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Black Holes in Supergravity: the non-BPS Branch

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Abstract

We construct extremal, spherically symmetric black hole solutions to 4D supergravity with charge assignments that preclude BPS-saturation. In particular, we determine the ground state energy as a function of charges and moduli. We find that the mass of the non-BPS black hole remains that of a marginal bound state of four basic constituents throughout the entire moduli space and that there is always a non-zero gap above the BPS bound.

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1 Introduction

The construction of regular black hole solutions in supergravity has been a major research area for many years. This effort has given a very complete understanding of the BPS black holes and their non-extremal generalizations. However, there are many assignments of asymptotic charges which do not correspond to regular, spherically symmetric BPS black holes. The black holes describing such configurations are qualitatively different from their BPS relatives. This makes them interesting, but also more complicated than the BPS solutions. In fact, there are many simple cases where the black hole solutions have not even been constructed. This paper seeks to fill this gap.

The shortcoming of the standard inventory of solutions can be put in context by a well-known example. Consider the D0/D4 black hole solution with asymptotic moduli taking canonical values. In this case the BPS black holes correspond to one sign of the D0-brane charge (in our conventions $Q_0 > 0$) while the non-BPS solutions correspond to the other sign (alas, $Q_0 < 0$). Moreover, as long as we consider the simplest assigment of moduli, the BPS and non-BPS solution are related by analytical continuation: simply invert the sign of the gauge field coupling to D0-brane charge, keeping the geometry and all scalar fields invariant.

The point we wish to make is that this simple D0/D4 example is non-generic. In a more general situation there are further charges present in the configuration or, equivalently, nontrivial background moduli have been turned on. Either way, it is no longer possible to continue analytically between the BPS and the non-BPS solutions. In fact, the two types of solutions depend on charges so differently that their relation is nonanalytic. In this sense the non-BPS class of solutions are reminiscent of a different phase or, at least, a different branch of configuration space.

We focus on $N = 8$ supergravity for definiteness and consider a type IIA duality frame where all charges correspond to D-branes[1]. The general distinction between the two branches is encoded in the quartic invariant which in the present context can be written as [1, 2] ($\Sigma = 0..3, i = 1..3$):

$$I_4 = 4Q_0 P^1 P^2 P^3 - 4P^0 Q_1 Q_2 Q_3 - \left(\sum_\Sigma P^\Sigma Q_\Sigma \right)^2 + 4 \sum_{i<j} P^i Q_i P^j Q_j \ . \qquad (1.1)$$

The charge configurations with $I_4 > 0$ have a BPS limit whereas those with $I_4 < 0$ do not. Thus the configuration space of non-BPS solutions is as large as that of the BPS solutions, in that they have the same number of continuous parameters. The entropy of the black holes on either branch is given by [3]:

$$S = \frac{\pi}{G_N} \sqrt{|I_4|} \ . \qquad (1.2)$$

The entropies of the two branches are therefore related in a simple way. However, the solutions have no simple relation.

The most general spherically symmetric black hole solution in $N = 8$ (or $N = 4$) supergravity can be generated by acting with dualities on a seed solution with at least five charges

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[1]Most results apply to $N = 2$ and $N = 4$ supergravity after obvious changes of notation.

[4, 5]. Five parameter generating solutions were constructed in the BPS-case long time ago [6, 7, 8, 9], but on the non-BPS branch only four parameter solutions have been constructed so far[2]. The solutions we construct are the general seed solutions.

The charge assignments we focus on do in fact permit BPS solutions, at least in some cases. Those BPS solutions are the multicenter solutions, which have been the subject of much recent interest [12, 13]. These multicenter solutions have the same asymptotic charges as those we construct, but only exist for a limited set of possible background moduli. Nevertheless, since the multicenter solutions are BPS, and so have less energy than the solutions constructed here, we suspect that there is an interesting interplay between the two classes of solutions. We hope to return to this point elsewhere.

In this work we focus on the extremal case for conceptual clarity, but the non-BPS branch of solutions include generalizations of these with more energy, and with angular momentum. The solutions we construct represent ground states, since they are extremal. From this perspective the interesting output of the solution is the non-BPS mass formula. In the D0/D6 duality frame the mass takes the form [14, 15, 3]:

$$M = \frac{1}{\sqrt{8}G_4} \left(Q^{2/3} + P^{2/3} \right)^{3/2} .$$

We determine the generalization of this formula that includes B-fields on the world-volume of the D6. More generally, we suspect that the non-BPS formula reflects interesting and rather generic data that probes supersymmetry breaking in the gravitational sector.

This paper is organized as follows. Section 2 gives an overview of our conventions, the equations that need to be solved and of the various types of attractors. Section 3 is a review of known results for BPS attractor flows and uses a D0-D4-D4-D4 charge vector. Section 4 gives new results for the non-BPS extremal attractor flows in terms of one seed solution with $\overline{\text{D0}}$-D4-D4-D4 charge, with some important group theory features. Section 5 dualizes these results to get a $D6 - D0$ non-BPS attractor flow. Finally, section 6 closes with a brief discussion. The details of our non-BPS solution are derived in an appendix.

While this paper was in preparation, some overlapping and complementary results appeared in [16].

2   The Setting

We want to be specific about our notation and so we begin with a small review of our setting.

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[2]To our knowledge, the first examples of non-BPS extremal solutions were found in [10, 11].

2.1 The Theory

We work in the framework of $N = 2$ supergravity coupled to a number of vector multiplets. The bosonic action terms in the action are [17]:

$$S = \frac{1}{8\pi G_N} \int d^4x \mathcal{L} = \frac{1}{8\pi G_N} \int d^4x \left[-\frac{R}{2} + G_{a\bar{b}} \partial_\mu z^a \partial^\nu \bar{z}^b + \text{Im}\left(\mathcal{N}_{\Lambda\Sigma} \mathcal{F}_{\mu\nu}^{-\Lambda} \mathcal{F}_{\mu\nu}^{-\Sigma}\right) \right] , \quad (2.1)$$

where $\mathcal{F}_{\mu\nu}^{\pm\Lambda} = \mathcal{F}_{\mu\nu}^\Lambda \pm \frac{i}{2} \varepsilon_{\mu\nu\rho\sigma} \mathcal{F}^{\Lambda\,\mu\nu}$.

We focus on the $\mathcal{N} = 2$ theory known as the STU-model [18, 19, 20]. We will interpret the model in terms of IIA theory on a T^6 of the form $T^2 \times T^2 \times T^2$. The D0/D2/D4/D6-branes wrapping the various T^2's give four magnetic and four electric charges. The STU-model in $\mathcal{N} = 2$ theory captures the essential features of extremal black holes in the $\mathcal{N} = 4, 8$ theories and many of it's features also generalize well to extremal black holes in other $\mathcal{N} = 2$ theories (such as those from CY compactifications).

In $\mathcal{N} = 2$ theory it is convenient to use the language of special geometry. In the STU model the prepotential and its derivative are:

$$F = \frac{X^1 X^2 X^3}{X^0} , \qquad F_\Sigma = \frac{\partial F}{\partial X^\Sigma} . \qquad (2.2)$$

We gauge fix the projective coordinates X^Λ ($\Lambda = 0, 1, 2, 3$) so $X^0 = 1$ and then write $X^i = z^i = x^i - iy^i$ ($i = 1, 2, 3$).[3] The Kähler potential is:

$$K = -\log i(\bar{F}_\Sigma X^\Sigma - F_\Sigma \bar{X}^\Sigma) = -\ln(8y_1 y_2 y_3) . \qquad (2.3)$$

The corresponding metric and connection on moduli space are:

$$G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K = \frac{\delta_{i\bar{j}}}{(2y^i)^2} , \qquad \Gamma_{ii}^i = \frac{i}{y^i} . \qquad (2.4)$$

Here i is not summed over. The central charge of the $\mathcal{N} = 2$ superalgebra is written in terms of the superpotential W, as:

$$Z = e^{K/2} W = e^{K/2} \left[X^\Lambda Q_\Lambda - F_\Lambda P^\Lambda \right] . \qquad (2.5)$$

2.2 Charges

The electric and magnetic charges are defined as:

$$P^\Lambda = \frac{1}{4\pi} \int_{S_\infty^2} \mathcal{F}^\Lambda , \qquad Q_\Sigma = \frac{1}{4\pi} \int_{S_\infty^2} \mathcal{G}_\Sigma , \qquad (2.6)$$

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[3]Some authors use $z^i = x^i + iy^i$. Then in order to keep the Kähler metric positive, the sign of the pre-potential F is opposite. The resulting scalars are the complex conjugate of ours, as is the central charge. The sign of the electric charges Q_Σ are the opposite.

where the symplectic dual field strength is:

$$\mathcal{G}^{\mu\nu}_{\pm\Lambda} = -i\frac{\delta\mathcal{L}}{\delta\mathcal{F}^{\pm\Lambda}_{\mu\nu}} = \overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{+\Sigma\,\mu\nu} \ . \tag{2.7}$$

The physical charges (2.6) are organized in symplectic pairs:

$$\Gamma \equiv (P^\Lambda, Q_\Sigma) \ . \tag{2.8}$$

They have units of length and are related to dimensionless quantized charges by some dressing factors. We will normalize the asymptotic volume moduli so $y^i|_\infty = 1$ but keep the asymptotic B-fields $x^i_\infty = B^i = \frac{1}{V_i}\int_{V_i} B$ as free variables[4]. Then the dressing factors are just numerical factors

$$P^\Lambda = C^\Lambda p^\Lambda \ , \qquad Q_\Sigma = C_\Sigma q_\Sigma \ , \tag{2.10}$$

which are essentially the masses of the underlying branes:

$$C^0 = 2^{3/2}G_N M_{D6} = \sqrt{G_N v_6} \ , \qquad C^i = 2^{3/2}G_N M_{D4} = \sqrt{G_N v_6}\cdot\frac{1}{v_i} \ , \tag{2.11}$$

$$C_0 = 2^{3/2}G_N M_{D0} = \sqrt{\frac{G_N}{v_6}} \ , \qquad C_i = 2^{3/2}G_N M_{D2} = \sqrt{\frac{G_N}{v_6}}\cdot v_i \ .$$

Here v_i are the volumes of the T^2's measured in string units $v_i = V_i/(2\pi l_s)^2$. The overall compactification volume is $v_6 = v_1 v_2 v_3$ and the $D = 4$ Newton's constant $G_N = l_s^2 g_s^2/8v_6$.

