

# Black Holes in Higher-Derivative Gravity

H. Lü<sup>1</sup>, A. Perkins<sup>2</sup>, C.N. Pope<sup>3,4</sup> and K.S. Stelle<sup>2</sup>

<sup>1</sup> *Center for Advanced Quantum Studies, Department of Physics,  
Beijing Normal University, Beijing 100875, China*

<sup>2</sup> *The Blackett Laboratory, Imperial College London, Prince Consort Road, London SW7 2AZ, UK*

<sup>3</sup> *George P. & Cynthia W. Mitchell Institute for Fundamental Physics and Astronomy,  
Texas A&M University, College Station, TX 77843-4242, USA*

<sup>4</sup> *DAMTP, Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge, CB3 0WA, UK*

Extensions of Einstein gravity with higher-order derivative terms arise in string theory and other effective theories, as well as being of interest in their own right. In this paper we study static black-hole solutions in the example of Einstein gravity with additional quadratic curvature terms. A Lichnerowicz-type theorem simplifies the analysis by establishing that they must have vanishing Ricci scalar curvature. By numerical methods we then demonstrate the existence of further black-hole solutions over and above the Schwarzschild solution. We discuss some of their thermodynamic properties, and show that they obey the first law of thermodynamics.

The well-known problem of the non-renormalisability of Einstein gravity has given rise to many attempts to view it as an effective low-energy theory that will receive higher-order corrections that become important as the energy scale increases (see, for example, [1]). In string theory, the Einstein-Hilbert action is just the first term in an infinite series of gravitational corrections built from powers of the curvature tensor and its derivatives. In other approaches, only a finite number of additional terms might be added. It was shown in [2] that if one adds all possible quadratic curvature invariants to the usual Einstein-Hilbert action one obtains a renormalisable theory, albeit at the price of introducing ghost-like modes. Arguments have been given for why these might not be fatal to the theory (for example, see [3] for a recent discussion). In any case, it is worthwhile to study in detail the properties of the theory of Einstein gravity with added quadratic curvature terms, in order shed light on the question of whether it has irredeemable pathologies or whether they can be controlled in some manner.

Black holes are the most fundamental objects in a theory of gravity, and they provide powerful probes for studying some of the more subtle global aspects of the theory. It is therefore of considerable interest to investigate the structure of black-hole solutions in theories of gravity with higher-order curvature terms. In this paper, we report on some investigations of the static, spherically-symmetric black-hole solutions in four-dimensional Einstein-Hilbert gravity with added quadratic curvature terms, for which the most general action can be taken to be

$$I = \int d^4x \sqrt{-g} (\gamma R - \alpha C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \beta R^2), \quad (1)$$

where  $\alpha$ ,  $\beta$  and  $\gamma$  are constants and  $C_{\mu\nu\rho\sigma}$  is the Weyl

tensor. We shall work in units where we set  $\gamma = 1$ , and the equations of motion following from (1) are then

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - 4\alpha B_{\mu\nu} + 2\beta R (R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}) + 2\beta (g_{\mu\nu} \square R - \nabla_\mu \nabla_\nu R) = 0, \quad (2)$$

where  $B_{\mu\nu} = (\nabla^\rho \nabla^\sigma + \frac{1}{2} R^{\rho\sigma}) C_{\mu\rho\nu\sigma}$  is the Bach tensor, which is tracefree.

In general, the theory describes a system with a massive spin-2 mode with mass-squared  $m_2^2 = 1/(2\alpha)$  and a massive spin-0 mode with mass-squared  $m_0^2 = 1/(6\beta)$ , in addition to the massless spin-2 graviton. These massive modes will be associated with rising and falling Yukawa type behaviour in the metric modes near infinity [4], of the form  $\frac{1}{r} e^{\pm m_2 r}$  and  $\frac{1}{r} e^{\pm m_0 r}$ . In particular, one can expect that if generic initial data is set at some small distance, the rising exponentials will eventually dominate, leading to singular asymptotic behaviour. In seeking black-hole solutions, the question then arises as to whether the rising exponentials can be avoided for appropriately finely-tuned initial data.

