-3)\mu_2(r) - \left(C_1 r^{4-D} + r - \frac{r^3 \Lambda_D}{D-1} \right) \lambda_2'(r) \\ - C_3 C_4 (D-2)(D-1)(-\Lambda_D r^2 + D-3)r^{3-D} - C_2[(D-1)(C_1 r^{3-D} + 1) - \Lambda_D r^2] \\ \times \left[2\Lambda_D g'(R_0)^2 + g(R_0) \left(\frac{2D\Lambda_D g''(R_0)}{D-2} - g'(R_0) \right) - \frac{4D\Lambda_D^2 g'(R_0)g''(R_0)}{D-2} \right] r^2 \\ - C_4[C_1(D-1)r^{3-D} + 2][2\Lambda_D g'(R_0) - g(R_0)]r^2 = 0\end{aligned}\quad (43)$$

whose solutions are

$$\begin{aligned}\lambda_2(r) &= C_6 + \frac{C_6 C_1 + (C_3 C_4 - C_2 C_5)(D-2)(D-1)}{r^{D-3}} \\ &+ \left[-\frac{C_6 \Lambda_D}{D-1} + (g(R_0) - 2\Lambda_D g'(R_0)) \left(C_4 + C_2 g'(R_0) - \frac{2C_2 D \Lambda_D g''(R_0)}{D-2} \right) \right] r^2\end{aligned}\quad (44)$$

$$\mu_2(r) = \frac{C_5}{r^{D-3}} + \frac{(g(R_0) - 2\Lambda_D g'(R_0))(2D\Lambda_D g''(R_0) - (D-2)g'(R_0))}{(D-2)^2(D-1)} r^2. \quad (45)$$

Further orders in $\alpha^{3,4,\dots}$ can be obtained by inserting previous results in the order $3, 4, \dots$ ones to get $\{\lambda_{3,4,\dots}(r), \mu_{3,4,\dots}(r)\}$, but of course the corresponding equations become increasingly complicated.

Notice that from the obtained results up to second order in α , the corresponding metric has constant scalar curvature for any value of the parameters C_1, C_2, \dots, C_6 . As a matter of fact, this metric is nothing but the standard

Schwarzschild-AdS geometry, and can be easily rewritten in the usual form by making a trivial time reparametrization as follows:

$$\begin{aligned}\bar{\lambda}(r) &\equiv \lambda(r)[-C_2(D^2 + 3D - 2) \\ &\quad + C_4(D^2 - 3D + 2)\alpha + C_6\alpha^2 + \mathcal{O}(\alpha^3)] \\ \bar{\mu}(r) &\equiv \mu(r).\end{aligned}\quad (46)$$

Therefore, at least up to second order, the only static, spherically symmetric solutions which are analytical in α are the standard Schwarzschild-AdS space-times.

On the other hand, taking the inverse point of view, if we assume the solutions to be of the AdS BH type at any order in the α expansion we can write

$$\lambda(r) \equiv \mu(r) = 1 + \left(\frac{\bar{R}_S}{r}\right)^{D-3} + Jr^2 \quad (47)$$

as a solution for the Einstein equations (34) and (35) with the gravitational Lagrangian (33) and

$$\bar{R}_S = R_S + \sum_{i=1}^{\infty} C_i \alpha^i \quad J = -\frac{\Lambda_D}{(D-1)} + \sum_{i=1}^{\infty} J_i \alpha^i \quad (48)$$

where R_S and C_i are arbitrary constants and the J_i coefficients can be determined from (7)

$$R - (D-2)\Lambda_D + \alpha g(R) + 2(D-1)J(1 + \alpha g'(R)) = 0 \quad (49)$$

with $R = -D(D-1)J$. Expanding the previous equation in powers of α , it is possible to find a recurrence equation for the J_i coefficients, namely, for the J_l (with $l > 0$) coefficient. We find

$$\begin{aligned}(2-D)(D-1)J_l + \sum_{i=0}^{l-1} \sum_{\text{cond1}} \frac{1}{i_1! i_2! \dots i_{l-1}!} \\ \times (J_1)^{i_1} (J_2)^{i_2} \dots (J_{l-1})^{i_{l-1}} g^{(i)}(R_0) + 2(D-1) \\ \times \sum_{k=0}^{l-1} J_k \sum_{i=0}^{l-k-1} \sum_{\text{cond2}} \frac{1}{i_1! i_2! \dots i_{l-k-1}!} \\ \times (J_1)^{i_1} (J_2)^{i_2} \dots (J_{l-k-1})^{i_{l-k-1}} g^{(i+1)}(R_0) = 0\end{aligned} \quad (50)$$

with $R_0 = -D(D-1)J_0 \equiv D\Lambda_D$, where the first sum is done under condition 1 given by

$$\sum_{m=1}^{l-1} i_m = i, \quad i_m \in \mathbb{N} \cup \{0\}, \quad \text{and} \quad \sum_{m=1}^{l-1} m i_m = l-1 \quad (51)$$

and the second one under condition 2:

$$\begin{aligned}\sum_{m=1}^{l-k-1} i_m = i, \quad i_m \in \mathbb{N} \cup \{0\}, \\ \text{and} \quad \sum_{m=1}^{l-k-1} m i_m = l-k-1.\end{aligned} \quad (52)$$

