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Date deposited: 23rd February 2012

Version of file: Author final

Peer Review Status: Peer reviewed

Citation for item:

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Black holes with current loops revisited

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(Dated: June 8, 2011)

The electromagnetic field around a Kerr black hole inside a current loop is sometimes used as the basis of a toy model for discussing the properties of particle orbits near astrophysical black holes. The motivation for the present paper is to correct the published solution to Maxwell's equations with a charged current loop. Dipole approximations and closed-form expressions in the extreme Kerr limit are also presented. Using the corrected solution, it turns out that imposing a vanishing electromotive force produces a loop with a potential which is finite everywhere outside the black hole. Ring solutions can be combined into solutions with multiple rings or current discs.

I.   INTRODUCTION

Attempts have been made over the years to explain the generation of untrarelativistic particles through physical effects around black holes located inside accretion disks (see [1] for a review). In some situations, large electric potentials can be produced by dynamo action if the black hole is rotating in a magnetic field generated in the accretion disk. In the original Blandford-Znajek model, for example, the induced electric fields drive currents and act as the source of energy fluxes [2, 3].

In the context of some of these models, it becomes important to study particle orbits and particle interactions in electric and magnetic fields close to the black hole. These studies are increasingly being replaced by magnetohydrodynamics (MHD), but the particle picture is needed when a fluid description is not appropriate. The study of circular orbits, for example, gives the location for the inner edge of the accretion disk for free charges [4–6]. In the presence of the magnetic field, the equations governing the charged particle orbits are no longer seperable, and the particle motion can be come quite complicated outside of the equatorial plane [7, 8]. This may well have an effect on issues such as charge separation and the validity of fluid approximations.

The large potential around the black hole implies that Penrose processes can be an important source of untrarelativistic particles [9]. Familiar limits on the efficiency of the Penrose process [10, 11] are not valid for charged particles, and the particles emerging from the ergoregion can have energies close to the available electric potential [12]. For example, magnetic fields of 10^4 Gauss around 10^9 solar mass black holes can produce positrons and electrons with energies as high as 10^{20}eV through reactions such as $\gamma\gamma \to e^+e^-$ inside the ergoregion.

All of the discussions of particle motion mentioned above have used the field from a current loop, or its dipole limit, as a toy model to represent the magnetic field. The solution to Maxwell's equations for the uncharged current loop was obtained a long time ago by Chitre and Vishveshwara [13], and generalised to a charged current loop by Petterson [14]. The general class of stationary solutions was analysed by Bičák and Dvořák [15], who also considered some special cases including current loops, dipoles and point charges. Surprisingly, the result of Petterson is not consistent with the other two papers, which is unfortunate because this is the only published solution which solves for the vector potentials that are needed to study the particle dynamics.

In order to explain the differences further, consider the general vacuum axisymmetric solution to Maxwell's equations, with a set of four arbitrary constants for each angular mode. These constants can be related to the current and charge by matching vacuum solutions inside and outside the current loop. The inconsistency between the three papers mentioned above is due to an error in Ref [14] when matching the solutions. (There is also a minor typographical error in the constants of Ref. [14]). Correcting the values of these constants leads to physical consequences, for example the potential at the horizon of the black hole changes and the zero EMF condition used in some astrophysical scenarios occurs for a different value of the black hole charge. The errors in Ref. [14] have subsequently been reproduced in standard texts [1]. The motivation for the present paper is to present the correct electromagnetic potentials for a charged current loop.

The current loop solutions are appropriate when there are no free charges surrounding the black hole and there is no radial current flow. In real situations the region surrounding the black hole may contain particle-antiparticle pairs which conduct current freely. This type of model could be built up by superimposing radial currents with the

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ring solutions. The simplest possibility is to have radial current flowing out along the equatorial plane and then back along the symmetry axis, as in the toy model by Li [16].

There are problems with the simple Blandford-Znajek model, for example it is likely that most magnetic field lines pass through the accretion disk [17] and the disk is then subject to a variety of MHD instabilities [18]. On the other hand, a similar model may still hold in systems with relatively low accretion rates where the inner parts of the disk are effectively Keplerian orbits [19]. These issues are not considered further here, and the results below can be used when the electromagnetic fields generated by the accretion disk can be replaced by the fields generated by current loops.

II.   MAXWELL FIELDS IN THE KERR BACKGROUND

The general solution to Maxwell's equations on a Kerr background is described in the textbook by Chandrasekhar [20]. In principle, we ought to solve the full Einstein-Maxell equations, but the gravitational back-reaction of the magnetic field can be ignored when $BM << 1$ in gravitational units, or

$$B << 2 \times 10^{10} \left(\frac{M}{10^9 M_.} \right) \text{ G} \tag{1}$$

in Gauss. In typical astrophysical situations, the electromagnetic field strengths are likely to be insufficiently large to have any significant effect and the Kerr metric for a rotating black hole can be used consistently.