2.3 The Equations of Motion

For the spherically symmetric, extremal solutions we are interested in, the metric takes the form:

$$ds^2 = -e^{2U(\tau)}dt^2 + e^{-2U(\tau)}d\vec{x}^2 \ , \tag{2.12}$$

where the warp factor is a function of $\tau = 1/|\vec{x}|$ only. The functional form of the gauge fields is fixed in terms of this warp factor and the charges and generates an effective potential for the scalars of the form [21, 22]:

$$\begin{aligned} V_{BH} &= |Z|^2 + \sum_i |\mathcal{D}_i Z|^2 \\ &= e^K\left(|W(z_1, z_2, z_3)|^2 + |W(\bar{z}_1, z_2, z_3)|^2 + |W(z_1, \bar{z}_2, z_3)|^2 + |W(z_1, z_2, \bar{z}_3)|^2\right) \ . \end{aligned} \tag{2.13}$$

Spherically symmetric solutions extremize the Lagrangian of the equivalent mechanics problem:

$$\mathcal{L}_{\text{eff}} = (\dot{U})^2 + G_{i\bar{j}}\dot{z}^i\dot{\bar{z}}^{\bar{j}} + e^{2U}V_{BH} \ , \tag{2.14}$$

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[4]We can change to conventions where $y^i|_\infty = v_i$ is nontrivial by taking $\tilde{z}^i = z^i v_i$. The effective potential (introduced below) satisfies:

$$V_{BH}(P^\Lambda, Q_\Sigma, z^i) = G_N \tilde{V}_{BH}(p^\Lambda, q_\Sigma, \tilde{z}^i) \ , \tag{2.9}$$

so it is natural to associate the dressed charges P^Λ, Q_Σ with the unit normalized z^i's and the quantized charges p^Λ, q_Σ with volume normalized $\tilde{z}^i$'s.

which amounts to solving the Euler-Lagrange equations:

$$\ddot{U} = e^{2U} V_{BH} \ , \qquad \ddot{z}^i + \Gamma^i_{jk} \dot{z}^j \dot{z}^k = e^{2U} \partial^i V_{BH} \ . \tag{2.15}$$

Solutions must also satisfy the Hamiltonian constraint:

$$\dot{U}^2 + G_{i\bar{j}} \dot{z}^i \dot{z}^{\bar{j}} - e^{2U} V_{BH} = 0 \ . \tag{2.16}$$

In all these equations dots denote derivatives with respect to τ. In the appendix we make these equations explicit for the STU-model.

2.4 Attractors

Black hole solutions are characterized by their conserved charges. In our setting the asymptotic data is just the charge vector $\Gamma = (P^\Lambda, Q_\Sigma)$ because we assume spherical symmetry (so angular momentum vanishes) and extremality (so the mass is determined by the charge as the minimal one giving a regular black hole).

In $\mathcal{N} = 8$ theory there are qualitatively different classes of black hole solutions, classified by the quartic invariant $I_4(\Gamma)$: if $I_4(\Gamma) > 0$ the solution is BPS but if $I_4(\Gamma) < 0$ the single center solution cannot be BPS. If the invariant is null, the solution is BPS but preserves more than the minimum 1/8 SUSY [23, 24].

The $\mathcal{N} = 2$ STU-theory inherits the quartic invariant from the $\mathcal{N} = 8$ theory:

$$I_4(\Gamma) \ = \ 4Q_0 P^1 P^2 P^3 - 4P^0 Q_1 Q_2 Q_3 - \left(P^\Sigma Q_\Sigma \right)^2 + 4 \sum_{i<j} P^i Q_i P^j Q_j \ . \tag{2.17}$$

It may happen that some of the $I_4(\Gamma) > 0$ solutions do not preserve any of the $\mathcal{N} = 2$ SUSY even though they do preserve some of the $\mathcal{N} = 8$ SUSY (see [25, 24]). Although such solutions are non-BPS in the $\mathcal{N} = 2$ theory they have essentially the same properties as the BPS solutions. Our interest are in the soutions with $I_4(\Gamma) < 0$ which are non-BPS whether in $\mathcal{N} = 2$ or $\mathcal{N} = 8$.

The extremal black holes, be they BPS or not, all exhibit an attractor mechanism. One can solve for these attractors values by minimizing $V_{BH}(\Gamma, z^i)$ as a function of the z^i's with fixed Γ. For the BPS solutions of the STU model, the vector-multiplet moduli, z^i, are *all* completely fixed at the horizon and given by the following expression:

$$z^i_{\text{fix}} = \frac{P^i + i\partial_{Q_i} I_4^{1/2}(\Gamma)}{P^0 + i\partial_{Q_0} I_4^{1/2}(\Gamma)} \ . \tag{2.18}$$

The non-BPS attractors with $I_4 < 0$ are qualitatively different: out of the six real moduli in the STU model, there are only four fixed scalars and two flat directions. The expression for the attractor values for the four fixed scalars is now somewhat more complicated due to certain subtle phases [26]. The appearance of flat directions is most easily appreciated in the $D0 - D6$ duality frame where the relative size of the T^2's remain undetermined upon extremization of V_{BH}. We will return to this point in much more detail.

The entropy (from the horizon area) of all extremal solutions, whether BPS or not, is essentially the effective potential (2.14) evaluated at the extremum:

$$S = \frac{A}{4G_N} = \frac{\pi}{G_N} \, V_{\mathrm{BH}}\big|_{\mathrm{ext}} \ , \tag{2.19}$$

which can be shown to give

$$S = \frac{\pi}{G_N} \sqrt{|I_4(\Gamma)|} \ . \tag{2.20}$$

3 BPS Solutions

Before describing new extremal non-BPS solutions to the STU-model and the $\mathcal{N} = 8$ theory we review the familiar BPS solutions [4, 27, 28, 29].

3.1 A Simple BPS Solution: D0-D4 Without B-fields

A good benchmark solution is the case of a $D0 - D4 - D4 - D4$ black hole with $Q_0 > 0$ and $P^i > 0$ but $P^0 = Q_i = 0$. This charge configuration is BPS. Thus its attractor values are given by (2.18) which become:

$$z^j_{\mathrm{fix}} = -i\sqrt{\frac{2Q_0 \, P^j}{s_{jkl} P^k \, P^l}} \ , \tag{3.1}$$

where we introduced $s_{jkl} = |\epsilon_{jkl}|$. These attractor values give a natural way to write down the full solution in the simple case where the B-fields, encoded in the moduli x^i, vanish asymptotically. One starts with four harmonic functions:

$$H^i = \frac{1}{\sqrt{2}} + P^i \tau \ , \qquad H_0 = \frac{1}{\sqrt{2}} + Q_0 \tau \ , \tag{3.2}$$

and then the solution to our effective Lagrangian is:

$$e^{-4U} = 4H_0 H^1 H^2 H^3 \ , \tag{3.3}$$

$$z^j = -i\sqrt{\frac{2H_0 \, H^j}{s_{jkl} H^k \, H^l}} \ . \tag{3.4}$$

Inspecting the limit $\tau \to \infty$ we recover the attractor values (3.1) and the black hole entropy (2.20). It is also evident that for this solution the mass formula is just the marginal sum:

$$G_N \, M = \frac{1}{2\sqrt{2}} \Big(Q_0 + \sum_i P^i\Big) = Z\big|_{\tau=0} \ . \tag{3.5}$$

This also follows from the fact that without B-fields the phases of the individual central charges for the D0-brane and D4-branes:

$$Z_{D0} = 2^{-3/2} \, Q_0 \ , \qquad Z_{D4^i} = 2^{-3/2} \, P^i \ , \tag{3.6}$$

are completely aligned (such marginal bound states in terms of more basic constituents were initially explored in [30, 31]).

3.2 The Most General BPS Solution

By expanding the framework above one can accommodate a wider range of asymptotic data, including the appearance of B-fields and a more general charge vector Γ (see *e.g.* [12, 32, 33, 34]). In addition to the four harmonics H_0, H^i we need four more harmonic functions H^0, H_i. Defining the constant terms:

$$\Gamma_\infty = (\bar{P}^\Sigma, \bar{Q}_\Lambda) = (H^\Sigma|_{\tau=0}, H_\Lambda|_{\tau=0}) \ , \tag{3.7}$$

we can write the whole set of harmonic functions compactly, as a single charge-vector valued function:

$$\mathcal{H}(\tau) = \Gamma_\infty + \Gamma\,\tau \ . \tag{3.8}$$

The constant terms (3.7) are subject to two conditions:

$$I_4(\Gamma_\infty) = 1 \ , \qquad \langle \Gamma, \Gamma_\infty \rangle = P^\Sigma \bar{Q}_\Sigma - Q_\Lambda \bar{P}^\Lambda = 0 \ . \tag{3.9}$$

As long as these are satisfied we can completely describe any $I_4(\Gamma) > 0$ solution succinctly using a generalization of the entropy formula to an entropy function and a generalization of the attractor equations called the *stabilization* equations [32]. The warp factor is obtained by inserting our charge-valued harmonic function into the entropy formula in (2.17):

$$e^{-4U(\tau)} = I_4(\mathcal{H}(\tau)) \ . \tag{3.10}$$

Similarly the solution of our scalars z^i is obtained by generalizing the attractor equations to:

$$z^i(\tau) = \frac{H^i(\tau) + i\partial_{H_i} I_4^{1/2}(\mathcal{H}(\tau))}{H^0(\tau) + i\partial_{H_0} I_4^{1/2}(\mathcal{H}(\tau))} \ . \tag{3.11}$$

It is straightforward to verify that this formalism recovers the simple solution in (3.3).

As a special case of (3.11) we note that the constants in the harmonic function, Γ_∞, satisfy the attractor equations. This is the attractor at infinity [35], a map between the 6 real moduli and the 8 constants in the harmonic functions which are subject to the 2 constraints (3.9). The fact that BPS attractors completely fix the scalars is crucial for our ability to write the full solution in the attractor form (3.10-3.11).