For instance we have

$$\begin{aligned}J_1 &= \frac{A(g; D, \Lambda_D)}{(D-2)(D-1)} \\ J_2 &= -\frac{A(g; D, \Lambda_D)[(D-2)g'(R_0) - 2D\Lambda_D g''(R_0)]}{(D-2)^2(D-1)}\end{aligned} \quad (53)$$

where $A(g; D, \Lambda_D) \equiv g(R_0) - 2\Lambda_D g'(R_0)$.

Now we can consider the possibility of removing Λ_D from the action from the very beginning and are still getting an AdS BH solution with an effective cosmological constant depending on $g(R)$ and its derivatives evaluated at $R_0 \equiv 0$. In this case the results, order by order in α up to order α^2 , are

$$\begin{aligned}J_0(\Lambda_D = 0) &= 0 \\ J_1(\Lambda_D = 0) &= \frac{g(0)}{(D-2)(D-1)} \\ J_2(\Lambda_D = 0) &= -\frac{g(0)g'(0)}{(D-2)(D-1)}.\end{aligned} \quad (54)$$

As we see, in the context of $f(R)$ gravities, it is possible to have a BH in an AdS asymptotic space even if the initial cosmological constant Λ_D vanishes.

To end these two sections, we can summarize by saying that in the context of $f(R)$ gravities the only spherically symmetric and static solutions of negative constant curvature are the standard BH in AdS space. The same result applies in the general case (without imposing constant curvature) in perturbation theory up to second order. However, the possibility of having static and spherically symmetric solutions with nonconstant curvature cannot be excluded in the case of $f(R)$ functions which are not analytical in α .

VI. BLACK HOLE THERMODYNAMICS

In order to consider the different thermodynamic quantities for the $f(R)$ black holes in AdS, we start from the temperature. In principle there are two different ways of introducing this quantity for the kind of solutions we are considering here. First, we can use the definition coming from Euclidean quantum gravity [19]. In this case one introduces the Euclidean time $\tau = it$ and the Euclidean metric ds_E^2 is defined as

$$-ds_E^2 = -d\sigma^2 - r^2 d\Omega_{D-2}^2 \quad (55)$$

where

$$d\sigma^2 = e^{-2\Phi(r)}A(r)d\tau^2 + A^{-1}(r)dr^2. \quad (56)$$

The metric corresponds only to the region $r > r_H$ where r_H is the outer horizon position with $A(r_H) = 0$. Expanding $d\sigma^2$ near r_H we have

$$d\sigma^2 = e^{-2\Phi(r_H)}A'(r_H)\rho d\tau^2 + \frac{d\rho^2}{A'(r_H)\rho} \quad (57)$$

where $\rho = r - r_H$. Now we introduce the new coordinates \tilde{R} and θ defined as

$$\theta = \frac{1}{2}e^{-\Phi(r_H)}A'(r_H)\tau \quad \tilde{R} = 2\sqrt{\frac{\rho}{A'(r_H)}} \quad (58)$$

so that

$$d\sigma^2 = \tilde{R}^2 d\theta^2 + dR^2. \quad (59)$$

According to the Euclidean quantum gravity prescription, τ belongs to the interval defined by 0 and $\beta_E = 1/T_E$. On the other hand, in order to avoid conical singularities, θ must run between 0 and 2π . Thus it is found that

$$T_E = \frac{1}{4\pi}e^{-\Phi(r_H)}A'(r_H). \quad (60)$$

Another possible definition of temperature was first proposed in [20] stating that temperature can be given in terms of the horizon gravity \mathcal{K} as

$$T_{\mathcal{K}} \equiv \frac{\mathcal{K}}{4\pi} \quad (61)$$

where \mathcal{K} is given by

$$\mathcal{K} = \lim_{r \rightarrow r_H} \frac{\partial_r g_{tt}}{\sqrt{|g_{tt}g_{rr}|}}. \quad (62)$$

Then it is straightforward to find

$$T_{\mathcal{K}} = T_E. \quad (63)$$

Therefore both definitions give the same result for this kind of solution. Notice also that in any case the temperature depends only on the behavior of the metric near the horizon but it is independent from the gravitational action. By this we mean that different actions having the same solutions also have the same temperature. This is not the case for other thermodynamic quantities as we will see later. Taking into account the results in previous sections, for simplicity we will concentrate only on constant curvature AdS BH solutions with $\Phi = 0$ as a natural choice, and