The Kerr metric in Boyer-Lindquist coordinates (t, r, θ, ϕ) is given by

$$ds^2 = -\frac{\Delta}{\Sigma} \omega_1^2 + \frac{\sin^2 \theta}{\Sigma} \omega_2^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \tag{2}$$

where

$$\omega_1 = dt - a \sin \theta d\phi, \quad \omega_2 = (r^2 + a^2) d\phi - a dt \tag{3}$$
$$\Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 + a^2 - 2mr \tag{4}$$

The Maxwell field strength $F_{\mu\nu}$ is represented by three complex scalar quantities

$$\phi_0 = F_{\mu\nu} l^\mu m^\nu \tag{5}$$
$$\phi_1 = F_{\mu\nu} (l^\mu n^\nu + m^{\mu*} m^\nu)/2 \tag{6}$$
$$\phi_0 = F_{\mu\nu} m^{\mu*} n^\nu \tag{7}$$

where the Newman-Penrose tetrad vectors are

$$l^\mu = \frac{1}{\Delta} \left(r^2 + a^2, \Delta, 0, a \right) \tag{8}$$
$$n^\nu = \frac{1}{2\Sigma} \left(r^2 + a^2, -\Delta, 0, a \right) \tag{9}$$
$$m^\mu = \frac{1}{\bar{\rho}\sqrt{2}} \left(ia \sin \theta, 0, 1, i\text{cosec}\theta \right) \tag{10}$$

and $\bar{\rho} = r + i \cos \theta$.

The stationary axisymmetric solutions to Maxwell's equations with vanishing sources are determined by solving the Teukolsky equation

$$-\Delta \Psi_{,rr} - \text{cosec}\theta (\sin \theta \Psi_{,\theta})_{,\theta} + (\cot^2 \theta + 1)\Psi = 0 \tag{11}$$

where commas denote derivatives and the field $\Psi = -\Delta \phi_0$. The general solution for ϕ_0 can be expressed in terms of angular mode functions $P_l{}^1(\cos \theta)$,

$$\phi_0 = \frac{1}{\sqrt{2}} \sum_{l=1}^{\infty} \left(\alpha_l P_{l,r}(u) P_l{}^1(\cos \theta) + \beta_l Q_{l,r}(u) P_l{}^1(\cos \theta) \right) \tag{12}$$

The coefficients α_l and β_l are complex constants and $P_l(u)$ and $Q_l(u)$ are Legendre functions of $u = (r - M)/(r_+ - M)$, where r_+ is the radial coordinate of the event horizon. The solution can be checked using the identity

$$(\Delta P_{l,r}(u))_{,r} = l(l + 1) P_l(u). \tag{13}$$

The terms involving $P_l(u)$ are regular at the horizon and the terms involving $Q_l(u)$ are regular at spatial infinity. Typically, we have different inner and outer solutions separated by a region with charges and currents.

The reconstruction of the gauge potentials given in ref [20] is incomplete, but with a little extra work it is possible to extend the method used there to rederive the inner solution given by Petterson [14]

$$A_t = \alpha_t - \frac{rQ}{\Sigma} - \sum_{l=1}^{\infty} \left\{ \frac{\Delta}{\Sigma} P_{l,r}(u) P_l(\cos\theta)(r\alpha_l^r - a\cos\theta\alpha_l^i) + \frac{a}{\Sigma} P_l(u)\sin\theta P_l^{\,1}(\cos\theta)(a\cos\theta\alpha_l^r + r\alpha_l^i) \right\} \quad (14)$$

$$A_\phi = \frac{arQ}{\Sigma}\sin^2\theta + \sum_{l=1}^{\infty} \left\{ \frac{a\Delta}{\Sigma} P_{l,r}(u) P_l(\cos\theta)\sin^2(\theta)(r\alpha_l^r - a\cos\theta\alpha_l^i) \right. \quad (15)$$

$$\left. + \frac{r^2 + a^2}{\Sigma} P_l(u)\sin\theta P_l^{\,1}(\cos\theta)(a\cos\theta\alpha_l^r + r\alpha_l^i) - \frac{\Delta}{l(l+1)} P_{l,r}(u) P_l^{\,1}(\cos\theta)\sin\theta\alpha_l^i \right\} \quad (16)$$

where Q is the black hole charge, α_t is a constant and $\alpha_l = \alpha_l^i + i\alpha_l^r$. The possibility of a constant term in A_ϕ is excluded by the regularity of the vector potential along the axis of rotation.