3.3 D0-D4 Revisited: Non-trivial B-fields

We now apply this machinery to our D0-D4 BPS solution and use it to include non-trivial B-fields. Thus we introduce general constant terms for the harmonic function H_0, H^i and also allow the harmonic functions H^0, H_i to have non-zero constant terms as well. All these parameters are fixed in part by our choice that the asymptotic volume moduli $y^i_\infty = 1$. We parameterize the remaining freedom in terms of the asymptotic B-field densities, $B^i = x^i_\infty$,

and a phase α which we will explain shortly:

$$\bar{P}^0 = \frac{\sin\alpha}{\sqrt{2}} , \quad \bar{P}^i = \frac{1}{\sqrt{2}} \left[B^i \sin\alpha + \cos\alpha \right] , \tag{3.12}$$

$$\bar{Q}_1 = -\frac{1}{\sqrt{2}} \left[\sin\alpha(1 - B_2 B_3) - \cos\alpha(B_2 + B_3) \right] , \quad \text{(and cyclic permutations)} ,$$

$$\bar{Q}_0 = \frac{1}{\sqrt{2}} \left[(\sum_i B^i - \prod_i B^i) \sin\alpha + (1 - \sum_{i<j} B^i B^j) \cos\alpha \right] .$$

These expressions were constructed so that the first constraint in (3.9) $I_4(\Gamma_\infty) = 1$ is satisfied. To satisfy the second constraint we must choose α so that($s_{jk}^i = |\epsilon_{ijk}|$):

$$\langle \Gamma, \Gamma_\infty \rangle = P^i \bar{Q}_i - Q_0 \bar{P}^0$$
$$= \text{Im} \left[\left(Q_0 + \sum_i P^i \frac{s_{ijk}}{2} (1 + iB^j)(1 + iB^k) \right) \frac{e^{-i\alpha}}{\sqrt{2}} \right]$$
$$= 2\,\text{Im} \left[Ze^{-i\alpha} \right] = 0 . \tag{3.13}$$

In addition to specifying α in the solution we therefore find the interpretation of α: it is the phase of the central charge Z.[5] If we insert our harmonics into (3.10) we get:

$$e^{-4U} = 1 + \sqrt{2} \left[(Q_0 + \sum_i P^i (1 - s_{ijk} \frac{B^j B^k}{2}) \cos\alpha + \sum_i s_{ijk} B^j P^k \sin\alpha \right] \tau \tag{3.14}$$
$$+ \mathcal{O}(\tau^2) + \mathcal{O}(\tau^3) + 4Q_0 P^1 P^2 P^3 \tau^4 .$$

This gives the correct BPS mass and black hole entropy:

$$M = G_N^{-1}|Z| = G_N^{-1} \text{Re} \left[Ze^{-i\alpha} \right] , \tag{3.15}$$

$$S = \frac{2\pi}{G_N} \sqrt{Q_0 P^1 P^2 P^3} . \tag{3.16}$$

Expanding the general stabilisation equation (3.11), the scalars take the form:

$$z^1 = \frac{-H_1 H^1 + H_0 H^0 + H_2 H^2 + H_3 H^3 - ie^{-2U}}{2(H^2 H^3 - H^0 H_1)} , \quad \text{(and cyclic permutations)} . \tag{3.17}$$

The moduli exhibit the correct asymptotic behavior, namely:

$$z^j|_{\tau=0} \to B^j - i , \quad z^j|_{\tau=\infty} = -i\sqrt{\frac{Q^0\,P^j}{\frac{1}{2}s_{jkl}P^k\,P^l}} . \tag{3.18}$$

In other words, we satisfy the boundary conditions we wanted at $r = \infty$ and flow to the previously determined attractor values at the horizon $r = 0$.

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[5]This is up to a shift by π. This ambiguity is resolved by the mass formula

4 Non-BPS solutions: the $\overline{\text{D0}}$-D4 case

We now get to the core of our results, the non-BPS black hole solutions to the STU-model and the $\mathcal{N} = 8$ supergravity theory. We first briefly review how a non-BPS solution can be constructed by analytical continuation and then present the more general solution that cannot be obtained this way. We then discuss the non-BPS mass formula and the action of the dualities on our solution.

4.1 The Simple Non-BPS Solution

Once again we use the canonical representative, a D0-D4-D4-D4 charge vector. In the non-BPS case we assume $Q_0 < 0$, $P^i > 0$ so $I_4(\Gamma) < 0$. As mentioned in the introduction, we can derive some simple solutions by analytic continuation from the BPS case [7, 26, 36]. Thus we can start from the harmonic functions:

$$H^i = \frac{1}{\sqrt{2}} + P^i \tau \ , \qquad H_0 = -\frac{1}{\sqrt{2}} + Q_0 \tau \ , \tag{4.1}$$

and immediately write the non-BPS solution:

$$e^{-4U} = |4H_0 H^1 H^2 H^3| \ , \quad z^i = -i\sqrt{\frac{-H_0 H^i}{\frac{1}{2} s_{ijk} H^j H^k}} \ . \tag{4.2}$$

In particular this gives the attractor values:

$$z^i = -i\sqrt{\frac{-Q_0 P^i}{\frac{1}{2} s_{ijk} P^j P^j}} \ . \tag{4.3}$$

Our goal is to generalize the canonical non-BPS solution (4.2) to situations where the asymptotic moduli are more general and/or there are more charges present.

4.2 The Seed Solution: Non-BPS Black Holes with 5 Parameters

The experience with BPS black hole solutions suggests several strategies which all appear to encounter difficulties:

- We could determine the general attractor equations, like (2.18) for BPS, and then try to generate the full flow from appropriate stabilizer equations, like (3.11) for BPS. This approach was suggested for non-BPS solutions [26], but it does not seem to work in the general case where an extra phase appears in the attractor equations. This phase difficulty can be circumvented in the first order formulation for non-BPS black holes proposed in [37]. When considering the same superpotential as in this work, their flow equations describe the change in the phase from its value at asymptotic infinity to its value at the horizon. When attempting to extend this formulation for general superpotentials, the proposal in [38] fails to cover this phase change.

- Another strategy is to exploit the uplift to five dimensions where the situation is simpler [39]. The difficulty with this approach is the specific assumptions for the lift: the natural *ansatz* assumes that the 5D geometry is a time fibration over a Hyper-Kähler base. This assumption does not hold for known solutions like the D0-D6 solution in [14, 3]. However, as it turns out, for the D2-D2-D2-D6 charge vector, the extremal non-BPS black hole solution found in [39] is U-dual to the extremal D0-D6 black hole we will present in the next section.

- The solution generating technique acting on a Schwarzschild (or Kerr) seed solution gives the most general nonextremal solution, at least in principle. All solutions with a BPS limit were generated this way a long time ago [6, 7]. The technique works in principle for the non-BPS branch as well: it generated the D0-D6 solution in [3]. However, the solutions derived this way are parameterized in an unilluminating manner that has so far resisted extraction of the general non-BPS extremal solutions.

Instead of attempting to generalize the approaches used for the BPS solutions we find our seed solution by direct integration of the equations of motion, generalizing another recent computation [40]. The solution identified this way has arbitrary D0 and D4-brane charges as well as equal B-field densities, $B^i = B$, on all three T^2's.

We leave the full derivation of the integration of the equations of motion to appendix A and present the final solution in its simplest form. Once again we have four harmonic functions:

$$H^i = \frac{1}{\sqrt{2}} + P^i \tau , \qquad H_0 = -\frac{1}{\sqrt{2}}(1 + B^2) + Q_0 \tau , \qquad (4.4)$$

with which we can write the solution as:

$$e^{-4U} = -4H_0 H^1 H^2 H^3 - B^2 , \qquad z^i = \frac{B - ie^{-2U}}{s_{ijk} H^j H^k} . \qquad (4.5)$$

We have already mentioned that for the non-BPS solutions $Q_0 < 0$, $P^i > 0$. With these assignments $H_0 < 0$ and $H^i > 0$; so the first term in the warp factor e^{-4U} is positive definite. The constant terms in H_0, H^i are such that $e^{-4U} > 1$ during the entire flow $0 < \tau < \infty$.

The non-BPS solution with B-field (4.5) gives the same attractor values for the scalars and also the same black hole entropy as the simple non-BPS solution without a B-field (4.2). These aspects are therefore reproduced correctly by analytical continuation from the BPS-solution discussed in section 3.3. However, we emphasize again that the full radial flow cannot be obtained in this way.

Although we have discussed our solutions in the setting of the STU-theory, they are readily embedded also in $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supergravity. In these contexts they serve as seed solutions which generate the most general spherically symmetric black hole solutions upon acting with dualities. It also appears that our solutions generalize to other $\mathcal{N} = 2$ theories with cubic prepotential. For such theories $s_{ijk} = |\epsilon_{ijk}|$ should be replaced by the structure constants c_{ijk}, and $H^1 H^2 H^3$ should be replaced by the invariant $\frac{1}{6} c_{ijk} H^i H^j H^k$.

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4.3 Duality Orbits

The solution we have given above depends on exactly 5 parameters, four charges and a B-field, and so it is adequate to generate the most general black hole solution. We now review how this works in principle [5]. Explicit examples are postponed to the next section.

The theories we consider have a continuous duality group G which is spontaneously broken by the scalar fields taking values on some coset G/H. Starting from a seed solution with some canonical values of the asymptotic scalars we can generate whatever more general values of the asymptotic scalars we desire by acting with G. Subsequently we can act with H, which leaves the scalars invariant, to bring the charges to the values we want to realize. That *all* solutions are generated this way depends on the details of the theory.

In the STU-model $G = SL(2,\mathbb{R})^3$ and $H = U(1)^3$. Starting from our seed solution with asymptotic moduli $z^j = B - i$ for $j = 1, 2, 3$ we act with $SL(2,\mathbb{R})$ on each z^j to realize general moduli. There is some redundancy in this: since the seed solution already has one explicit modulus, B, there is a diagonal (the same in all three $SL(2,\mathbb{R})$'s) duality transformation that is not needed to cover moduli space. Having transformed to the desired point in moduli space, the next step is to realize all charge vectors without further changing the moduli. Since the moduli actually belong to $(SL(2,\mathbb{R})/U(1))^3$, the $U(1)^3$ leaves moduli space invariant. These $U(1)$'s act on the *relative* phases of the the central charge Z and it's covariant derivatives $\mathcal{D}_{\hat{i}}Z$ with corresponding actions on the charge vectors (p^Λ, q_Λ) ($\Lambda = 0, 1, 2, 3$). In order to realize all charge configurations we also need to act on the *overall* phase of the central charge and it's covariant derivatives. This is precisely what the redundant $(SL(2,\mathbb{R})/U(1))^3$ duality transformation can accomplish. We see that the fifth parameter, which we parameterize as a diagonal B-field, is exactly what is needed in order that the most general solution can be generated.