$$A(r) = 1 - \frac{R_S^{D-3}}{r^{D-3}} + \frac{r^2}{l^2}. \quad (64)$$

Then, both definitions of temperature lead to

$$\beta = 1/T = \frac{4\pi l^2 r_H}{(D-1)r_H^2 + (D-3)l^2}. \quad (65)$$

Notice that the temperature is a function of r_H only, i.e., it

depends only on the BH size. In the limit r_H going to zero the temperature diverges as $T \sim 1/r_H$ and for r_H going to infinite T grows linearly with r_H . Consequently T has a minimum at

$$r_{H0} = l\sqrt{\frac{D-3}{D-1}} \quad (66)$$

corresponding to a temperature:

$$T_0 = \frac{\sqrt{(D-1)(D-3)}}{2\pi l}. \quad (67)$$

The existence of this minimum was established in [10] for $D = 4$ by Hawking and Page a long time ago and it is well known. More recently, Witten extended this result to higher dimensions [11]. The minimum is important in order to set the regions with different thermodynamic behaviors and stability properties. For $D = 4$, an exact solution can be found for r_H :

$$r_H = l \frac{2^{1/3} \left(9 \frac{R_S}{l} + \sqrt{12 + 81 \frac{R_S^2}{l^2}} \right)^{2/3} - (24)^{1/3}}{6^{2/3} \left(9 \frac{R_S}{l} + \sqrt{12 + 81 \frac{R_S^2}{l^2}} \right)^{1/3}}. \quad (68)$$

Thus, in the $R_S \ll l$ limit, we find $r_H = R_S$, whereas in the opposite case $l \ll R_S$, we get $r_H = (l^2 R_S)^{1/3}$. For the particular case $D = 5$, r_H can also be exactly found to be

$$r_H^2 = \frac{l^2}{2} \left(\sqrt{1 + \frac{4R_S^2}{l^2}} - 1 \right) \quad (69)$$

which goes to R_S^2 for $R_S \ll l$ and to lR_S for $l \ll R_S$. Notice that for any $T > T_0$, we have two possible BH sizes: one corresponding to the small BH phase with $r_H < r_{H0}$ and the other corresponding to the large BH phase with $r_H > r_{H0}$.

In order to compute the remaining thermodynamic quantities, the Euclidean action

$$S_E = -\frac{1}{16\pi G_D} \int d^D x \sqrt{g_E} (R + f(R)) \quad (70)$$

is considered. When the previous expression is evaluated on some metric with a periodic Euclidean time with period β , it equals β times the free energy F associated to this metric. Extending to the $f(R)$ theories, the computation by Hawking and Page [10], generalized to higher dimensions by Witten [11], we compute the difference of this action evaluated on the BH and the AdS metric which can be written as

$$\Delta S_E = -\frac{R_0 + f(R_0)}{16\pi G_D} \Delta V \quad (71)$$

where $R_0 = -D(D-1)/l^2$ and ΔV is the volume difference between both solutions which is given by

$$\Delta V = \frac{\beta \mu_{D-2}}{2(D-1)} (l^2 r_H^{D-3} - r_H^{D-1}) \quad (72)$$

so that

$$\Delta S_E = -\frac{(R_0 + f(R_0))\beta \mu_{D-2}}{36\pi(D-1)G_D} (l^2 r_H^{D-3} - r_H^{D-1}) = \beta F. \quad (73)$$

Notice that from this expression it is straightforward to obtain the free energy F . We see that provided $-(R_0 + f(R_0)) > 0$, which is the usual case in EH gravity, we have $F > 0$ for $r_H < l$ and $F < 0$ for $r_H > l$. The temperature corresponding to the horizon radius $r_H = l$ will be denoted T_1 and it is given by

$$T_1 = \frac{D-2}{2\pi l}. \quad (74)$$

Notice that for $D > 2$ we have $T_0 < T_1$.

On the other hand, the total thermodynamical energy may now be obtained as

$$E = \frac{\partial \Delta S_E}{\partial \beta} = -\frac{(R_0 + f(R_0))Ml^2}{2(D-1)} \quad (75)$$

where M is the mass defined in (29). This is one of the possible definitions for the BH energy for $f(R)$ theories; see for instance [21] for a more general discussion. For the EH action we have $f(R) = -(D-2)\Lambda_D$ and then it is immediate to find $E = M$. However this is not the case for general $f(R)$ actions. Notice that positive energy in AdS space-time requires $R_0 + f(R_0) < 0$. Now the entropy S can be obtained from the well-known relation:

$$S = \beta E - \beta F. \quad (76)$$

Then one gets

$$S = -\frac{(R_0 + f(R_0))l^2 A_{D-2}(r_H)}{8(D-1)G_D} \quad (77)$$

where $A_{D-2}(r_H)$ is the horizon area given by $A_{D-2}(r_H) = r_H^{D-2} \mu_{D-2}$. Notice that once again positive entropy requires $R_0 + f(R_0) < 0$. For the EH action we have $R_0 + f(R_0) = -2(D-1)/l^2$ and then we get the famous Hawking-Bekenstein result [22]

$$S = \frac{A_{D-2}(r_H)}{4G_D}. \quad (78)$$

Finally, we can compute the heat capacity C which can be written as

$$C = \frac{\partial E}{\partial T} = \frac{\partial E}{\partial r_H} \frac{\partial r_H}{\partial T}. \quad (79)$$