The outer solution uses the alternate set of Legendre functions, but it is also important to account for the charge of the source region q. Since this will be important later, it is worth noting the full outer solution,

$$A_t = -\frac{r(Q+q)}{\Sigma} - \sum_{l=1}^{\infty} \left\{ \frac{\Delta}{\Sigma} Q_{l,r}(u) P_l(\cos\theta)(r\beta_l^r - a\cos\theta\beta_l^i) + \frac{a}{\Sigma} Q_l(u)\sin\theta P_l^{\,1}(\cos\theta)(a\cos\theta\beta_l^r + r\beta_l^i) \right\} \quad (17)$$

$$A_\phi = \frac{ar(Q+q)}{\Sigma}\sin^2\theta + \sum_{l=1}^{\infty} \left\{ \frac{a\Delta}{\Sigma} Q_{l,r}(u) P_l(\cos\theta)\sin^2(\theta)(r\beta_l^r - a\cos\theta\beta_l^i) \right. \quad (18)$$

$$\left. + \frac{r^2 + a^2}{\Sigma} Q_l(u)\sin\theta P_l^{\,1}(\cos\theta)(a\cos\theta\beta_l^r + r\beta_l^i) - \frac{\Delta}{l(l+1)} Q_{l,r}(u) P_l^{\,1}(\cos\theta)\sin\theta\beta_l^i \right\} \quad (19)$$

The gauge has been chosen so that A_t vanishes at spatial infinity.

III. CHARGED CURRENT LOOP

The field due to the charged current loop is obtained by matching the inner and outer solutions at the position of the loop. The safest way to do this is to use the Maxwell scalars ϕ_0 or ϕ_2, because these are readily expressed in terms of orthogonal polynomials in $\cos\theta$. Matching ϕ_1, as was attempted in Ref [14], leads to errors because ϕ_1 contains both $P_l(\cos\theta)$ and $P_l^{\,1}(\cos\theta)$, and these are not orthogonal functions.

The current density for the charged current loop in the Boyer-Lindquist coordinates is given by [15]

$$J^\mu = \left(\frac{q}{2\pi r_0^2}\delta(r-r_0)\delta(\cos\theta), 0, 0, \frac{I}{r_0^2}\delta(r-r_0)\delta(\cos\theta) \right). \quad (20)$$

The parameter q is equal to the total charge of the loop, as defined by an integration of the charge density over a surface S of constant time,

$$\int_S J^\mu d\Sigma_\mu = q. \quad (21)$$

Gauss' law, applied to a large sphere, implies that this is the same charge which appears in the potential (18).

The relationship between the Maxwell scalars and the source is obtained through the corresponding Teukolsky equation,

$$-\Delta\Psi_{,rr} - \mathrm{cosec}\,\theta(\sin\theta\Psi_{,\theta})_{,\theta} + (\cot^2\theta + 1)\Psi = 4\pi J \quad (22)$$

The source function is related to the current density by

$$J = \Sigma(\Delta + \mu)\Sigma^{-1}J_{m^*} - \Sigma(\delta^* + \pi - \tau^*)\Sigma^{-1}J_n, \quad (23)$$

where Δ and δ^* are directional derivatives along n and m^* respectively, and

$$\mu = -\frac{\Delta}{2\Sigma\bar{\rho}^*}, \quad \pi = \frac{ia\sin\theta}{\bar{\rho}^{*2}\sqrt{2}}, \quad \tau = -\frac{ia\sin\theta}{\Sigma\sqrt{2}}. \quad (24)$$

For axisymmetric sources, we may express the fields in angular modes with components R_l, where

$$R_l = \int_{-1}^1 \Psi P_l{}^1(\cos\theta) d\cos\theta, \tag{25}$$

and similarly for J_l. These components satisfy

$$R_{l,rr} - \frac{l(l+1)}{\Delta} R_l = -8\pi \frac{J_l}{\Delta}. \tag{26}$$

In the source-free regions, ϕ_0 is given by Eq. (12), and the integration (25) yields

$$R_l = \frac{-\Delta}{\sqrt{2}} \frac{2l(l+1)}{2l+1} \alpha_l P_{l,r}(u) \text{ inner region} \tag{27}$$

$$= \frac{-\Delta}{\sqrt{2}} \frac{2l(l+1)}{2l+1} \beta_l Q_{l,r}(u) \text{ outer region} \tag{28}$$

When the current density (20) is substituted into eq. (23), the differential equation (26) for R_l becomes

$$R_{l,rr} - \frac{l(l+1)}{\Delta} R_l = F(r)\delta(r-r_0)_{,r} + G(r)\delta(r-r_0), \tag{29}$$

where

$$F(r) = -\frac{i}{\sqrt{2}} \frac{4\pi I(r_0^2 + a^2) - 2aq}{r_0} P_l{}^1(0) \tag{30}$$

$$G(r) = -\frac{i}{\sqrt{2}} 4\pi I P_l{}^1(0) + \frac{l(l+1)}{\sqrt{2}} \frac{4\pi a I - 2q}{r_0} P_l(0) \tag{31}$$

Integrating Eq. (29) through the delta function gives

$$-\frac{\Delta}{\sqrt{2}} \frac{2l(l+1)}{2l+1} \left(\beta_l Q_{l,r} - \alpha_l P_{l,r}\right) = F(r_0) \tag{32}$$

$$-\frac{\Delta}{\sqrt{2}} \frac{2l(l+1)}{2l+1} \left(\beta_l Q_l - \alpha_l P_l\right) = \frac{\Delta}{l(l+1)} G(r_0). \tag{33}$$