Let us also consider the general $\mathcal{N} = 8$ theory where $G = E_{7(7)}$ and $H = SU(8)$. Here we first act with $E_{7(7)}$ on the moduli in the seed solution, thus reaching a generic point in moduli space. We then transform the charges with $H = SU(8)$, which leaves moduli space invariant. To be more precise, the central charges can be organized in an antisymmetric 8×8 matrix $x^{ab} + iy_{ab}$ ($a, b = 1, \ldots, 8$) with skew-eigenvalues Z_Λ ($\Lambda = 0, 1, 2, 3$). The $SU(8)$ duality group transforms the central charges in the antisymmetric representation and one can show that it generates the most general charge vector from four real magnitudes and the overall phase of the skew-eigenvalues (left invariant because the $SU(8)$ has unit determinant) [5]. Again, the extra parameter we have in our solution is equivalent to this overall phase.

We also note that virtually identical considerations apply to the $\mathcal{N} = 4$ theory which has duality group $G = SO(m, n)$ and $H = SO(m) \times SO(n)$.

The duality orbits are not special to the non-BPS branch, nor to the extremal case. We have merely reviewed why, and in what sense, seed solutions in four dimensions must have five parameters. From this point of view our contribution is to advocate a particular duality frame which make the seed solutions particularly simple. The group theoretic distinction between the BPS and non-BPS branches appears when we consider the attractor mechanism, the subject of the next subsection.

4.4 Non-BPS Attractors with Flat Directions

Suppose we have followed the procedure just outlined and reached our desired duality frame, *i.e.* the asymptotic scalars have been set to realize a specific vacuum, and the charge vector has been transformed to $\gamma = (p^\Sigma, q_\Lambda)$. We then ask: what subgroup $\hat{H}$ of the duality group G leaves the charge vector γ invariant ?

We have defined $\hat{H}$ so that it leaves our charges invariant, but it generally acts non-trivially on the scalars. Denoting by $\hat{h}_0 \subset \hat{H}$ the subgroup of the duality group that leaves both moduli and charges invariant we see that the coset $\hat{H}/\hat{h}_0$ is the nontrivial scalar manifold generated by duality transformations that leave the charges invariant. The physical significance of this coset is that it corresponds to new solutions with the same charges but different asymptotic moduli. Since the transformations act on the entire orbit these solutions can also have different attractor values for z^i. The attractor values of the scalars are therefore *not* uniquely determined by the charges if the coset $\hat{H}/\hat{h}_0$ is nontrivial at the horizon.

We defined $\hat{h}_0$ as the elements in the duality group G that leaves both the charge vector γ and the moduli invariant. The group leaving moduli invariant (but not necessarily the charge vectors) is H, the maximal compact subgroup of the full duality group G. Since $\hat{h}_0 \subset H$ we find in particular that $\hat{h}_0$ is compact. For typical values of the scalars, h_0 will be trivial, but at the horizon $\hat{h}_0$ is enhanced to the maximal compact subgroup H_0 of $\hat{H}$ [6]. Therefore, if $\hat{H}$ is non-compact, there will be $\dim\hat{H} - \dim\hat{H}_0$ flat directions of the black hole potential for the given charge vector and the corresponding moduli are fixed by spontaneous symmetry breaking rather than dynamics. This set of flat directions is parameterized by the coset $\hat{H}/\hat{H}_0$ (see [41, 42] for more details).

The important point we wish to make is that $\hat{H}/\hat{H}_0$ is trivial for BPS black holes, but non-trivial on the non-BPS branch. In other words: the attractor mechanism determines all moduli for a BPS charge vector, but leaves flat directions if the charge vector is non-BPS. The key distinction between BPS and non-BPS can be appreciated by contemplating the quartic invariant $I_4(\Gamma)$ (2.17). Only for non-BPS charge configurations $I_4(\Gamma) < 0$ is it possible to have just two non-vanishing charges (P^0, Q_0), and in this frame there are clearly exceptional duality transformations which remain symmetries because (P^i, Q_i) vanish. These additional symmetries persist for other non-BPS charge configurations. We will be very explicit about how this works when we examine the D6-D0 realization of our non-BPS solutions in the next section.

We end this discussion by reviewing the explicit expressions for the various groups. For the STU-model the duality group is $G = SL(2,\mathbb{R})^3$ with maximal compact subgroup $H = U(1)^3$. In this case the non-BPS charge configurations are left invariant by $\hat{H} = SO(1,1)^2$ which has only the trivial compact subgroup $\hat{H}_0 = 1$. Therefore $\hat{H}/\hat{H}_0 = SO(1,1)^2$ parameterize two flat directions which decouple from the attractor mechanism.

For $\mathcal{N} = 8$ supergravity we need $G = E_{7(7)}$, $H = SU(8)$. BPS charge configurations are

---

[6]At first it seems naively like this enhancement should always hold, but U-duality elements acting on the charges act with a left action on the scalars represented as right-cosets of G, so compact elements of $\hat{H}$ need not always be in $\hat{h}_0$.

left invariant by $\hat{H} = E_{6(2)}$ with the compact subgroup $\hat{H}_0 = SU(2) \times SU(6)$ leaving the BPS charge vector invariant as well. The coset $\hat{H}/\hat{H}_0 = E_{6(2)}/SU(2) \times SU(6)$ parameterizes 40 flat directions, from the $\mathcal{N} = 2$ point of view these are the decoupled hypermultiplet scalars. Non-BPS charge configurations are left invariant by $\hat{H} = E_{6(6)}$ with the maximal compact subgroup $\hat{H}_0 = USp(8)$. The coset $\hat{H}/\hat{H}_0 = E_{6(6)}/USp(8)$ parameterizes 42 flat directions, which is just large enough to contain the forty hypermultiplet scalars and the two flat directions we see in the STU theory.

4.5 The Non-BPS Mass Formula and more General Moduli

Now that we understand the symmetries which allow to easily generate other solutions from our seed solution, we would like to see how the extremal non-BPS black hole mass compares to the BPS bound for any moduli.

We can appreciate the differences between BPS and non-BPS extremal black holes better by working out their masses for the seed solution. The non-BPS mass formula is also useful in physical applications.

Expanding the warp factor (4.5) we find the mass:

$$2G_N M_{\text{Non-BPS}} = \frac{1}{\sqrt{2}} \left(|Q_0| + \sum_i P^i(1 + B^2) \right) . \tag{4.6}$$

There is a simple interpretation of this expression: the mass is just the sum of the masses of the D0 and D4-branes individually, with the B-field taken into account for each constituent independently. Interestingly, this indicates that the non-BPS black hole is a *marginal* bound state. In the special case where all B-fields vanish the mass formulae are related by analytical continuation from $Q_0 < 0$ to $Q_0 > 0$. However, the more general expressions with B-fields turned on are not related in this way. This indicates that the physics of the two branches is qualitatively different, in a manner reminiscent of a system with distinct phases.

It is instructive to compare our non-BPS mass formula to the BPS bound $M_{\text{BPS}} = |Z_\Gamma|/G_N$:

$$2G_N M_{\text{BPS}} = 2|Z_\Gamma| = \frac{1}{\sqrt{2}} \left| Q_0 + \sum_i P^i (1 + iB)^2 \right| . \tag{4.7}$$

If we consider the gap between the squares of the two masses, we get that

$$\Delta = 8G_N^2 (M^2 - M_{BPS}^2) = 4|Q_0| \sum_i P^i > 0 . \tag{4.8}$$

Thus the additional energy associated with a non-BPS state is always strictly positive.

We have control over the general situation, with more charges and/or moduli turned on, due to the dualities spelled out above, in section 4.3. The masses are *invariant* under such transformations and so we immediately find:

- The existence of a gap between BPS and non-BPS branch holds quite generally.

- The mass of a non-BPS extremal bound state is always the sum of the masses of four 1/2-BPS (in the N=8 language) constituents. However, the quantum numbers of these constituents will generally be complicated.

As a special instance of these considerations one might consider the $\bar{D}0 - D4$ bound state and contemplate adding on general B-fields. This can be accomplished concretely by acting with the duality group $\hat{H}$ which leaves charges invariant and acts on moduli alone. The non-BPS formula for this case can deduced explicitly this way but it does not seem to be simple, since the stabilizer group which keeps the quantized charge invariant scales the volumes of the various T^2's so that the dressed charges vary. The physical origin of these difficulties is that, if some of the B-fields are not equal, the constituents will not be just $\bar{D}0$-branes and D4-branes but also D6-branes with fluxes.

5 The D0-D6 Solution with B-fields

In this section we work out the explicit example of a non-BPS black hole with only D6-brane and D0-brane charge but arbitrary moduli. There are several motivations for doing this:

- We would like to compare our seed solutions to previously known non-BPS solutions [14, 3].

- In the D6-D0 frame the flat directions that decouple from the attractor flow are manifest: they correspond to adjusting the volumes of the individual T^2's without adjusting the overall T^6 volume. By working out the duality transformations explicitly we can exhibit the flat directions in other non-BPS systems as well, including our original $\overline{D0}$-D4 frame.

- The D0-D6 system has interesting physical properties [43, 15, 3, 44, 45, 46, 36, 47]. Our supergravity solutions add new and interesting facts about this system.

Our strategy is as follows: we determine the duality transformation relating the D6-D0 U-duality frame to the D0-D4 frame and then use this to find the D6-D0 solution. We move in steps of increasing complexity, starting with no B-fields on the D0/D6, then 3 identical B-fields, and finally the more complicated case of 3 different B-fields. Along the way we take the opportunity to revisit the important group $\hat{H}$, introduced as the stabilizer of the charge vector. We will make the group more explicit and further explain its significance.