Then it is easy to find

$$C = \frac{-(R_0 + f(R_0))(D-2)\mu_{D-2}r_H^{D-2}l^2}{8G_D(D-1)} \times \frac{(D-1)r_H^2 + (D-3)l^2}{(D-1)r_H^2 - (D-3)l^2}. \quad (80)$$

For the particular case of the EH action we find

$$C = \frac{(D-2)\mu_{D-2}r_H^{D-2}}{4G_D} \frac{(D-1)r_H^2 + (D-3)l^2}{(D-1)r_H^2 - (D-3)l^2}. \quad (81)$$

In the Schwarzschild limit l going to infinity this formula gives

$$C = -\frac{(D-2)\mu_{D-2}r_H^{D-2}}{4G_D} < 0 \quad (82)$$

which is the negative well-known result for standard BH. In the general case, assuming like in the EH case $(R_0 + f(R_0)) < 0$, we find $C > 0$ for $r_H > r_{H0}$ (the large BH region) and $C < 0$ for $r_H < r_{H0}$ (the small BH region). For $r_H \sim r_{H0}$ (T close to T_0) C is divergent. Notice that in EH gravity, $C < 0$ necessarily implies $F > 0$ since $T_0 < T_1$.

In any case, for $f(R)$ theories with $R_0 + f(R_0) < 0$, we have found a scenario similar to the one described in full detail by Hawking and Page in [10] a long time ago for the EH case.

For $T < T_0$, the only possible state of thermal equilibrium in an AdS space is pure radiation with negative free energy and there are no stable BH solutions. For $T > T_0$ we have two possible BH solutions; the small (and light) BH and the large (heavy) BH. The small one has negative heat capacity and positive free energy as the standard Schwarzschild BH. Therefore it is unstable under Hawking radiation decay. For the large BH we have two possibilities; if $T_0 < T < T_1$ then both the heat capacity and the free energy are positive and the BH will decay by tunneling into radiation, but if $T > T_1$ then the heat capacity is still positive but the free energy becomes negative. In this case the free energy of the heavy BH will be less than that of pure radiation. Then pure radiation will tend to tunnel or to collapse to the BH configuration in equilibrium with thermal radiation.

In general $f(R)$ theories, one could also in principle consider the possibility of having $R_0 + f(R_0) > 0$. However, in this case, the mass and the entropy would be negative and therefore in such theories the AdS BH solutions would be unphysical. Therefore $R_0 + f(R_0) < 0$ can be regarded as a necessary condition for $f(R)$ theories in order to support AdS BH solutions. Using (7), this condition implies $1 + f'(R_0) > 0$. This last condition has a clear physical interpretation in $f(R)$ gravities (see [23] and references therein). Indeed, it can be interpreted as the condition for the effective Newton's constant $G_{\text{eff}} = G_D/(1 + f'(R_0))$ to be positive. It can also be interpreted

from the quantum point of view as the condition which prevents the graviton from becoming a ghost.

VII. PARTICULAR EXAMPLES

In this section we will consider some particular $f(R)$ models in order to calculate the heat capacity C and the free energy F as the relevant thermodynamical quantities for local and global stability of BHs. For these particular models, R_0 can be calculated exactly by using (7). For the sake of simplicity, we will fix the D -dimensional Schwarzschild radius in (29) as $R_S^{D-3} = 2$. The models we have considered are as follows.

A. Model I: $f(R) = \alpha(-R)^\beta$

Substituting in (7) for an arbitrary dimension we get

$$R \left[\left(1 - \frac{2}{D}\right) - \alpha(-R)^{\beta-1} \left(1 - \frac{2}{D}\beta\right) \right] = 0. \quad (83)$$

We will only consider nonvanishing curvature solutions, thus we find

$$R_0 = - \left[\frac{2-D}{(2\beta-D)\alpha} \right]^{1/(\beta-1)}. \quad (84)$$

Since D is assumed to be larger than 2, the condition $(2\beta-D)\alpha < 0$ provides well-defined scalar curvatures R_0 . Two separated regions have thus to be studied: Region 1 $\{\alpha < 0, \beta > D/2\}$ and region 2 $\{\alpha > 0, \beta < D/2\}$. For this model we also get

$$1 + f'(R_0) = \frac{D(\beta-1)}{2\beta-D}. \quad (85)$$

Notice that in region 1, $1 + f'(R_0) > 0$ for $D > 2$, since in this case $\beta > 1$ is straightforwardly accomplished. In region 2, we find that for $D > 2$, the requirement $R_0 + f(R_0) < 0$, i.e., $1 + f'(R_0) > 0$, fixes $\beta < 1$, since this is the most stringent constraint over the parameter β in this