Combining these equation gives the solution for β_l,

$$\beta_l = \frac{l+1/2}{l(l+1)} \frac{1}{w} \left\{ \frac{\sqrt{2} G(r_0)}{l(l+1)} P_{l,r}(u_0) - \frac{\sqrt{2} F(r_0)}{\Delta(r_0)} P_l(u_0) \right\}, \tag{34}$$

where w is the Wronskian,

$$w = P_l(u_0) Q_{l,r}(u_0) - Q_l(u_0) P_{l,r}(u_0) = \frac{r_+ - M}{\Delta(r_0)}. \tag{35}$$

The corresponding solution for α_l is obtained by replacing P_l by Q_l. The real and imaginary parts of β_l can be identified by using Eqs. (30) and (31),

$$\beta_l^r = \frac{l+1/2}{l(l+1)} \frac{1}{w} \frac{4\pi a I - 2q}{r_0} P_{l,r}(u_0) P_l(0) \tag{36}$$

$$\beta_l^i = \frac{l+1/2}{l(l+1)} \frac{1}{w} \left\{ \frac{4\pi I(r_0^2 + a^2) - 2aq}{r_0 \Delta(r_0)} P_l(u_0) - \frac{4\pi I}{l(l+1)} P_{l,r}(u_0) \right\} P_l{}^1(0) \tag{37}$$

Similarly, for α_l,

$$\alpha_l^r = \frac{l+1/2}{l(l+1)} \frac{1}{w} \frac{4\pi a I - 2q}{r_0} Q_{l,r}(u_0) P_l(0) \tag{38}$$

$$\alpha_l^i = \frac{l+1/2}{l(l+1)} \frac{1}{w} \left\{ \frac{4\pi I(r_0^2 + a^2) - 2aq}{r_0 \Delta(r_0)} Q_l(u_0) - \frac{4\pi I}{l(l+1)} Q_{l,r}(u_0) \right\} P_l{}^1(0) \tag{39}$$

These results are consistent with Chitre and Vishveshwara [13] in the $q = 0$ limit, allowing for the different definition of I. The real parts differ from Petterson [14, 21] by a factor of r_0, but the imaginary parts are totally different.

The constant term α_t remains to be evaluated, and for this we use the field along the axis of rotation,

$$A_t = -\frac{\Delta}{r^2 + a^2} \sum_{l=1}^{\infty} \left(r\alpha_l^r - a\alpha_l^i \right) P_{l,r}(u) - \frac{rQ}{r^2 + a^2} + \alpha_t \tag{40}$$

$$A_t = -\frac{\Delta}{r^2 + a^2} \sum_{l=1}^{\infty} \left(r\beta_l^r - a\beta_l^i \right) Q_{l,r}(u) - \frac{r(Q + q)}{r^2 + a^2} \tag{41}$$

For α_t, we require continuity of A_t at $r = r_0$. From Eq. (32), we get

$$\alpha_t = -\frac{r_0 q}{r_0^2 + a^2} - \frac{\sqrt{2}aF^i}{r_0^2 + a^2} \sum_{l=1}^{\infty} \frac{l + 1/2}{l(l + 1)}. \tag{42}$$

Evaluating the sum and using Eq. (30) gives

$$\alpha_t = \frac{2\pi I a - q}{r_0}. \tag{43}$$

Note that the term α_t represents the value of the potential at the horizon of an uncharged black hole.

