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BLASCHKE PRODUCTS AND RATIONAL FUNCTIONS WITH SIEGEL DISKS

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Abstract. Let m be a positive integer. We show that for any given real number $\alpha \in [0,1]$ and complex number μ with $|\mu| \leq 1$ which satisfy $e^{2\pi i\alpha}\mu^m \neq 1$, there exists a Blaschke product B of degree $2m+1$ which has a fixed point of multiplier μ^m at the point at infinity such that the restriction of the Blaschke product B on the unit circle is a critical circle map with rotation number α. Moreover if the given real number α is irrational of bounded type, then a modified Blaschke product of B is quasiconformally conjugate to some rational function of degree $m+1$ which has a fixed point of multiplier μ^m at the point at infinity and a Siegel disk whose boundary is a quasicircle containing its critical point.

1. Introduction

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function of degree $d \geq 2$. In the theory of the complex dynamics, there are two important sets called the *Fatou set* and the *Julia set.* The Fatou set $F(f)$ is the set of normality in the sense of Montel for the family $\{f^n\}_{n=1}^{\infty}$, where $f^n = f \circ \cdots \circ f$ is n iterates of f. The Julia set $J(f)$ is the complement $\widehat{\mathbb{C}} \setminus F(f)$. A solution z_0 of the equation $f(z) = z$ is called a *fixed point* of f and $\lambda = f'(z_0)$ is called the *multiplier* of z_0 if $z_0 \in \mathbb{C}$. The multiplier of $z_0 = \infty$ is defined as the multiplier of the origin for $\psi \circ f \circ \psi^{-1}$, where $\psi(z) = 1/z$. The fixed point z_0 is attracting, repelling or indifferent if its multiplier λ satisfies that $|\lambda| < 1$, $|\lambda| > 1$ or $|\lambda| = 1$ respectively. Attracting fixed points belong to the Fatou set and repelling fixed points belong to the Julia set. In the case that z_0 is indifferent, the classification is more complicated. The fixed point z_0 is parabolic, a Siegel point or a Cremer point if its multiplier is a root of unity, $z_0 \in F(f)$ or $z_0 \in J(f)$ respectively. Parabolic fixed points belong to the Julia set. The Fatou component containing a Siegel point is called a *Siegel disk* centered at z_0. Non-repelling fixed points "capture" at least one critical point of f, which is a solution of the equation $f'(z) = 0$.

In this paper, we investigate rational functions with Siegel disks. Let f : $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function of degree $d \geq 2$ with a fixed point of multiplier

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$e^{2\pi i\alpha}$ at the origin, where $\alpha \in [0,1]$ is irrational. If the origin is a Siegel point, then there exists a local holomorphic change of coordinate $\Phi : \mathbb{D} \to \mathbb{C}$ with $0 = \Phi(0)$ such that $\Phi^{-1} \circ f \circ \Phi(z) = e^{2\pi i\alpha}z$, where $\mathbb{D}$ is the unit disk. The Siegel disk Δ centered at the origin contains $\Phi(\mathbb{D})$.

For the irrational number α, we consider the continued fraction expansion

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

of α and then a sequence of rational numbers

$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$$

converges to α, where a_n is a positive integer uniquely determined by α for all $n \in \mathbb{N}$. The irrational number α is a *Diophantine number of order* $\kappa \geq 2$ if there exists $\varepsilon > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| > \frac{\varepsilon}{q^\kappa}$$

for all rational numbers p/q. The class of Diophantine numbers of order κ is denoted by $\mathcal{D}_\kappa$. The irrational number α belongs to $\mathcal{D}_\kappa$ if and only if the sequence

$$\left\{\frac{q_{n+1}}{q_n^{\kappa-1}}\right\}_{n=1}^\infty$$

is bounded. In the case that $\kappa = 2$, the sequence $\{a_n\}_{n=1}^\infty$ is bounded if and only if $\{q_{n+1}/q_n\}_{n=1}^\infty$ is bounded. Therefore Diophantine numbers of order 2 are said to be of *bounded type.* The irrational number α is a *Bryuno number* if the sum

$$\sum_{n=1}^\infty \frac{\log q_{n+1}}{q_n}$$

converges. The class of Bryuno numbers is denoted by $\mathcal{B}$. Note that for $\kappa > 2$, $\mathcal{D}_2 \subsetneq \mathcal{D}_\kappa \subsetneq \mathcal{B}$ and $\mathcal{D}_\kappa$ has full measure on $\mathbb{R}/\mathbb{Z}$ (see [7] or [11]). Bryuno showed that if α is a Bryuno number, then f is linearizable at the origin. Yoccoz showed that if a quadratic polynomial $P_\alpha(z) = z^2 + e^{2\pi i\alpha}z$ is linearizable at the origin, then α is a Bryuno number, that is, P_α is linearizable at the origin if and only if α is a Bryuno number. Moreover the following theorem holds if α is of bounded type (see [10] or [11]).

Theorem 1.1 (Ghys-Douady-Herman-Shishikura-Świątek). *If an irrational number $\alpha \in [0,1]$ is of bounded type, then the boundary of the Siegel disk Δ of P_α centered at the origin is a quasicircle containing its critical point $-e^{2\pi i\alpha}/2$.*

Moreover if the irrational number α is of bounded type, then the following holds:

(a) (Petersen). The Julia set $J(P_\alpha)$ of P_α is locally connected and has measure zero.
(b) (McMullen). The Hausdorff dimension of $J(P_\alpha)$ is less than 2.
(c) (Graczyk-Jones). The Hausdorff dimension of $\partial\Delta$ is greater than 1.

Conversely, Petersen showed that if $\partial\Delta$ is a quasicircle containing the finite critical point $-e^{2\pi i\alpha}/2$ of P_α, then $\alpha \in [0,1]$ is of bounded type. Zakeri extended Theorem 1.1 to the case of cubic polynomials.

Theorem 1.2 ([12]). *Let P be a cubic polynomial with fixed point of multiplier $e^{2\pi i\alpha}$ at the origin. If an irrational number $\alpha \in [0,1]$ is of bounded type, then the boundary of the Siegel disk Δ of P centered at the origin is a quasicircle containing one or both critical points.*

Geyer showed the following theorem which is extended to some polynomials. Let $Q_m(z) = e^{2\pi i\alpha}z(1+z/m)^m$. Note that P_α is conformally conjugate to Q_1.

Theorem 1.3 ([4]). *Let $m \geq 1$ be a positive integer. If an irrational number $\alpha \in [0,1]$ is of bounded type, then the boundary of the Siegel disk Δ of Q_m centered at the origin is a quasicircle containing its critical point $-m/(m+1)$.*

Let $F_{\lambda,\mu}(z) = z(z+\lambda)/(\mu z+1)$ with $\lambda\mu \neq 1$. The origin and the point at infinity are fixed points of $F_{\lambda,\mu}$ of multiplier λ and μ respectively. In the case that $\mu = 0$, $F_{\lambda,0}(z) = \lambda z + z^2$. Therefore the quadratic rational function $F_{\lambda,\mu}$ is considered as a perturbation of the quadratic polynomial $z \mapsto \lambda z + z^2$. In the case that $\lambda = e^{2\pi i\alpha}$ and α is irrational of bounded type, the author showed the following theorem which is a generalization of Theorem 1.1.

Theorem 1.4 ([5]). *If an irrational number $\alpha \in [0,1]$ is of bounded type, $\lambda = e^{2\pi i\alpha}$ and $\mu \in \overline{\mathbb{D}}$ with $\lambda\mu \neq 1$, then the boundary of the Siegel disk Δ of $F_{\lambda,\mu}$ centered at the origin is a quasicircle containing its critical point.*

For complex numbers λ and μ with $\lambda\mu \neq 1$ and a positive integer m, let

$$F_{\lambda,\mu,m}(z) = z\left(\frac{z+\lambda}{\mu z+1}\right)^m .$$

Note that $F_{\lambda,\mu,1} = F_{\lambda,\mu}$. The origin and the point at infinity are fixed points of $F_{\lambda,\mu,m}$ of multiplier λ^m and μ^m respectively. In the case that $\mu = 0$,

$$F_{\lambda,0,m}(z) = z\,(z+\lambda)^m .$$

Therefore the rational function $F_{\lambda,\mu,m}$ of degree $m+1$ is considered as a perturbation of the polynomial $F_{\lambda,0,m}$ of degree $m+1$. Note that $F_{\lambda,0,m}$ is conformally conjugate to Q_m if $\lambda^m = e^{2\pi i\alpha}$. We show the following theorem which contains Theorem 1.4.