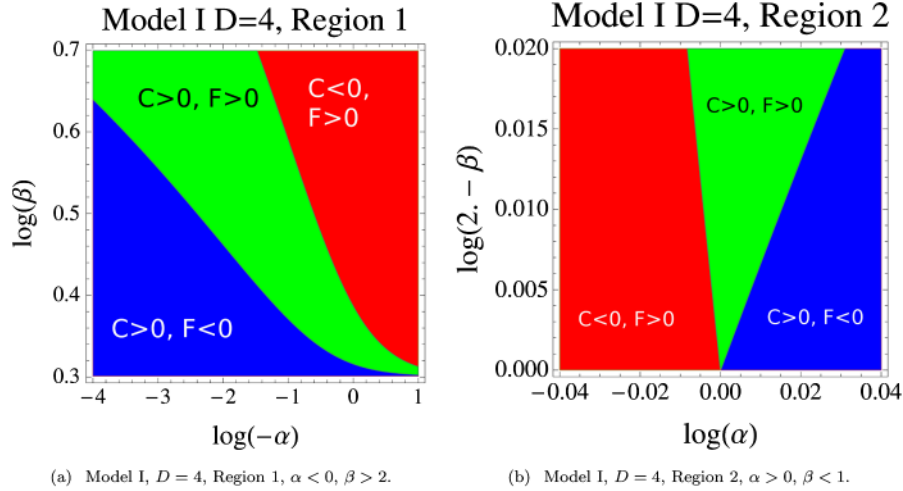


FIG. 1 (color online). Thermodynamical regions in the (α, β) plane for Model I in $D = 4$. Region 1 (left), Region 2 (right).

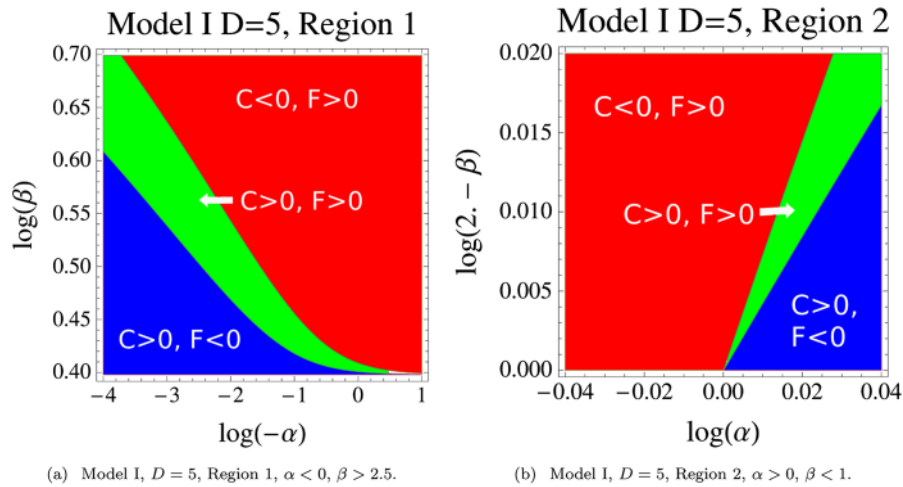
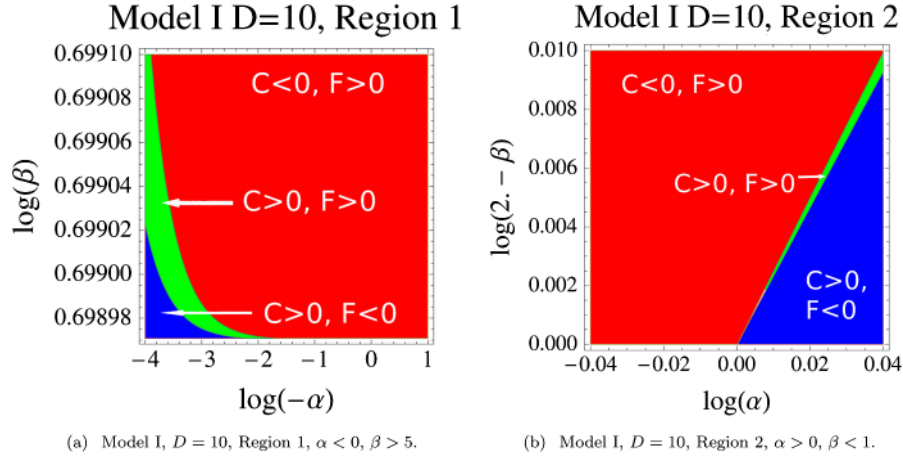


FIG. 2 (color online). Thermodynamical regions in the (α, β) plane for Model I in $D = 5$. Region 1 (left), Region 2 (right).

FIG. 3 (color online). Thermodynamical regions in the (α, β) plane for Model I in $D = 10$. Region 1 (left), Region 2 (right).

region. Therefore the physical space of parameters in region 2 is restricted to be $\{\alpha > 0, \beta < 1\}$.

In Figs. 1–3, we plot the physical regions in the parameter space (α, β) corresponding to the different signs of (C, F) .