The magnetic field lines (in any synchronous frame) can be traced by the contours of constant A_ϕ, and a couple of examples are shown in Fig. 1. The plot on the left shows the field lines in a vertical plane through the axis for an extreme Kerr black hole with a current loop at $r = 6M$, and the plot on the right shows the field lines for an extreme Kerr-Newman black hole (with a small charge). The former case is an example where the black hole expels the magnetic flux at the horizon. This black hole version of the Meisner effect has been noted before [22–24]. The plots shown here demonstrate clearly that extreme black holes can have magnetic flux across the horizon when they are charged. In fact, $A_\phi = Q/2$ at the black hole equator and $A_\phi = 0$ at the north pole.
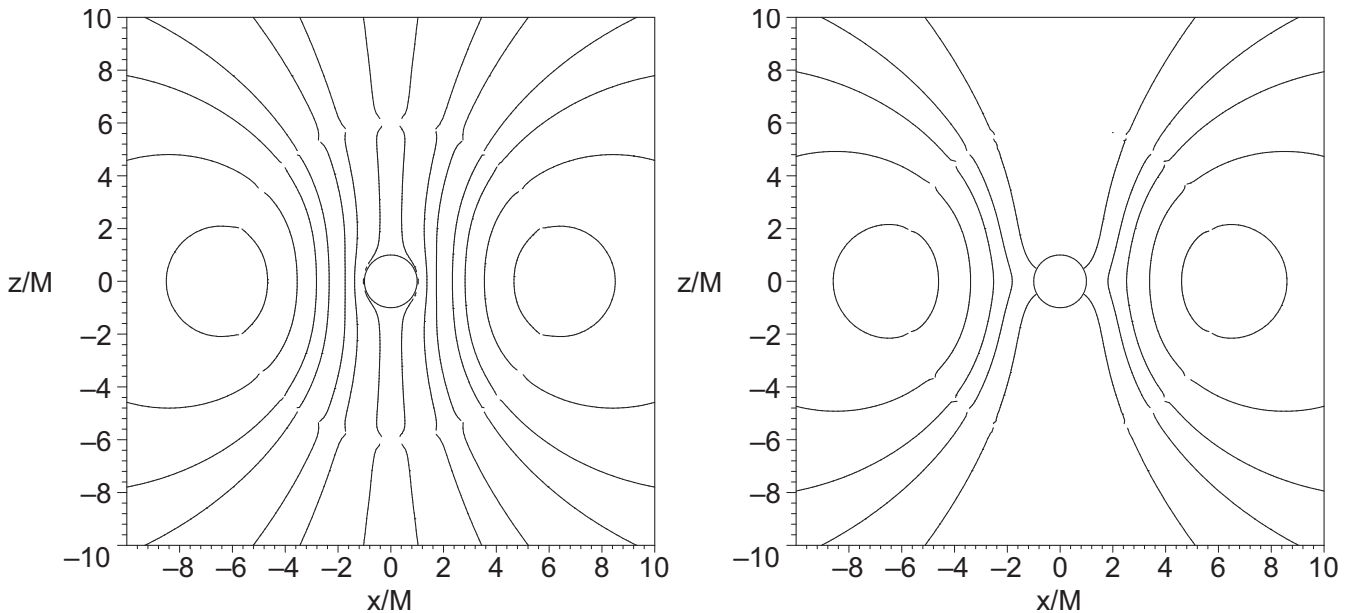

FIG. 1: The magnetic field lines are shown in a vertical plane through the axis for an extreme Kerr black hole with an equatorial current loop at $r = 6M$ (left), and for an extreme Kerr-Newman black hole (right). The mode sum has been truncated at $l = 16$, and the small irregularities ar $r = 6M$ are due to this truncation.

IV. CHARGE ACCRETION

The physical interpretation of the horizon potential is that the rotation of the black hole combined with the magnetic field of the loop generates a large electromotive force (EMF). In an astrophysical context, as first pointed out by Wald [25], free charges can migrate along the magnetic field lines to cancel the EMF, resulting in a charged black hole.

If the total system of black hole plus current loop is to remain neutral, then charges should also migrate to the current loop. We shall refer to the resulting solution as the zero-EMF solution with vanishing total charge. The zero-EMF and vanishing total charge conditions where analysed in Ref. [14], but these results used an incorrect form for the electromagnetic potential. The charge on the current loop was corrected by Linet [26, 27], but without obtaining the vector potential. (Note that Linet uses a different definition of the current I.) The full solution is given below for the first time.

A.   Zero-EMF solution with vanishing total charge

It will be sufficient to consider the electrostatic potential (40) along the axis of rotation. The gauge has been chosen so that the potential vanishes at infinity, and so the total EMF is given by the potential $-A_t$ at the horizon r_+. It vanishes when

$$\alpha_t = \frac{Q}{2M}. \tag{44}$$

The quantity α_t was evaluated in (43). If we require vanishing total charge then $q = -Q$ and the loop has a charge

$$q = -\frac{4\pi MIa}{r_0 - 2M}. \tag{45}$$

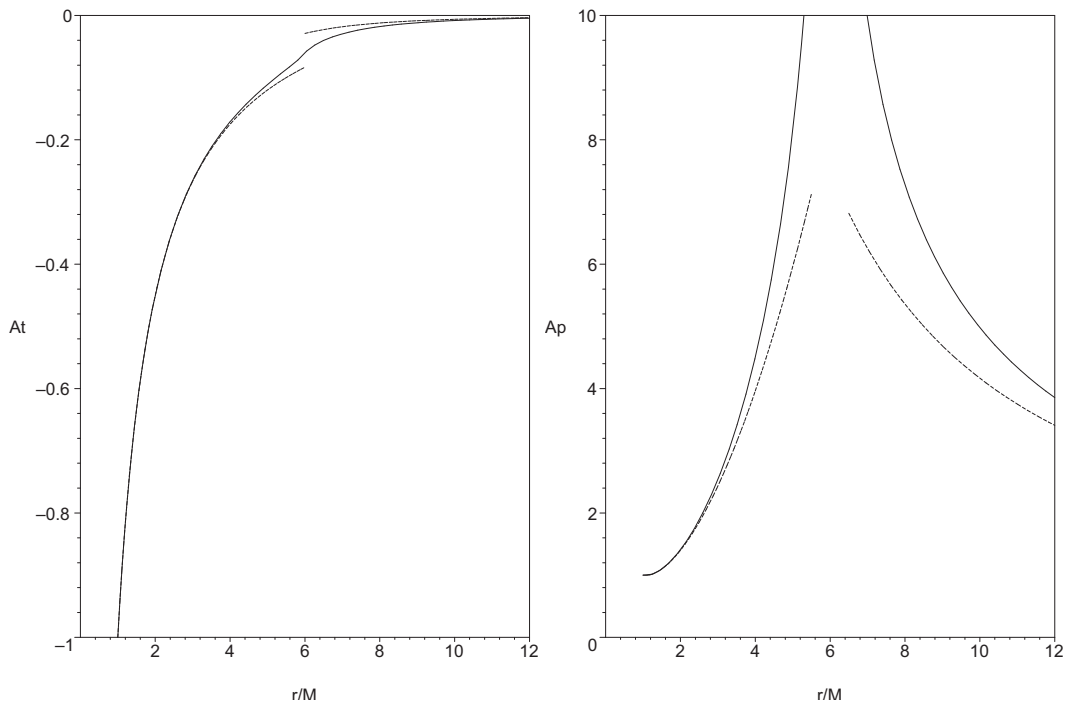Examples of the vector potentials for the zero-EMF solution are shown in Fig. 2.