Theorem 1.5. *Let $m \geq 1$ be a positive integer and let $\mu \in \overline{\mathbb{D}}$. If an irrational number $\alpha \in [0,1]$ is of bounded type and $e^{2\pi i\alpha}\mu^m \neq 1$, then there exist suitable pairs $\{(\lambda_j, \mu_j)\}_{j=1}^m$ with*

(i) $\lambda_j^m = e^{2\pi i\alpha}$, $\mu_j^m = \mu^m$ *and* $\lambda_j\mu_j \neq 1$ *for* $j \in \{1, \ldots, m\}$

(ii) $\lambda_j \neq \lambda_k$ *if* $j \neq k$

such that for each $j \in \{1, \ldots, m\}$, the boundary of the Siegel disk Δ of $F_{\lambda_j,\mu_j,m}$ centered at the origin is a quasicircle containing its critical point.

Theorem 1.5	
$m = 1,\ \mu = 0$	Theorem 1.1
$m = 1$	Theorem 1.4
$\mu = 0$	Theorem 1.3

TABLE 1. Special cases of Theorem 1.5

Theorem 1.5 contains Theorems 1.1, 1.3, and 1.4. Moreover we obtain the following corollary.

Corollary 1.6. *Let $m \geq 1$ be a positive integer, $\alpha \in [0,1]$ be an irrational number of bounded type, $\mu^m = e^{2\pi i\beta}$ with $e^{2\pi i\alpha}\mu^m \neq 1$ and $\{(\lambda_j, \mu_j)\}_{j=1}^m$ be as in Theorem 1.5. If $\beta \in [0,1]$ is an irrational number of bounded type, then the boundaries of Siegel disks Δ and Δ_∞ of $F_{\lambda_j,\mu_j,m}$ centered at the origin and the point at infinity respectively are quasicircles containing its critical point.*

2. Blaschke products with a critical point on the unit circle

2.1. Existence of Blaschke products

Let $m \geq 1$ be a positive integer. We consider a Blaschke product

$$B(z) = e^{2\pi i m\theta} z \left(\frac{z-a}{1-\bar{a}z}\right)^m \left(\frac{z-b}{1-\bar{b}z}\right)^m$$

of degree $2m+1$ with $a\bar{b} \neq 1$ and $0 < |a| \leq |b| < \infty$. Let $\lambda = abe^{2\pi i\theta}$ and let $\mu = \bar{a}\bar{b}e^{-2\pi i\theta}$. The derivative B' of B is

$$B'(z) = \frac{e^{2\pi i m\theta}}{(1-\bar{a}z)^2(1-\bar{b}z)^2} \left(\frac{z-a}{1-\bar{a}z}\right)^{m-1} \left(\frac{z-b}{1-\bar{b}z}\right)^{m-1} g(z),$$

where

$$\begin{aligned} g(z) = \bar{a}\bar{b}z^4 &+ \left\{-(m+1)(\bar{a}+\bar{b}) + (m-1)\bar{a}\bar{b}(a+b)\right\}z^3 \\ &+ \left\{2m+1-(2m-1)|ab|^2 + |a+b|^2\right\}z^2 \\ &+ \left\{-(m+1)(a+b) + (m-1)ab(\bar{a}+\bar{b})\right\}z + ab. \end{aligned}$$

So multipliers of fixed points $z = 0$ and $z = \infty$ are λ^m and μ^m respectively. Let c_1, c_2, $c_3 = 1/\bar{c}_2$ and $c_4 = 1/\bar{c}_1$ be solutions of the equation $g(z) = 0$. Therefore critical points of B are a, $1/\bar{a}$, b, $1/\bar{b}$ c_1, c_2, c_3 and c_4 and multiplicities of critical points a, $1/\bar{a}$, b and $1/\bar{b}$ are $m-1$. Since c_1, c_2, c_3 and c_4 are solutions of $g(z) = 0$, we obtain that

$$\begin{aligned} g(z) &= \bar{a}\bar{b}(z-c_1)(z-c_2)(z-c_3)(z-c_4) \\ &= \bar{a}\bar{b}\Big\{z^4 - C_3 z^3 + C_2 z^2 - C_1 z + C_0\Big\}, \end{aligned}$$

where

$$\begin{aligned} C_3 &= c_1 + \frac{1}{\bar{c}_1} + c_2 + \frac{1}{\bar{c}_2}, \\ C_2 &= \frac{c_1}{\bar{c}_1} + \frac{c_2}{\bar{c}_2} + \left(c_1 + \frac{1}{\bar{c}_1}\right)\left(c_2 + \frac{1}{\bar{c}_2}\right), \\ C_1 &= \frac{c_1}{\bar{c}_1}\left(c_2 + \frac{1}{\bar{c}_2}\right) + \frac{c_2}{\bar{c}_2}\left(c_1 + \frac{1}{\bar{c}_1}\right), \\ C_0 &= \frac{c_1 c_2}{\bar{c}_1 \bar{c}_2}. \end{aligned}$$

Comparing coefficients of two representations of $g(z)$ implies that

$$c_1 + \frac{1}{\bar{c}_1} + c_2 + \frac{1}{\bar{c}_2} = \frac{(m+1)(\bar{a}+\bar{b}) - (m-1)(a+b)\bar{a}\bar{b}}{\bar{a}\bar{b}}, \tag{1}$$

$$\frac{c_1}{\bar{c}_1} + \frac{c_2}{\bar{c}_2} + \left(c_1 + \frac{1}{\bar{c}_1}\right)\left(c_2 + \frac{1}{\bar{c}_2}\right) = \frac{2m+1-(2m-1)|ab|^2 + |a+b|^2}{\bar{a}\bar{b}}, \tag{2}$$

$$\frac{c_1}{\bar{c}_1}\left(c_2 + \frac{1}{\bar{c}_2}\right) + \frac{c_2}{\bar{c}_2}\left(c_1 + \frac{1}{\bar{c}_1}\right) = \frac{(m+1)(a+b) - (m-1)(\bar{a}+\bar{b})ab}{\bar{a}\bar{b}}, \tag{3}$$

$$\frac{c_1 c_2}{\bar{c}_1 \bar{c}_2} = \frac{ab}{\bar{a}\bar{b}}. \tag{4}$$

Eliminating c_1 and $\bar{c}_1$ from equations (1), (2), and (4) gives that

$$\begin{aligned} |a+b|^2 &- (m+1)\left(c_2 + \frac{1}{\bar{c}_2}\right)(\bar{a}+\bar{b}) - \left(\frac{\bar{c}_2}{c_2}\right)ab \\ &+ \left\{\left(c_2 + \frac{1}{\bar{c}_2}\right)^2 - \frac{c_2}{\bar{c}_2}\right\}\bar{a}\bar{b} + (m-1)\left(c_2 + \frac{1}{\bar{c}_2}\right)(a+b)\bar{a}\bar{b} \\ &+ 2m+1-(2m-1)|ab|^2 = 0 \end{aligned} \tag{5}$$

and eliminating c_1 and $\bar{c}_1$ from equations (1), (3), and (4) gives that

$$(6)\quad \begin{aligned}&\frac{\bar{c}_2}{c_2}\left(c_2+\frac{1}{\bar{c}_2}\right)ab+(m+1)\left(\frac{c_2}{\bar{c}_2}\right)(\bar{a}+\bar{b})-(m-1)\left(\frac{c_2}{\bar{c}_2}\right)(a+b)\bar{a}\bar{b}\\ &=\frac{c_2}{\bar{c}_2}\left(c_2+\frac{1}{\bar{c}_2}\right)\bar{a}\bar{b}+(m+1)(a+b)-(m-1)(\bar{a}+\bar{b})ab.\end{aligned}$$