B. Model II: $f(R) = -(-R)^\alpha \exp(q/R) - R$

In this case, a vanishing curvature solution appears provided $\alpha > 1$. In addition, we also have

$$R_0 = \frac{2q}{2\alpha - D}. \quad (86)$$

To get $R_0 < 0$ the condition $q(2\alpha - D) < 0$ must hold and two separated regions will be studied: Region 1 $\{q > 0, \alpha < D/2\}$ and region 2 $\{q < 0, \alpha > D/2\}$.

In Figs. 4–6, we plot the regions in the parameter space (α, q) corresponding to the different signs of (C, F) .

C. Model III: $f(R) = R(\log \alpha R)^q - R$

A vanishing curvature solution also appears in this model. The nontrivial one is given by

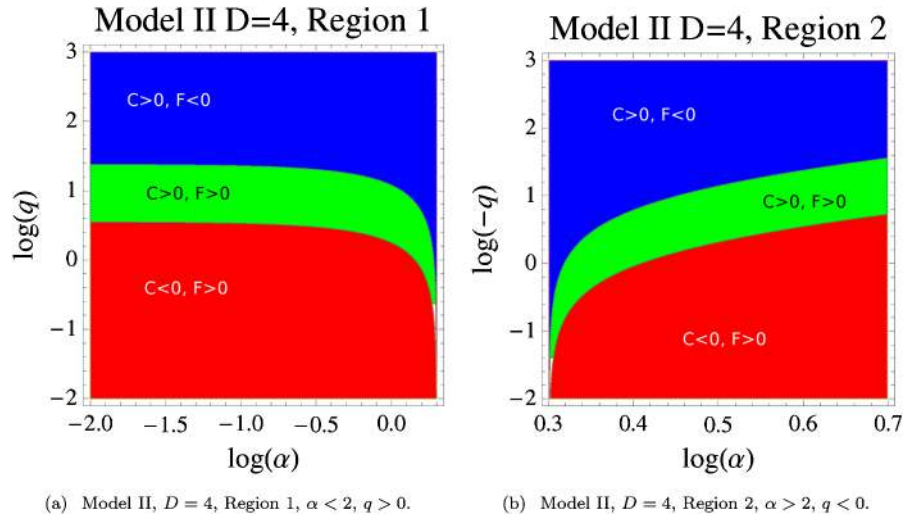
$$R_0 = \frac{1}{\alpha} \exp\left(\frac{2q}{D-2}\right). \quad (87)$$

