Ark. Mat., **50** (2012), 269–278
DOI: 10.1007/s11512-011-0151-7

Blaschke condition and zero sets in weighted Dirichlet spaces

Dominique Guillot

Abstract. Let $D(\mu)$ be the Dirichlet space weighted by the Poisson integral of the positive measure μ. We give a characterization of the measures μ equal to a countable sum of atoms for which the Blaschke condition is a necessary and sufficient condition for a sequence to be a zero set for $D(\mu)$.

1. Introduction

The *Dirichlet space* D is the space of analytic functions on the unit disc $\mathbb{D}$ having a finite *Dirichlet integral*

$$D(f) := \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2 \, dA(z).$$

It is a subspace of H^2 that has been extensively studied, notably by Beurling and Carleson. Nevertheless, many natural questions about D are still open today. For example, there is still no complete characterization of the invariant subspaces of the shift operator $Sf(z)=zf(z)$ on D. In this paper, we are interested in another important question: the characterization of the zero sets of D. A sequence $\{z_n\}_{n=1}^{\infty}\subset\mathbb{D}$ is a *zero set* for D if there exists a function $f\in D$, $f\not\equiv 0$, such that $f(z_n)=0$. According to a theorem of Carleson [1], later improved by Shapiro and Shields [8], if

$$\sum_{n=1}^{\infty}\left(\log\frac{1}{1-r_n}\right)^{-1} < \infty$$

for some sequence of radii $\{r_n\}_{n=1}^{\infty}$, then $\{r_n e^{i\theta_n}\}_{n=1}^{\infty}$ is a zero set for D for any choice of arguments $\{\theta_n\}_{n=1}^{\infty}$. On the other hand, by a result of Nagel, Rudin and

Work supported by scholarships from NSERC (Canada) and FQRNT (Quebec).

Shapiro [5], if the series is divergent, there exists a sequence of arguments such that $\{r_n e^{i\theta_n}\}_{n=1}^{\infty}$ is not a zero set.

A complete characterization of the zero sets for D is still an open problem. Contrary to the H^2 case, to find such a characterization, one must take into consideration the arguments of the points z_n, making the problem much more difficult. For a review of the recent results, see [4] and the references cited therein.

In studying the invariant subspaces for the shift on D, Richter and Sundberg [6], [7], have introduced weighted versions of D. For a positive Borel measure μ on $\mathbb{T}$, the space $D(\mu)$ is the space of analytic functions on $\mathbb{D}$ with finite μ-Dirichlet integral

$$D_\mu(f) = \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2 P\mu(z)\, dA(z),$$

where $P\mu(z)$ is the Poisson integral of μ:

$$P\mu(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|e^{it}-z|^2}\, d\mu(e^{it}).$$

The space $D(\mu)$ is a Hilbert space with the norm $\|f\|_\mu^2 = \|f\|_2 + D_\mu(f)$. In particular, if $\mu=0$, we get the Hardy space H^2, and for $\mu=m$, we obtain the Dirichlet space D. The spaces $D(\mu)$ are interesting since they give intermediate examples of spaces between the well-known H^2 case and the difficult Dirichlet case. Moreover, the study of particular examples (like the case where μ is a Dirac measure) is often easier and gives good insights about what can happen in the most difficult cases.

In this paper, we study the zero sets of functions in $D(\mu)$ in the case where μ is a countable sum of Dirac measures $\mu = \sum_{n\ge 1} c_n \delta_{\zeta_n}$, where $\{\zeta_n\}_{n=1}^{\infty} \subset \mathbb{T}$. Recall that a sequence $\{z_n\}_{n=1}^{\infty}$ is a zero set for H^2 if and only if the sequence satisfies the *Blaschke condition* $\sum_{n\ge 1}(1-|z_n|) < \infty$. Since $D(\mu) \subset H^2$, the Blaschke condition is the minimum required for a sequence to be a zero set in $D(\mu)$. We give a characterization of the measures $\mu = \sum_{n\ge 1} c_n \delta_{\zeta_n}$ for which this condition is necessary and sufficient. We also prove that, given a sequence of points $\{\zeta_n\}_{n=1}^{\infty} \subset \mathbb{T}$, we can always find a way to distribute a mass on $\{\zeta_n\}_{n=1}^{\infty}$ such that the Blaschke condition is necessary and sufficient to get a zero sequence in $D(\mu)$.