FIG. 2: The vector potential solutions in the equatorial plane for a loops with radius $6M$ around an extreme Kerr black hole. The dipole solution for a field of strength B is also shown. The fields are scaled by the modulus of the horizon value.

It is interesting to note that there is a lower limit to the radius of the loop in the zero-EMF solution when $a > M/2$. Suppose that the loop contains N_+ particles of charge e and N_- particles of charge $-e$, so that

$$q = (N_+ - N_-)e. \tag{46}$$

The current is bounded by having all of the charges moving at close to the speed of light, hence [30]

$$|I| < \frac{(N_+ + N_-)e}{2\pi r_0}. \tag{47}$$

The charge $q = -Q$ is related to the current by (45), and the limit (47) implies that

$$r_0 > M + (2Ma - M^2)^{1/2} \tag{48}$$

when $a > M/2$. This is larger than the radius of the minimal circular stable orbit for black holes close to the extremal limit.

B. Finitenes of the potential

It is natural for the electromagnetic fields to diverge in the vicinity of a charged current loop, and it is remarkable that the electrostatic potential plotted in Fig. 2 appears to be finite. This turns out to be a general feature of the zero-EMF solution, where both the electrostatic potential and the electric field remain finite at the loop in the Boyer-Lindquist coordinate system. As a consequence, loops of this type are useful when combined into an equatorial disc of constant potential.

Consider the leading terms in the multipole expansion (18) which give a divergence as r approaches r_0 in the equatorial plane,

$$A_t \sim -\frac{\Delta(r_0)^2 f}{r_0^2(r_+ - M)} \sum_{l=1}^{\infty} \frac{l+1/2}{l(l+1)} Q_{l,r}(u_0) P_{l,r}(u_0) P_l(0)^2 x^l - \frac{ag}{r_0^2(r_+ - M)} \sum_{l=1}^{\infty} \frac{l+1/2}{l(l+1)} Q_l(u_0) P_l(u_0) P_l^1(0)^2 x^l \tag{49}$$

where $f = 4\pi Ia - 2q$, $g = 4\pi I(r_0^2 + a^2) - 2aq$ and x is a dummy variable which has $x \to 1$ as $r \to r_0$. Standard formulae for the large l limits of the Legendre functions give,

$$P_l(u_0)Q_l(u_0) \sim \frac{1}{2l+1}(r_+ - M)\Delta(r_0)^{-1/2} \tag{50}$$

$$P_{l,r}(u_0)Q_{l,r}(u_0) \sim \frac{l(l+1)}{2l+1}(r_+ - M)\Delta(r_0)^{-3/2} \tag{51}$$

The divergent terms in the sum are then

$$A_t \sim -\frac{\Delta(r_0)^{1/2} f}{2r_0^2} \sum_{l=1}^{\infty} P_l(0)^2 x^l - \frac{ag}{2\Delta(r_0)^{1/2} r_0^2} \sum_{l=1}^{\infty} \frac{P_l^1(0)^2}{l(l+1)} x^l \tag{52}$$

The sums can be expressed in terms of hypergeometric functions,

$$\sum_{l=0}^{\infty} P_l(0)^2 x^l = F(1/2, 1/2; 1; x^2) \tag{53}$$

$$\sum_{l=1}^{\infty} \frac{P_l^1(0)^2}{l(l+1)} x^l = xF(1/2, 1/2; 1; x^2) - \frac{x}{2}F(1/2, 1/2; 2; x^2). \tag{54}$$

The hypergeometric function $F(1/2, 1/2; 1; x^2)$ diverges as $x \to 1$. The potential is convergent if, and only if, the coefficient of this hypergeometric function vanishes, and that happens when

$$q = -\frac{4\pi IMa}{r_0 - 2M} \tag{55}$$

Consequently, in view of Eq. (45), the zero-EMF solution with vanishing total charge has a finite potential.