We will find is useful to move back and forth between two normalizations of our scalar fields, z^i and $\tilde{z}^i = v_i z^i$, using the former for dressed charges and the latter for quantized charges. Duality transformations are simplest in terms of the quantized charges (p^Λ, q_Σ) but we will revert to the use of dressed charges when presenting our mass formulas.

5.1 Duality Transformations

We want to transform between the non-BPS $\overline{\text{D0}}$-D4 charges $(q_0 = q, p^i)$ used hitherto and the D0-D6 charges which we denote (q_0, p^0). To so we recall that the charges of the STU-model transform in the $(2, 2, 2)$ of the $[SL(2, \mathbb{R})]^3$ duality symmetry. We can make the transformation properties of the charge vector (p^Λ, q_Σ) manifest by introducing the notation $\{a_{ijk}\}$:

$$
\begin{aligned}
p^0 &= a_{111} & , && q_0 &= -a_{000} \, , \\
p^1 &= a_{011} & , && q_1 &= a_{100} \, , \\
p^2 &= a_{101} & , && q_2 &= a_{010} \, , \\
p^3 &= a_{110} & , && q_3 &= a_{001} \, .
\end{aligned}
\tag{5.1}
$$

The duality transformations then become:

$$
a'_{i'j'k'} = (M_1)_{i'}^{\ i} \, (M_2)_{j'}^{\ j} \, (M_3)_{k'}^{\ k} \, a_{ijk} \, .
\tag{5.2}
$$

We have introduced three independent $SL(2, \mathbb{R})$ transformations, M_j $(j = 1, 2, 3)$, whose action on the complex moduli $\tilde{z}^j = \tilde{x}^j - i \, \tilde{y}^j$ is:

$$
M_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} : \quad \tilde{z}^j \longrightarrow \frac{a_j \, \tilde{z}^j + b_j}{c_j \, \tilde{z}^j + d_j} \, .
\tag{5.3}
$$

The M_i's that dualize from the D6-D0 frame to the $\overline{\text{D0}}$-D4 frame must satisfy the eight relations:

$$
\begin{aligned}
-q &= -a_1 a_2 a_3 \, q_0 + b_1 b_2 b_3 \, p^0 \, , \\
0 &= -a_1 a_2 \, c_3 \, q_0 + b_1 b_2 \, d_3 \, p^0 \, , \text{(and cyclic permutations)} \, , \\
p^i &= -\frac{1}{2} s_{ijk} \, a_i c_j c_k \, q_0 + \frac{1}{2} s_{ijk} \, b_i d_j d_k \, p^0 \, , \\
0 &= -c_1 c_2 c_3 \, q_0 + d_1 d_2 d_3 \, p^0 \, ,
\end{aligned}
\tag{5.4}
$$

where $s_{ijk} = |\epsilon_{ijk}|$. There are no solutions to these equations unless the product $q \, p^1 \, p^2 \, p^3$ is negative, as expected because we must be in the non-BPS branch ($I_4 < 0$) in order to dualize to the D0-D6 system. The D0/D6 charges p^0 and q_0 can have any signs, which is also as expected since $I_4 < 0$ independently of those. Without loss of generality, we take $\{q < 0, p^i > 0\}$ and $\{p^0, q_0 > 0\}$. With these assignments, the $SL(2, \mathbb{R})$ matrices that map the D0-D6 charge vector into the $\overline{\text{D0}}$-D4 configuration are:

$$
M_i = \frac{-1}{\sqrt{2\lambda \rho_i}} \begin{pmatrix} \rho_i \lambda & -\rho_i \\ \lambda & 1 \end{pmatrix} \, , \quad \rho_i = \sqrt{\frac{-qp^i}{\frac{1}{2} s_{ijk} p^j p^k}} \, , \quad \lambda = \left(\frac{p^0}{q_0} \right)^{1/3} \, .
\tag{5.5}
$$

The duality invariant I_4 is the same in either frame:

$$
I_4 = 4 \, q \, p^1 \, p^2 \, p^3 = -(p^0 \, q_0)^2 < 0 \, .
\tag{5.6}
$$

This is necessary for the transformations M_i to belong to $\mathrm{SL}(2, \mathbb{R})$ and for the consistency of the relations (5.4) with the matrix (5.5). We will also need the inverse matrices, mapping the $\overline{\mathrm{D0}}$-D4 system into D0-D6:

$$M_i^{-1} = \frac{-1}{\sqrt{2\lambda\,\rho_i}} \begin{pmatrix} 1 & \rho_i \\ -\lambda & \rho_i\,\lambda \end{pmatrix}. \tag{5.7}$$

The transformation matrix (5.5) is not the most general one mapping the D0-D6 charge vector into the $\overline{\mathrm{D0}}$-D4 configuration. There exists a two parameter family of transformations by considering different λ_i ($i = 1, 2, 3$) subject to the constraint $\lambda_1\,\lambda_2\,\lambda_3 = p^0/q_0$. The existence of such general transformations agrees with the conclusion in our previous duality group orbit discussion.

5.2   Flat Directions Made Explicit

We now have the explicit formulae needed to make the abstract discussion of flat direction in the previous subsection more explicit. By definition, the subgroup $\hat{H}$ is the subgroup of $[\mathrm{SL}(2, \mathbb{R})]^3$ that leaves a given charge vector Γ invariant. This subgroup is particularly simple to characterize for the D0-D6, since Γ has only two non-vanishing components.

The explicit $SL(2, \mathbb{R})^3$ transformation is (5.4), but now with the D0-D6 charges on the left hand side as well as the right hand side. The solutions of the equations are the elements of $\hat{H}$. We find $\mathrm{SL}(2, \mathbb{R})$ matrices of the form:

$$N_i^{(60)} = \begin{pmatrix} e^{\alpha_i} & 0 \\ 0 & e^{-\alpha_i} \end{pmatrix}, \qquad \sum_i \alpha_i = 0. \tag{5.8}$$

Thus, in the D0-D6 frame, the action of this subgroup on the complex moduli is equivalent to a T^6 volume preserving rescaling. That is, each moduli $\tilde{z}^i$ is rescaled:

$$\tilde{z}^i \to e^{2\alpha_i}\,\tilde{z}^i, \tag{5.9}$$

keeping the product $\tilde{z}^1\,\tilde{z}^2\,\tilde{z}^3$ invariant since $\sum_i \alpha_i = 0$.

The $\hat{H}$ action in the $\overline{\mathrm{D0}}$-D4 frame is more complicated, but it can be obtained by mapping the charge vector $\Gamma_{\overline{\mathrm{D0}}-D4}$ to Γ_{D0-D6}, acting with $N_i^{(60)}$ in that frame, and mapping the charge vector back to the original $\overline{\mathrm{D0}}$-D4 frame. In other words, the action of $\hat{H}$ is given by the conjugated matrices $N_i^{(40)} = M_i \cdot N_i^{(60)} \cdot M_i^{-1}$:

$$N_i^{(40)} = \begin{pmatrix} \cosh\alpha_i & \rho_i \sinh\alpha_i \\ \rho_i^{-1} \sinh\alpha_i & \cosh\alpha_i \end{pmatrix}, \qquad \sum_i \alpha_i = 0. \tag{5.10}$$

One can explicitly check that the $\overline{\mathrm{D0}}$-D4 charges are left invariant under these transformations. This action does mix the volume and metrics, as can be seen by writing the explicit action on the complex moduli fields $\tilde{z}^j = \tilde{x}^j - i\,\tilde{y}^j$:

$$\tilde{z}^j \to \frac{\cosh 2\alpha_j\,\tilde{x}^j + (1/2)\,\sinh 2\alpha_j\,[\rho_j + \rho_j^{-1}\,((\tilde{x}^j)^2 + (\tilde{y}^j)^2)] - i\,\tilde{y}_j}{\cosh^2\alpha_j + \rho_j^{-2}\,\sinh^2\alpha_j\,((\tilde{x}^j)^2 + (\tilde{y}^j)^2) + \rho_j^{-1}\,\sinh 2\alpha_j\,\tilde{x}^j}. \tag{5.11}$$

This action applies to the entire flow. For example, we can act on the simple non-BPS solution (4.2). This will modify the simple attractor behavior (4.3) which, in particular will include B-fields after this transformation. This is despite the fact that charges have not changed. Thus the transformed attractor is sensitive to data beyond the charges, namely the asymptotic moduli (which are computed by (5.11) acting on the asymptotic moduli).

5.3 D0-D6 with no B Fields

Let us start with the seed solution written with undressed charges. Define a a new B-field, $b = B/G_N$, and the rescaled "undressed" harmonics $h_0 = C_0^{-1} H_0$ and $h^i = (C^i)^{-1} H^i$. Then the seed solution can be written as:

$$\tilde{z}^i = \frac{b - i\sqrt{-4h_0 h^1 h^2 h^3 - b^2}}{s_{ijk} h^j h^k} \ . \tag{5.12}$$

In order to have zero B-fields in the dual D0-D6 frame, we want the transformed moduli, $\tilde{\phi}^i = M_i^{-1}(\tilde{z}^i)$, to be purely imaginary. Inspection of (5.7) reveals this can only occur when:

$$|\tilde{z}^i| = \rho^i \quad \forall \tau \qquad \Leftrightarrow \qquad h^i = p\,(h + \tau)\ , \quad h_0 = q\,(h + \tau)\ , \tag{5.13}$$

for some constant h. Thus, the D0-D6 scalars are:

$$\tilde{\phi}^i = \lambda^{-1} \frac{1 + \rho_i^{-1} \tilde{z}^i}{1 - \rho_i^{-1} \tilde{z}^i} = -i\lambda^{-1} \sqrt{\frac{|p^0 q_0|\,(h+\tau)^2 + b}{|p^0 q_0|\,(h+\tau)^2 - b}} = -i\lambda^{-1} \sqrt{\frac{|P^0 Q_0|\,(h+\tau)^2 + B}{|P^0 Q_0|\,(h+\tau)^2 - B}}\ , \tag{5.14}$$

and the warp factor becomes:

$$e^{-4U} = \sqrt{(P^0 Q_0)^2 (h+\tau)^4 - B^2}\ . \tag{5.15}$$

All the volumes v_i are equal on the $D0 - D6$ side. This feature can easily be relaxed by acting with $\hat{H}$ (5.9), which will make the v_i general, while keeping the overall volume $v_1 v_2 v_3$ fixed.