2. The Blaschke condition

As we shall see later the existence of zero sequences in $D(\mu)$ is closely related to the existence of outer functions vanishing on subsets of $\mathbb{T}$.

Theorem 2.1. *Let μ be a positive finite Borel measure on $\mathbb{T}$. If there exists $F \in D(\mu)$ such that $F=0$ μ-a.e., then*

$$\int_{\mathbb{T}} \log V_2\mu(e^{it})\,dt < \infty,$$

where $V_2\mu$ is the Newtonian potential *associated with μ:*

$$V_2\mu(e^{it}) = \int_{\mathbb{T}} \frac{d\mu(\zeta)}{|e^{it}-\zeta|^2}.$$

Proof. Suppose $F=0$ μ-a.e. According to [6], Proposition 2.2,

$$D_\mu(F) = \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|F(e^{it})|^2}{|e^{it}-\zeta|^2}\frac{dt}{2\pi}\,d\mu(\zeta).$$

Thus, if $F \in D(\mu)$, by applying Fubini's theorem and Jensen's inequality

$$\begin{aligned}\infty > \log D_\mu(F) &\geq \int_{\mathbb{T}} \log\biggl(|F(e^{it})|^2 \int_{\mathbb{T}} \frac{d\mu(\zeta)}{|e^{it}-\zeta|^2}\biggr)\frac{dt}{2\pi} \\ &= \int_{\mathbb{T}} 2\log|F(e^{it})|\,\frac{dt}{2\pi} + \int_{\mathbb{T}} \log\biggl(\int_{\mathbb{T}} \frac{d\mu(\zeta)}{|e^{it}-\zeta|^2}\biggr)\frac{dt}{2\pi}.\end{aligned}$$

Since $F \in H^2$, it follows that $\log|F|$ is integrable and so

$$\int_{\mathbb{T}} \log\biggl(\int_{\mathbb{T}} \frac{d\mu(\zeta)}{|e^{it}-\zeta|^2}\biggr)dt < \infty.$$

This completes the proof. □

In the case where μ is a countable sum of Dirac measures, the preceding condition is also sufficient.

Theorem 2.2. *Let $\mu = \sum_{n=1}^{\infty} c_n\delta_{\zeta_n}$ with $\sum_{n=1}^{\infty} c_n < \infty$, $c_n \geq 0$ and $\zeta_n \in \mathbb{T}$. Then the following are equivalent*:

(1) *There exists $F \in D(\mu)$ such that $F=0$ μ-a.e.;*
(2) *There exists $F \in D(\mu)$ such that $\|zF\|_\mu = \|F\|_\mu$;*
(3) *We have*

$$\int_{\mathbb{T}} \log V_2\mu(e^{it})\,dt < \infty.$$

Proof. According to a result of Richter and Sundberg ([6], Corollary 2.3), for $F \in D(\mu)$,

$$\|zF\|_\mu^2 - \|F\|_\mu^2 = \int_{\mathbb{T}} |F|^2 \, d\mu.$$

This shows the equivalence between (1) and (2). We will prove that (1) is equivalent to (3). According to Theorem 2.1, we have (1) implies (3). Now assume that (3) is true. Define F by

$$F(z) = \exp\left(- \int_{\mathbb{T}} \frac{e^{it}+z}{e^{it}-z} \log V_2\mu(e^{it}) \frac{dt}{2\pi} \right).$$

According to (3), F is well defined. Moreover, remark that

$$\frac{1}{|F(e^{it})|} = V_2\mu(e^{it}) \geq \tfrac{1}{4}\mu(\mathbb{T}) \quad \text{a.e.}$$

