Blaschke-Santaló diagram for volume, perimeter and first Dirichlet eigenvalue

Ilias Ftouhi, Jimmy Lamboley

Institutions: Institut de Mathématiques de Jussieu

Published on: 24 Mar 2021 - Siam Journal on Mathematical Analysis (Society for Industrial and Applied Mathematics)

Topics: Dirichlet eigenvalue, Diagram, Isoperimetric inequality, Regular polygon and Boundary (topology)

Blaschke-Santaló diagram for volume, perimeter and first Dirichlet eigenvalue

Ilias Ftouhi, Jimmy Lamboley

Blaschke-Santaló diagram for volume, perimeter and first Dirichlet eigenvalue

Ilias Ftouhi, Jimmy Lamboley

November 11, 2020

Abstract

We are interested in the study of Blaschke-Santaló diagrams describing the possible inequalities involving the first Dirichlet eigenvalue, the perimeter and the volume, for different classes of sets. We give a complete description of the diagram for the class of open sets in $\mathbb{R}^d$, basically showing that the isoperimetric and Faber-Krahn inequalities form a complete system of inequalities for these three quantities. We also give some qualitative results for the Blaschke-Santaló diagram for the class of planar convex domains: we prove that in this case the diagram can be described as the set of points contained between the graphs of two continuous and increasing functions. This shows in particular that the diagram is simply connected, and even horizontally and vertically convex. We also prove that the shapes that fill the upper part of the boundary of the diagram are smooth ($C^{1,1}$), while those on the lower one are polygons (except for the ball). Finally, we perform some numerical simulations in order to have an idea on the shape of the diagram; we deduce both from theoretical and numerical results some new conjectures about geometrical inequalities involving the functionals under study in this paper.

Contents

1 Introduction

In this paper, we are interested in describing all possible geometrical inequalities that are invariant under homotheties and involving the three following quantities: the volume, the perimeter, and the first Dirichlet eigenvalue of a given shape.

A Blaschke-Santaló diagram is a tool that allows to visualize all possible inequalities between three quantities depending on the shape of a set: it was named as a reference to [5] and [52], where the authors were looking for the description of inequalities involving three geometrical quantities for a given convex set. Usually in convex geometry, Blaschke-Santaló diagrams are studied for purely geometrical fuctionals. We refer to [33] for more details and to [4, 22, 23] for some recent results in this purely geometrical setting.

More recently, some interest has grown for geometrical inequalities involving the spectral quantities of a given shape $\Omega \subset \mathbb{R}^d$, like the eigenvalues of the Laplacian on the set Ω with Dirichlet boundary conditions on $\partial\Omega$: therefore, the approach by Blaschke-Santaló diagrams has been applied in this context, see for example [14] and [3], see also [57, 43].

In the present paper, we propose to study an example mixing geometric and spectral quantities. In order to be more precise, let us define the Blaschke-Santaló diagrams we are interested in in this paper: given $\mathcal{C}$ a class of open sets of

$\mathbb{R}^d$, we define

$$\begin{aligned}\mathcal{D}_{\mathcal{C}} &= \Big\{(x,y)\in\mathbb{R}^2,\ \ \exists\Omega\in\mathcal{C} \text{ such that } |\Omega|=1, P(\Omega)=x, \lambda_1(\Omega)=y\Big\}\\ &:= \Big\{\ \big(P(\Omega),\lambda_1(\Omega)\big),\ \Omega\in\mathcal{C},\ |\Omega|=1\ \Big\},\end{aligned}$$

where $|\Omega|$ denotes the volume of the set Ω, $P(\Omega)=\mathcal{H}^{d-1}(\partial\Omega)$ is its perimeter, and $\lambda_1(\Omega)$ is its first Dirichlet eigenvalue, which can be quickly defined with the following variational formulation:

$$\lambda_1(\Omega):=\min\left\{\frac{\displaystyle\int_\Omega|\nabla u|^2dx}{\displaystyle\int_\Omega u^2dx},\ u\in H_0^1(\Omega)\setminus\{0\}\right\}, \tag{1}$$

where $H_0^1(\Omega)$ denotes the completion for the H^1-norm of the space $C_c^\infty(\Omega)$ of infinitely differentiable functions of compact support in Ω.

Remark 1 *We recall the following behavior with respect to homothety:*

$$\forall t>0,\quad \lambda_1(t\Omega)=\frac{\lambda_1(\Omega)}{t^2},\quad |t\Omega|=t^d|\Omega|\quad \textit{and}\quad P(t\Omega)=t^{d-1}P(\Omega).$$

This allows us to give a scaling invariant formulation of the diagram: if $\mathcal{C}$ is a class of nonempty and bounded open sets in $\mathbb{R}^d$, then

$$\begin{aligned}\mathcal{D}_{\mathcal{C}} &= \Big\{(x,y)\in\mathbb{R}^2,\ \ \exists\Omega\in\mathcal{C} \textit{ such that } P(\Omega)/|\Omega|^{\frac{d-1}{d}}=x,\ \ |\Omega|^{\frac{2}{d}}\lambda_1(\Omega)=y\Big\}\\ &:= \left\{\ \left(\frac{P(\Omega)}{|\Omega|^{\frac{d-1}{d}}}, |\Omega|^{\frac{2}{d}}\lambda_1(\Omega)\right),\ \ \Omega\in\mathcal{C}\right\}.\end{aligned}$$

We are now in position to state the first main result in this paper:

Theorem 1.1 *Let $\mathcal{O}$ be the class of C^∞ open sets in $\mathbb{R}^d$, we have:*

$$\mathcal{D}_{\mathcal{O}}=\Big(\big(P(B),+\infty\big)\times\big(\lambda_1(B),+\infty\big)\Big)\cup\big\{\big(P(B),\lambda_1(B)\big)\big\},$$

where B is a ball of volume 1.

Let us give a few comments on this result:

- the most famous inequalities in this framework are the isoperimetric and the Faber-Krahn inequalities, stating that

$$\forall\Omega\in\mathcal{O}\text{ such that }|\Omega|=1,\quad P(\Omega)\geq P(B)\qquad\text{and}\qquad \lambda_1(\Omega)\geq\lambda_1(B). \tag{2}$$

  In terms of the diagram, it says that $\mathcal{D}_{\mathcal{O}}$ is included in the "up-right" quadrant defined by the point $(P(B),\lambda_1(B))$. Theorem 1.1 asserts that the diagram is in fact exactly this quadrant (see the next point for a discussion about whether the boundary of the quadrant should be included or not in the diagram); in other words, inequalities given in (2) are exhaustive in the sense that any other inequality that is invariant with homotheties and only involves the three quantities $(P,\lambda_1,|\cdot|)$ are already taken into account in (2); we say that this is a complete system of inequalities in the class $\mathcal{O}$.

- one could wonder why we chose to work with C^∞ domains: the main reason is that there are several definitions of perimeter, that all agree for smooth enough sets (say Lipschitz sets) but may disagree for nonsmooth sets. In the smooth framework, the equality cases in (2), up to translations, occurs if and only if Ω is the ball B (see for example [47, Section 2] and [35, Example 2.11] respectively for the first and second inequalities). It explains why the boundary of the quadrant (except the point $(P(B),\lambda_1(B))$) is not included in the diagram. Also, it shows that Theorem 1.2 is the strongest statement in the sense that for any subclass of Lipschitz domains that contains C^∞-domains, the diagram is the same.

  However, when working with nonsmooth domains, equality in (2) may happen for sets different from a ball. If we choose to work with the De Giorgi's perimeter for example, one has $P(B)=P(B\backslash K)$, for any Borel set K with zero Lebesgue measure. On the other hand, for the Faber-Krahn inequality, we have $\lambda_1(B)=\lambda_1(B\backslash K)$ as soon as K is a set of zero capacity, see for example [31, Remark 3.2.2]. In remark 2.1 we deduce from Theorem 1.1 a full description of the diagram for the class of non necessarily smooth sets when P is the perimeter of De Giorgi: this description could be different for another definition of the perimeter, but as shown by Theorem 1.1, this can only affect the boundary of the diagram.

It is now natural to restrict the class of sets, so that the corresponding Blaschke-Santaló diagram becomes more challenging to understand: a natural class that has been extensively studied in the purely geometrical context is the class of planar convex sets. The Blaschke-Santaló diagram of $(P, \lambda_1, |\cdot|)$ in this specific case has been first numerically studied by P. Antunes and P. Freitas in [2]. We would like to give a theoretical description of the diagram, in the same spirit of [14, 4] . We obtain the following main result:

Theorem 1.2 *Let $\mathcal{K}^2$ be the class of convex planar open sets:*

$$\mathcal{K}^2 = \Big\{\Omega \subset \mathbb{R}^2,\ \ \Omega \text{ is convex and open}\Big\}.$$

We denote $x_0 = P(B) = 2\sqrt{\pi}$, where B is a disk of area 1. Then there exist two functions $f : [x_0, +\infty) \to \mathbb{R}$ and $g : [x_0, +\infty) \to \mathbb{R}$ such that

1. *the diagram $\mathcal{D}_{\mathcal{K}^2}$ is made of all points in $\mathbb{R}^2$ lying between the graphs of f and g, more precisely:*

$$\mathcal{D}_{\mathcal{K}^2} = \Big\{(x, y) \in \mathbb{R}^2,\ x \geq x_0 \ \text{ and } \ f(x) \leq y \leq g(x)\Big\}, \tag{3}$$

2. *the functions f and g are continuous and strictly increasing,*
3. *for every $x > x_0$, let $\Omega \in \mathcal{K}^2$ such that $|\Omega| = 1$ and $\lambda_1(\Omega) = x$, then*
   - *if $P(\Omega) = g(x)$, then Ω is $C^{1,1}$,*
   - *if $P(\Omega) = f(x)$, then Ω is a polygon.*
4. $f(x) \underset{x\to\infty}{\sim} \frac{\pi^2}{16}x^2, \qquad g(x) \underset{x\to\infty}{\sim} \frac{\pi^2}{4}x^2, \qquad f'(x_0) = 0$ *and* $\displaystyle\limsup_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} \geq \frac{\lambda_1(B)}{3\sqrt{\pi}}\left(\frac{\lambda_1(B)}{\pi} - 2\right).$

Let us comment about this result and its proof:

- this result gives a good understanding of the shape of the diagram $\mathcal{D}_{\mathcal{K}^2}$: it says in particular that it is simply connected, and even horizontally and vertically convex.
- in other words, the knowledge of f and g is enough to describe all possible (scaling invariant) inequalities involving the three quantities $(P, \lambda_1, |\cdot|)$, in the class of convex sets of $\mathbb{R}^2$: these functions quantify in which way one can improve inequalities (2) if one knows that the shape Ω is convex, and not only just an open set.
- of course, it is not expected to have an explicit formula for functions f and g. Up to our knowledge, Theorem 1.2 is one of the first qualitative and complete description of a Blaschke-Santaló diagram while we do not have a good knowledge of the shapes that achieve the boundary of the diagram. Compare with [57, 43] where it is still an open problem whether the Blaschke-Santaló diagram for the triplet $(\lambda_1, T, |\cdot|)$ where T denotes the torsional rigidity, is simply connected (or horizontally and vertically convex, which is a stronger statement), both for the class of open domains (see [57, Problem 3]) and for the class of convex domains ([43, Conjecture 2]).
- the proof of the first two points in Theorem 1.2 is therefore the most involved part of this paper (see also Section 3.2.1); it relies in particular on a perturbation lemma (see Lemma 3.5) among convex sets and involving functionals P, λ_1 and $|\cdot|$, which states that if we denote $\mathcal{K}^2_1$ the set of planar convex domains with unit area endowed with the Hausdorff distance d^H, then

  1. the ball is the only local minimizer of the perimeter, as well as the only local minimizer of λ_1, in $(\mathcal{K}^2_1, d^H)$
  2. however, there is no local maximizer of the perimeter in $(\mathcal{K}^2_1, d^H)$, and no local maximizer of λ_1 that is $C^{1,1}$.