A less trivial task is to rewrite B, h in terms of physical quantities in the D0-D6 frame. First, from the normalization of the volume moduli at infinity, we get:

$$\Lambda^2 \equiv \left(\frac{P^0}{Q_0}\right)^{2/3} = \lambda^2 v_i^2 = \frac{|P^0 Q_0|\,h^2 + B}{|P^0 Q_0|\,h^2 - B}\ . \tag{5.16}$$

Second, requiring that the warp factor (5.15) asymptotes to Minkowski space we find:

$$(P^0 Q_0)^2\,h^4 - B^2 = 1\ . \tag{5.17}$$

This allows us to solve for $\{B,\,h\}$ in terms of the D0-D6 dressed charges $\{Q_0,\,P^0\}$:

$$B = \frac{1}{2}(\Lambda - \Lambda^{-1}) = \frac{1}{2(P^0 Q_0)^{1/3}}[(P^0)^{2/3} - (Q_0)^{2/3}]\ , \tag{5.18}$$

$$h = \frac{(\Lambda + \Lambda^{-1})^{1/2}}{\sqrt{2}(P^0 Q_0)^{1/2}} = \frac{1}{\sqrt{2}(P^0 Q_0)^{2/3}}[(P^0)^{2/3} + (Q_0)^{2/3}]^{1/2}\ . \tag{5.19}$$

With these identifications the D0-D6 solution (5.14-5.15) agrees with the one that was constructed directly in [14, 3].

Expanding the warped factor at first order in τ gives the mass of the D0-D6 system:

$$2^{3/2} G_N M = 2^{3/2} (P^0 Q_0)^2 h^3 = [(P^0)^{2/3} + (Q_0)^{2/3}]^{3/2} \; . \tag{5.20}$$

Note that we can rewrite the mass formula as:

$$2^{3/2} G_N M = 4 \left(\frac{P^0}{4} \right) [1 + \Lambda^{-2}]^{3/2} \; , \tag{5.21}$$

which is exactly the sum of the masses of four D6-branes with charge $P^0/4$ and fluxes Λ^{-1} on each T^2 with signs $(+++), (-++), (+-+), (++-)$ as already seen in [44, 36]. This is an example of the more general phenomenon that the non-BPS mass can be interpreted as a marginal bound state of elementary constituents.

5.4 D0-D6 with Equal B-fields

To turn on equal B-fields we need a unique non-vanishing real part for the transformed $\tilde{\phi}^i$ for $i = 1, 2, 3$. This is achieved by considering the choice of harmonic functions:

$$h^i = p \, (h + \tau), \quad i = 1, 2, 3 \quad \text{and} \quad h_0 = q(k + \tau) \, . \tag{5.22}$$

Using the inverse matrices (5.7) and the above form for the harmonic functions, the transformed complex moduli $\tilde{\phi}^j = \tilde{x}^j - i \, \tilde{y}^j$ are:

$$\tilde{x}^j = \lambda^{-1} \frac{(h - k)(h + \tau)}{(h + \tau)(h + k + 2\tau) - 2\bar{b}} \; , \tag{5.23}$$

$$\tilde{y}^j = 2\lambda^{-1} \frac{\sqrt{(k + \tau)(h + \tau)^3 - \bar{b}^2}}{(h + \tau)(h + k + 2\tau) - 2\bar{b}} \; , \tag{5.24}$$

$$\tag{5.25}$$

where we introduced $\bar{b} = b/\sqrt{|I_4|}$.

Let us proceed as in the vanishing B field configurations. First, let us make sure the metric is asymptotically Minkowski, by requiring the warped factor to vanish at infinity:

$$U(\tau = 0) = 0 \quad \Leftrightarrow \quad |I_4| \, k \, h^3 - b^2 = 1 \, . \tag{5.26}$$

Second, let us normalize the moduli at infinity:

$$\lim_{\tau \to 0} z^j = B - i \quad \Rightarrow \quad B = \sqrt{|I_4|} \, h \chi \, , \quad \frac{1}{2} \sqrt{|I_4|} \, h(h + k) - b = \Lambda^{-1} \, . \tag{5.27}$$

Above, we did introduce the notation $\chi = (h - k)/2$, we used (5.26) when necessary, and we already took into account the scaling relation $\tilde{z}^j = z^j \, v_j$.

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Equations (5.26)-(5.27) provide three constraints that allow us to fix the constants $\{k, h, b\}$ in terms of the physical parameters $\{B, Q_0, P^0\}$. First, from the second equation in (5.27):

$$(b + \Lambda^{-1})^2 = |I_4| h^2 (h - \chi)^2 = |I_4| h^3 (h - 2\chi) + |I_4| h^2 \chi^2 = 1 + b^2 + B^2 , \qquad (5.28)$$

where, in the last step, we used (5.26) and the first equation in (5.27). The above determines b:

$$b = \frac{1}{2} \left[\Lambda (1 + B^2) - \Lambda^{-1} \right] . \qquad (5.29)$$

Inserting this back into the second equation in (5.27) we get:

$$\sqrt{|I_4|} \, h^2 - B = \Lambda^{-1} + b \qquad \Rightarrow \qquad \sqrt{|I_4|} \, h^2 = B + \frac{1}{2} \left[\Lambda(1 + B^2) + \Lambda^{-1} \right] . \qquad (5.30)$$

This fixes h, and so we can use the first equation in (5.27) to determine χ (or k).

The mass can be obtained, as usual, by studying the first order asymptotic correction to the warped factor. This gives:

$$2 G_N M = \frac{1}{2} |I_4| h^2 \, (3k + h) = \sqrt{|I_4|} \, h \left(2\sqrt{|I_4|} \, h^2 - 3\sqrt{|I_4|} \, h \chi \right)$$

$$= \sqrt{\frac{\sqrt{|I_4|}}{2}} \left[\Lambda(1 + B^2) + \Lambda^{-1} + 2B \right]^{1/2} \left[\Lambda(1 + B^2) + \Lambda^{-1} - B \right] . \qquad (5.31)$$

This can be written in terms of the dressed charges $\{Q_0, P^0\}$ as:

$$2 G_N M = \frac{1}{\sqrt{2}} \left[(P^0)^{2/3}(1 + B^2) + (Q_0)^{2/3} + 2B(P^0 Q_0)^{1/3} \right]^{1/2}$$

$$\times \left[(P^0)^{2/3}(1 + B^2) + (Q_0)^{2/3} - B(P^0 Q_0)^{1/3} \right] . \qquad (5.32)$$

This formula make look a bit perplexing at first. Once again, however, it has a natural interpretation sum of the mass of four D6-branes with fluxes $|F_i| = \Lambda^{-1}$ coming with signs $(+ + +), (- + +), (+ - +), (+ + -)$. The mass formula above is then re-written as:

$$
\begin{aligned}
2^{3/2} G_N M \;=\; & \frac{P^0}{4} \left(1 + (\Lambda^{-1} + B)^2 \right)^{1/2} \left(1 + (\Lambda^{-1} + B)^2 \right)^{1/2} \left(1 + (\Lambda^{-1} + B)^2 \right)^{1/2} \\
+\; & \frac{P^0}{4} \left(1 + (\Lambda^{-1} + B)^2 \right)^{1/2} \left(1 + (\Lambda^{-1} - B)^2 \right)^{1/2} \left(1 + (\Lambda^{-1} - B)^2 \right)^{1/2} \quad (5.33) \\
+\; & \frac{P^0}{4} \left(1 + (\Lambda^{-1} - B)^2 \right)^{1/2} \left(1 + (\Lambda^{-1} + B)^2 \right)^{1/2} \left(1 + (\Lambda^{-1} - B)^2 \right)^{1/2} \\
+\; & \frac{P^0}{4} \left(1 + (\Lambda^{-1} - B)^2 \right)^{1/2} \left(1 + (\Lambda^{-1} - B)^2 \right)^{1/2} \left(1 + (\Lambda^{-1} + B)^2 \right)^{1/2} .
\end{aligned}
$$

The variables P^0 and Λ only depend on the product of the three T^2 volumes, so adjusting the volumes while keeping their product invariant will leave our mass formula invariant, once again the action of $\hat{H}$.

5.5 D0-D6 with Non-equal B-fields: the Most General Solution

To turn on three different B fields, we need to consider the most general set of harmonic functions:

$$h^i = p^i \left(k^i + \tau\right), \quad i = 1, 2, 3 \quad \text{and} \quad h_0 = q \left(a + \tau\right). \tag{5.34}$$

Using the inverse matrices (5.7) and the h_I defined above, the transformed complex moduli $\tilde{\phi}^j = \tilde{x}^j - i\,\tilde{y}^j$ are:

$$\tilde{x}^i = \lambda^{-1} \frac{(k^j + \tau)(k^l + \tau) - (a + \tau)(k^i + \tau)}{(k^j + \tau)(k^l + \tau) + (a + \tau)(k^i + \tau) - 2\bar{b}}, \tag{5.35}$$

$$\tilde{y}^i = 2\lambda^{-1} \frac{\sqrt{(a + \tau)(k^1 + \tau)(k^2 + \tau)(k^3 + \tau) - \bar{b}^2}}{(k^j + \tau)(k^l + \tau) + (a + \tau)(k^i + \tau) - 2\bar{b}}. \tag{5.36}$$

$$\tag{5.37}$$

where we introduced $\bar{b} = b/\sqrt{|I_4|}$.

Let us map the constants in h_I to the dressed charges and B_i fields. Requiring the warped factor to vanish at infinity is equivalent to:

$$U(\tau = 0) = 0 \quad \Leftrightarrow \quad |I_4|\,a\,k^1\,k^2\,k^3 - b^2 = 1. \tag{5.38}$$

The normalisation of the moduli fields at infinity gives rise to:

$$\lim_{\tau \to 0} z^j = B_j - i \quad \Rightarrow \quad B_j = \frac{1}{2}\sqrt{|I_4|}\left(\frac{1}{2}s_{ijl}\,k^j\,k^l - a\,k^i\right), \tag{5.39}$$

$$k^j\,k^l + a\,k^i - 2\bar{b} = 2\frac{\Lambda_i^{-1}}{\sqrt{|I_4|}}, \tag{5.40}$$

where we defined $\Lambda_i = \lambda\,v_i$. To derive these expressions, we used (5.38) when necessary, and we already took into account the scaling relation $\tilde{z}^j = z^j\,v_j$.