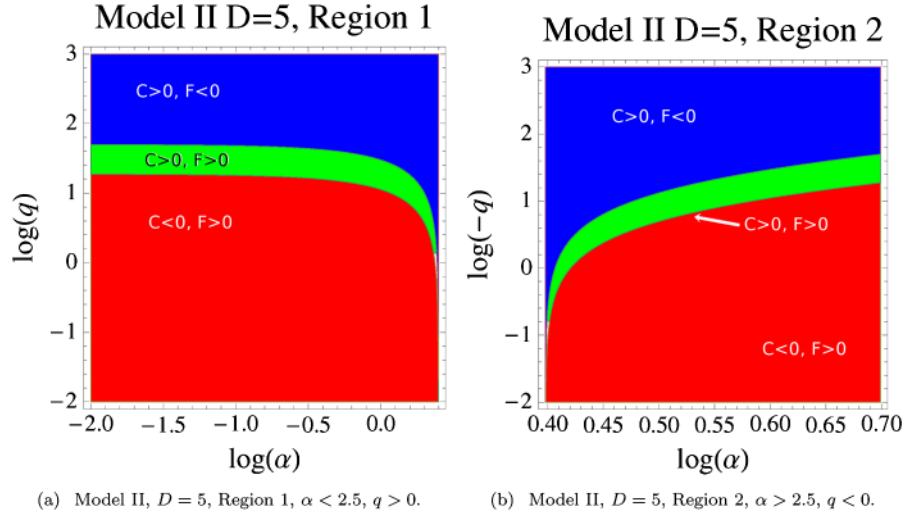
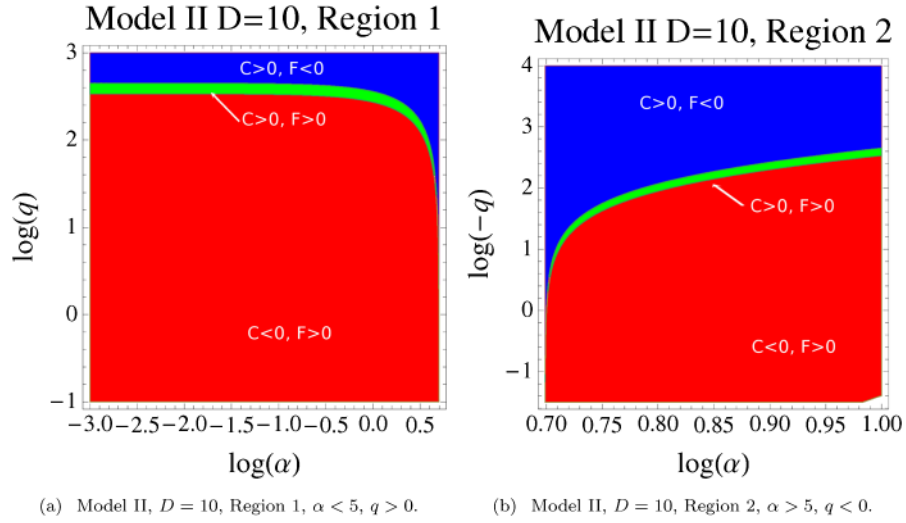
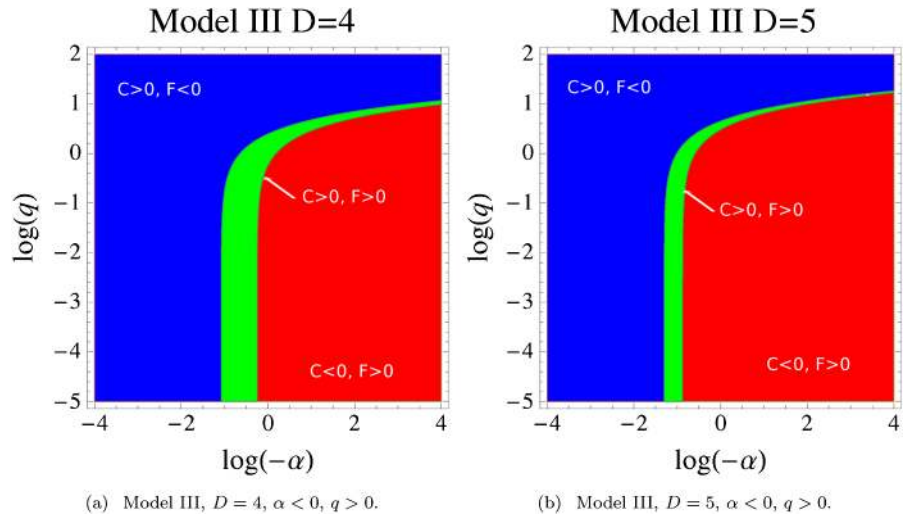
Since R_0 has to be negative, α must be negative as well, accomplishing $\alpha R_0 > 0$, and since αR , and therefore αR_0 , has to be bigger than one to have a positive number powered to q , which imposes $q > 0$ as can be read from the argument of the exponential in the previous equation. Therefore there exists a unique accessible region for parameters in this model: $\alpha < 0$ and $q > 0$.

In Figs. 7 and 8, we plot the regions in the parameter space (α, q) corresponding to the different signs of (C, F) .

D. Model IV: $f(R) = -\alpha \frac{c_1(\frac{R}{\alpha})^n}{1 + \beta(\frac{R}{\alpha})^n}$

This model has been proposed in [24] as cosmologically viable. Throughout this section, we consider $n = 1$ for this

FIG. 4 (color online). Thermodynamical regions in the (α, q) plane for Model II in $D = 4$. Region 1 (left), Region 2 (right).

FIG. 5 (color online). Thermodynamical regions in the (α, q) plane for Model II in $D = 5$. Region 1 (left), Region 2 (right).FIG. 6 (color online). Thermodynamical regions in the (α, q) plane for Model II in $D = 10$. Region 1 (left), Region 2 (right).FIG. 7 (color online). Thermodynamical regions in the (α, q) plane for Model III in $D = 4$ (left) and $D = 5$ (right).

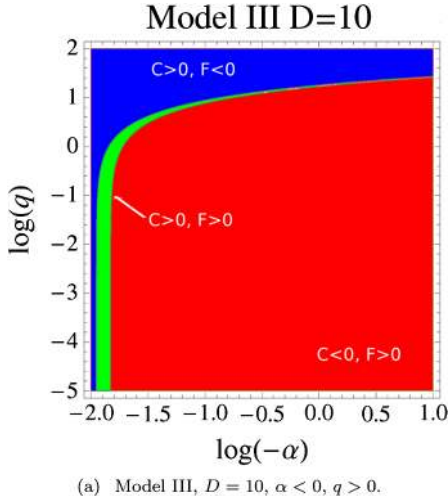


FIG. 8 (color online). Thermodynamical regions in the (α, q) plane for Model III in $D = 10$.

model. Hence imposing $f'(R_0) = \epsilon$ we get

$$c_1 = -\frac{(D - 2(1 + \epsilon))^2}{D^2 \epsilon}. \quad (88)$$

Hence a relation between c_1 , D , and ϵ can be imposed and therefore this model would only depend on two parameters α and β . A vanishing curvature solution also appears in this model and two nontrivial curvature solutions are given by

$$R_0^\pm = \frac{\alpha \left[(c_1 - 2)D + 4 \pm \sqrt{c_1} \sqrt{c_1 D^2 - 8D + 16} \right]}{2\beta(D - 2)}. \quad (89)$$