It can easily be seen that any solution of pure Einstein gravity will also be a solution of (2), and so in particular the usual Schwarzschild black hole continues to be a solution in the higher-order theory. The question we wish to address, then, is whether there exist any other static black hole solutions, over and above the Schwarzschild solution.

Static, spherically-symmetric black-hole solutions have been investigated in [5], using generalisations of the Lichnerowicz and Israel theorems for Einstein gravity. Since we will arrive at somewhat different conclusions, we shall briefly summarise the key elements in [5], although derived in a different notation. We consider static metrics of the form  $ds_4^2 = -\lambda^2 dt^2 + h_{ij} dx^i dx^j$ , where  $\lambda$

and  $h_{ij}$  are functions only of the three spatial coordinates  $x^i$ . Taking the trace of the field equations (2) gives  $\beta(\square - m_0^2)R = 0$ . We then multiply this by  $\lambda R$  and integrate over the spatial domain from a putative horizon out to infinity. Expressed in terms of the covariant derivative  $D_i$  with respect to the spatial 3-metric  $h_{ij}$ , this gives

$$\int \sqrt{h} d^3x \left[ D^i(\lambda R D_i R) - \lambda(D_i R)^2 - m_0^2 \lambda R^2 \right] = 0. \quad (3)$$

Since  $\lambda$  vanishes on the horizon, it follows that if  $D_i R$  goes to zero sufficiently rapidly at spatial infinity the total derivative (*i.e.* surface term) gives no contribution, and the non-positivity of the remaining terms then implies  $R = 0$ . In other words, as shown in [5], any static black-hole solution of (1) must have vanishing Ricci scalar. This leads to a great simplification, and it means that one can, without loss of generality, study the case of pure Einstein-Weyl gravity (*i.e.* (1) with  $\beta = 0$ ), since obviously the term quadratic in  $R$  makes no contribution to the field equations for a configuration with  $R = 0$ . Furthermore, the trace of the field equations (2) for Einstein-Weyl gravity immediately implies  $R = 0$ . In fact, the two differential equations for  $h$  and  $f$  are both now of only second order in derivatives.

The second stage of the discussion in [5] then involved looking at the remaining content of (2), *i.e.* the non-trace part. According to [5], this led to another integral identity that then implied, under certain assumptions, that  $R_{\mu\nu} = 0$ . If this were correct, then the conclusion would be that the usual Schwarzschild solution is the only static black hole solution of the theory described by (1). However, we find that there are sign errors in the expression given in [5]. Setting  $R = 0$ , as already argued above, multiplying (2) by  $\lambda R^{\mu\nu}$ , and then integrating over the spatial region outside the horizon gives

$$\begin{aligned} \int \sqrt{h} d^3x \left[ D^i W_i - \frac{1}{4} \lambda (D_i \bar{R} - 4 D^j R_{ij})^2 + 4 \lambda (D^j R_{ij})^2 \right. \\ \left. - 4 \lambda (D_{[i} R_{j]k})^2 + \lambda (D_i R_{jk})^2 - \frac{1}{4} \lambda \bar{R}^2 (m_2^2 + \bar{R}) \right. \\ \left. - \lambda (m_2^2 R^{ij} R_{ij} - 2 R^{ij} R_{jk} R^k{}_i) \right] = 0, \quad (4) \end{aligned}$$

where  $W_i = \lambda R^{jk} D_i R_{jk} + \frac{1}{4} \lambda \bar{R} D_i \bar{R} - 2 \lambda R^{jk} D_j R_{ik} - \lambda \bar{R} D_j R_i{}^j$ , and  $\bar{R}$  is the Ricci scalar of the spatial metric  $h_{ij}$ . Although the surface term will give zero, the mix of positive and negative signs in the bulk terms precludes one from obtaining any kind of vanishing theorem for the Ricci tensor of the four-dimensional metric. This raises the intriguing possibility that there might in fact exist static, spherically symmetric black-hole solutions over and above the Schwarzschild solution.