  We believe this Lemma is interesting in itself; its second part is not easy to prove at all. It uses the tools and results of shape optimization under convexity constraints studied in [38, 40, 41]. It mainly explains the restriction to dimension 2. Up to our knowledge, the results given in Theorem 1.2 or in Lemma 3.5 are open in dimension 3 or higher. We also denote $\mathcal{K}^d$ the set of convex open subsets of $\mathbb{R}^d$ when $d \geq 3$, and $\mathcal{D}_{\mathcal{K}^d}$ denotes the associated Blaschke-Santaló diagram. Notice that some results from Section 3.2 are stated and proved in arbitrary dimensions (see Propositions 3.2 and 3.7). In Section 4.2, we discuss the case of higher dimensions and conjecture that as for the planar case, $\mathcal{D}_{\mathcal{K}^d}$ is given by the set of points contained between two continuous and increasing curves, see Conjecture 2.
- the third assertion provides some regularity (or non-regularity) properties for domains lying on the boundary of the diagram. If follows from results of [40], see Corollary 3.13. We note that to be able to apply [40], we have to prove a Serrin's type lemma on convex sets, where no regularity assumption is made: see Lemma 3.11, which is given for arbitrary dimensions and is rather interesting in itself. The $C^{1,1}$ regularity of the upper optimal domains allows us to restrict the fourth assertion of the perturbation Lemma 3.5 to the case of smooth domains, which is easier to prove, see also [39].

- though it is not expected to compute explicitly f and g, the last point in Theorem 1.2 provides some results about the asymptotic behavior of f and g near $+\infty$ and near $x_0 = P(B)$, see Proposition 3.7, Corollary 3.17 (which are stated and proved in arbitrary dimensions), and Corollary 3.22 (which is proved only in dimension 2). We actually provide an improvement to the result $f'(x_0) = 0$, which is the main novelty about these asymptotics: more precisely, investigating the lower part of the diagram for x close to $P(B)$ is related to the following question: for what exponent α may we expect that there is an inequality of the form

$$\lambda_1(\Omega) - \lambda_1(B) \geq c\,(P(\Omega) - P(B))^{\alpha}$$

for $\Omega \in \mathcal{K}_1^2$ close to the ball and for some $c > 0$ independent of Ω. We show in the second part of Theorem 3.19 that α must necessarily be greater or equal to $3/2$ for such an inequality to be valid, and we show evidence that such an inequality is likely to be true with $\alpha = 3/2$ (see the first part of Theorem 3.19 and Proposition 3.21) even though we are not yet in position to prove it, see Section 4.1. Finally, we compare the conjectured inequality (with the exponent $3/2$) with the sharp quantitative Faber-Krahn inequality proved in [11], see Remark 4.1.

In the following section, we give a proof of Theorem 1.1. In Section 3, we focus on the case of convex planar domains: we first recall theoretical information that was known about the diagram, and provide numerical simulations. We then prove the main lemma about perturbation results in the class of convex sets in $\mathbb{R}^2$ (Lemma 3.5), and then deduce that the boundary of the diagram is made of the graph of two increasing and continuous functions (see Theorem 3.9), and furthermore that the diagram is simply connected (see Theorem 3.14). This eventually leads to the proof of Theorem 1.2. We also describe the asymptotics of f and g near $+\infty$ and x_0, see Proposition 3.7 and Section 3.3. In the last Section, we discuss related problems and new possible conjectures.

2 Proof of Theorem 1.1

As explained below the statement of Theorem 1.1, the inclusion

$$\mathcal{D}_{\mathcal{O}} \subset \Big(\big(P(B), +\infty\big) \times \big(\lambda_1(B), +\infty\big)\Big) \cup \big\{\big(P(B), \lambda_1(B)\big)\big\}$$

is due to the isoperimetric and Faber-Krahn inequalities with equality cases, see for example [47, Section 2] and [35, Example 2.11]. It remains to show the reverse inclusion.

<u>**Step 1**</u>: we first show, using a homogenization strategy, that for any $\mu \in (0, +\infty)$, there exists a sequence $(\Omega_n)_{n\in\mathbb{N}}$ of C^∞ open sets with unit area such that:

$$P(\Omega_n) \underset{n\to+\infty}{\longrightarrow} P(B) \tag{4}$$

and

$$\lambda_1(\Omega_n) \underset{n\to+\infty}{\longrightarrow} \lambda_1(B) + \mu. \tag{5}$$

Let $n \in \mathbb{N}^*$, we cover $\mathbb{R}^d$ by cubes $(P_i^n)_{i\in\mathbb{N}}$ of size $2/n$. From each cube P_i^n such that $P_i^n \subset B$ we remove the ball T_i^n of radius $a_{d,n}$ centered at the center of the cube, where:

$$a_{d,n} = \begin{cases} C_d n^{-d/(d-2)} & \text{if } d \geq 3, \\ \exp(-C_2 n^2) & \text{if } d = 2 \end{cases} \qquad \text{and} \qquad C_d = \begin{cases} \left(\frac{2^d \mu}{d(d-2)\omega_d}\right)^{\frac{1}{d-2}} & \text{if } d \geq 3, \\ 2\mu/\pi & \text{if } d = 2, \end{cases}$$

with ω_d classically denoting the volume of the unit ball.

We consider n sufficiently big so that $a_{d,n} < \frac{1}{n}$. Let us define $\Lambda_n := B \setminus \bigcup_{i\in I_n} T_i^n$, Where $I_n := \{i \in \mathbb{N} \mid P_i^n \subset B\}$.

In order to preserve the total measure, we use the sets $\Omega_n = \Lambda_n \cup \bigcup_{i\in I_n} (v_d + T_i^n)$ which are smooth and with unit volume, where $v_d \in \mathbb{R}^d$ is chosen such that $B \cap \bigcup_{i\in I_n} (v_d + T_i^n) = \emptyset$ (see Figure 1).

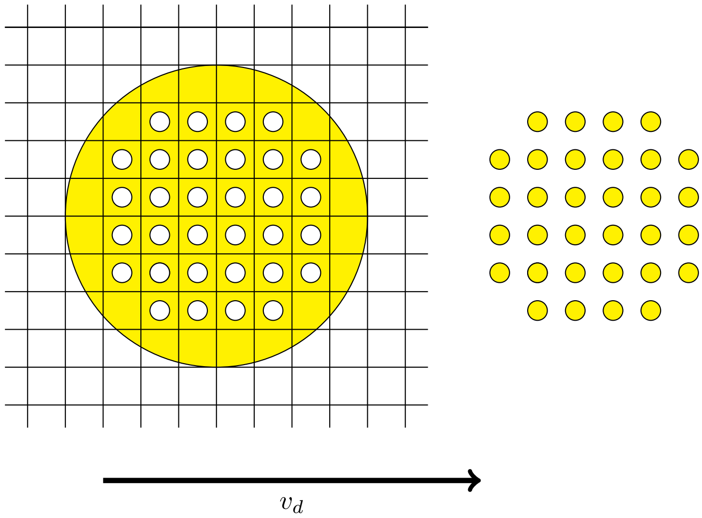

Figure 1: The domains Ω_n

We have:

$$P(\Omega_n) = P(B) + 2 \times \text{Card}(I_n) P(T_1^n) \leq P(B) + M_d n^d a_{d,n}^{d-1} \underset{n\to\infty}{\longrightarrow} P(B),$$

where M_d dimensional constant.

Let $A_n : L^2(B) \to L^2(B)$ be the resolvent operator of the Dirichlet Laplacian on Λ_n, which associates to $f \in L^2(B)$ the unique solution $u \in H_0^1(\Lambda_n)$ to $-\Delta u = f$, extended by zero outside Λ_n.

[16, Theorem 1.2] shows that, for every $f \in L^2(B)$, $A_n(f)$ strongly converges to $A(f)$ in $L^2(B)$, where A is the resolvent operator of $-\Delta + \mu$ in $H^1(B)$ with Dirichlet boundary conditions on ∂B. In particular, in view of [31, Theorem 2.3.2], the eigenvalues of A_n converge to the corresponding eigenvalue of A; as a consequence, we have $\lim\limits_{n\to+\infty} \lambda_1(\Lambda_n) = \lambda_1(B) + \mu$. This implies $\lim\limits_{n\to+\infty} \lambda_1(\Omega_n) = \lim\limits_{n\to+\infty} \lambda_1(\Lambda_n) = \lambda_1(B) + \mu$.

Step 2: In this step, we analyze the effect on the perimeter and the first Dirichlet eigenvalue of adding a flat ellipsoid to a given open set, rescaled so that the total volume remains 1.

Given Ω a smooth open set of volume 1, as well as $\varepsilon \in (0,1)$ and $\alpha \in (-\infty, 1]$, we consider $\Omega^{\varepsilon,\alpha} := \left[(1-\varepsilon^d)^{\frac{1}{d}}\,\Omega\right] \cup E^{\varepsilon,\alpha}$, where $E^{\varepsilon,\alpha}$ is a translated and rescaled version of

$$\left\{(x_1, ..., x_d) \in \mathbb{R}^d \;\middle|\; \frac{x_1^2}{\varepsilon^{2(1-\alpha)d+2\alpha}} + \frac{1}{\varepsilon^{2\alpha}} \sum_{k=2}^{d} x_k^2 < 1\right\}$$

so that $\left[(1-\varepsilon^d)^{\frac{1}{d}}\,\Omega\right] \cap E^{\varepsilon,\alpha} = \emptyset$ and $|E^{\varepsilon,\alpha}| = \varepsilon^d$. Note first that $|\Omega^{\varepsilon,\alpha}| = (1-\varepsilon^d) + \varepsilon^d = 1$.
Then for every $\alpha \leq 1$ and $\varepsilon \in (0,1)$ we have by Faber-Krahn inequality:

$$\lambda_1(E^{\varepsilon,\alpha}) \geq \lambda_1\left(\varepsilon \frac{B_1}{|B_1|^{1/d}}\right) = |B_1|^{2/d} \lambda_1(B_1) \times \frac{1}{\varepsilon^2},$$

where B_1 is a ball of unit radius.