We obtain that

$$(7)\quad \begin{aligned}&|a+b|^2-2(m+1)e^{2\pi i\varphi}(\bar{a}+\bar{b})-e^{2\pi i(-2\varphi)}ab+3e^{2\pi i\cdot 2\varphi}\bar{a}\bar{b}\\ &+2(m-1)e^{2\pi i\varphi}(a+b)\bar{a}\bar{b}+2m+1-(2m-1)|ab|^2=0\end{aligned}$$

and

$$(8)\quad \begin{aligned}&e^{2\pi i(-2\varphi)}ab+\frac{m+1}{2}e^{2\pi i\varphi}(\bar{a}+\bar{b})-\frac{m-1}{2}e^{2\pi i\varphi}(a+b)\bar{a}\bar{b}\\ &=e^{2\pi i\cdot 2\varphi}\bar{a}\bar{b}+\frac{m+1}{2}e^{2\pi i(-\varphi)}(a+b)-\frac{m-1}{2}e^{2\pi i(-\varphi)}(\bar{a}+\bar{b})ab\end{aligned}$$

by substituting $c_2 = e^{2\pi i\varphi}$ into equations (5) and (6). Eliminating ab from equations (7) and (8) gives that

$$(9)\quad \begin{aligned}&|a+b|^2-\frac{3}{2}(m+1)e^{2\pi i\varphi}(\bar{a}+\bar{b})\\ &-\frac{m+1}{2}e^{2\pi i(-\varphi)}(a+b)+2e^{2\pi i\cdot 2\varphi}\bar{a}\bar{b}+\frac{m-1}{2}e^{2\pi i(-\varphi)}(\bar{a}+\bar{b})ab\\ &+\frac{3}{2}(m-1)e^{2\pi i\varphi}(a+b)\bar{a}\bar{b}+2m+1-(2m-1)|ab|^2=0.\end{aligned}$$

Let $\zeta = a+b$. Then

$$(10)\quad \begin{aligned}&|\zeta|^2-\frac{3}{2}(m+1)e^{2\pi i\varphi}\bar{\zeta}-\frac{m+1}{2}e^{2\pi i(-\varphi)}\zeta+2e^{2\pi i\cdot 2\varphi}\bar{a}\bar{b}+\frac{m-1}{2}e^{2\pi i(-\varphi)}ab\bar{\zeta}\\ &+\frac{3}{2}(m-1)e^{2\pi i\varphi}\bar{a}\bar{b}\zeta+2m+1-(2m-1)|ab|^2=0.\end{aligned}$$

The real part of the left side of the equation (10) is

$$(11)\quad \begin{aligned}&x^2+y^2-2x\Big\{(m+1)\cos 2\pi\varphi-(m-1)r\cos 2\pi(\varphi+\theta+\omega)\Big\}\\ &-2y\Big\{(m+1)\sin 2\pi\varphi+(m-1)r\sin 2\pi(\varphi+\theta+\omega)\Big\}\\ &+2r\cos 2\pi(2\varphi+\theta+\omega)+2m+1-(2m-1)r^2=0\end{aligned}$$

and the imaginary part of the left side of the equation (10) is

$$(12)\quad \begin{aligned}&y\Big\{(m+1)\cos 2\pi\varphi+(m-1)r\cos 2\pi(\varphi+\theta+\omega)\Big\}\\ &-x\Big\{(m+1)\sin 2\pi\varphi-(m-1)r\sin 2\pi(\varphi+\theta+\omega)\Big\}\\ &+2r\sin 2\pi(2\varphi+\theta+\omega)=0,\end{aligned}$$

where $\zeta = x + iy$ and $\mu = \bar{a}\bar{b}e^{-2\pi i\theta} = re^{2\pi i\omega}$. The solutions of simultaneous equations (11) and (12) are

$$x = \left\{(m+1)^2 + (m-1)^2r^2 + 2(m^2-1)r\cos 2\pi(2\varphi+\theta+\omega)\right\}^{-1}$$
$$\times \left\{C_4\cos 2\pi\varphi + C_5\cos 2\pi(\varphi+\theta+\omega) + C_6\cos 2\pi(3\varphi+\theta+\omega) + C_7\cos 2\pi(3\varphi+2\theta+2\omega)\right\}$$

and

$$y = \left\{(m+1)^2 + (m-1)^2r^2 + 2(m^2-1)r\cos 2\pi(2\varphi+\theta+\omega)\right\}^{-1}$$
$$\times \left\{C_4\sin 2\pi\varphi - C_5\sin 2\pi(\varphi+\theta+\omega) + C_6\sin 2\pi(3\varphi+\theta+\omega) - C_7\sin 2\pi(3\varphi+2\theta+2\omega)\right\},$$

where

$$C_4 = (m+1)^2(2m+1) - 2m(m^2-1)r^2,$$
$$C_5 = 2m(m^2-1)r - (m-1)^2(2m-1)r^3,$$
$$C_6 = -(m+1)^2r,$$
$$C_7 = -(m-1)^2r^2.$$

Hence $\zeta = x+iy$ satisfies the equation (10). Conversely, we show the following theorem.

Theorem 2.1. *Let $\mu = re^{2\pi i\omega} \in \overline{\mathbb{D}}$ and let $a = a(\theta,\varphi)$ and $b = b(\theta,\varphi)$ with $|a| \le |b|$ be complex numbers satisfying relations $a+b = x+iy$ and $ab = re^{-2\pi i(\theta+\omega)}$, that is, a and b are the solutions of the equation*

$$(\dagger) \qquad Z^2 - (x+iy)Z + re^{-2\pi i(\theta+\omega)} = 0,$$

where x and y are as above and $(\theta,\varphi) \in [0,1]^2$. Then the following holds:

(a) *In the case that $r = 0$, solutions of the equation $(\dagger)$ are $a = 0$ and $b = (2m+1)e^{2\pi i\varphi}$.*
(b) *In the case that $0 < r < 1$, the equation $(\dagger)$ does not have double roots. Moreover $0 < |a| < 1 < |b| < \infty$.*
(c) *In the case that $r = 1$ and $2\varphi+\theta+\omega \equiv 0 \pmod 1$, the equation $(\dagger)$ has double roots and $a = b = e^{2\pi i\varphi}$.*
(d) *In the case that $r = 1$ and $2\varphi+\theta+\omega \not\equiv 0 \pmod 1$, the equation $(\dagger)$ does not have double roots. Moreover $0 < |a| < 1 < |b| < \infty$.*
(e) *In the case* (a), (b) *or* (d),

$$B(z) = B_{\theta,\varphi,m}(z) = e^{2\pi im\theta}z\left(\frac{z-a}{1-\bar{a}z}\right)^m\left(\frac{z-b}{1-\bar{b}z}\right)^m$$

is a Blaschke product of degree $2m+1$ *and the point at infinity is a fixed point of* B *with multiplier* μ^m. *Moreover* $z = e^{2\pi i\varphi}$ *is a critical point of* B *and* $B|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ *is a homeomorphism, where* $\mathbb{T}$ *is the unit circle.*

Proof. First, we show the following lemma.