The equations of motion following from (1) are too complicated to be able to solve explicitly, even for the case of the static, spherically-symmetric ansatz

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (5)$$

In our work, we have therefore carried out a numerical investigation of the solutions. To do this, we begin by supposing that there exists a black-hole horizon at some radius  $r = r_0 > 0$ , at which the metric functions  $h$  and  $f$  vanish, and we then obtain near-horizon Taylor expansions for  $h(r)$  and  $f(r)$ , of the form

$$\begin{aligned} h(r) &= c \left[ (r - r_0) + h_2 (r - r_0)^2 + h_3 (r - r_0)^3 + \dots \right], \\ f(r) &= f_1 (r - r_0) + f_2 (r - r_0)^2 + f_3 (r - r_0)^3 + \dots \quad (6) \end{aligned}$$

Substituting into the equations of motion (2), with  $\beta$  set to zero for the reasons discussed above, the coefficients  $h_i$  and  $f_i$  for  $i \geq 2$  can be solved for in terms of the two non-trivial free parameters  $r_0$  and  $f_1$ . There is also a ‘‘trivial’’ parameter, corresponding to the freedom to rescale the time coordinate, which we have accordingly written in the form of an overall scaling of  $h(r)$ . Thus we have

$$h_2 = \frac{1 - 2f_1 r_0}{f_1 r_0^2} + \frac{1 - f_1 r_0}{8\alpha f_1^2 r_0}, \quad f_2 = \frac{1 - 2f_1 r_0}{r_0^2} - \frac{3(1 - f_1 r_0)}{8\alpha f_1 r_0},$$

and so on. (We used Taylor expansions to  $\mathcal{O}((r - r_0)^9)$  in our numerical integrations.) The Schwarzschild solution corresponds to  $f_1 = 1/r_0$ , and so it is convenient to parameterise  $f_1$  as

$$f_1 = \frac{1 + \delta}{r_0}, \quad (7)$$

with non-vanishing  $\delta$  characterising the extent to which the near-horizon solution deviates from Schwarzschild.

We use the expansions to set initial data at a radius  $r_i$  just outside the horizon, and then use numerical routines in Mathematica to integrate the equations out to large radius. Generically, one finds that for a given choice of the parameters  $r_0$  and  $\delta$  the solution rapidly becomes singular as one integrates outwards from  $r = r_i$ , as expected in view of our earlier observations about the rising Yukawa terms in the asymptotic form for the metric. If we fix a particular value for  $r_0$ , we can then use the shooting method to try to home in on a special value of  $\delta$  for which the outward integration can proceed without encountering a singularity. Of course in practice, because of accuracy limitations in the integrations, the solution will always eventually become singular at large enough  $r$ . The signal for a good black-hole solution is that  $f(r)$  and  $h(r)$  should approach constants as  $r$  increases, (in fact Ricci scalar flatness implies  $f(r)$  must approach 1), and that by stepping up the accuracy and precision goals in the calculations one can extend at will the maximum upper limit  $r = r_f$  for which the smooth behaviour can be achieved. In practice, by running the routines with accuracy and precision goals of order 20 decimal places, we have been able to obtain very clean and trustworthy solutions out to at least 60 times the horizon radius.

Our findings are that there exists a range of values for the horizon radius, bounded below by a certain multiple

of the length  $\sqrt{\alpha}$ , for which we can obtain precisely one static black-hole solution in addition to the Schwarzschild solution. In order to make the statement of our results in the most concise possible way, it is convenient, without loss of mathematical generality, to make a specific choice for the value of  $\alpha$  in (1). We shall take

$$\alpha = \frac{1}{2}. \quad (8)$$

We then find that for each choice of  $r_0 > r_0^{\min}$ , where

$$r_0^{\min} \approx 0.876, \quad (9)$$

we can find a non-Schwarzschild static black hole. For each such  $r_0$ , there is a corresponding value  $\delta = \delta^*$  of the “non-Schwarzschild parameter” that yields the black-hole solution without a singularity at spatial infinity. As  $r_0$  is taken closer and closer to the value  $r_0^{\min}$ , the required value  $\delta^*$  becomes smaller and smaller, tending to zero at  $r_0 = r_0^{\min}$ . Thus the Schwarzschild and the non-Schwarzschild black holes “coalesce” as  $r_0 = r_0^{\min}$  is approached.