As a consequence, $|F(e^{it})| \leq 4/\mu(\mathbb{T})$ a.e. and so $F \in H^\infty$. We claim that $F(\zeta_k)=0$ for each k. To prove this claim, recall that the Poisson integral of a measure ν has a radial limit equal to ∞ at e^{it} whenever the symmetric derivative of ν at e^{it},

$$D\nu(e^{it}) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} d\mu(e^{it})$$

exists and is equal to ∞. Notice that $|F(z)| = \exp(-P(\log V_2\mu))(z)$. Therefore, to prove that $F(\zeta_k)=0$, it suffices to show that the symmetric derivative of $\log V_2\mu$ at ζ_k is ∞. Let $\varepsilon>0$ and let $\zeta_k = e^{i\alpha}$. Then

$$\begin{aligned}
\frac{1}{2\varepsilon} \int_{\alpha-\varepsilon}^{\alpha+\varepsilon} \log V_2\mu(e^{it}) \, dt &\geq \frac{1}{2\varepsilon} \int_{\alpha-\varepsilon}^{\alpha+\varepsilon} \log\left(\frac{c_k}{|e^{it}-\zeta_k|^2} \right) dt \\
&\geq \frac{C}{\varepsilon} \int_0^\varepsilon \log\left(\frac{c_k}{t^2} \right) dt \\
&= C \log c_k - \frac{2C}{\varepsilon} \int_0^\varepsilon \log t \, dt \\
&= C \log c_k - \frac{2C}{\varepsilon} (\varepsilon \log \varepsilon - \varepsilon) \\
&= C \log c_k + 2C - 2C \log \varepsilon,
\end{aligned}$$

where C is some positive constant. By letting $\varepsilon \to 0^+$, we obtain $D \log V_2\mu(\zeta_k) = \infty$ proving the claim.

Let us compute $D_\mu(F)$. We have

$$D_\mu(F)=\iint_{\mathbb{T}}\frac{|F(e^{it})|^2}{|e^{it}-\zeta|^2}\,\frac{dt}{2\pi}\,d\mu(\zeta)=\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{1}{|e^{it}-\zeta|^2}\,d\mu(\zeta)|F(e^{it})|^2\,\frac{dt}{2\pi}$$
$$=\int_{\mathbb{T}}|F(e^{it})|\,\frac{dt}{2\pi}\leq\int_{\mathbb{T}}\frac{4}{\mu(\mathbb{T})}\,\frac{dt}{2\pi}<\infty.$$

Hence $F\in D(\mu)$ and $F(\zeta_k)=0$ for every k. □

Lemma 2.3. *Let $\{\zeta_n\}_{n=1}^\infty\subset\mathbb{T}$. Then there exists a sequence $\{z_n\}_{n=1}^\infty\subset\mathbb{D}$ such that $\sum_{n=1}^\infty(1-|z_n|)<\infty$ and*

$$\sum_{n=1}^\infty\frac{1-|z_n|^2}{|\zeta_k-z_n|^2}=\infty$$

for every k.

Proof. We can easily construct such a sequence. For example, we may let $r_{n,k}:=1-1/n^2k^2$ and consider the points $\{r_{n,k}\zeta_k\}_{n,k=1}^\infty$. Let $\{z_m\}_{m=1}^\infty$ be an enumeration of the points of this set. For each k, we have

$$\sum_{m=1}^\infty\frac{1-|z_m|^2}{|\zeta_k-z_m|^2}=\infty,$$

since an infinite number of the points of the sequence $\{z_m\}_{m=1}^\infty$ are on the ray from the origin to ζ_k and

$$\frac{1-|r\zeta_k|^2}{|\zeta_k-r\zeta_k|^2}\to\infty$$

as $r\to1^-$. Moreover,

$$\sum_{m=1}^\infty(1-|z_m|)=\sum_{n=1}^\infty\sum_{k=1}^\infty\frac{1}{n^2}\,\frac{1}{k^2}<\infty.$$

Therefore, the sequence $\{z_m\}_{m=1}^\infty$ satisfies the conditions of the lemma. □

Theorem 2.4. *Let $\mu=\sum_{n=1}^\infty c_n\delta_{\zeta_n}$ where $c_n\geq0$, $\sum_{n=1}^\infty c_n<\infty$ and $\zeta_n\in\mathbb{T}$. The following are equivalent*:

(1) *For each Blaschke sequence $\{z_n\}_{n=1}^\infty$, there exists $f\in D(\mu)$ and $f\not\equiv0$, such that $f(z_n)=0$ for every n;*