In an inertial frame, Gauss' law implies that the electric field of a charged loop diverges, and so a loop can only have a finite potential if there exists an inertial frame in which the charge of the loop vanishes. The appropriate frame is the one with orthogonal basis vectors

$$e_t{}' = (1, 0, 0, 0) \tag{56}$$

$$e_\phi{}' = \left(\frac{-2Mar\sin^2\theta}{\Sigma - 2Mr}, 0, 0, 1\right) \tag{57}$$

This is the stationary frame, rather than the usual co-rotating frame of the Kerr black hole. We conclude that the zero-EMF solution with vanishing total charge also has vanishing local charge in the stationary frame.

V.   SPECIAL LIMITS

The charged ring solution is likely to be most useful for giving the electromagnitic fields in vacuum regions close to the hole. In these regions a dipole truncation of the electromagnetic field can be a used as an approximation to the full field. The solution is especially useful for locating the innermost stable circular orbits [4] and for examining the production of energetic particles by the Penrose process.

The effective ergosphere in which the Penrose process can take place extends far out along the equatorial plane with a relatively narrow height along the rotation axis [28]. Therefore, the fields in the equatorial plane can be used for studies of both stable orbits and the Penrose process. For a current loop around an extreme Kerr solution it is possible to obtain the vector potential in the equatorial plane in a useful closed form given below.

A.   Dipole and quadrupole truncations

Starting with the inner solution, the Coulomb and Dipole terms combine into

$$A_t = -\frac{aMBr}{\Sigma}\sin^2\theta + (Q - 2aMB)\left(\frac{1}{2M} - \frac{r}{\Sigma}\right) \tag{58}$$

$$A_\phi = \frac{BA}{\Sigma}\sin^2\theta + (Q - 2aMB)\frac{ar}{\Sigma}\sin^2\theta \tag{59}$$

where $A = (r^2 + a^2)^2 - \Delta a^2\sin^2\theta$ and

$$B = \frac{\alpha_1^i}{r_+ - M}. \tag{60}$$

The B-dependent terms in each field component represent the fields of a black hole immersed in a constant magnetic field of strenght B [25]. Fig. 2 shows the dipole approximation for the zero-EMF solution with an extremal hole and a loop of radius $6M$. The approximation agrees remarkably well for the A_t component and reasonably well for A_ϕ.

Dipole truncations of the inner and outer general solutions for the vector field do not match smoothly on the sphere at $r = r_0$. However, a combination of the inner and outer dipole truncations of the loop solution will at least give a reasonable approximation away from $r = r_0$. For large r, with zero net charge the leading terms in the vector potential are a combination of a magnetic dipole and a charge quadrupole,

$$A_t \sim \frac{d}{r^3}P_2(\cos\theta) \tag{61}$$

$$A_\phi \sim \frac{m}{r}\sin^2\theta \tag{62}$$

The dipole and quadrupole strenghts are

$$m = -\frac{2}{3}(r_+ - M)^2\beta_1^i, \quad d = \frac{2}{5}(r_+ - M)^3\beta_2^r - \frac{2}{3}a(r_+ - M)^2\beta_1^i. \tag{63}$$

For the zero-EMF solution, these are given by

$$m = \frac{\pi I r_0 \Delta(r_0)}{r_0 - 2M} \tag{64}$$

$$d = -\frac{\pi I a M \Delta(r_0)}{r_0 - 2M} \tag{65}$$

For large loops with $r_0 >> r_+$, these reduce to the ordinary magnetic dipole strength $m = I\pi r_0^2$ and an electric quadrupole moment $d = -\pi IaMr_0$.

B. Extreme Kerr limit in the equatorial plane

In the extremal Kerr limit the parameter $u \to \infty$ and the asymptotic forms of the Legendre functions in the general loop solution can be used. The mode summations can be done in a similar way to those of Sect. IV B, and the results expressed in terms of hypergeometric functions,

$$F_c(r) = F\left(\frac{1}{2}, \frac{1}{2}; c; \left(\frac{r-M}{r_0-M}\right)^2\right). \tag{66}$$

In the equatorial plane for $r < r_0$,

$$A_t = \left(1 - \frac{M}{r}\right)(\zeta(r_0)F_1(r) + \eta(r_0)F_2(r)) + \frac{2\pi IM^2}{r_0-2M}\frac{1}{r} - \frac{Q}{r}, \tag{67}$$

where

$$\zeta(r_0) = -\frac{2\pi IM^2 + q(r_0-2M)}{(r_0-M)^2}, \tag{68}$$

$$\eta(r_0) = \frac{2\pi IM^2(r_0+M) - qM^2}{2r_0(r_0-M)^2}. \tag{69}$$

The outer solution for $r > r_0$ is given by the same expression provided that values of the hypergeometric function on the branch cut are defined by taking the average of values just above and below the branch cut. The zero-EMF solutions with vanishing total charge can be obtained by fixing the black hole charge to $Q = -q$ and using Eq. (45),

$$A_t = \frac{\pi IM^2 F_2(r)}{(r_0-2M)(r_0-M)}\left(1 - \frac{M}{r}\right) - \frac{2\pi IM^2}{r_0-2M}\frac{1}{r}. \tag{70}$$

This is finite at the ring, as we discussed in the previous section. The simple form of the solution accounts for the high accuracy of the dipole approximation.