The corresponding $1 + f'(R_0)$ values for (89) are

$$1 + f'(R_0^\pm) = 1 - \frac{4(D - 2)^2}{(\sqrt{c_1} D^2 - 8D + 16 \pm \sqrt{c_1} D)^2} \quad (90)$$

where $c_1 > 0$ and $c_1 > (8D - 16)/D^2$ are required for real

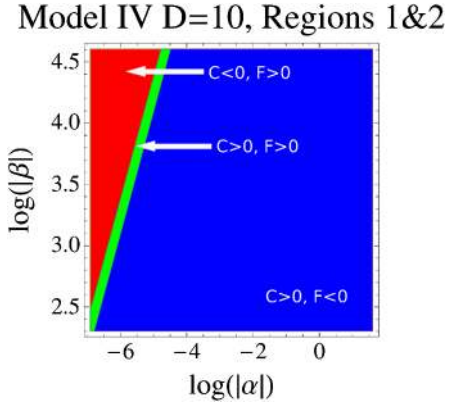


FIG. 10 (color online). Thermodynamical regions in the $(|\alpha|, |\beta|)$ plane for Model IV in $D = 10$.

R_0 solutions. Since $1 + f'(R_0) > 0$ is required, that means that $\text{sign}(R_0^\pm) = \text{sign}(\alpha\beta)$. It can be shown that $1 + f'(R_0^-)$ is not positive for any allowed value of c_1 and therefore this curvature solution R_0^- is excluded for our study.

$1 + f'(R_0^+) > 0$ only requires $c_1 > 0$ for dimension $D \geq 4$ and therefore $\epsilon < 0$ is required according to (88). Therefore only two accessible regions need to be studied: Region 1 $\{\alpha > 0, \beta < 0\}$ and region 2, $\{\alpha < 0, \beta > 0\}$.

In Figs. 9 and 10, we plot the thermodynamical regions in the parameter space (α, β) for a chosen $\epsilon = -10^{-6}$. Note that $1 + f'(R_0^+)$ does depend neither on α nor on β and that R_0^+ only depends on the quotient α/β for a fixed c_1 .

VIII. CONCLUSIONS

In this work we have considered static spherically symmetric solutions in $f(R)$ theories of gravity in arbitrary dimensions. After discussing the constant curvature case (including charged black holes in 4 dimensions), we have studied the general case without imposing, *a priori*, the condition of constant curvature. We have performed a

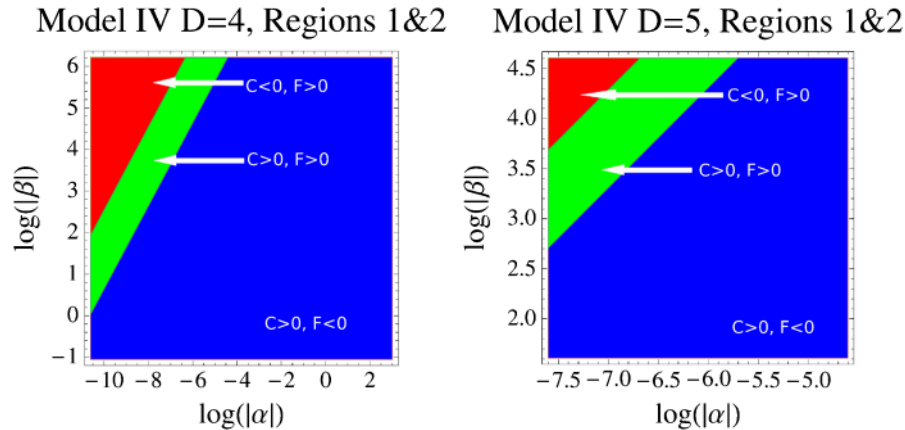


FIG. 9 (color online). Thermodynamical regions in the $(|\alpha|, |\beta|)$ plane for Model IV in $D = 4$ (left) and $D = 5$ (right).

perturbative analysis around the EH case which makes it possible to study those solutions which are regular in the perturbative parameter α . We have found explicit expressions up to second order for the metric coefficients, which give rise to constant curvature (Schwarzschild-AdS) solutions as in the EH case.

On the other hand, we have also calculated thermodynamical quantities for the AdS black holes and considered the issue of the stability of this kind of solutions. We have found that the condition for a $f(R)$ theory of gravity to support this kind of black holes is given by $R_0 + f(R_0) < 0$ where R_0 is the constant curvature of the AdS space-time. This condition has been seen to imply also that the effective Newton's constant is positive and that the graviton does not become a ghost. For these $f(R)$ gravities, the qualitative thermodynamic behavior of the BH is the

same as the one found by Hawking and Page for the AdS BH but the value of some thermodynamic magnitudes is different for different $f(R)$ gravities.

Finally, we have considered several explicit examples of $f(R)$ functions and studied the parameter regions in which BH in such theories are locally stable and globally preferred, finding the same qualitative behavior as in standard EH gravity.

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