(2) *We have*

$$\int_{\mathbb{T}}\log V_2\mu(e^{it})\,dt<\infty.$$

Proof. According to Lemma 2.3, there exists a Blaschke sequence such that

$$\sum_{n=1}^{\infty} \frac{1-|z_n|^2}{|\zeta_k - z_n|^2} = \infty$$

for every k, and, according to the hypothesis of the theorem, there exists $f \in D(\mu)$ such that $f(z_n)=0$ for every n. We can decompose f as $f=BSF$, where B is the Blaschke product associated with the sequence $\{z_n\}_{n=1}^{\infty}$, S is a singular inner function and F is an outer function. According to a formula of Carleson [3] generalized by Richter and Sundberg [6], Theorem 3.1, $BF \in D(\mu)$ and so, without loss of generality, we can assume that $S=1$. Also,

$$D_\mu(BF) = D_\mu(F) + \sum_{k=1}^{\infty} c_k |F(\zeta_k)|^2 \sum_{n=1}^{\infty} \frac{1-|z_n|^2}{|\zeta_k - z_n|^2} < \infty.$$

Since the right-hand sum is infinite for every k, we must have $F(\zeta_k)=0$ for every k. So $F=0$ μ-a.e. It follows from Theorem 2.2 that

$$\int_{\mathbb{T}} \log V_2\mu(e^{it})\, dt < \infty.$$

Conversely, suppose that the preceding integral is convergent and let $\{z_n\}_{n=1}^{\infty}$ be a Blaschke sequence. Then there exists $F \in D(\mu)$ such that $F=0$ μ-a.e. If B is the Blaschke product associated with the sequence $\{z_n\}_{n=1}^{\infty}$, then $D_\mu(BF)=D_\mu(F)$. So $f=BF \in D(\mu)$ and $f(z_n)=0$ for every n. □

3. Carleson's condition is not necessary

Definition 3.1. A closed set $E \subset \mathbb{T}$ is a *Carleson set* if

$$\int_{\mathbb{T}} \log\left(\frac{1}{d(e^{it}, E)}\right) dt < \infty.$$

If E is a Carleson set, then according to a classical result of Carleson ([2], Theorem 1), there exists a holomorphic function F such that $F=0$ on E and F is continuously differentiable on $\mathbb{T}$. As a consequence, if $\{\zeta_n\}_{n=1}^{\infty}$ is Carleson, there exists $F \in D(\mu)$ such that $F(\zeta_n)=0$ for every n. Theorem 2.2 gives another proof of this result, as follows.

Corollary 3.2. *Let* $\mu = \sum_{n=1}^{\infty} c_n \delta_{\zeta_n}$, *where* $c_n \geq 0$, $\sum_{n=1}^{\infty} c_n < \infty$ *and* $\zeta_n \in \mathbb{T}$. *Assume that* $E := \{\zeta_n\}_{n=1}^{\infty}$ *is a Carleson set. Then there exists* $F \in D(\mu)$ *such that* $F(\zeta_n)=0$ *for every* n.

Proof. It suffices to remark that

$$\int_{\mathbb{T}} \log V_2\mu(e^{it})\,dt = \int_{\mathbb{T}} \log\left(\int_{\mathbb{T}} \frac{1}{|e^{it}-\zeta|^2}\,d\mu(\zeta)\right) dt \le \int_{\mathbb{T}} \log \frac{\mu(\mathbb{T})}{d(e^{it},E)^2}\,dt. \qquad \square$$

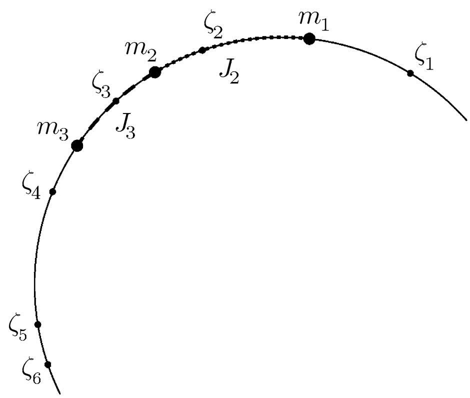
Therefore, if $E=\{\zeta_n\}_{n=1}^{\infty}$ is a Carleson set, then every Blaschke sequence is a zero sequence in $D(\mu)$ if $\mu=\sum_{n\ge1} c_n\delta_{\zeta_n}$ for a summable sequence $\{c_n\}_{n=1}^{\infty}$. The following theorem shows that we can give a mass to almost any sequence $\{\zeta_n\}_{n=1}^{\infty}$ in such a way that the Blaschke condition becomes a necessary and sufficient condition to have a zero sequence. Therefore, Carleson's condition is not necessary.