Lemma 2.2. $|x+iy| \geq 2$. *Moreover the equality holds if and only if* $r=1$ *and* $2\varphi+\theta+\omega \equiv 0 \pmod 1$ *hold.*

Proof of Lemma 2.2. We calculate that

$$
\begin{aligned}
&|x+iy|^2 \\
&= \Big\{(m+1)^2 + (m-1)^2 r^2 + 2(m^2-1) r \cos 2\pi(2\varphi+\theta+\omega)\Big\}^{-2} \\
&\quad \times \left| C_4 e^{2\pi i\varphi} + C_5 e^{-2\pi i(\varphi+\theta+\omega)} + C_6 e^{2\pi i(3\varphi+\theta+\omega)} + C_7 e^{-2\pi i(3\varphi+2\theta+2\omega)} \right|^2 \\
&= \Big\{(m+1)^2 + (m-1)^2 r^2 + 2(m^2-1) r \cos 2\pi(2\varphi+\theta+\omega)\Big\}^{-2} \\
&\quad \times \left\{ C_4 e^{2\pi i\varphi} + C_5 e^{-2\pi i(\varphi+\theta+\omega)} + C_6 e^{2\pi i(3\varphi+\theta+\omega)} + C_7 e^{-2\pi i(3\varphi+2\theta+2\omega)} \right\} \\
&\quad \times \left\{ C_4 e^{-2\pi i\varphi} + C_5 e^{2\pi i(\varphi+\theta+\omega)} + C_6 e^{-2\pi i(3\varphi+\theta+\omega)} + C_7 e^{2\pi i(3\varphi+2\theta+2\omega)} \right\} \\
&= \Big\{(m+1)^2 + (m-1)^2 r^2 + 2(m^2-1) r \cos 2\pi(2\varphi+\theta+\omega)\Big\}^{-2} \\
&\quad \times \left\{ C_4^2 + C_5^2 + C_6^2 + C_7^2 + \Big(2C_4C_5 + 2C_4C_6 + 2C_5C_7\Big) \cos 2\pi(2\varphi+\theta+\omega) \right. \\
&\quad \left. + \Big(2C_4C_7 + 2C_5C_6\Big) \cos 2\pi\cdot 2(2\varphi+\theta+\omega) + 2C_6C_7 \cos 2\pi\cdot 3(2\varphi+\theta+\omega) \right\} \\
&= \Big\{(m+1)^2 + (m-1)^2 r^2 + 2(m^2-1) r \cos 2\pi(2\varphi+\theta+\omega)\Big\}^{-2} \\
&\quad \times \left\{ C_4^2 + C_5^2 + C_6^2 + C_7^2 - 2C_4C_7 - 2C_5C_6 \right. \\
&\quad + 2\Big(C_4C_5 + C_4C_6 + C_5C_7 - 3C_6C_7\Big) \cos 2\pi(2\varphi+\theta+\omega) \\
&\quad \left. + 4\Big(C_4C_7 + C_5C_6\Big) \cos^2 2\pi(2\varphi+\theta+\omega) + 8C_6C_7 \cos^3 2\pi(2\varphi+\theta+\omega) \right\}
\end{aligned}
$$

since

$$\cos 2\pi\cdot 2(2\varphi+\theta+\omega) = 2\cos^2 2\pi(2\varphi+\theta+\omega) - 1$$

and

$$\cos 2\pi\cdot 3(2\varphi+\theta+\omega) = 4\cos^3 2\pi(2\varphi+\theta+\omega) - 3\cos 2\pi(2\varphi+\theta+\omega).$$

Therefore

$$
\begin{aligned}
&|x+iy|^2 \\
&= \Big\{(m+1)^2+(m-1)^2r^2+2(m^2-1)r\cos 2\pi(2\varphi+\theta+\omega)\Big\}^{-2} \\
&\quad\times \Bigg[4m^6+20m^5+41m^4+44m^3+26m^2+8m+1 \\
&\quad+\left(-4m^6-12m^5-5m^4+12m^3+14m^2+8m+3\right)r^2 \\
&\quad+\left(-4m^6+12m^5-5m^4-12m^3+14m^2-8m+3\right)r^4 \\
&\quad+\left(4m^6-20m^5+41m^4-44m^3+26m^2-8m+1\right)r^6 \\
&\quad+\Big\{\left(8m^6+16m^5-10m^4-48m^3-44m^2-16m-2\right)r \\
&\quad+\left(-16m^6+44m^4-24m^2-4\right)r^3 \\
&\quad+\left(8m^6-16m^5-10m^4+48m^3-44m^2+16m-2\right)r^5\Big\}\cos 2\pi(2\varphi+\theta+\omega) \\
&\quad+\Big\{\left(-16m^5-20m^4+16m^3+24m^2-4\right)r^2 \\
&\quad+\left(16m^5-20m^4-16m^3+24m^2-4\right)r^4\Big\}\cos^2 2\pi(2\varphi+\theta+\omega) \\
&\quad+\left(8m^4-16m^2+8\right)r^3\cos^3 2\pi(2\varphi+\theta+\omega)\Bigg] \\
&= \Big\{(m+1)^2+(m-1)^2r^2+2(m^2-1)r\cos 2\pi(2\varphi+\theta+\omega)\Big\}^{-2} \\
&\quad\times\Bigg[\Big[(m+1)^2+(m-1)^2r^2+2(m^2-1)r\cos 2\pi(2\varphi+\theta+\omega)\Big] \\
&\quad\times\Big[(m+1)^2(2m+1)^2-2(4m^4-5m^2-1)r^2+(m-1)^2(2m-1)^2r^4 \\
&\quad+\Big\{-4m(m+1)(2m+1)r+4m(m-1)(2m-1)r^3\Big\}\cos 2\pi(2\varphi+\theta+\omega) \\
&\quad+4(m^2-1)r^2\cos^2 2\pi(2\varphi+\theta+\omega)\Big]\Bigg] \\
&= \Big\{(m+1)^2+(m-1)^2r^2+2(m^2-1)r\cos 2\pi(2\varphi+\theta+\omega)\Big\}^{-1} \\
&\quad\times\Bigg[(m+1)^2(2m+1)^2-2(4m^4-5m^2-1)r^2+(m-1)^2(2m-1)^2r^4 \\
&\quad+4mr\Big\{-(m+1)(2m+1)+(m-1)(2m-1)r^2\Big\}\cos 2\pi(2\varphi+\theta+\omega) \\
&\quad+4(m^2-1)r^2\cos^2 2\pi(2\varphi+\theta+\omega)\Bigg].
\end{aligned}
$$

Let $X = \cos 2\pi(2\varphi + \theta + \omega)$ and we consider the function

$$f(X) = \Big\{(m+1)^2 + (m-1)^2 r^2 + 2(m^2-1)rX\Big\}^{-1}$$
$$\times \Bigg[(m+1)^2(2m+1)^2 - 2(4m^4 - 5m^2 - 1)r^2 + (m-1)^2(2m-1)^2 r^4$$
$$+ 4mr\Big\{-(m+1)(2m+1) + (m-1)(2m-1)r^2\Big\}X + 4(m^2-1)r^2X^2\Bigg].$$

Then the function f is monotone decreasing on $[-1,1]$ and

$$f(1) = \Big\{2m+1-(2m-1)r\Big\}^2.$$

In the case that $0 \le r < 1$, we obtain that

$$|x+iy| \ge \sqrt{f(1)} = 2m+1-(2m-1)r > 2.$$

In the case that $r = 1$ and $2\varphi + \theta + \omega \not\equiv 0 \pmod 1$, we obtain that

$$|x+iy| > \sqrt{f(1)} = 2m+1-(2m-1)\cdot 1 = 2.$$

Moreover in the case that $r = 1$ and $2\varphi + \theta + \omega \equiv 0 \pmod 1$, we obtain that

$$|x+iy| = \sqrt{f(1)} = 2m+1-(2m-1)\cdot 1 = 2. \qquad \square$$

Proof of (a). It is clear.

Proof of (b). By Lemma 2.2, $|a+b| = |x+iy| > 2$. In the case that $0 < r < 1$, either $0 < |a| < 1 \le |b| < \infty$ or $0 < |a| \le |b| \le 1$ hold since $|a||b| = r$. If $0 < |a| \le |b| \le 1$, then

$$2 < |a+b| \le |a| + |b| \le 2.$$

This is a contradiction and hence the situation $0 < |a| < 1 \le |b| < \infty$ happens. If $|b| = 1$, then

$$2 < |a+b| \le |a| + |b| = |a| + 1 < 2.$$

This is a contradiction. Therefore the equation (†) does not have double roots and $0 < |a| < 1 < |b| < \infty$.

Proof of (c). By assumptions, we obtain that $x+iy = 2e^{2\pi i\varphi}$ and $re^{-2\pi i(\theta+\omega)} = e^{2\pi i\cdot 2\varphi}$. Therefore the equation (†) is

$$Z^2 - 2e^{2\pi i\varphi}Z + e^{2\pi i\cdot 2\varphi} = 0$$

and hence $a = b = e^{2\pi i\varphi}$.