The axial part is not quite so simple. The axial vector potential is

$$A_\phi = \frac{C_1(r,r_0)}{r_0(r_0-M)^2}\frac{F_1(r)}{r} - \frac{C_2(r,r_0)}{r_0(r_0-M)^2}\left(1 - \frac{M}{r}\right)F_2(r) + \frac{(2\pi IM - q)M}{r_0 r} - \frac{QM}{r} + 2\pi IM \tag{71}$$

where

$$C_1(r) = 2\pi Ir_0(r_0-M)r^3 + \left(2\pi IM^2 - 2\pi Ir_0^2 + 4\pi IMr_0 - qM\right)Mr^2$$
$$+2\pi Ir_0^2(r_0-M)Mr - 4\pi Ir_0M^4 - q(r_0-2M)r_0M^2 \tag{72}$$
$$C_2(r) = 2\pi Ir_0(r_0-1)r^2 + \left(2\pi IM^2 - 2\pi Ir_0^2 + 4\pi IMr_0 - qM\right)Mr + 2\pi I(r_0+M)M^3 - qM^3 \tag{73}$$

C. Extreme Kerr with disks

The availability of a closed form solution for equatorial current loops around extreme Kerr black holes allows us to examine what happens with thin equatorial disks. To construct models of this general type we must impose some physical condition on the disc. A suitable choice would be for the electric field to vanish is some specially chosen rotating frame with angular speed $\omega(r)$,

$$A_{t,r} + \omega A_{\phi,r} = 0 \tag{74}$$

This allows us to solve for the charge density $q(r)$ given a current distribution $I(r)$. as an example, we shall examine the simplest case, $\omega = 0$, and find the charge distribution in a disk which has zero potential in the Boyer-Lindquist frame.

We can obtain the solution inside the disc by combining the solutions (67) for rings,

$$A_t = \left(1 - \frac{M}{r}\right) \int_{r_1}^{\infty} \left[\zeta(r')F_1(r) + \eta(r')F_2(r)\right] dr' + \frac{1}{r} \int_{r_1}^{\infty} \frac{2\pi I M^2}{r_0 - 2M} dr' - \frac{Q}{r}, \tag{75}$$

where r_1 is the inner edge of the ring. A zero potential solution can be obtained by having the first integral vanish and by fixing the black hole's charge

$$Q = \int_{r_1}^{\infty} \frac{2\pi I M^2}{r_0 - 2M} dr'. \tag{76}$$

In order to find out when the first integral vanishes, we use the identity $4F_1 = (xF_2)' + 2F_2$ to eliminate F_1. After integration by parts, this leads to two conditions,

$$\zeta(r_1) = 0, \quad 4\zeta(r) + 4\eta(r) + (r - M)\zeta(r)_{,r} = 0. \tag{77}$$

These can be regarded as equations which determine the charge density for a given current density, by substituting Eqs. (68) and (68). The first condition is the finite-potential condition,

$$q(r_1) = -\frac{4\pi M^2 I(r_1)}{r_1 - 2M}. \tag{78}$$

The second condition is a first order differential equation,

$$\left[r(r - 2M)^2 q\right]_{,r} + M^2(r - 2M)(4\pi r I)_{,r} = 0. \tag{79}$$

The solution is

$$q(r) = -\frac{4\pi M^2 I(r)}{r - 2M} + \frac{4\pi M^2}{r(r - 2M)^2} \int_{r_1}^{r} I(r')r' dr'. \tag{80}$$

This is only valid for $r_1 > 2M$. Unfortunately, this appears to be the only case of (74) which seems solvable analytically.

VI. CONCLUSION

The main result of this paper has been to correct the published solutions to Maxwell's equations for the electromagnetic vector potential near to a Kerr black hole surrounded by a charged current loop. The vector potential is a necesary ingredient for the determination of charged particle orbits close to a black hole immersed in a magnetic field. These orbits can be used to determine the inner edge of an accretion disc. They could also be used to extend recent work on energy flux generation using the Penrose pair creation $\gamma\gamma \to e^+ e^-$ and Penrose Compton scattering [29] to include magnetic fields.

Current loops can be superposed to form non-vacuum solutions [16]. In these situations the loops will be charged. For example, an equatorial conducting disk with angular speed ω_D would have a zero charge density in the disk frame, but a non-zero charge density in the Boyer-Lindquist coordinate frame. This can be modeled by a superposition of charged current loops in a similar way to the analysis in Sect V C. It may also be of interest to combine the fields due to a current along the rotation axis with the current rings to produce a toy model which has a current circulating out along the equatorial plane and back down the rotation axis.

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