Theorem 3.3. *Let $E:=\{\zeta_n\}_{n=1}^{\infty}\subset\mathbb{T}$. There exists a sequence $\{c_n\}_{n=1}^{\infty}$ such that $c_n\ge0$, $\sum_{n=1}^{\infty} c_n<\infty$ and $c_n>0$ for an infinite number of n and*

$$\int_{\mathbb{T}} \log V_2\mu(e^{it})\,dt<\infty,$$

where $\mu=\sum_{n=1}^{\infty} c_n\delta_{\zeta_n}$. As a consequence, for this measure μ, the Blaschke condition is necessary and sufficient to have a zero sequence in $D(\mu)$.

Proof. Set $\zeta_n:=e^{i\alpha_n}$ and suppose, without loss of generality, that $\zeta_n\to\zeta$ and that $0<\alpha_1<\alpha_2<\alpha_3<...<2\pi$ (if this is not the case, extract such a sequence and let $c_n=0$ for the other points of E). Denote by m_j the middle point of the arc (ζ_j,ζ_{j+1}). Let m_0 be the middle point of the arc contained between ζ and ζ_1 containing no other point of E. Denote by J_j the arc (m_{j-1},m_j), $j=1,2,...$, and let $J:=\bigcup_{j=1}^{\infty} J_j$ (see figure below).



Moreover, denote by $d_j:=d(J_j,E\setminus\{\zeta_j\})$, the distance between the arc J_j and the other points of E. Consider the sequence $c_n:=d_n^2/2^n$.

We will show that this sequence satisfies the hypothesis of the theorem. First, we have $\sum_{n=1}^{\infty} d_n < \infty$ since $d_n = \min(d(m_n, \zeta_n), d(\zeta_n, m_{n-1})) \leq |J_n|$ and $\sum_{n=1}^{\infty} |J_n| \leq 2\pi$ since the arcs J_j are disjoint. Therefore, $\sum_{n=1}^{\infty} c_n < \infty$. Next, to estimate the integral, remark that

$$\int_{\mathbb{T}\setminus J} \log V_2\mu(e^{it})\,dt \leq C + \int_{\mathbb{T}\cap J} \log V_2\mu(e^{it})\,dt,$$

for a constant $C>0$, since the integral over $B(\zeta,\varepsilon)\cap J$ is bigger than the integral over $B(\zeta,\varepsilon)\cap J^c$. Therefore, it is sufficient to bound the integral over J:

$$\begin{aligned}\int_J \log V_2\mu(e^{it})\,dt &= \int_J \log\biggl(\sum_{k=1}^{\infty} \frac{c_k}{|e^{it}-\zeta_k|^2}\biggr)\,dt = \int_J \log\biggl(\sum_{k=1}^{\infty} \frac{d_k^2/2^k}{|e^{it}-\zeta_k|^2}\biggr)\,dt \\ &\leq \sum_{n=1}^{\infty} \int_{J_n} \log\Biggl[\sum_{\substack{k=1\\k\neq n}}^{\infty} \biggl(\frac{1}{2^k}\biggr) + \frac{d_n^2/2^n}{|e^{it}-\zeta_n|^2}\Biggr]\,dt,\end{aligned}$$

because $|e^{it}-\zeta_k| \geq d_k$ if $e^{it} \in J_n$ and $k \neq n$. Hence,

$$\int_J \log V_2\mu(e^{it})\,dt \leq \sum_{n=1}^{\infty} \int_{J_n} \log\biggl(1 + \frac{d_n^2/2^n}{|e^{it}-\zeta_n|^2}\biggr)\,dt \leq \sum_{n=1}^{\infty} \int_{J_n} \log\biggl(1 + \frac{d_n^2}{|e^{it}-\zeta_n|^2}\biggr)\,dt.$$