As  $r_0$  is increased above  $r_0^{\min}$ , the Schwarzschild and non-Schwarzschild black-hole solutions separate more from one another (and in particular the required value of  $\delta$  increases). The mass of the Schwarzschild black hole is simply  $\frac{1}{2}r_0$ , and thus it increases linearly as  $r_0$  increases. By contrast, the mass of the non-Schwarzschild black hole decreases as  $r_0$  increases, until at  $r_0 = r_0^{m=0}$  it becomes massless, where

$$r_0^{m=0} \approx 1.143. \quad (10)$$

(The definition of mass in higher-derivative theories was discussed in [6, 7]. For asymptotically-flat black holes it is just  $\frac{1}{2}$  the coefficient of  $1/r$  in  $g_{tt}$  (assuming  $g_{tt}$  is normalised canonically at infinity.) Interestingly, if  $r_0$  is increased beyond  $r_0^{m=0}$ , one can still obtain a non-Schwarzschild black-hole solution for an appropriate choice of  $\delta$ , but now the mass is actually negative. In other words there is still a regular horizon, and the metric is asymptotically flat at large distances, but the metric function  $f$  now rises above 1 as  $r$  increases from  $r_i$ , before sinking down to 1 again in the asymptotic region. Figure 1 shows the masses of the Schwarzschild and non-Schwarzschild black holes as a function of  $r_0$ , and their masses as functions of their Hawking temperatures. The maximum possible mass for the non-Schwarzschild black hole, attained when  $r_0 = r_0^{\min}$ , is given by  $M^{\max} = \frac{1}{2}r_0^{\min} \approx 0.438$ . From the slope of  $M(T)$  it can be seen that the specific heat  $C = \partial M/\partial T$  is negative for both black holes, and more negative for the non-Schwarzschild black hole at a given temperature.

The plots of the metric functions  $f$  and  $h$  for the examples of a positive-mass black hole with  $r_0 = 1$ , and a negative-mass black hole, with  $r_0 = 2$ , are shown in Figure 2.

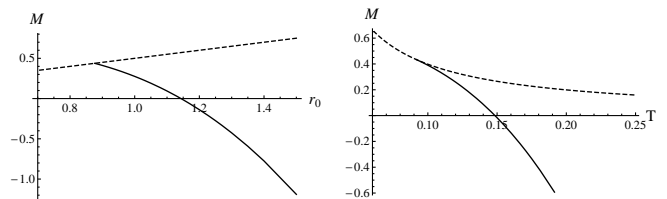


FIG. 1: The masses as functions of  $r_0$  (left plot), and as functions of the Hawking temperatures (right plot) for the Schwarzschild (dashed line) and non-Schwarzschild (solid line) black holes.

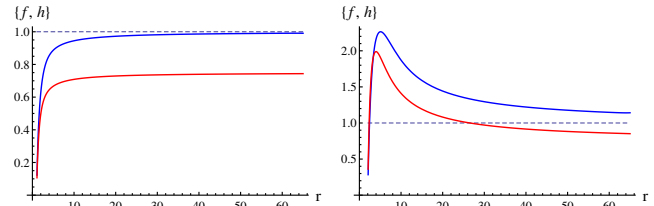


FIG. 2: The non-Schwarzschild black hole for  $r_0 = 1$  (left plot) and  $r_0 = 2$  (right plot). In each plot the upper curve is  $f(r)$  and the lower curve is  $h(r)$ . For clarity we have chosen a rescaling of  $h$  so that it approaches  $\frac{3}{4}$ , rather than 1, to avoid an asymptotic overlap of the curves.

Having established the existence of the non-Schwarzschild black holes, it is instructive to study some of their thermodynamic properties, and to compare these with the properties of the Schwarzschild black holes. In order to do this, we have collected the numerical results for a sequence of black-hole solutions with  $r_0$  in the range  $r_0^{\min} \approx 0.876 < r_0 < 1.5$ , and then fitted the data to appropriate polynomials. Because we are working with a higher-derivative theory, the entropy is not simply given by one quarter of the area of the event horizon, and instead we need to use the formula derived by Wald [8, 9]. This has been evaluated for the ansatz (5) in quadratic curvature gravities in [10], and applied to our case with  $\beta = 0$  and  $\gamma = 1$  in (1) this gives  $S = \pi r_0^2 + 4\pi\alpha(1 - f_1 r_0) = \pi r_0^2 - 4\pi\alpha\delta^*$ . (There is a freedom to add a constant multiple of the Gauss-Bonnet invariant to the Lagrangian, which shifts the entropy by a parameter-independent constant without affecting the equations of motion. We have used this to ensure the entropy of the Schwarzschild black hole vanishes when the mass vanishes.) We then find that the mass and the temperature of these non-Schwarzschild black holes, as a function of the entropy, take the form