Proof of (d). By Lemma 2.2, $|a+b| = |x+iy| > 2$. In the case that $r = 1$, either $0 < |a| < 1 < |b| < \infty$ or $|a| = |b| = 1$ hold since $|a||b| = 1$. If $|a| = |b| = 1$, then

$$2 < |a+b| \le |a| + |b| = 2.$$

This is a contradiction. Therefore the equation (†) does not have double roots and $0 < |a| < 1 < |b| < \infty$.

Proof of (e). Let

$$u(z) = \left(\frac{z-a}{1-\bar{a}z}\right)\left(\frac{z-b}{1-\bar{b}z}\right) = \frac{z^2-(a+b)z+ab}{\bar{a}\bar{b}z^2-(\bar{a}+\bar{b})z+1}.$$

The necessary and sufficient condition that the degree of the Blaschke product B be $2m+1$ is that the function u be not constant. So the necessary and sufficient condition that the degree of the Blaschke product B be 1 is that the function u be constant. In the case that $r=0$, the function u is not constant since

$$u(z) = \frac{z^2-(2m+1)e^{2\pi i\varphi}z}{-(2m+1)e^{-2\pi i\varphi}z+1}.$$

If $r \neq 0$, then

$$u(z) = \frac{1}{\bar{a}\bar{b}} \cdot \frac{\bar{a}\bar{b}z^2-\bar{a}\bar{b}(a+b)z+|ab|^2}{\bar{a}\bar{b}z^2-(\bar{a}+\bar{b})z+1}.$$

In the case that $0<r<1$, the degree of the Blaschke product B is $2m+1$ since $|ab|=r<1$. In the case that $r=1$, we obtain that

$$\bar{a}\bar{b}(a+b)-(\bar{a}+\bar{b}) = \frac{-2me^{-2\pi i(3\varphi+\theta+\omega)}\left\{e^{2\pi i(2\varphi+\theta+\omega)}-1\right\}^3}{m^2+1+(m^2-1)\cos 2\pi(2\varphi+\theta+\omega)}.$$

Therefore in the case that $r=1$ and $2\varphi+\theta+\omega \not\equiv 0 \pmod 1$, the degree of the Blaschke product B is $2m+1$. On the other hand, if $r=1$ and $2\varphi+\theta+\omega \equiv 0 \pmod 1$, then

$$u(z) = \frac{1}{\bar{a}\bar{b}} = e^{2\pi i\cdot 2\varphi}$$

and the degree of the Blaschke product B is 1. It is clear that the point at infinity is a fixed point of B with multiplier μ^m. Moreover it is clear that $g(e^{2\pi i\varphi})=0$ and hence $z=e^{2\pi i\varphi}$ is a critical point of B, where

$$B'(z) = \frac{e^{2\pi im\theta}}{(1-\bar{a}z)^2(1-\bar{b}z)^2}\left(\frac{z-a}{1-\bar{a}z}\right)^{m-1}\left(\frac{z-b}{1-\bar{b}z}\right)^{m-1} g(z)$$

and

$$\begin{aligned} g(z) = \bar{a}\bar{b}z^4 &+ \left\{-(m+1)(\bar{a}+\bar{b})+(m-1)\bar{a}\bar{b}(a+b)\right\}z^3 \\ &+ \left\{2m+1-(2m-1)|ab|^2+|a+b|^2\right\}z^2 \\ &+ \left\{-(m+1)(a+b)+(m-1)ab(\bar{a}+\bar{b})\right\}z+ab. \end{aligned}$$

Finally we show that two critical points of B other than a, $1/\bar{a}$, b, $1/\bar{b}$ (if $m \geq 2$) and $e^{2\pi i\varphi}$ are in $\widehat{\mathbb{C}} \setminus \mathbb{T}$. In the case that $r=0$, we obtain that

$$g(z) = -(m+1)(2m+1)e^{-2\pi i\varphi}z\left(z-e^{2\pi i\varphi}\right)^2.$$

Therefore critical points of B are b, $1/\bar{b}$ (if $m \geq 2$), 0, ∞ and $e^{2\pi i\varphi}$. In the case that $r \neq 0$, let

$$h(z) = z^2 + \frac{e^{2\pi i\varphi}}{C_{10}} \left\{ C_9 e^{-2\pi i\cdot 2(2\varphi+\theta+\omega)} + C_8 e^{-2\pi i(2\varphi+\theta+\omega)} + C_9 \right\} z + e^{-2\pi i\cdot 2(\varphi+\theta+\omega)},$$

where

$$C_8 = -(m+1)^3(2m+1) + 2(2m^4 - m^2 - 1)r^2 - (m-1)^3(2m-1)r^4,$$
$$C_9 = (m+1)^3 r - (m-1)^3 r^3,$$
$$C_{10} = (m+1)^2 r + (m-1)^2 r^3 + 2(m^2-1)r\cos 2\pi(2\varphi+\theta+\omega).$$

Then we can factor $r^{-1}e^{-2\pi i(\theta+\omega)}g(z)$ as

$$\frac{1}{r}\cdot e^{-2\pi i(\theta+\omega)}\cdot g(z) = \left(z - e^{2\pi i\varphi}\right)^2 \cdot h(z).$$

Let

$$h_1(z) = \frac{e^{2\pi i\varphi}}{C_{10}} \left\{ C_9 e^{-2\pi i\cdot 2(2\varphi+\theta+\omega)} + C_8 e^{-2\pi i(2\varphi+\theta+\omega)} + C_9 \right\} z$$

and

$$h_2(z) = z^2 + e^{-2\pi i\cdot 2(\varphi+\theta+\omega)}.$$

For $z \in \mathbb{T}$, $|h_2(z)| \leq 2$.

Lemma 2.3. $|h_1(z)| > 2$ *on* $\mathbb{T}$.

Proof of Lemma 2.3. In the case that $0 < r < 1$, we obtain that

$$\begin{aligned} |h_1(z)| &= \frac{1}{|C_{10}|}\left| C_9 e^{-2\pi i\cdot 2(2\varphi+\theta+\omega)} + C_8 e^{-2\pi i(2\varphi+\theta+\omega)} + C_9 \right| \\ &\geq \frac{|C_8| - 2|C_9|}{|C_{10}|} \\ &= \frac{-C_8 - 2C_9}{|C_{10}|} \\ &\geq v(m,r) \end{aligned}$$

on $\mathbb{T}$, where

$$\begin{aligned} v(m,r) = &\left\{ (3m-1)(m+1)r + (m-1)^2 r^3 \right\}^{-1} \\ &\times \Big\{ (m+1)^3(2m+1) - 2(m+1)^3 r - 2(2m^4 - m^2 - 1)r^2 \\ &\quad + 2(m-1)^3 r^3 + (m-1)^3(2m-1)r^4 \Big\}. \end{aligned}$$

Since the function $r \mapsto v(m,r)$ is monotone decreasing on $(0,1]$ and $v(m,1) = 2$, we obtain that $|h_1(z)| > 2$ on $\mathbb{T}$. In the case that $r = 1$ and $2\varphi + \theta + \omega \not\equiv 0$