Set $J_n := (\alpha_n - a_n, \alpha_n + b_n)$ with $a_n, b_n \in [0, 2\pi)$. Then

$$\begin{aligned}\int_J \log V_2\mu(e^{it})\,dt &\asymp \sum_{n=1}^{\infty} \int_{\alpha_n - a_n}^{\alpha_n + b_n} \log\biggl(1 + \frac{d_n^2}{(t-\alpha_n)^2}\biggr)\,dt \\ &= \sum_{n=1}^{\infty} \int_{-a_n}^{b_n} \log\biggl(1 + \frac{d_n^2}{t^2}\biggr)\,dt \\ &= \sum_{n=1}^{\infty} \biggl[t \log\biggl(1 + \frac{d_n^2}{t^2}\biggr) + 2d_n \arctan\biggl(\frac{t}{d_n}\biggr)\biggr]_{-a_n}^{b_n} \\ &\leq \sum_{n=1}^{\infty} 8\pi d_n + \sum_{n=1}^{\infty} \log\biggl[\biggl(1 + \frac{d_n^2}{b_n^2}\biggr)^{b_n} \biggl(1 + \frac{d_n^2}{a_n^2}\biggr)^{a_n}\biggr].\end{aligned}$$

Note that $d_n = \min(a_n, b_n)$, as m_n is the middle point of the arc (ζ_n, ζ_{n+1}). Thus,

$$\int_J \log V_2\mu(e^{it})\,dt \leq \sum_{n=1}^{\infty} 8\pi d_n + \sum_{n=1}^{\infty} (a_n + b_n) \log 2.$$

The first sum is convergent since $\sum_{n=1}^{\infty} d_n<\infty$. For the second sum, notice that $a_n+b_n=|J_n|$. As the J_n are disjoint, it follows that $\sum_{n=1}^{\infty}|J_n|\le 2\pi$. This completes the proof. □

In particular, let $E:=\{\zeta_n\}_{n=1}^{\infty}$ be a monotone sequence such that $\zeta_n\to\zeta$ and such that $\overline{E}$ is not a Carleson set. According to the theorem, we can choose masses $c_n>0$ such that the Blaschke sequence is necessary and sufficient to have a zero sequence in $D(\mu)$ with $\mu=\sum_{n=1}^{\infty} c_n\delta_{\zeta_n}$. So we have the following corollary.

Corollary 3.4. *There exists a sequence $E:=\{\zeta_n\}_{n=1}^{\infty}$ and a sequence $\{c_n\}_{n=1}^{\infty}$ such that $\overline{E}$ is not a Carleson set, $c_n>0$ for every n, $\sum_{n=1}^{\infty} c_n<\infty$, and if $\mu=\sum_{n=1}^{\infty} c_n\delta_{\zeta_n}$, then the Blaschke condition is a necessary and sufficient condition for having a zero sequence in $D(\mu)$.*

Acknowledgements. This work is part of the author's doctoral dissertation. The author would like to thank Professor Thomas Ransford for having read the paper and for his invaluable remarks and suggestions.

References

1. Carleson, L., On the zeros of functions with bounded Dirichlet integrals, *Math. Z.* **56** (1952), 289–295.
2. Carleson, L., Sets of uniqueness for functions regular in the unit circle, *Acta Math.* **87** (1952), 325–345.
3. Carleson, L., A representation formula for the Dirichlet integral, *Math. Z.* **73** (1960), 190–196.
4. Mashreghi, J. and Shabankhah, M., Zero sets and uniqueness sets with one cluster point for the Dirichlet space, *J. Math. Anal. Appl.* **357** (2009), 498–503.
5. Nagel, A., Rudin, W. and Shapiro, J., Tangential boundary behavior of functions in Dirichlet-type spaces, *Ann. of Math.* **116** (1982), 331–360.
6. Richter, S. and Sundberg, C., A formula for the local Dirichlet integral, *Michigan Math. J.* **38** (1991), 355–379.
7. Richter, S. and Sundberg, C., Multipliers and invariant subspaces in the Dirichlet space, *J. Operator Theory* **28** (1992), 167–186.

8. Shapiro, H. S. and Shields, A. L., On the zeros of functions with finite Dirichlet integral and some related function spaces, *Math. Z.* **80** (1962), 217–299.

Dominique Guillot
Département de mathématiques et de statistique
Université Laval
Québec, QC G1V 0A6
Canada
dominique.guillot.1@ulaval.ca

Received April 26, 2010
published online August 27, 2011