Since $\left[(1-\varepsilon^d)^{\frac{1}{d}}\,\Omega\right] \cap E^{\varepsilon,\alpha} = \emptyset$, we have that $\lambda_1(\Omega^{\varepsilon,\alpha}) = \min\Big(\lambda_1\big((1-\varepsilon^d)^{\frac{1}{d}}\Omega\big), \lambda_1(E^{\varepsilon,\alpha})\Big)$, which leads to the following fact:

$$\text{if } \varepsilon \text{ is such that } \frac{\lambda_1(\Omega)}{(1-\varepsilon^d)^{\frac{2}{d}}} \leq |B_1|^{2/d} \lambda_1(B_1) \times \frac{1}{\varepsilon^2}, \text{ then } \lambda_1(\Omega^{\varepsilon,\alpha}) = \lambda_1\big((1-\varepsilon^d)^{\frac{1}{d}}\Omega\big) = \frac{\lambda_1(\Omega)}{(1-\varepsilon^d)^{2/d}}. \quad (6)$$

On the other hand, given $\varepsilon \in (0,1)$, it is clear that the function $\alpha \in (-\infty, 1] \mapsto P(\Omega^{\varepsilon,\alpha})$ is continuous, and we have

$$P(\Omega^{\varepsilon,1}) = P((1-\varepsilon^d)^{1/d}\Omega) + P(E^{\varepsilon,1}) = (1-\varepsilon^d)^{1-1/d} P(\Omega) + \gamma_d \varepsilon^{d-1} \quad \text{and} \quad P(\Omega^{\varepsilon,\alpha}) \underset{\alpha\to-\infty}{\longrightarrow} +\infty, \quad (7)$$

where γ_d is the perimeter of the unit ball.

Conclusion: let $x > P(B)$ and $y > \lambda_1(B)$. We want to prove that there exists Ω a smooth open set of unit volume such that $P(\Omega) = x$ and $\lambda_1(\Omega) = y$. To that end, we use the previous steps, and will adjust the parameters $\mu \in (0, +\infty)$, $n \in \mathbb{N}$, $\varepsilon \in (0,1)$, $\alpha \in (-\infty, 1]$.

- First, we use step 1 above that leads to the existence of a sequence of open sets $(\Omega_n)_{n\in\mathbb{N}}$ of unit volume and such that $P(\Omega_n)$ converges to $P(B)$ and $\lambda_1(\Omega_n)$ converges to $\lambda_1(B) + \mu$ where μ will be chosen later.

- For each $n \in \mathbb{N}$, we then use the second step to obtain $\Omega_n^{\varepsilon,\alpha}$ for $\varepsilon \in (0,1)$ and $\alpha \in (-\infty, 1]$. We notice now that

$$\text{if } \lambda_1(\Omega_n) < y \text{ then one can find } \varepsilon_n = \varepsilon_n(y) \in (0,1) \text{ such that } y = \frac{\lambda_1(\Omega_n)}{(1-\varepsilon_n^d)^{2/d}}.$$

  We therefore assume from now on that $\mu < y - \lambda_1(B)$ and n is large enough so that $\lambda_1(\Omega_n) < y$. By assuming also that μ is close to $y - \lambda_1(B)$, we have $\lambda_1(\Omega_n)$ as close to y as we want for n large enough, and then clearly ε_n is close to 0.

  In particular this implies $\frac{\lambda_1(\Omega_n)}{(1-\varepsilon_n^d)^{\frac{2}{d}}} \leq \frac{|B_1|^{2/d}\lambda_1(B_1)}{\varepsilon_n^2}$, so using (6), this leads to

$$\lambda_1(\Omega_n^{\varepsilon_n,\alpha}) = y,$$

  independently of $\alpha \in (-\infty, 1]$.

- Finally, as we just noticed that one can assume ε_n as small as we want, and as $P(\Omega_n)$ is close to $P(B)$ if n is large, the first formula in (7) shows that $P(\Omega_n^{\varepsilon_n,1}) \leq x$, and therefore by continuity of $\alpha \in (-\infty, 1] \mapsto P(\Omega^{\varepsilon_n,\alpha})$ and using the second part of (7), we deduce that there exists α such that $P(\Omega_n^{\varepsilon_n,\alpha}) = x$.

This concludes the proof. □

Remark 2.1 *As explained in the introduction (comments on Theorem 1.1), if $\mathcal{O}'$ is a class of open domains that may contain nonsmooth sets, say for example the class of open subsets of $\mathbb{R}^d$, the diagram $\mathcal{D}_{\mathcal{O}'}$ depends on the choice of the perimeter. For example, if we consider the distributional (De Giorgi's) perimeter, we are able to prove*

$$\mathcal{D}_{\mathcal{O}'} = \Big(\big[P(B), +\infty\big) \times \big(\lambda_1(B), +\infty\big) \Big) \cup \big\{ \big(P(B), \lambda_1(B)\big) \big\},$$

which differs from $\mathcal{D}_{\mathcal{O}}$ as it contains the vertical half-line $\{(P(B), \ell), \quad \ell > \lambda_1(B)\}$.

- *if we take $\Omega \in \mathcal{O}'$ such that $\lambda_1(\Omega) = \lambda_1(B)$, then the H^1-capacity of the symmetrical difference $\Omega \Delta B$ is equal to zero, which also implies that its d-dimensional Lebesgue measure is also equal to zero. Thus since the distributional perimeter doesn't detect sets with zero d-dimensional Lebesgue measure we have $P(\Omega) = P(B)$, and thus the horizontal half line $(P(B), +\infty) \times \{\lambda_1(B)\}$ is not in the diagram.*

- *On the other hand, if we take $\ell > \lambda_1(B)$, we are able to construct a set $K_\ell \in \mathcal{O}'$ with unit measure such that $P(K_\ell) = P(B)$ and $\lambda_1(K_\ell) = \ell$. Let us introduce $r_0, r_1 > 0$, such that:*

  - *r_1 is the radius of the ball $B \subset \mathbb{R}^d$ of unit measure.*
  - *r_0 is chosen such that $\lambda_1\big(\{x \in \mathbb{R}^d, \ \ \|x\| < r_0\}\big) = \ell$, so in particular $r_0 < r_1$.*

  One can then choose $N_\ell \in \mathbb{N}^*$ large enough so that

  - $\forall k \in [\![0, N_\ell - 1]\!], \quad \lambda_1\left(\left\{x \in \mathbb{R}^d, \ \ r_0 + \frac{k(r_1-r_0)}{N_\ell} < \|x\| < r_0 + \frac{(k+1)(r_1-r_0)}{N_\ell}\right\}\right) > \ell.$

  We take

$$K_\ell := B \backslash \bigcup_{k=0}^{N_\ell - 1} \left\{ x \in \mathbb{R}^d, \ \ \|x\| = r_0 + \frac{k(r_1 - r_0)}{N_\ell} \right\}.$$

  We have $\lambda_1(K_\ell) = \lambda_1\big(\{x \in \mathbb{R}^d, \ \ \|x\| < r_0\}\big) = \ell$ and $P(K_\ell) = P(B)$, because the d-dimensional Hausdorff measure of $\bigcup_{k=0}^{N_\ell-1} \big\{x \in \mathbb{R}^d, \ \ \|x\| = r_0 + k \times (r_1 - r_0)/N_\ell\big\}$ is equal to zero and thus not detected by the De Giorgi's perimeter.

3 The case of convex domains

Finding estimates of λ_1 via geometric quantities is a question that has interested various communities. If Theorem 1.1 shows that Faber-Krahn and isoperimetric inequalities form a complete system of inequalities in the case of open sets, this is no longer the case if one restricts the class of domains to convex or simply connected ones. We focus in this section on the case of convex sets, see Section 4.3 for some comments on the case of simply connected sets.

3.1 Known inequalities and numerical simulations

Let us recall the well-known inequalities providing estimates of λ_1 in terms of perimeter and volume:

1. One early result in this direction is due to G. Polya who proved in [49] (1959) that for any convex planar domain Ω one has:
$$\lambda_1(\Omega) < \frac{\pi^2}{4}\left(\frac{P(\Omega)}{|\Omega|}\right)^2. \tag{8}$$
This inequality actually holds for simply connected planar sets, see [48]. It is also sharp, as equality is attained asymptotically by a family of vanishing thin rectangles. It is noticed in [34] that Polya's proof of inequality (8) holds for convex sets in higher dimensions, and the authors extend it to a larger class of sets. Recently, a generalization for $p \in (1, +\infty)$ in the case of the first p-Laplacian eigenvalue was obtained, see [21, 10].

2. Another classical result is proven by E. Makai in [44] (1960): it gives a lower estimate of the fundamental frequency of a planar convex set Ω:
$$\lambda_1(\Omega) > \frac{\pi^2}{16}\left(\frac{P(\Omega)}{|\Omega|}\right)^2. \tag{9}$$
The inequality is sharp, as equality is attained asymptotically by a family of vanishing thin triangles. This result was recently extended to higher dimensions by L. Brasco [9, Corollary 5.1.]: for $d \geq 2$, he proves:
$$\forall \Omega \in \mathcal{K}^d, \qquad \lambda_1(\Omega) \geq \left(\frac{\pi}{2d}\right)^2\left(\frac{P(\Omega)}{|\Omega|}\right)^2, \tag{10}$$
which is also sharp, as equality is attained asymptotically by a certain family of "collapsing pyramids". Note that [9] also generalizes such an inequality for the first p-Laplacian eigenvalue, where $p \in (1, +\infty)$.

3. The Payne-Weinberger's inequality [48] states that for every planar, open and simply connected set Ω, one has:
$$\lambda_1(\Omega) - \lambda_1(B) \leq \lambda_1(B)\left(\frac{1}{J_1^2(j_{01})} - 1\right)\left(\frac{P(\Omega)^2}{4\pi|\Omega|} - 1\right), \tag{11}$$
where B is the disk of same measure as Ω and J_1 is the Bessel function of the first kind of order one and j_{01} is the first zero of the Bessel function of the first kind and of order zero. Moreover, equality is achieved only when Ω is a disk. For large values of $P(\Omega)$, this inequality is weaker than (8). But for values of $P(\Omega)$ close to $P(B)$, (11) provides a quantitative estimate of the Faber-Krahn deficit $\lambda_1(\Omega) - \lambda_1(B)$ by the isoperimetric deficit. It shows in particular that when the perimeter of Ω is close to the perimeter of the ball with the same measure, then the eigenvalues are also close to each other. One can find results in the same spirit for convex domains in arbitrary dimensions in [8, 21].

The stated inequalities give an explicit region in $\mathbb{R}^2$ which contains $\mathcal{D}_{\mathcal{K}^2}$ and is, up to our knowledge, the smallest known set containing $\mathcal{D}_{\mathcal{K}^2}$, see Figure 2.