Using (5.40):

$$\left(b + \Lambda_i^{-1}\right)^2 = \frac{|I_4|}{2}\left(\frac{1}{2}s_{ijl}k^j\,k^l - a\,k^i + 2a\,k^i\right)^2 \tag{5.41}$$

$$= B_i^2 + |I_4|\,a\,k^i\,\frac{1}{2}s_{ijl}\,k^j\,k^l = 1 + b^2 + B_i^2, \tag{5.42}$$

where in the last step we used (5.38), allows us to determine b:

$$b = \frac{1}{2}\left(\Lambda_i\left(1 + B_i^2\right) - \Lambda_i^{-1}\right). \tag{5.43}$$

Since the left hand side is a given number (b), we do explicitly see that for a given value of the B_i fields, the volumes v_i are entirely fixed at this point. The above equation gives rise to two conditions:

$$\Lambda_1\left(1 + B_1^2\right) - \Lambda_1^{-1} = \Lambda_2\left(1 + B_2^2\right) - \Lambda_2^{-1} = \Lambda_3\left(1 + B_3^2\right) - \Lambda_3^{-1}. \tag{5.44}$$

There exists a third one coming from the fact that:

$$\Lambda_1 \Lambda_2 \Lambda_3 = \Lambda^3 . \tag{5.45}$$

These equations provide an implicit map between the "fluxes" Λ_i and the B_i fields. A better parameterisation can be achieved by introducing a new set of parameters β_i as follows:

$$\Lambda_i = \frac{e^{\beta_i}}{\sqrt{1 + B_i^2}} . \tag{5.46}$$

In terms of these, the above conditions are:

$$\sqrt{1 + B_1^2} \sinh \beta_1 = \sqrt{1 + B_2^2} \sinh \beta_2 = \sqrt{1 + B_3^2} \sinh \beta_3 , \tag{5.47}$$

$$\Lambda^3 = \frac{e^{\beta_1 + \beta_2 + \beta_3}}{\sqrt{1 + B_1^2} \sqrt{1 + B_2^2} \sqrt{1 + B_3^2}} . \tag{5.48}$$

Plugging (5.43) into (5.40) allows us to fix $a \, k^i$:

$$\sqrt{|I_4|} \, a \, k^i = \frac{1}{2} \left(\Lambda_i^{-1} + \Lambda_i \left(1 + B_i^2 \right) - 2 B_i \right) = \sqrt{1 + B_i^2} \cosh \beta_i - B_i . \tag{5.49}$$

Using this into (5.39) fixes $s_{ijl} k^j \, k^l$ to be:

$$\sqrt{|I_4|} \frac{1}{2} s_{ijl} k^j \, k^l = \frac{1}{2} \left(\Lambda_i^{-1} + \Lambda_i \left(1 + B_i^2 \right) + 2 B_i \right) = \sqrt{1 + B_i^2} \cosh \beta_i + B_i . \tag{5.50}$$

The mass can be obtained, as usual, by studying the first order asymptotic correction to the warped factor. This gives:

$$2 G_N M = \frac{1}{2} |I_4| \left(k^1 \, k^2 \, k^3 + a \left(k^1 \, k^2 + k^1 \, k^3 + k^2 \, k^3 \right) \right) . \tag{5.51}$$

To write this in terms of the physical charges and B moduli, let us first multiply the three independent equations (5.50). This allows us to determine the product $k^1 \, k^2 \, k^3$:

$$k^1 \, k^2 \, k^3 = |I_4|^{-3/4} \prod_{i+1}^{3} \left[B_i + \sqrt{1 + B_i^2} \cosh \beta_i \right]^{1/2} . \tag{5.52}$$

If we now multiply (5.49) with (5.52) and divide by (5.50), we can determine $a \, k^i \, k^j$:

$$a \, k^1 \, k^2 = |I_4|^{-3/4} \left[\sqrt{1 + B_1^2} \cosh \beta_1 - B_1 \right]^{1/2} \cdot \left[B_2 + \sqrt{1 + B_2^2} \cosh \beta_2 \right]^{1/2}$$
$$\cdot \left[\sqrt{1 + B_3^2} \cosh \beta_3 - B_3 \right]^{1/2} , \tag{5.53}$$

with cyclic permutation expressions for the analogous remaining terms appearing in the mass formula. We stress that to derive the above relation it may be convenient to use the identity:

$$\left(\sqrt{1+B_i^2}\cosh\beta_i - B_i\right)\left(\sqrt{1+B_i^2}\cosh\beta_i + B_i\right) =$$
$$\left(\sqrt{1+B_j^2}\cosh\beta_j - B_j\right)\left(\sqrt{1+B_j^2}\cosh\beta_j + B_j\right) , \quad (5.54)$$

for any pair $\{i,j\} = \{1,2,3\}$, which is a consequence of conditions (5.47).

To simplify the final mass formula, it is convenient to use:

$$\sqrt{1+B_i^2}\cosh\beta_i \pm B_i = \frac{\Lambda_i}{2}\left[1+(\Lambda_i^{-1}\pm B_i)^2\right] . \quad (5.55)$$

This allows us to write the mass formula for arbitrary B_i fields as:

$$
\begin{aligned}
2^{3/2}\,G_N\,M \;=\; & \frac{P^0}{4}\left(1+(\Lambda_1^{-1}+B_1)^2\right)^{1/2}\left(1+(\Lambda_2^{-1}+B_2)^2\right)^{1/2}\left(1+(\Lambda_3^{-1}+B_3)^2\right)^{1/2} \\
+ & \frac{P^0}{4}\left(1+(\Lambda_1^{-1}+B_1)^2\right)^{1/2}\left(1+(\Lambda_2^{-1}-B_2)^2\right)^{1/2}\left(1+(\Lambda_3^{-1}-B_3)^2\right)^{1/2} \quad (5.56) \\
+ & \frac{P^0}{4}\left(1+(\Lambda_1^{-1}-B_1)^2\right)^{1/2}\left(1+(\Lambda_2^{-1}+B_2)^2\right)^{1/2}\left(1+(\Lambda_3^{-1}-B_3)^2\right)^{1/2} \\
+ & \frac{P^0}{4}\left(1+(\Lambda_1^{-1}-B_1)^2\right)^{1/2}\left(1+(\Lambda_2^{-1}-B_2)^2\right)^{1/2}\left(1+(\Lambda_3^{-1}+B_3)^2\right)^{1/2} .
\end{aligned}
$$

Once again, the mass remains marginal and can still be interpreted as the sum of the masses of four D6-branes with the appropriate fluxes. In this case, the fluxes are determined implicitly as a function of the B fields.

As we emphasized above, the volumes of the different T^2's are fixed in the current solution. We can generate a more general solution in which these volumes are generic, but having a fixed T^6 volume, as required by the attractor mechanism. This is achieved by the action of $\hat{H}$. Since the mass M only depends on the charges $\{P^0, Q_0\}$ and the three B^i, and all these are left invariant by $\hat{H}$, the mass will remain marginal along all the moduli space in the D0-D6 system. Actually, using U-duality, we can extend this marginality claim on the $\overline{D0} - D4$ side even for non-equal B-fields.

6 Discussion

In this paper we presented constructions of the extremal non-BPS black holes in the STU-model along with their embeddings into the $\mathcal{N} = 8$ and $\mathcal{N} = 4$ supergravity theories. In addition to our detailed formulae, we highlight several qualitative lessons:

- The extremal mass formula for the non-BPS black holes is generally *not* related by analytical continuation to the BPS mass formula. For generic charge vector and/or generic moduli the mass formula differ qualitatively between the two branches. This may indicate that analysis of the non-BPS black holes relying on analytical continuation is misleading.

- The non-BPS charge configurations have a canonical split into four subparts, $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_2 + \Gamma_3$, realized via U-duality to the $\bar{D}0 - D4 - D4 - D4$ frame. In this duality frame the non-BPS mass formula takes the form of a marginal bound state of the subparts. This is quite different from the experience with BPS-black holes at generic points in moduli space.

- The flat directions previously noticed for non-BPS attractors extend to the whole flow. This is due to the existence of a nontrivial subgroup $\hat{H}$ of the duality group G which leaves non-BPS charge vectors invariant, while acting non-trivially on the scalars in G/H.

We expect all these results to extend to all symmetric supergravity theories by extension from the STU case. On the other hand, the flat directions we see are intricately tied to the symmetries of our scalar manifold so we don't expect to see such flat directions occur naturally in non-symmetric $\mathcal{N} = 2$ theories.

The marginality property of our non-BPS mass formula begs for an explanation. Such an explanation might possibly be found by following up on the surprising observation in [48] that the near horizon regions for our non-BPS black holes can be lifted to super-symmetric five dimensional solutions, with "supersymmetry without supersymmetry" as in [49, 50]. In any case, an explanation is beyond the scope of this paper.

One might also wonder if the marginality property extends to non-BPS extremal black holes (with or without wrapped D6-brane charge) in other $\mathcal{N} = 2$ theories, especially those with cubic prepotentials. Finally, it would be interesting to know how the marginality property survives α' corrections to the action. In this context, it would be particularly interesting to look at cases where Γ is properly quantized but the subparts Γ_μ are not.

Acknowledgements

A Non-BPS D0-D4 with General Moduli: Derivation

We consider a non-BPS system with D0-brane charge $Q_0 < 0$ and three D4-brane charges $P^i > 0$ for $i = 1, 2, 3$. In order to be sufficiently general we include arbitrary complex moduli fields $z_j = x_j - i\, y_j$ $(j = 1, 2, 3)$ at the outset.