(mod 1), we obtain that

$$\begin{aligned}
|h_1(z)| &= \frac{|C_9|}{|C_{10}|}\left|e^{-2\pi i\cdot 2(2\varphi+\theta+\omega)} + \frac{C_8}{C_9}e^{-2\pi i(2\varphi+\theta+\omega)} + 1\right| \\
&= \frac{C_9}{|C_{10}|}\left|\left\{e^{-2\pi i(2\varphi+\theta+\omega)} + 1\right\}^2 + \left(\frac{C_8}{C_9} - 2\right)e^{-2\pi i(2\varphi+\theta+\omega)}\right| \\
&\geq \frac{C_9}{|C_{10}|}\left|\left|e^{-2\pi i(2\varphi+\theta+\omega)} + 1\right|^2 - \left|\frac{C_8}{C_9} - 2\right|\right| \\
&= \frac{C_9}{|C_{10}|}\left|\left|e^{-2\pi i(2\varphi+\theta+\omega)} + 1\right|^2 - \frac{4(4m^2+1)}{3m^2+1}\right| \\
&\geq \frac{3m^2+1}{2m^2}\left\{\frac{4(4m^2+1)}{3m^2+1} - \left|e^{-2\pi i(2\varphi+\theta+\omega)} + 1\right|^2\right\} \\
&> \frac{3m^2+1}{2m^2}\left\{\frac{4(4m^2+1)}{3m^2+1} - 4\right\} \\
&= 2
\end{aligned}$$

on $\mathbb{T}$. □

By the Rouché's theorem, the number of roots of $h(z) = h_1(z) + h_2(z)$ on $\mathbb{D}$ is one since $|h_1(z)| > 2 \geq |h_2(z)|$ on $\mathbb{T}$ and the number of roots of $h_1(z)$ on $\mathbb{D}$ is one. So one of critical points of B other than a, $1/\bar{a}$, b, $1/\bar{b}$ (if $m \geq 2$) and $e^{2\pi i\varphi}$ is in $\mathbb{D}$. Since critical points of a Blaschke product are symmetric with respect to the unit circle, the other one critical point of B is in $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. In this case, the inverse image $B^{-1}(\mathbb{T})$ of the unit circle $\mathbb{T}$ is the union of $\mathbb{T}$ and a figure eight 8 which crosses at $z = e^{2\pi i\varphi}$. Refer to Figure 1. Then $B|_8 : 8 \to \mathbb{T}$ is a $2m$-to-1 map and therefore $B|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is a homeomorphism. □

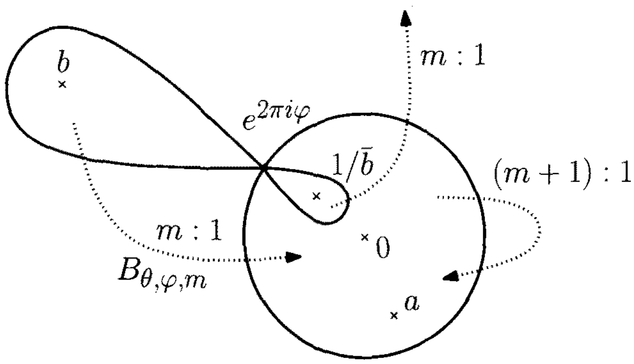

FIGURE 1. The inverse image $B^{-1}_{\theta,\varphi,m}(\mathbb{T})$ of the unit circle $\mathbb{T}$.

Remark 2.4. Two complex numbers $a = a(\theta, \varphi)$ and $b = b(\theta, \varphi)$ satisfy that

$$a(\theta + 1, \varphi) = a(\theta, \varphi) = a(\theta, \varphi + 1)$$

and

$$b(\theta + 1, \varphi) = b(\theta, \varphi) = b(\theta, \varphi + 1).$$

2.2. Rotation numbers of Blaschke products

Let $f : \mathbb{T} \to \mathbb{T}$ be an orientation preserving homeomorphism and let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be a lift of f via $x \mapsto e^{2\pi i x}$ which satisfies $\tilde{f}(x+1) = \tilde{f}(x) + 1$ for all $x \in \mathbb{R}$. The lift $\tilde{f}$ of f is unique up to addition of an integer constant. The rotation number $\rho(\tilde{f})$ of $\tilde{f}$ is defined as

$$\rho(\tilde{f}) = \lim_{n \to \infty} \frac{\tilde{f}^n(x)}{n},$$

which is independent of $x \in \mathbb{R}$. The rotation number $\rho(f)$ is defined as the residue class of $\rho(\tilde{f})$ modulo $\mathbb{Z}$. Poincaré showed that the rotation number is rational with denominator q if and only if f has a periodic point with period q. The following theorem is important (see [6]).

Theorem 2.5. *Let $\mathcal{F}$ be the set of all orientation preserving homeomorphisms from the unit circle onto itself with the topology of uniform convergence. Then the rotation number function $\rho : \mathcal{F} \to \mathbb{R}/\mathbb{Z}$ defined as $f \mapsto \rho(f)$ is continuous.*

Let $a(\theta, \varphi)$ and $b(\theta, \varphi)$ be as in Theorem 2.1. We define a map $\Gamma_m : [0,1]^3 \to \mathbb{T}$ as

$$\Gamma_m(x, \theta, \varphi) = \left(\frac{e^{2\pi i x} - a(\theta, \varphi)}{1 - \overline{a(\theta, \varphi)} e^{2\pi i x}} \right)^m \left(\frac{e^{2\pi i x} - b(\theta, \varphi)}{1 - \overline{b(\theta, \varphi)} e^{2\pi i x}} \right)^m$$

and a map $H_m : [0,1]^4 \to \mathbb{T}$ as

$$H_m(x, \theta, \varphi, t) = \left(\frac{e^{2\pi i x} - a(\theta, \varphi, t)}{1 - \overline{a(\theta, \varphi, t)} e^{2\pi i x}} \right)^m \left(\frac{e^{2\pi i x} - b(\theta, \varphi, t)}{1 - \overline{b(\theta, \varphi, t)} e^{2\pi i x}} \right)^m,$$

where

$$a(\theta, \varphi, t) = (1-t) a(\theta, \varphi) + t e^{2\pi i \varphi}$$

and

$$b(\theta, \varphi, t) = (1-t) b(\theta, \varphi) + t e^{2\pi i \varphi}.$$

Note that $\Gamma_m(x, \theta, \varphi) = e^{2\pi i \cdot 2m\varphi}$ if $r = 1$ and $2\varphi + \theta + \omega \equiv 0 \pmod 1$. The following three lemmas play important roles in the proof of Theorem 2.9.

Lemma 2.6. *A map $H_m(\,\cdot\,, \theta, \varphi, \,\cdot\,) : [0,1]^2 \to \mathbb{T}$ is a homotopy between a loop $x \mapsto \Gamma_m(x, \theta, \varphi)$ and a constant loop $x \mapsto e^{2\pi i \cdot 2m\varphi}$ for any $(\theta, \varphi) \in [0,1]^2$.*

Proof. It is clear since $H_m(\cdot, \theta, \varphi, 0) = \Gamma_m(\cdot, \theta, \varphi)$ and $H_m(\cdot, \theta, \varphi, 1) = e^{2\pi i \cdot 2m\varphi}$. □

Lemma 2.7. *A map $H_m(x, \,\cdot\,, \varphi, \,\cdot\,) : [0,1]^2 \to \mathbb{T}$ is a homotopy between a loop $\theta \mapsto \Gamma_m(x, \theta, \varphi)$ and a constant loop $\theta \mapsto e^{2\pi i \cdot 2m\varphi}$ for any $(x, \varphi) \in [0,1]^2$.*

Proof. It is clear since $H_m(x,\cdot,\varphi,0) = \Gamma_m(x,\cdot,\varphi)$ and $H_m(x,\cdot,\varphi,1) = e^{2\pi i\cdot 2m\varphi}$. □

Lemma 2.8. *A map $H_m(x,\theta,\cdot,\cdot) : [0,1]^2 \to \mathbb{T}$ is a homotopy between a loop $\varphi \mapsto \Gamma_m(x,\theta,\varphi)$ and a loop $\varphi \mapsto e^{2\pi i\cdot 2m\varphi}$ for any $(x,\theta) \in [0,1]^2$.*

Proof. It is clear since $H_m(x,\theta,\cdot,0) = \Gamma_m(x,\theta,\cdot)$ and $H_m(x,\theta,\cdot,1) = e^{2\pi i\cdot 2m\varphi}$. □

Lemma 2.6 and Lemma 2.7 imply that

$$\arg(\Gamma_m(x+1,\theta,\varphi)) = \arg(\Gamma_m(x,\theta,\varphi)) = \arg(\Gamma_m(x,\theta+1,\varphi))$$

and Lemma 2.8 implies that

$$\frac{1}{2\pi}\arg(\Gamma_m(x,\theta,\varphi+1)) = \frac{1}{2\pi}\arg(\Gamma_m(x,\theta,\varphi)) + 2m.$$