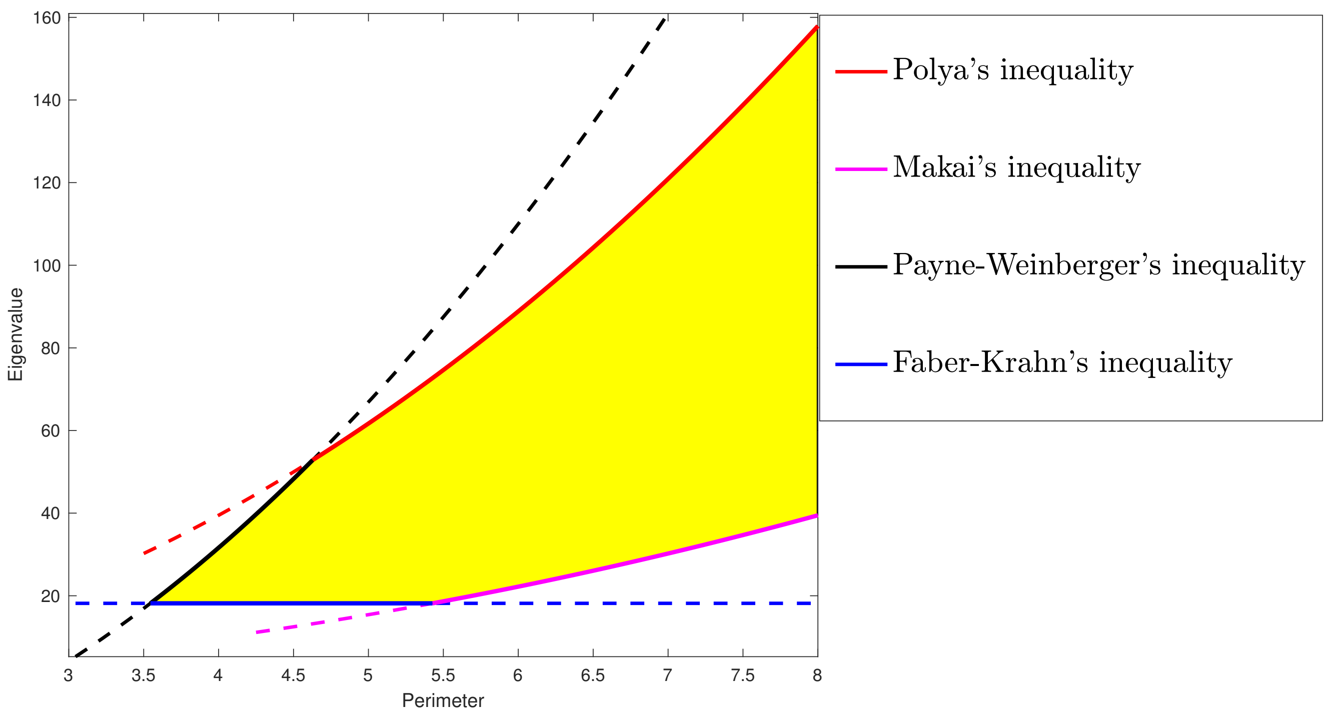

Figure 2: The smallest known domain that contains the diagram (in yellow).

In order to have an idea on the shape of $\mathcal{D}_{\mathcal{K}^2}$, P. Antunes and P. Freitas [2] generated random convex polygons of unit area and whose number of sides is between 3 and 8. In this paper, we first get a slight improvement of the numerical diagram by generating 10^5 polygons whose numbers of sides are between 3 and 30, see Figure 3. Note that the problem of generating convex polygons is rather interesting in itself: in [51], one can find a brief introduction and an efficient method of generating random convex polygons, the algorithm is based on a work of P. Valtr [56]. We notice that with these random polygons we get a quite good description of the lower boundary of the diagram, in contrast with the upper part of the diagram part which seems more "sparse". This may be explained by the fact that the domains which lay on the lower boundary are polygons while those on the upper one are smooth (see Corollary 3.13). We also notice:

- on one hand, that regular polygons lay on the lower boundary of the diagram as well as superequilateral triangles (that is, an isosceles triangle whose aperture (angle between its two equal sides) is greater that $\pi/3$).
- on the other hand, that we expect thin stadiums (domains obtained by adding two half disks to the extremities of a rectangle) to be a good approximation of domains describing the upper part of the diagram: it is easy to prove that they realize asymptotically equality in (8), and they are better candidates that any random polygons or shapes we have tested.

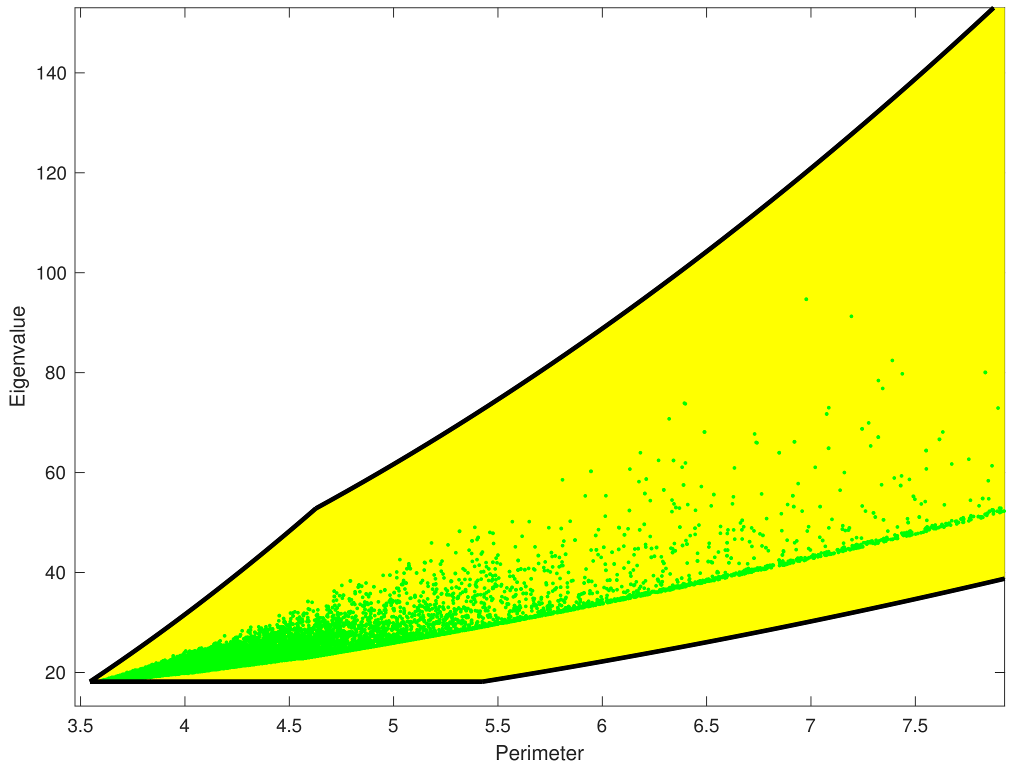

Figure 3: Blaschke-Santaló diagram obtained by generating 10^5 random convex polygons with at most 30 edges.

By adding the later special shapes to the diagram, one can obtain an improved version of $\mathcal{D}_{\mathcal{K}^2}$, see Figure 4: indeed we note that thanks to Theorem 3.14, we can say that it contains the surface lying between the lowest points of the diagram (given by random polygons) and the one given by the stadiums: this zone provides an improved numerical estimation of the diagram, see Figure 4. Actually, since the problem of theoretically finding the extremal shapes (those on the boundary of the diagram) is most certainly challenging (see Section 4.1) and actually likely unreachable, it is interesting to try to provide numerical computation of optimal shapes. Then, once a precise description of the upper and lower boundaries is obtained, from Theorem 1.2 this implies a precise description of the diagram. As mentioned before, we prove in Corollary 3.13 that the domains realizing the lower boundary of the diagram are polygons while those realizing the upper one are quite smooth ($C^{1,1}$): this suggests that we should use two different shape optimization approaches. We refer to [26] for a more detailed numerical study of the optimal shapes describing the boundary of $\mathcal{D}_{\mathcal{K}^2}$ and also a numerical study of other Blaschke-Santaló diagrams.

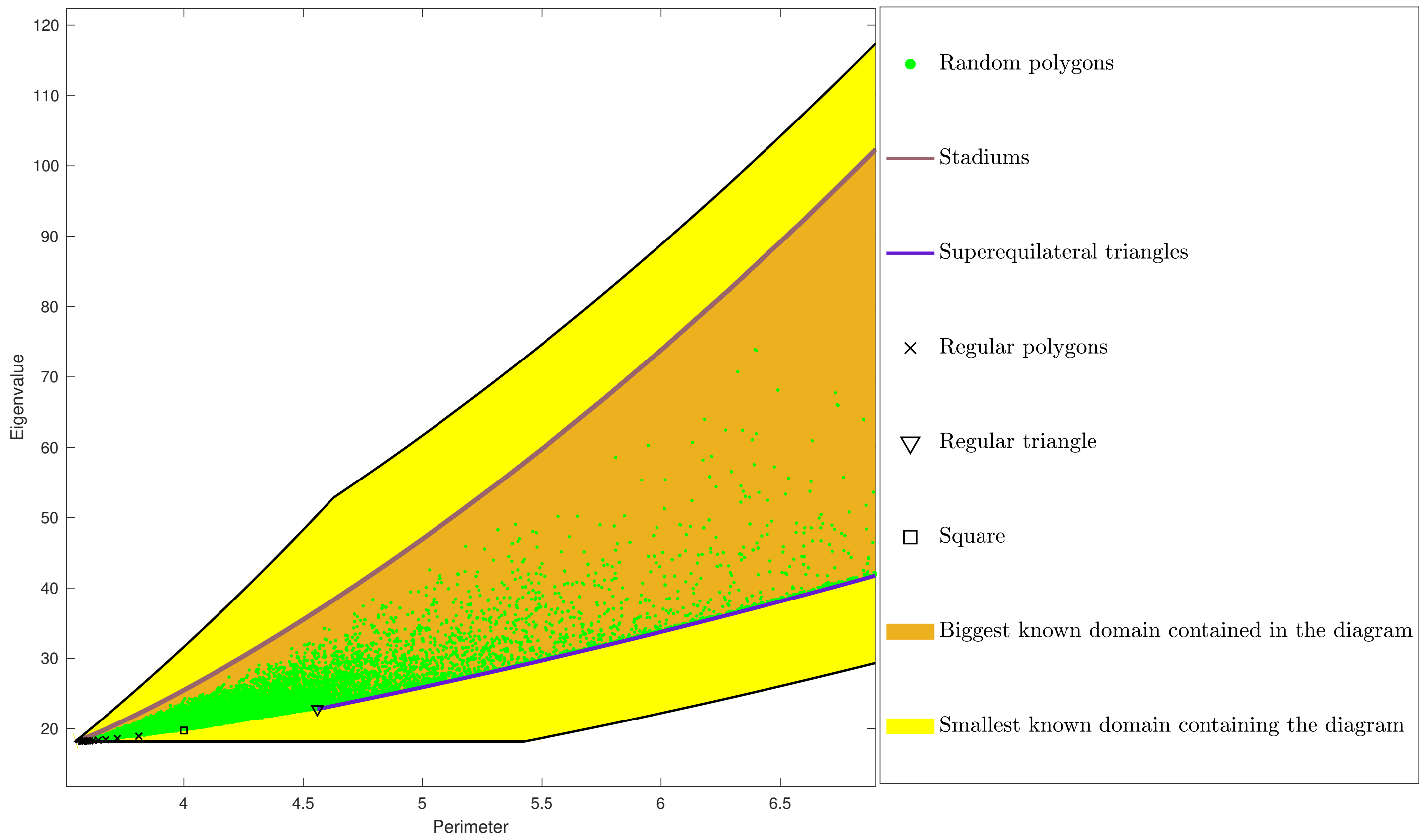

Figure 4: An improved description of the diagram.