$$\begin{aligned} M &\approx 0.168 + 0.131 S - 0.00749 S^2 - 0.000139 S^3 + \dots, \\ T &\approx 0.131 - 0.0151 S - 0.000428 S^2 + \dots. \end{aligned} \quad (11)$$

It can be seen that  $\partial M/\partial S \approx 0.131 - 0.0150 S - 0.000417 S^2$ , which is very close to the expression for the

temperature. Thus the non-Schwarzschild black holes are seen to obey the first law  $dM = TdS$  to quite a high precision. Note that the expressions for  $M$  and  $T$  as a function of  $S$  for the Schwarzschild black holes are very different in form, with  $M = (S/4\pi)^{1/2}$  and  $T = \frac{1}{4}(\pi S)^{-1/2}$ .

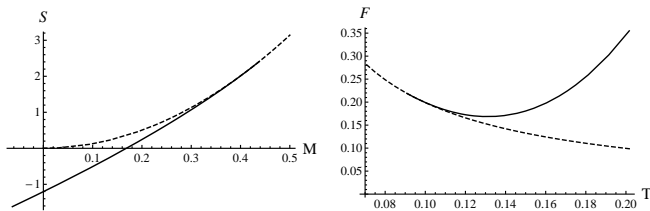


FIG. 3: The first plot shows the entropy as a function of mass, and the second shows the free energy  $F = M - TS$  as a function of  $T$ , for the Schwarzschild (dashed line) and non-Schwarzschild (solid line) black holes.

It is interesting to note that the entropy of the non-Schwarzschild black hole of a given mass is always less than the entropy of the Schwarzschild black hole of the same mass. The two entropies approach each other asymptotically as  $r_0$  approaches  $r_0^{\min}$ . This can be seen in the left-hand plot in Figure 3. It is also of interest to look at the free energy  $F = M - TS$  as a function of temperature. This is shown in the right-hand plot in Figure 3. It can be seen that the free energy is always larger for the non-Schwarzschild black hole at a given temperature, with the two curves again meeting at the lower limit when  $r_0 = r_0^{\min}$ .

In this paper, we have used black holes to probe some of the consequences of interpreting the action (1) as complete classical action in its own right. We have seen that there exists a second branch of static, spherically symmetric black holes, over and above the Schwarzschild solutions. These are not Ricci flat, although they do have vanishing Ricci scalar. Restoring the factors of  $\alpha$  and  $\gamma$  that we fixed in our numerical simulations, the second branch of black holes have masses, which can become negative, bounded approximately by  $M \leq 0.438\sqrt{2\alpha\gamma}$ . Thus in a regime where  $\alpha$  is small, which one might hope would correspond to a small correction to Einstein gravity, the second branch of black holes will be tiny, and will actually have very large curvature near the horizon, thus tending to invalidate the requirement that the curvature-squared should be small. The fact that their mass can be negative, violating the usual positive-mass theorem of standard Einstein gravity, indicates that the ghost-like nature of the quadratically-corrected action is becoming dominant in this regime; one may view the negative-mass black holes as condensates dominated by the contribution of the ghost-like massive spin-2 modes. It could be viewed as a satisfactory outcome of our investigation that the only indications of the existence of black holes with potentially pathological properties in the

quadratic-curvature theories occur in a regime where yet higher-order corrections, as in string theory, are going to be important also. It would be interesting to obtain analytical proofs of the existence of the numerical solutions we have found. Although this could be challenging, it might perhaps be easier to obtain restricted no-hair theorems that confirmed the apparent absence of non-Schwarzschild black holes outside the parameter range where we have found them.

Naturally, the fourth-order equations of motion (2) have wider classes of solutions than the black-hole solutions with horizons that we have considered here. These were initially investigated in [4] and will be given a more detailed analysis, along with a more extensive analysis of the black hole solutions, in [11].

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