Theorem 2.9. *Let $\alpha \in [0,1]$ and let $\mu = re^{2\pi i\omega} \in \overline{\mathbb{D}}$. Besides let $a = a(\theta,\varphi)$ and $b = b(\theta,\varphi)$ be as in Theorem 2.1. Then for the Blaschke product*

$$B_{\theta,\varphi,m}(z) = e^{2\pi i m\theta} z \left(\frac{z-a}{1-\bar{a}z}\right)^m \left(\frac{z-b}{1-\bar{b}z}\right)^m,$$

$B_{\theta,\varphi,m}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism. Moreover

(a) *In the case that $0 \le r < 1$, there exists $(\theta_0,\varphi_0) \in [0,1]^2$ such that*

$$\rho(B_{\theta_0,\varphi_0,m}|_{\mathbb{T}}) = \alpha.$$

(b) *In the case that $r = 1$, if $\alpha + m\omega \not\equiv 0 \pmod 1$, then there exists $(\theta_0,\varphi_0) \in [0,1]^2$ such that $\rho(B_{\theta_0,\varphi_0,m}|_{\mathbb{T}}) = \alpha$ and $2\varphi_0 + \theta_0 + \omega \not\equiv 0 \pmod 1$.*

Proof. In the case that $r = 1$ and $2\varphi + \theta + \omega \equiv 0 \pmod 1$,

$$B_{\theta,\varphi,m}(z) = e^{2\pi i m(2\varphi+\theta)}z = e^{2\pi i(-m\omega)}z.$$

Therefore $B_{\theta,\varphi,m}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism and its rotation number satisfies that $\rho(B_{\theta,\varphi,m}|_{\mathbb{T}}) \equiv -m\omega \pmod 1$. In the other cases, we consider a lift

$$\widetilde{B}_{\theta,\varphi,m}(x) = m\theta + x + \frac{1}{2\pi}\arg(\Gamma_m(x,\theta,\varphi))$$

of $B_{\theta,\varphi,m}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ via $x \mapsto e^{2\pi i x}$. By Lemma 2.6,

$$\widetilde{B}_{\theta,\varphi,m}(x+1) = m\theta + x + 1 + \frac{1}{2\pi}\arg(\Gamma_m(x+1,\theta,\varphi)) = \widetilde{B}_{\theta,\varphi,m}(x) + 1$$

for all $x \in \mathbb{R}$. This implies that $B_{\theta,\varphi,m}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism. Consequently the rotation number of $\rho(\widetilde{B}_{\theta,\varphi,m})$ is well defined. By Lemma 2.7, we obtain that $\widetilde{B}^n_{1,\varphi,m}(x) = \widetilde{B}^n_{0,\varphi,m}(x) + mn$ and hence

$$\rho(\widetilde{B}_{1,\varphi,m}) = \rho(\widetilde{B}_{0,\varphi,m}) + m. \tag{13}$$

Moreover by Lemma 2.8, we obtain that $\widetilde{B}^n_{\theta,1,m}(x) = \widetilde{B}^n_{\theta,0,m}(x) + 2mn$ and hence

$$\rho(\widetilde{B}_{\theta,1,m}) = \rho(\widetilde{B}_{\theta,0,m}) + 2m. \tag{14}$$

These two equations (13) and (14) imply that

$$\rho(\widetilde{B}_{1,1,m}) = \rho(\widetilde{B}_{0,0,m}) + 3m.$$

Therefore in the case that $0 \leq r < 1$, there exists $(\theta_0, \varphi_0) \in [0,1]^2$ such that

$$\alpha = \rho(B_{\theta_0,\varphi_0,m}|_{\mathbb{T}}) \equiv \rho(\widetilde{B}_{\theta_0,\varphi_0,m}) \pmod 1$$

since the rotation number function $(\theta, \varphi) \mapsto \rho(B_{\theta,\varphi,m}|_{\mathbb{T}})$ is continuous. In the case that $r = 1$, if $2\varphi + \theta + \omega \equiv 0 \pmod 1$, then $\rho(B_{\theta,\varphi,m}|_{\mathbb{T}}) \equiv -m\omega \pmod 1$. Hence if $\alpha + m\omega \not\equiv 0 \pmod 1$, then there exists $(\theta_0, \varphi_0) \in [0,1]^2$ such that

$$\alpha = \rho(B_{\theta_0,\varphi_0,m}|_{\mathbb{T}}) \equiv \rho(\widetilde{B}_{\theta_0,\varphi_0,m}) \pmod 1$$

and $2\varphi_0 + \theta_0 + \omega \not\equiv 0 \pmod 1$. □

Remark 2.10. By Theorem 2.1, the degree of $B_{\theta_0,\varphi_0,m}$ is $2m+1$.

3. Rational functions with Siegel disks

In this section, we show Theorem 1.5. Let $f : \mathbb{R} \to \mathbb{R}$ be a homeomorphism. If there exists $k \geq 1$ such that

$$\frac{1}{k} \leq \left| \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \right| \leq k$$

for all $x \in \mathbb{R}$ and all $t \geq 0$, then f is called *k-quasisymmetric.* A homeomorphism $h : \mathbb{T} \to \mathbb{T}$ is *k-quasisymmetric* if its lift $\tilde{h} : \mathbb{R} \to \mathbb{R}$ is k-quasisymmetric. By the theorem of Beurling and Ahlfors, any k-quasisymmetric homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is extended to a K-quasiconformal map $F : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$, where $\mathbb{H}$ is the upper half plain (More precisely $F : \mathbb{C} \to \mathbb{C}$). The dilatation K of F depends only on k. Therefore if a homeomorphism $h : \mathbb{T} \to \mathbb{T}$ is k-quasisymmetric, then we can extend h to a K-quasiconformal map $H : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ whose dilatation depends only on k.

Theorem 3.1 (Herman-Świątek). *The rotation number $\rho(f)$ of a real analytic orientation preserving homeomorphism $f : \mathbb{T} \to \mathbb{T}$ is of bounded type if and only if f is quasisymmetrically linearizable, that is, there exits a quasisymmetric homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such that $h \circ f \circ h^{-1}(z) = e^{2\pi i \rho(f)} z$.*

Recall that

$$B_{\theta,\varphi,m}(z) = e^{2\pi i m\theta} z \left(\frac{z-a}{1-\bar{a}z} \right)^m \left(\frac{z-b}{1-\bar{b}z} \right)^m$$