We note that by taking advantage of (2), it is also classical to represent Blaschke-Santaló diagram as subset of $[0,1]^2$, in our situation, this means to consider the set $\{(P(B)/P(\Omega), \lambda_1(B)/\lambda_1(\Omega)) \mid \Omega \in \mathcal{K}_1^2\}$, see Figure 5 below.

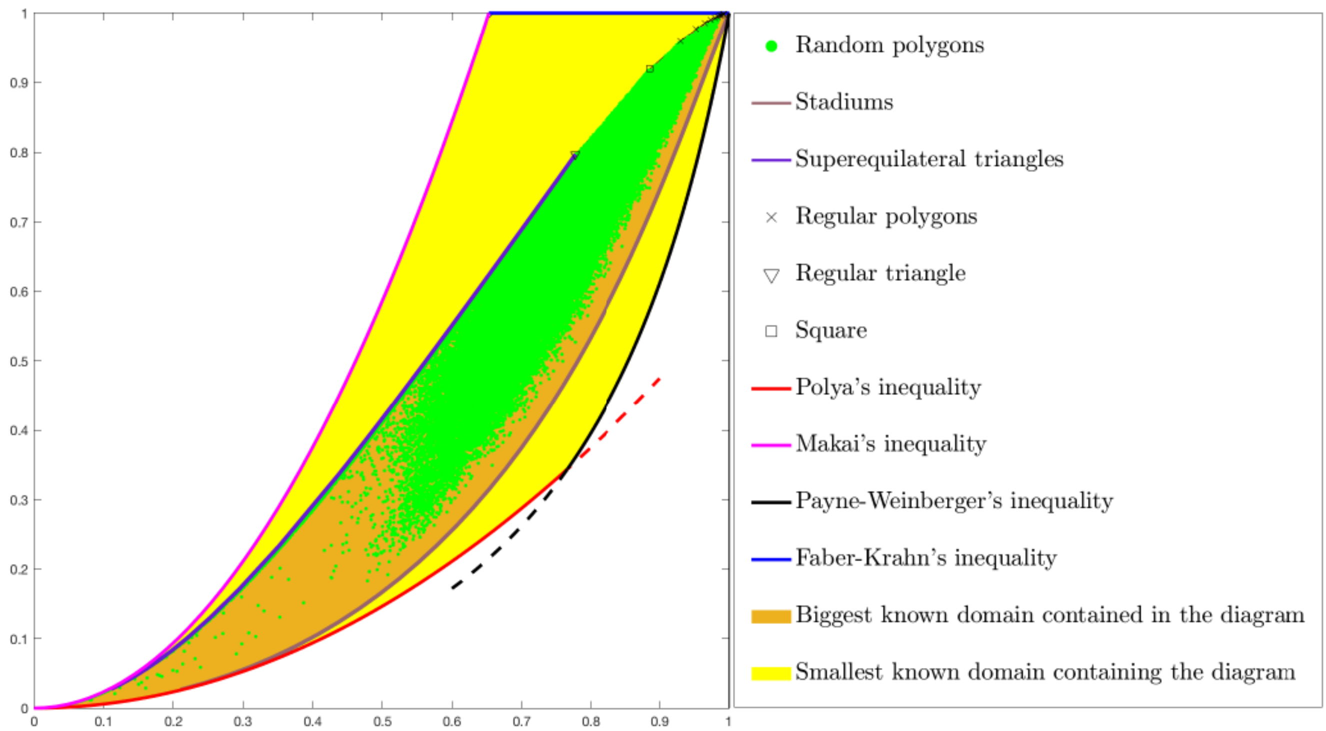

Figure 5: Blaschke-Santaló diagram represented in $[0,1]^2$.

3.2 Proof of Theorem 1.2

As the proof of Theorem 1.2 is quite involved, we proceed in several paragraphs: we first prove that the diagram is closed and path-connected, which rely on the use of Hausdorff convergence and classical results, see Proposition 3.2. Then in Section 3.2.3 we state and prove the main perturbation lemma (Lemma 3.5). With these preliminaries, we are in position to prove the four assertions of Theorem 1.2:

1. see Theorem 3.14,
2. see Theorem 3.9,
3. see Corollary 3.13,
4. see Proposition 3.7 and Corollaries 3.17 and 3.20,

where the proofs of Theorems 3.9 and 3.14 are both using Lemma 3.5. Finally, we note that, as their proofs do not rely on the perturbation lemma, Propositions 3.2 and 3.7 and Corollary 3.17 are stated and proved for arbitrary dimension $d \geq 2$.

3.2.1 Strategy of proof of the first and second assertions of Theorem 1.2

Before detailing the proofs, let us first give a few comments on the strategy of proof of the first two assertions of Theorem 1.2, which are consequences of Lemma 3.5. We decompose the proof into two main steps (both steps use Lemma 3.5): in Theorem 3.9 we define f and g as the lower and upper parts of the diagram (see (14) and (15)), and show that these functions are continuous and strictly increasing. Then in Theorem 3.14 we show (3) which implies that $\mathcal{D}_{\mathcal{K}^2}$ is simply connected: as already mentioned in the introduction the simple connectedness property of a Blaschke-Santaló diagram may be rather complicated to prove. If we were able to find explicitly the extremal domains (those who are on the upper and lower boundaries of the diagram) then we could use them to construct continuous paths via Minkowski sums, relating the upper boundary to the lower one, and prove that this process fills all the surface between the upper and lower curves (in fact it is because one can observe that these explicit optimal sets have a continuous dependence in the abscissa x: this seems to be a difficult statement to achieve without knowing explicitly these optimal shapes). In our situation, finding the explicit extremal domains is at least very challenging (see Conjecture 1 for example) and very likely impossible. Nevertheless, we manage to overpass this difficulty and give a proof of the simple-connectedness of the diagram without knowing the extremal sets (see the proof of Theorem 3.14): the proof is also based on the construction of suitable Minkowski paths and the use of the perturbation Lemma 3.5. We believe that our approach can be generalized and applied to other diagrams, in the sense that once a similar perturbation lemma is achieved for a triplet of functionals (instead of $(P, \lambda_1, |\cdot|)$), a similar strategy can be used to obtain qualitative results for its Blaschke-Santaló diagram.

3.2.2 The diagram is closed

We recall the following definition:

Definition 3.1 *The Minkowski sum of two subsets X and Y of $\mathbb{R}^d$ is the set $X + Y := \{x + y,\ (x, y) \in X \times Y\}$.*

Proposition 3.2 *Take $d \geq 2$, the diagram $\mathcal{D}_{\mathcal{K}^d}$ is a closed and connected by arcs subset of $\mathbb{R}^2$.*

Proof.

- Let $(x_n, y_n)_n$ a sequence of elements of $\mathcal{D}_{\mathcal{K}^d}$ converging to (x, y) in $\mathbb{R}^2$. Let us show that $(x, y) \in \mathcal{D}_{\mathcal{K}^d}$.

  We have, by definition, the existence of a sequence $(\Omega_n)_n$ of convex open sets such that

  $$\forall n \in \mathbb{N}, |\Omega_n| = 1,\ \ P(\Omega_n) = x_n \ \text{ and } \ \lambda_1(\Omega_n) = y_n.$$

  We recall that for any $\Omega \in \mathcal{K}^d$, one has the following inequality:

  $$d(\Omega) < C_d \frac{P(\Omega)^{d-1}}{|\Omega|^{d-2}}, \tag{12}$$

  where $d(\Omega)$ denotes the diameter of Ω and C_d is a dimensional constant, see [24, Lemma 4.1].

  In particular, the sequence $(P(\Omega_n))_n$ is bounded (because it is convergent), since the sets $(\Omega_n)_n$ are in $\mathcal{K}_1^d$, by (12), $(d(\Omega_n))_n$ is also bounded, and given that the considered functionals are invariant by translation, we can assume that the domains $(\Omega_n)_n$ are contained in a bounded box. Then by Blaschke selection Theorem (see for example [53, Th. 1.8.7]), there exists a convex domain Ω^* such that (Ω_n) converges up to a subsequence (for which we keep the notation (Ω_n)) for the Hausdorff distance to Ω^*.

  It is well known that the involved functionals (perimeter, volume and λ_1) are continuous for the Hausdorff distance among convex bodies, see for example [53] and [32, Theorem 2.3.17]. So we can write:

  $$\begin{cases} |\Omega^*| = \lim\limits_{n\to+\infty} |\Omega_n| = 1 \\ P(\Omega^*) = \lim\limits_{n\to+\infty} P(\Omega_n) = x \\ \lambda_1(\Omega^*) = \lim\limits_{n\to+\infty} \lambda_1(\Omega_n) = y \end{cases}$$

  and this concludes the proof.

- Take $\Omega_0,\Omega_1\in\mathcal{K}_1^d$, we denote $\Omega_t:=\frac{(1-t)\Omega_0+t\Omega_1}{|(1-t)\Omega_0+t\Omega_1|^{1/d}}$, since $t\in[0,1]\longmapsto(1-t)\Omega_0+t\Omega_1\in(\mathcal{K}^d,d^H)$ and the functionals $(|\cdot|,P,\lambda_1)$ are continuous for the Hausdorff distance, we have by composition that $t\in[0,1]\longmapsto\big(P(\Omega_t),\lambda_1(\Omega_t)\big)\in\mathcal{D}_{\mathcal{K}^d}\subset\mathbb{R}^2$ is also continuous and relates Ω_0 to Ω_1.

□

Corollary 3.3 *Take $d\geq 2$, for every $p\geq P(B)$ and $l\geq\lambda_1(B)$, the optimization problems*

$$\inf/\sup\big\{\lambda_1(\Omega)\ /\ \Omega\in\mathcal{K}^d,|\Omega|=1 \text{ and } P(\Omega)=p\big\}\quad\text{and}\quad\inf/\sup\big\{P(\Omega)\ /\ \Omega\in\mathcal{K}^d,|\Omega|=1 \text{ and } \lambda_1(\Omega)=l\big\}$$

have solutions.