The Lagrangian (2.14) for the analogue mechanics problem is:

$$\mathcal{L}_{\text{eff}} = (U')^2 + \frac{1}{4} \sum_{i=1}^{3} \left[\frac{(y_i')^2 + (x_i')^2}{y_i^2} \right] + e^{2U}\, V_{\text{BH}} \,, \tag{A.1}$$

where the potential V_{BH} in the case of the STU-model is:

$$
V_{\text{BH}}(x_i, y_i) = \frac{Q_0^2}{2} \frac{1}{y_1\, y_2\, y_3} + Q_0 \frac{P^3 x_1 x_2 + P^2 x_1 x_3 + P^1 x_2 x_3}{y_1\, y_2\, y_3} + \frac{1}{2} \left[y_1\, y_2\, y_3 \sum_{i=1}^{3} (P^i)^2 y_i^{-2} \right.
$$

$$
\left. + \frac{1}{y_1\, y_2\, y_3} \left((P^3 x_1 x_2 + P^2 x_1 x_3 + P^1 x_2 x_3)^2 + \sum_{j \neq k \neq i=1}^{3} (y_i)^2\, (P^k x_j + P^j x_k)^2 \right) \right]. \tag{A.2}
$$

Inspired by [40] we introduce the rescaled field parameters and variables ($s_{ijk} = |\epsilon_{ijk}|$):

$$M_0^2 = 2\sqrt{-Q_0 P^1 P^2 P^3}\,, \quad R_i = \sqrt{\frac{-Q_0 P^i}{\frac{1}{2} s_{ijk} P^j P^k}}\,, \tag{A.3}$$

$$x_i = R_i\, t_i\,, \quad y_i = R_i\, e^{\phi_i}\,. \tag{A.4}$$

In terms of these, the black hole potential V_{BH} simplifies to:

$$V_{BH}(t_i, \phi_i) = \frac{M_0^2}{4} e^{-\sum_r \phi_r} \left(\sum_{i<j\; k\neq i,\,j} \left(e^{2(\phi_i + \phi_j)} + (t_i + t_j)^2\, e^{2\phi_k} \right) + \left(1 + \sum_{i<j} t_i\, t_j \right)^2 \right). \tag{A.5}$$

The equations of motion are:

$$U'' = e^{2U}\, V_{BH}\,, \tag{A.6}$$

$$(t_i'\, e^{-2\phi_i})' = 2 e^{2U} \frac{\partial V_{BH}}{\partial t_i}\,, \tag{A.7}$$

$$\phi_i'' + (t_i')^2\, e^{-2\phi_i} = 2 e^{2U} \frac{\partial V_{BH}}{\partial \phi_i}\,, \tag{A.8}$$

and the constraint equation (2.16) is:

$$(U')^2 + \frac{1}{4} \sum_i \left[(\phi_i')^2 + (t_i')^2\, e^{-2\phi_i} \right] = e^{2U}\, V_{BH}\,. \tag{A.9}$$

Yet another set of field variables $\{\beta,\, \alpha_i\}$ (i=1,2,3) will help us to disentangle the coupled differential equations (A.6-A.9):

$$\alpha_i = U + \frac{1}{2}\phi_i \quad \Leftrightarrow \quad -2\phi_i = \beta + \sum_j \alpha_j - 4\alpha_i\,, \tag{A.10}$$

$$\beta = U - \frac{1}{2} \sum_i \phi_i \quad \Leftrightarrow \quad 4U = \beta + \sum_j \alpha_j\,. \tag{A.11}$$

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The dynamical equations now become:

$$\alpha_i'' + \frac{1}{2}(t_i')^2\, e^{-2\phi_i} \;=\; e^{2U}\left(V_{BH} + \frac{\partial V_{BH}}{\partial \phi_i}\right), \qquad (A.12)$$

$$\beta'' - \frac{1}{2}\sum_i (t_i')^2\, e^{-2\phi_i} \;=\; e^{2U}\left(V_{BH} - \sum_i \frac{\partial V_{BH}}{\partial \phi_i}\right), \qquad (A.13)$$

$$\sum_i [(\alpha_i')^2 - 2\alpha_i''] + (\beta')^2 + \sum_{i<j}(\alpha_i' - \alpha_j') \;=\; -2e^{2U}\left(V_{BH} + \sum_i \frac{\partial V_{BH}}{\partial \phi_i}\right), \quad (A.14)$$

as well as (A.7). For the potential (A.5) we have:

$$V_{BH} + \frac{\partial V_{BH}}{\partial \phi_k} \;=\; \frac{M_0^2}{2} e^{\phi_k - \phi_i - \phi_j}\left[(t_i + t_j)^2 + e^{2\phi_i} + e^{2\phi_j}\right], \quad i,j \neq k, \quad (A.15)$$

$$V_{BH} - \sum_k \frac{\partial V_{BH}}{\partial \phi_k} \;=\; \frac{M_0^2}{2} e^{-\sum_i \phi_i}\left[\sum_{i,j \neq k} e^{2\phi_r}(t_i + t_j)^2 + 2(1 + \sum_{i<j} t_i t_j)^2\right], \quad (A.16)$$

$$V_{BH} + \sum_k \frac{\partial V_{BH}}{\partial \phi_k} \;=\; -\frac{M_0^2}{2} e^{-\sum_i \phi_i}\left[(1 + \sum_{i<j} t_i t_j)^2 - \sum_{i<j} e^{2(\phi_i + \phi_j)}\right]. \qquad (A.17)$$

The dynamical equations still appear very complicated at this point but they are in fact integrable. It would be interesting to explore the structure that makes this possible but we will be content with simply finding the solutions, proceeding as follows[40]. Suppose we could find solutions α_i satisfying:

$$\alpha_i'' \;=\; \frac{M_0^2}{2} e^{2\alpha_i}\left[e^{2(\alpha_k - \alpha_j)} + e^{2(\alpha_j - \alpha_k)}\right], \quad i \neq j \neq k,$$

$$\sum_i [(\alpha_i')^2 - 2\alpha_i''] + \sum_{i<j}(\alpha_i' - \alpha_j') \;=\; -M_0^2 \sum_{i \neq j \neq k} e^{2(\alpha_j + \alpha_k - \alpha_i)}. \qquad (A.18)$$

If this is possible then (A.12) reduces to:

$$t_i' = \pm M_0\,(t_j + t_k)\, e^{3\alpha_i - \alpha_j - \alpha_k}, \quad j \neq k \neq i, \qquad (A.19)$$

and (A.14) reduces to:

$$\beta' = \pm M_0\, e^{\beta}\,(1 + \sum_{i<j} t_i t_j). \qquad (A.20)$$

One can check that (A.19) and (A.20) ensure that the remaining equations (A.13) and (A.7) are satisfied, provided we take the same sign in both equations. It is therefore sufficient to solve (A.18-A.20).

The first equation in (A.18) can be reorganized as:

$$(\alpha_1 + \alpha_2 - \alpha_3)'' = M_0^2 e^{2(\alpha_1 + \alpha_2 - \alpha_3)}, \quad \text{(and cyclic permutations)}, \qquad (A.21)$$

which is readily integrated as:

$$e^{-(\alpha_j + \alpha_k - \alpha_i)} = a_i + M_0 \tau \equiv \hat{h}_i, \quad \text{(and cyclic permutations)}, \qquad (A.22)$$

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where a_i is a positive integration constant. One can check that this solution automatically satisfies the second equation in (A.18).

Inserting the solution (A.22) for α_i in the moduli equations (A.19) we find:

$$t'_i = \pm M_0 \left(t_j + t_k \right) \frac{\hat{h}_i}{\hat{h}_j \, \hat{h}_k} \; , \quad \text{(and cyclic permutations)} \, . \tag{A.23}$$

It turns out that only the upper sign leads to regular solutions. For the lower sign we have the solution:

$$t_i = \frac{c}{\hat{h}_j \, \hat{h}_k} \; , \quad \text{(and cyclic permutations)} \, . \tag{A.24}$$

where c is a new integration constant.

We are finally left to integrate (A.20) which now takes the form:

$$\beta' = -M_0 e^\beta \left(1 + \frac{c^2}{\hat{h}_1 \, \hat{h}_2 \, \hat{h}_3} \left(\frac{1}{\hat{h}_1} + \frac{1}{\hat{h}_2} + \frac{1}{\hat{h}_3} \right) \right) \, . \tag{A.25}$$

We find:

$$e^{-\beta} = \left(a_0 + M_0 \tau \right) - \frac{c^2}{\hat{h}_1 \, \hat{h}_2 \, \hat{h}_3} \equiv -\hat{h}_0 - \frac{c^2}{\hat{h}_1 \, \hat{h}_2 \, \hat{h}_3} \, . \tag{A.26}$$

Here a_0 is yet another integration constant which must be chosen sufficiently positive so that $e^{-\beta} > 0$ for all $0 < \tau < \infty$.

We have thus found a solution in terms of four functions $\hat{h}_\Lambda$ ($\Lambda = 0, 1, 2, 3$) that are linear in τ or, equivalently, harmonic on a spatial slice of the black hole. There are five integration constants c, a_Λ. We transform to a more natural set of harmonic functions by asking that each one has a τ-dependence proportional to the corresponding dressed charge:

$$\hat{h}_0 = -\frac{M_0}{Q_0} \, H_0 \; , \qquad \hat{h}_i = \frac{M_0}{P^i} \, H^i \, . \tag{A.27}$$

If we use these new harmonic functions along with the changes of variables (A.3-A.4) and (A.10-A.11) we can present the solution in terms of the physical warp factor:

$$e^{-4U} = -\hat{h}_0 \, \hat{h}_1 \, \hat{h}_2 \, \hat{h}_3 - c^2 \; = 4 H_0 H^1 H^2 H^3 - c^2 \, , \tag{A.28}$$

and the complex moduli fields:

$$z_i = R_i \frac{c - i \, e^{-2U}}{\frac{1}{2} s_{ijk} \hat{h}_j \, \hat{h}_k} = \frac{c - i \, e^{-2U}}{s_{ijk} H^j \, H^k} \, . \tag{A.29}$$

We can now match the asymptotic conditions at infinity, $z^i \to B - i$ and $e^{2U} \to 1$, by choosing $c = B$ and taking the other constants to be:

$$a_0 = -\frac{M_0}{\sqrt{2} \, Q_0} \left(1 + B^2 \right) , \qquad a_i = \frac{M_0}{\sqrt{2} \, P^i} \, . \tag{A.30}$$

This completely specifies our seed solution.

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