and

$$F_{\lambda,\mu,m}(z) = z \left(\frac{z+\lambda}{\mu z + 1} \right)^m.$$

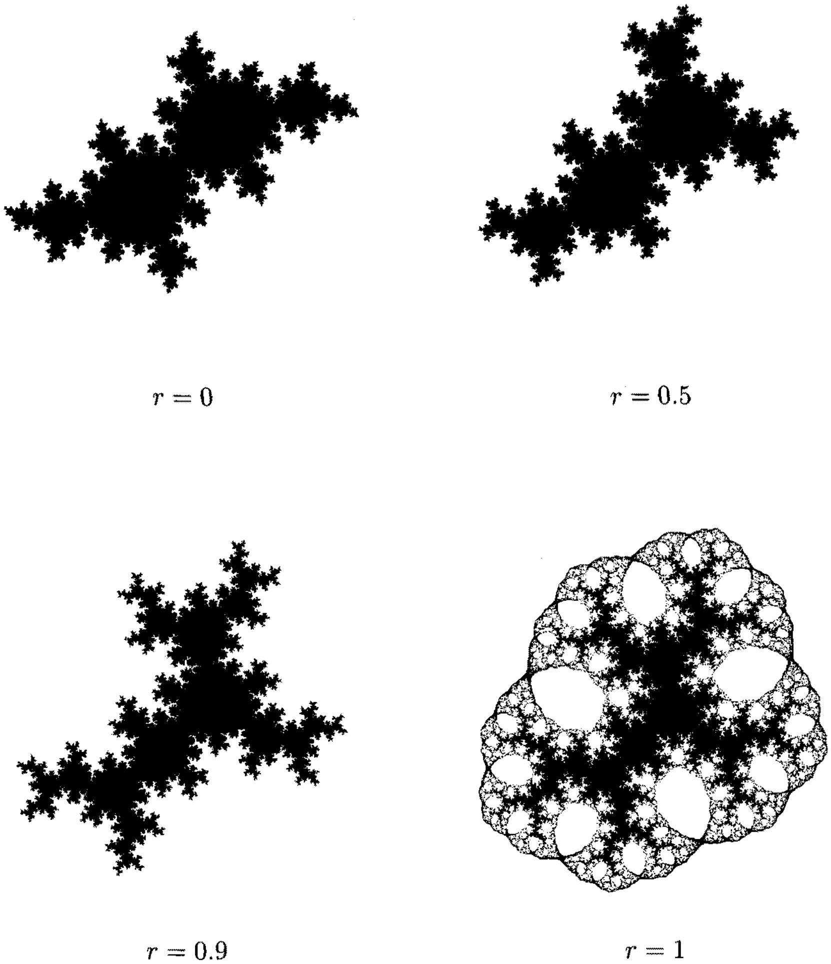

FIGURE 2. Golden Siegel disks of $F_{\lambda,\mu,1}$ centered at the origin, where $\lambda = e^{2\pi i\cdot(\sqrt{5}-1)/2}$ and $\mu = re^{2\pi i\cdot(\sqrt{5}-1)/2}$. In the case that $r = 1$, the point at infinity is the center of another golden Siegel disk.

Proof of Theorem 1.5. By Theorem 2.9, there exist $(\theta, \varphi) \in [0,1]^2$ such that the degree of $B_{\theta,\varphi,m}$ is $2m+1$ and $\rho(B_{\theta,\varphi,m}|_{\mathbb{T}}) = \alpha$. By Theorem 3.1, there exists a quasisymmetric homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such that $h \circ B_{\theta,\varphi,m}|_{\mathbb{T}} \circ h^{-1}(z) = R_\alpha(z) = e^{2\pi i\alpha}z$ since α is of bounded type. By the theorem of Beurling and Ahlfors, h has a quasiconformal extension $H : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ with $H(0) = 0$. We define a new map $\mathfrak{B}_{\theta,\varphi,m}$ as

$$\mathfrak{B}_{\theta,\varphi,m} = \begin{cases} B_{\theta,\varphi,m} & \text{on } \widehat{\mathbb{C}} \setminus \mathbb{D}, \\ H^{-1} \circ R_\alpha \circ H & \text{on } \mathbb{D}. \end{cases}$$

The map $\mathfrak{B}_{\theta,\varphi,m}$ is quasiregular on $\widehat{\mathbb{C}}$ since $\mathbb{T}$ is an analytic curve. Moreover $\mathfrak{B}_{\theta,\varphi,m}$ is a degree $m+1$ branched covering of $\widehat{\mathbb{C}}$. We define a conformal structure $\sigma_{\theta,\varphi,m}$ as

$$\sigma_{\theta,\varphi,m} = \begin{cases} H^*(\sigma_0) & \text{on } \mathbb{D}, \\ \left(\mathfrak{B}^n_{\theta,\varphi,m}\right)^* \circ H^*(\sigma_0) & \text{on } \mathfrak{B}^{-n}_{\theta,\varphi,m}(\mathbb{D}) \setminus \mathbb{D} \text{ for all } n \in \mathbb{N}, \\ \sigma_0 & \text{on } \widehat{\mathbb{C}} \setminus \bigcup_{n=1}^{\infty} \mathfrak{B}^{-n}_{\theta,\varphi,m}(\mathbb{D}), \end{cases}$$

where σ_0 is the standard conformal structure on $\widehat{\mathbb{C}}$. The conformal structure $\sigma_{\theta,\varphi,m}$ is invariant under $\mathfrak{B}_{\theta,\varphi,m}$ and its maximal dilatation is the dilatation of H since H is quasiconformal and $B_{\theta,\varphi,m}$ is holomorphic. By the measurable Riemann mapping theorem, there exists a quasiconformal homeomorphism $\Psi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\Psi^*\sigma_0 = \sigma_{\theta,\varphi,m}$. Therefore $\Psi \circ \mathfrak{B}_{\theta,\varphi,m} \circ \Psi^{-1}$ is a rational map of degree $m+1$. We normalize $\Psi = \Psi_j$ by $\Psi_j(0) = 0$, $\Psi_j(b) = -\lambda_j$ and $\Psi_j(\infty) = \infty$, where $\lambda_j = e^{2\pi i(\alpha+j)/m}$ for $j \in \{1, \ldots, m\}$.

Lemma 3.2. *If $\mu \neq 0$, then there exists μ_j with $\mu_j^m = \mu^m$ such that*

$$F_{\lambda_j,\mu_j,m} = \Psi_j \circ \mathfrak{B}_{\theta,\varphi,m} \circ \Psi_j^{-1}.$$

Proof of Lemma 3.2. Define ξ_j as $\xi_j = -\Psi_j(1/\bar{a})$. Note that $\lambda_j \neq \xi_j$ since such Ψ_j is unique. Since orders of zeros and poles are invariant under conjugation, we obtain that

$$\Psi_j \circ \mathfrak{B}_{\theta,\varphi,m} \circ \Psi_j^{-1}(z) = \eta_j z \left(\frac{z+\lambda_j}{z+\xi_j}\right)^m.$$

Since multipliers of fixed points are also invariant under conjugation, we obtain that

$$\left(\Psi_j \circ \mathfrak{B}_{\theta,\varphi,m} \circ \Psi_j^{-1}\right)'(0) = \frac{\eta_j \lambda_j^m}{\xi_j^m} = e^{2\pi i\alpha} \tag{15}$$

and

$$\frac{1}{\left(\Psi_j \circ \mathfrak{B}_{\theta,\varphi,m} \circ \Psi_j^{-1}\right)'(\infty)} = \frac{1}{\eta_j} = \mu^m. \tag{16}$$

By the equations (15) and (16), we obtain that $(\xi_j \mu)^m = 1$. Then there exists an m-th root of unity ν_j such that $\xi_j = \nu_j/\mu$. Therefore

$$\Psi_j \circ \mathfrak{B}_{\theta,\varphi,m} \circ \Psi_j^{-1}(z) = \frac{z}{\mu^m}\left(\frac{z+\lambda_j}{z+\nu_j/\mu}\right)^m = z\left(\frac{z+\lambda_j}{\mu z+\nu_j}\right)^m$$

$$= \frac{z}{\nu_j^m}\left(\frac{z+\lambda_j}{(\mu/\nu_j)\,z+1}\right)^m = z\left(\frac{z+\lambda_j}{\mu_j z+1}\right)^m = F_{\lambda_j,\mu_j,m}(z),$$

where $\mu_j = \mu/\nu_j$. □

Let $\mu_j = 0$ for all $j \in \{1, \ldots, m\}$ if $\mu = 0$. It is easy to check that the pairs $\{(\lambda_j, \mu_j)\}_{j=1}^m$ satisfies (i) and (ii). The map $F_{\lambda_j,\mu_j,m}$ has a Siegel disk $\Delta = \Psi_j(\mathbb{D})$ with a critical point $\Psi_j(e^{2\pi i\varphi}) \in \partial\Delta$. Moreover $\partial\Delta = \Psi_j(\mathbb{T})$ is a quasicircle since Ψ_j is quasiconformal. □

Proof of Corollary 1.6. Let $\mathcal{I}(z) = 1/z$. Then $F_{\lambda_j,\mu_j,m} = \mathcal{I} \circ F_{\mu_j,\lambda_j,m} \circ \mathcal{I}$. Let Δ and Δ_∞ be Siegel disks of $F_{\lambda_j,\mu_j,m}$ centered at the origin and the point at infinity respectively. By Theorem 1.5, the boundary of Δ contains a critical point of $F_{\lambda_j,\mu_j,m}$. On the other hand, $\mathcal{I}(\Delta_\infty)$ is the Siegel disk of $F_{\mu_j,\lambda_j,m}$ centered at the origin. By Theorem 1.5, the boundary of $\mathcal{I}(\Delta_\infty)$ contains a critical point of $F_{\mu_j,\lambda_j,m}$. Therefore the boundary of Δ_∞ contains a critical point of $F_{\lambda_j,\mu_j,m}$. □

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