Proof. Take $p\geq P(B)$, by inequalities (8) and (10) and the positivity of λ_1 and the perimeter, we have:

$$\forall y\in\mathbb{R}\text{ such that }(p,y)\in\mathcal{D}_{\mathcal{K}^d},\qquad 0\leq y\leq\frac{\pi^2}{4}p^2,$$

$$\forall x\in\mathbb{R}\text{ such that }(x,l)\in\mathcal{D}_{\mathcal{K}^d},\qquad 0\leq x\leq\frac{2d}{\pi}l,$$

this implies that the supremum and infimum of $\{y\ /\ (p,y)\in\mathcal{D}_{\mathcal{K}^d}\}$ (resp. $\{x\ /\ (x,l)\in\mathcal{D}_{\mathcal{K}^d}\}$) are finite. If $(y_n)_n$ (resp. (x_n)) is a minimizing or maximizing sequence (i.e. such that $\lim\limits_{n\to+\infty}y_n=\inf/\sup\{y\ /\ (p,y)\in\mathcal{D}_{\mathcal{K}^d}\}$ and $\lim\limits_{n\to+\infty}x_n=\inf/\sup\{x\ /\ (x,l)\in\mathcal{D}_{\mathcal{K}^d}\}$), then the sequence $(p,y_n)_n$ (resp. $(x_n,l)_n$) converges in $\mathbb{R}^2$ and thus by Proposition 3.2 the limit is in the closed set $\mathcal{D}_{\mathcal{K}^d}$, thus the existence of solutions of the problems in $\mathcal{K}^d$.

□

3.2.3 Main lemma

In the following, we will denote

$$\mathcal{K}_1^d:=\{\Omega\in\mathcal{K}^d,\ |\Omega|=1\},\qquad\text{and}\qquad\mathcal{K}_{1,p}^d:=\{\Omega\in\mathcal{K}^d,\ |\Omega|=1,P(\Omega)=p\},$$

for $d\geq 2$ and $p\geq P(B)$ with B being the ball of $\mathbb{R}^d$ of volume 1.

Before stating the perturbation lemma, we recall useful classical result on the volume of the Minkowski sum of convex sets. For more details on the Brunn-Minkowski theory, we refer for example to [53].

Proposition 3.4 *There exist $d+1$ bilinear (for Minkowski sum and dilatation) forms $W_k:\mathcal{K}^d\times\mathcal{K}^d\longrightarrow\mathbb{R}$, for $k\in[\![0;d]\!]$, named Minkowski mixed volumes, such that for every $K_1,K_2\in\mathcal{K}^d$ and $t_1,t_2\in\mathbb{R}^+$, we have:*

$$|t_1K_1+t_2K_2|=\sum_{k=0}^{d}\binom{d}{k}t_1^{d-k}t_2^kW_k(K_1,K_2).\tag{13}$$

Moreover, the W_k are continuous for the Hausdorff distance, in the sense that if two sequences of convex bodies $(K_1^n)_n$ and $(K_2^n)_n$ converge to some convex bodies K_1 and K_2 both for the Hausdorff distance, one has:

$$\lim_{n\to+\infty}W_k(K_1^n,K_2^n)=W_k(K_1,K_2).$$

Now, we state the perturbation Lemma.

Lemma 3.5 *(Perturbation Lemma) We endow the space of convex bodies with the Hausdorff distance. We have:*

1. *the ball is the only local minimizer of the perimeter in $\mathcal{K}_1^2$;*
2. *the ball is the only local minimizer of λ_1 in $\mathcal{K}_1^d$, where $d\geq 2$;*
3. *there is no local maximizer of the perimeter in $\mathcal{K}_1^2$;*
4. *a $C^{1,1}$ convex domain cannot be a local maximizer of λ_1 in $\mathcal{K}_1^2$.*

Remark 2 *Notice that one of the main difficulties for this lemma is to show that one can perturb a given convex domain in order to increase or decrease its perimeter or its eigenvalue, and still remain convex. Of course, if the domain is smooth and uniformly convex, such perturbations are easy to build. But it is a difficult task, in general, to build any perturbation of a general convex domain, see for example [37, 38]. This mainly explains why the first, third and fourth points are only given when $d=2$. Note that we trust that the first point could easily be obtained for the perimeter, using the same strategy as the for the second point: as we will use this result only in dimension 2, we chose to show a more elementary proof for the first point, that works well in dimension two but does not seem easy to adapt to higher dimension.*

Proof. We prove each assertion:

1. Let $\Omega \in \mathcal{K}_1^2\backslash\{B\}$. We use the Minkowski sum to build a perturbation of Ω that decreases the perimeter. We denote B_1 the ball of radius 1 (which is not the same as B whose volume is 1); then, given $s > 0$ sufficiently small, Steiner formulas give:
$$|\Omega + sB1| = |\Omega| + P(\Omega)s + |B_1|s^2, \qquad \text{and} \qquad P(\Omega + sB_1) = P(\Omega) + sP(B_1),$$
so considering
$$\Omega_s := \frac{(\Omega + sB_1)}{\sqrt{|\Omega + sB_1|}} \in \mathcal{K}_1^2,$$
where $s > 0$, we obtain
$$P(\Omega_s) = \frac{P(\Omega + sB_1)}{\sqrt{|\Omega + sB_1|}} = \frac{P(\Omega) + sP(B_1)}{\sqrt{|\Omega| + P(\Omega)s + |B_1|s^2}}.$$
By denoting $f : s \in [0, +\infty) \mapsto P(\Omega_s)$, a simple computation shows
$$f'(0) = \frac{P(B_1) - \frac{P(\Omega)^2}{2|\Omega|}}{\sqrt{|\Omega|}}$$
which is such that $f'(0) < 0$ by isoperimetric inequality $\frac{P^2(\Omega)}{|\Omega|} \geq 4\pi = 2P(B_1)$.
So for $s > 0$ small enough, we have $P(\Omega_s) < P(\Omega_0) = P(\Omega)$. Since $\Omega + sB$ converges to Ω when $s \to 0$ and the measure is continuous, both for the Hausdorff distance in $\mathcal{K}^2$, we have that $\Omega_s \underset{s\to 0}{\longrightarrow} \Omega$ for the Hausdorff distance. This shows that Ω is not a local minimizer of the perimeter in $\mathcal{K}_1^2$.

2. Let $\Omega \in \mathcal{K}_1^d\backslash\{B\}$. We now build a perturbation that decreases λ_1: as Ω is not a ball, there exists a hyperplane H such that Ω is not symmetric with respect to H. We choose coordinates so that $H = \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R}, y = 0\}$. We introduce the sets:
$$I_\Omega := \left\{x \in \mathbb{R}^{d-1} \ / \ \exists y \in \mathbb{R}, \ \ (x, y) \in \Omega\right\} \ \text{ and } \ J_\Omega^x := \{y \in \mathbb{R} \ / \ \ (x, y) \in \Omega\} \ \text{ where } x \in I_\Omega.$$
Since Ω is convex and of volume 1, it is bounded and non-empty, thus the sets I_Ω and J_Ω^x (where $x \in I_\Omega$) are also convex, bounded and non-empty. We can then introduce $y_1, y_2 : I_\Omega \to \mathbb{R}$ such that:
$$\forall x \in I_\Omega, \quad y_1(x) = \inf J_\Omega^x \ \text{ and } \ y_2(x) = \sup J_\Omega^x.$$
By convexity of Ω, we can write:
$$\Omega = \{(x, y) \in \mathbb{R}^{d-1} \times \mathbb{R}, \ x \in I_\Omega \ \text{ and } \ y_1(x) < y < y_2(x)\},$$
with y_1 convex and y_2 concave.
Now, we define a displacement field $V : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ by:
$$V(x, y) = \Big(0, -\frac{1}{2}\big(y_1(x) + y_2(x)\big)\Big).$$
Let $\Omega_t := (Id + tV)(\Omega)$, for $0 \leq t \leq 1$, where $Id : x \in \mathbb{R}^d \longmapsto x \in \mathbb{R}^d$ is the identity map. The process of deforming $\Omega = \Omega_0$ to the symmetric set Ω_1 through the path $t \longmapsto \Omega_t$ is a variant of the so called continuous Steiner symmetrization (see [13] for example). It is well known that the volume is preserved throughout this continuous process; moreover, we can show that convexity of domains is also preserved. Indeed, for every $t \in [0, 1]$:
$$\Omega_t = \left\{(x, y) \in \mathbb{R}^2, \ x \in I \ \text{ and } \ \left(1 - \frac{t}{2}\right) y_1(x) - \frac{t}{2} y_2(x) < y < -\frac{t}{2} y_1(x) + \left(1 - \frac{t}{2}\right) y_2(x)\right\}.$$
Yet, the facts that I_Ω is convex, the function $-\frac{t}{2}y_1 + \left(1 - \frac{t}{2}\right) y_2$ is concave and the function $\left(1 - \frac{t}{2}\right) y_1 - \frac{t}{2}y_2$ is convex yield that Ω_t is convex.
Moreover, we have that $\Omega_t \underset{t\to 0^+}{\longrightarrow} \Omega$ for the Hausdorff distance. Indeed
$$\begin{aligned} d^H(\Omega_t, \Omega) &= d^H(\partial\Omega_t, \partial\Omega) \\ &:= \max\big(\sup_{a\in\partial\Omega_t} \inf_{a'\in\partial\Omega} \|a - a'\|, \sup_{b\in\partial\Omega} \inf_{b'\in\partial\Omega_t} \|b - b'\|\big) \\ &\leq \frac{t}{2} \sup_{x\in I} |y_1(x) + y_2(x)| \underset{t\to 0^+}{\longrightarrow} 0. \end{aligned}$$
Finally, as Ω is not symmetric with respect to H, it was proven in [15, Lemma 3.1] that the continuous symmetrization strictly decreases the first eigenvalue, and since $\Omega_t \underset{t\to 0^+}{\longrightarrow} \Omega$ for the Hausdorff distance, we conclude that Ω is not a local minimizer of λ_1 is $\mathcal{K}_1^d$.

3. On one hand, from [38, Theorem 2.1, Remark 2.2], we deduce that any local maximizer of the perimeter under volume and convexity constraints must be a polygon (more precisely, in [38] they consider local minimum where the word local in understood for the $W^{1,\infty}$-norm on the so-called gauge function of the set ; but this is in particular the case if we consider local minimum for the Hausdorff distance).

   On the other hand, no polygon can be a local maximizer of the perimeter in $\mathcal{K}_1^2$: to prove this, we use a parallel chord movement. More precisely if Ω is a polygon, one can consider A, B, C three consecutive corners so that ABC forms a triangle. One can move B along the line passing through B and being parallel to the line (AC). This way, the volume is preserved, and the perimeter must increase when moving B away from the perpendicular bisector of $[A, C]$ (which is possible at least in one direction).

   From the two previous remarks, we deduce that there is no local maximizer of the perimeter under convexity and volume constraints.

4. In [39], the authors show that a local maximum of λ_1 in $\mathcal{K}_1^2$ is a polygon, this proves that any smooth (in particular $C^{1,1}$) domain is not a local maximizer of λ_1 in $\mathcal{K}_1^2$.

One can deduce the following result which gives a refinement of Lemma 3.5 concerning the perimeter functional:

Corollary 3.6 *Let $\Omega \in \mathcal{K}_1^2$. Then for any sequence (p_n) converging to $P(\Omega)$ such that $p_n \geq P(B)$ for all $n \in \mathbb{N}$, there exists a sequence (Ω_n) of elements of $\mathcal{K}_1^2$ converging to Ω for the Hausdorff distance, and such that $P(\Omega_n) = p_n$ for all $n \in \mathbb{N}$.*

Proof. If $\Omega \neq B$, by Lemma 3.5, we can build a sequence $(K_n)_n$ of elements of $\mathcal{K}_1^2$ converging to Ω for the Hausdorff distance, and such that $P(K_{2n}) < P(\Omega) < P(K_{2n+1})$ for every $n \in \mathbb{N}$ and since (p_n) is bounded, we can also assume K_0 and K_1 such that $p_n \in [P(K_0), P(K_1)]$ for all $n \in \mathbb{N}$. We will use this sequence (K_n) to build (Ω_n) : for $n \in \mathbb{N}$,

- if $p_n = P(\Omega)$, we take $\Omega_n = \Omega$ and define $\sigma(n) = n$.
- if $p_n > P(\Omega)$, then as $P(K_{2k+1})$ converges to $P(\Omega)$ from above and $p_n \leq P(K_1)$, we can define $\sigma(n) := \max\{2k+1 \ / \ P(K_{2k+1}) \geq p_n\}$and consider the function:

$$\phi_n : t \longmapsto P\left(\frac{tK_{\sigma(n)} + (1-t)\Omega}{\sqrt{\left|tK_{\sigma(n)} + (1-t)\Omega\right|}}\right).$$

  This function ϕ_n is continuous and since $p_n \in [\phi_n(0), \phi_n(1)] = [P(\Omega), P(K_{\sigma(n)})]$, by the intermediate value Theorem there exists $t_n \in [0,1]$ such that $\phi_n(t_n) = p_n$, we then take:

$$\Omega_n := \frac{t_n K_{\sigma(n)} + (1-t_n)\Omega}{\sqrt{\left|t_n K_{\sigma(n)} + (1-t_n)\Omega\right|}} \in \mathcal{K}^2_{1,p_n}.$$

- if $p_n < P(\Omega)$, we set $\sigma(n) := \max\{2k \mid P(K_{2k}) \leq p_n\}$ and choose Ω_n as in the previous case.

It remains to show that the sequence (Ω_n) converges to Ω for the Hausdorff distance. If the set $I := \{n \in \mathbb{N} \ / \ p_n \neq P(\Omega)\}$ is finite, then the sequence (Ω_n) is equal to Ω for n large enough. If on the other hand I is infinite, the fact that $P(K_n) \underset{n\to+\infty}{\longrightarrow} P(\Omega)$ implies that $\lim\limits_{n\to+\infty} \sigma(n) = +\infty$, which gives $\lim\limits_{n\to+\infty} K_{\sigma(n)} = \Omega$, thus $(t_n K_{\sigma(n)} + (1-t_n)\Omega) \underset{n\to+\infty}{\longrightarrow} \Omega$ for the Hausdorff distance, then by continuity of the measure, we get that $\Omega_n \underset{n\to+\infty}{\longrightarrow} \Omega$ for the Hausdorff distance.

If $\Omega = B$, one may reproduce the same strategy as above by considering a sequence $(K_n)_n$ of elements of $\mathcal{K}_1^2$ converging to B for the Hausdorff distance such that $P(B) < P(K_n)$ for every $n \in \mathbb{N}$ (second assertion of Lemma 3.5) and then use Minkowski sums and intermediate value Theorem to construct the sets (Ω_n). □

3.2.4 Study of the boundary of the diagram

We define functions f and g by:

$$\begin{array}{llll} f: & [P(B), +\infty) & \longrightarrow & \mathbb{R} \\ & p & \longmapsto & \min\left\{\lambda_1(\Omega), \Omega \in \mathcal{K}^d, |\Omega| = 1 \text{ and } P(\Omega) = p\right\} \end{array} \tag{14}$$

$$\begin{array}{llll} g: & [P(B), +\infty) & \longrightarrow & \mathbb{R} \\ & p & \longmapsto & \max\left\{\lambda_1(\Omega)\,,\ \Omega \in \mathcal{K}^d, |\Omega| = 1 \text{ and } P(\Omega) = p\right\} \end{array} \tag{15}$$

and we recall that these optimization problems admit solutions, see Corollary 3.3. By definition (and by the isoperimetric inequality), we have

$$\mathcal{D}_{\mathcal{K}^d} \subset \Big\{(x,y) \in \mathbb{R}^2 \mid x \geq P(B) \text{ and } f(x) \leq y \leq g(x)\Big\}. \tag{16}$$

In this section, we will first give the asymptotics of f and g near $+\infty$ for arbitrary dimension $d \geq 2$, then we prove the second part of Theorem 1.2, which is stated again below in Theorem 3.9. To obtain the first part of Theorem 1.2, we need to show the reverse inclusion of (16), which will be obtained with Theorem 3.14.

Proposition 3.7 *Take $d \geq 2$, we have*

$$g(x) \underset{x\to\infty}{\sim} \frac{\pi^2}{4}x^2 \quad \text{and} \quad f(x) \underset{x\to\infty}{\sim} \frac{\pi^2}{4d^2}x^2.$$

Proof. By inequalities (10) and (8), one has:

$$\forall K \in \mathcal{K}_1^d, \quad \frac{\pi^2}{4d^2}P(K)^2 \leq \lambda_1(K) < \frac{\pi^2}{4}P(K)^2.$$

Then:

$$\forall x \geq P(B), \quad \frac{\pi^2}{4d^2}x^2 \leq f(x) \leq g(x) < \frac{\pi^2}{4}x^2.$$

However, since the right- and left-hand-side inequalities are respectively attained in the limiting case of flat collapsing cuboids and collapsing pyramids (see [9, Corollary 5.1.]), we have the stated equivalences. □

Remark 3.8 *In this paper, all the study is done for shapes of volume 1. It is interesting to wonder about what would happen if one removes such constraint: we believe that in this case the diagram would be given by:*

$$\{(P(\Omega), \lambda_1(\Omega)) \mid \Omega \in \mathcal{K}^d\} = \left\{(x,y) \mid x > 0 \text{ and } y \geq \frac{\lambda_1(B)P(B)^{\frac{2}{d-1}}}{x^{\frac{2}{d-1}}}\right\},$$

where the boundary corresponds to balls. We note that the idea of "relaxing" the volume constraint has been successfully used in [43] to give some qualitative properties of the boundary of the diagram involving the first Dirichlet eigenvalue, the torsion and the volume.

Theorem 3.9 *Assume $d = 2$. Then functions f and g are continuous and strictly increasing.*

Remark 3.10 *Some of the properties of f and g come with minor efforts, namely the lower semicontinuity of f (or upper one of g). But to prove the full continuity and monotonicity, we use Lemma 3.5, and this explains why Theorem 3.9 is restricted to dimension 2. Compare to [43, Theorem 1.1] where the authors could not prove that the upper part of the diagram is the graph of a continuous and increasing function.*

Proof. We start by proving the continuity of f. Let $p_0 \in [P(B), +\infty[$.

For every $p \in [P(B), +\infty[$, by Corollary 3.3, there exists Ω_p a solution of the following minimization problem:

$$\min\left\{\lambda_1(\Omega) \mid \Omega \in \mathcal{K}_1^2 \text{ and } P(\Omega) = p\right\}.$$

- We first show an **inferior limit inequality**. Let $(p_n)_{n\geq 1}$ real sequence converging to p_0 such that

  $$\liminf_{p\to p_0} \lambda_1(\Omega_p) = \lim_{n\to+\infty} \lambda_1(\Omega_{p_n}).$$

  Up to translations, as the perimeter of $(\Omega_{p_n})_{n\in\mathbb{N}^*}$ is uniformly bounded, one may assume that the domains $(\Omega_{p_n})_{n\in\mathbb{N}^*}$ are included in a fixed ball: then by Blaschke selection Theorem, (Ω_{p_n}) converges to a convex set Ω^* for the Hausdorff distance, up to a subsequence that we denote p_n again for simplicity.

  By the continuity of the perimeter, the volume and λ_1 for the Hausdorff distance among convex sets, we have:

  $$\begin{cases} |\Omega^*| = \lim_{n\to+\infty} |\Omega_{p_n}| = 1, \\ P(\Omega^*) = \lim_{n\to+\infty} P(\Omega_{p_n}) = \lim_{n\to+\infty} p_n = p_0, \\ \lambda_1(\Omega^*) = \lim_{n\to+\infty} \lambda_1(\Omega_{p_n}) = \liminf_{p\to p_0} \lambda_1(\Omega_p). \end{cases}$$

  Then by definition of f (since $\Omega^* \in \mathcal{K}_1^2$ and $P(\Omega^*) = p_0$), we obtain:

  $$f(p_0) \leq \lambda_1(\Omega^*) = \lim_{n\to+\infty} \lambda_1(\Omega_{p_n}) = \liminf_{p\to p_0} \lambda_1(\Omega_p) = \liminf_{p\to p_0} f(p).$$

- It remains to prove **a superior limit inequality**. Let $(p_n)_{n\geq 1}$ be a real sequence converging to p_0 such that:

  $$\limsup_{p\to p_0} f(p) = \lim_{n\to+\infty} f(p_n).$$

  By Corollary 3.6, there exists a sequence $(K_n)_{n\geq 1}$ of $\mathcal{K}_1^2$ converging to Ω_{p_0} for the Hausdorff distance, and such that $P(K_n) = p_n$ for every $n \in \mathbb{N}^*$.

  Using the definition of f one can write

  $$\forall n \in \mathbb{N}^*, \quad f(p_n) \leq \lambda_1(K_n).$$

  Passing to the limit, we get:

  $$\limsup_{p\to p_0} f(p) = \lim_{n\to+\infty} f(p_n) \leq \lim_{n\to+\infty} \lambda_1(K_n) = \lambda_1(\Omega_{p_0}) = f(p_0).$$

As a consequence we finally get $\lim_{p\to p_0} f(p) = f(p_0)$, so f is continuous on $[P(B), +\infty[$. The same method can be applied to prove the continuity of g.

- We now prove that f **is strictly increasing**. Let us assume by contradiction that it is not the case: then by continuity of f and the fact that $\lim_{+\infty} f = +\infty$ (see Proposition 3.7), we deduce the existence of a local minimum of f at a point $p_0 > P(B)$. Using Corollary 3.3, this means there exists $\Omega^* \in \mathcal{K}_1^2$ and $\varepsilon > 0$ such that

$$P(\Omega^*) = p_0 \quad \text{and} \quad \forall p \in (p_0 - \varepsilon, p_0 + \varepsilon), \;\; \lambda_1(\Omega^*) = f(p_0) \leq f(p),$$

which implies

$$\forall \Omega \in \mathcal{K}_1^2 \text{ such that } P(\Omega) \in (p_0 - \varepsilon, p_0 + \varepsilon), \;\; \lambda_1(\Omega^*) \leq \lambda_1(\Omega).$$

Because of the continuity of the perimeter in $\mathcal{K}_1^2$, this would imply that Ω^* is a local minimum (for the Hausdorff distance) of λ_1 in $\mathcal{K}_1^2$, which, from the first point in Lemma 3.5 implies that Ω^* must be a ball, which in turn contradicts $P(\Omega^*) > P(B)$.

- We finally prove that g **is strictly increasing**. Assuming by contradiction that this is not the case, then there exist $p_2 > p_1 \geq P(B)$ such that $g(p_2) < g(p_1)$, and from the equality case in the isoperimetric inequality, we necessarily have $p_1 > P(B)$. Since g is continuous, it reaches its maximum on $[P(B), p_2]$ at a point $p^* \in (P(B), p_2)$, that is to say

$$\forall \Omega \in \mathcal{K}_1^2 \text{ such that } [P(B), p_2], \;\; g(p^*) \geq \lambda_1(\Omega). \tag{17}$$

Using Corollary 3.3 again, one knows that the problem

$$\min\{P(\Omega), \;\; \Omega \in \mathcal{K}_1^2 \;\; \text{and} \;\; \lambda_1(\Omega) = g(p^*)\} \tag{18}$$

admits a solution $K^* \in \mathcal{K}_1^2$.

On one hand, (17) implies that K^* is a local maximum (for the Hausdorff distance) of λ_1 in $\mathcal{K}_1^2$. From Lemma 3.5 we deduce that K^* cannot be $C^{1,1}$.

On the other hand, K^* is also a solution of (18). We want to apply the regularity result [40, Theorem 2] which shows that K^* is $C^{1,1}$, which is a contradiction. This theorem applies as, denoting $m(\Omega) = (\lambda_1(\Omega), |\Omega|) \in \mathbb{R}^2$ (which are the constraints in (18) besides the convexity constraint, the latter being dealt with by its own infinitely dimensional Lagrange multiplier, see the proof of [40, Theorem 2]), it is well known that the first order shape derivative (see for example [32] for definitions) writes:

$$\forall \xi \in C^\infty(\mathbb{R}^2, \mathbb{R}^2), \quad m'(K^*).\xi = \left(-\int_{\partial K^*} |\nabla u_1|^2 \xi \cdot n_{\partial K^*} d\sigma, \; \int_{\partial K^*} \xi \cdot n_{\partial K^*} d\sigma \right),$$

where u_1 is the first normalized Dirichlet eigenfunction on K^*: the convexity of K^* is used here to provide enough smoothness so that this formula is valid (indeed it is well-known that $u_1 \in H^2(\Omega)$ so its gradient has a trace on ∂K^*, see also [31, Theorem 2.5.1]). Therefore this shape derivative at K^* is in $L^\infty(\partial K^*)^2$ (see [40, Section 3.3] for the link between shape derivatives and derivatives in term of the gauge function as considered in [40, Theorem 2]), and also that it is onto: indeed, if it was not, we would have the existence of $c \geq 0$ such that $|\nabla u_1| = c$ on ∂K^*. With Lemma 3.11 proven just below, this would imply that K^* is a ball, which is again impossible. We conclude that g is strictly increasing, which ends the proof.

□

In the previous proof, we used the following lemma:[1]

Lemma 3.11 *Let Ω be an open and bounded convex set in $\mathbb{R}^d$, and u_1 solution of* (1)*, that is to say a first eigenfunction of the Dirichlet-Laplacian in Ω. We also assume that there exists $c \geq 0$ a constant such that*

$$|\nabla u_1| = c \text{ on } \partial\Omega. \tag{19}$$

Then Ω is a ball and $c > 0$.

Remark 3.12 *The result in Lemma 3.11 deals with a well-known problem that goes back to the famous result by J. Serrin [54]. The main difficulty here is that we do not assume regularity for Ω or u_1, except the one given by the convexity of Ω. There is an extensive literature on extensions of [54], some of which weakening these regularity assumptions, but we did not find a direct answer to the question raised in Lemma 3.11: the closest result we could find was [42, Theorem 1 and Remark (2) in Section 5]. Therefore, we adapt the regularity theory of free boundary problems by taking advantage of the convexity of Ω, which makes the context favorable.*

Proof. First note that from regularity theory, $u_1 \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$ (see for example [30]), so ∇u_1 has a trace on $\partial\Omega$, which shows that (19) has a meaning in the sense of traces. Also $u_1 \in C^0(\overline{\Omega})$ can be extended by 0 outside Ω, and then $u_1 \in C^0(\mathbb{R}^d)$.

[1]We thank Bozhidar Velichkov for helping us with the proof of this lemma.

- Let us first exclude the case $c = 0$. Assuming to the contrary that the hypotheses of the lemma are satisfied with $c = 0$, we have
$$\forall \varphi \in H^1(\Omega), \quad \int_\Omega \nabla u_1 \cdot \nabla \varphi dx = \lambda_1(\Omega) \int_\Omega u_1 \varphi dx + \int_{\partial\Omega} (\partial_n u_1)\varphi d\sigma.$$
As $\partial_n u_1 = 0$ on $\partial\Omega$ and applying this property with $\varphi \equiv 1$, we obtain $\lambda_1(\Omega) \int_\Omega u_1 dx = 0$, which is a contradiction as $u_1 > 0$ in Ω.

- Assume $c > 0$. In order to apply [54], we aim at proving that (19) implies regularity of the domain Ω. To that end, we use the theory of regularity for free boundaries: in our context, we want to apply [19, Theorem 1.2] with $f := \lambda_1(\Omega)u_1 \in C^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, which says that as Ω is a Lipschitz domain, if one can prove that (19) is valid in the sense of viscosity, then Ω must actually be $C^{1,\alpha}$ for some $\alpha > 0$. From there it is very classical with [36] that Ω is actually C^∞, which implies that $u_1 \in C^\infty(\overline{\Omega})$ and so [54] applies and provides the conclusion.

  Therefore, let us prove that $|\nabla u_1| = c$ in the sense of viscosity: this means that for every $x_0 \in \overline{\Omega}$ and every $\varphi \in C_c^2(\mathbb{R}^d)$,

  1. if $x_0 \in \Omega$, $\varphi(x_0) = u_1(x_0)$ and $\varphi \leq u_1$ (resp. $\varphi \geq u_1$), then $\Delta\varphi(x_0) \leq f(x_0)$ (resp. $\Delta\varphi(x_0) \geq f(x_0)$),
  2. if $x_0 \in \partial\Omega$, $\varphi(x_0) = u_1(x_0)$ and $\varphi_+ \leq u_1$ (resp. $\varphi_+ \geq u_1$), then $|\nabla\varphi(x_0)| \leq c$ (resp. $|\nabla\varphi(x_0)| \geq c$), where $\varphi_+ : x \in \mathbb{R}^d \longmapsto \max(\varphi(x), 0)$.

  For the first point, this follows from the regularity of u_1 inside Ω, namely $u_1 \in C^2(\Omega)$. Let us focus on the second point and take $x_0 \in \partial\Omega$ and $\varphi \in C_c^2(\mathbb{R}^d)$. In order to simplify the computations, we choose x_0 as the origin which allows to consider $x_0 = 0$: we will do a blow-up at x_0, so we denote
$$\Omega_r = \frac{\Omega}{r}, \qquad \text{and} \qquad \forall x \in \mathbb{R}^d,\ u_r(x) = \frac{u_1(rx)}{r}, \quad \varphi_r(x) = \frac{\varphi(rx)}{r}.$$
  We then claim:

  1. $(\Omega_r)_{r>0}$ is increasing and one can define
$$\Omega_0 := \bigcup_{r>0} \Omega_r$$
     which is a cone (it is the (interior of the) usual tangent of Ω at x_0 in the context of convex geometry). We also have that $(\partial\Omega_r)_{r>0}$ converges to $\partial\Omega_0$ locally in the Hausdorff sense.
  2. as $u_1 \in W^{1,\infty}(\Omega)$, up to a subsequence $(u_r)_{r>0}$ converges locally uniformly to a function u_0 defined and Lipschitz on $\mathbb{R}^d$. Moreover, as for $r > 0$, one has $-\Delta u_r(x) = rf(rx) - c\mathcal{H}^{d-1}_{|\partial\Omega_r}$ in the sense of distribution in $\mathbb{R}^d$, we have at the limit (using the previous point to justify the convergence):
$$\Delta u_0(x) = c\mathcal{H}^{d-1}_{|\partial\Omega_0}.$$
     As Ω_0 is a cone, this implies that u_0 is 1-homogeneous: indeed, for $\lambda \in (0, \infty)$, consider $u_0^\lambda : x \mapsto \frac{1}{\lambda}u_0(\lambda x)$. It is easy to see that u_0^λ has the same Laplacian as u_0 (in the sense of distribution in $\mathbb{R}^d$), so $v := u_0 - u_0^\lambda$ is harmonic in $\mathbb{R}^d$. As ∇v is bounded, from Liouville Theorem we deduce that v is affine, but as $v(0) = 0$ and $\nabla v(0) = 0$, we deduce that $v = 0$, which means u_0 is 1-homogeneous.
  3. as φ is smooth, $(\varphi_r)_{r>0}$ converges locally uniformly to an affine function $\varphi_0(x)$ that is such that, up to a choice of coordinates,
$$\forall x \in \mathbb{R}^d, \quad \varphi_0(x) := Ax_d$$
     where $A = |\nabla\varphi(x_0)|$.
  4. (a) Assume now $u_1 \geq \varphi_+$. Then $u_0(x) \geq \varphi_0(x) = Ax_d$ in $\mathbb{R}^d$. If $A = 0$ then $A \leq c$. Otherwise, we get $\{u_0 > 0\} \supset \{x_d > 0\}$. From convexity of Ω_0 we obtain equality of these two domains. Then as u_0 and $x \mapsto cx_d$ both satisfy the same Cauchy problem with conditions on $\partial\{x_d > 0\}$, we deduce that $u_0(x) = c(x_d)_+$, and then clearly $u_0 \geq \varphi_0$ implies $c \geq A$.
     (b) Assume finally that $u_1 \leq \varphi_+$. We reproduce here a proof similar to [50, Lemma 5.31]. Using that u_0 is 1-homogeneous and nonnegative, we get that the trace of u_0 on $\mathbb{S}^{d-1}$ is a first eigenfunction of the Laplace-Beltrami operator on $\Omega_0 \cap \mathbb{S}^{d-1}$ with Dirichlet boundary condition on $\partial\Omega_0 \cap \mathbb{S}^{d-1}$ corresponding to the eigenvalue $d-1$. As $\Omega_0 \cap \mathbb{S}^{d-1} \subset \mathbb{S}^{d-1}_+ := \mathbb{S}^{d-1} \cap \{x_d > 0\}$ and the Laplace-Beltrami of $\mathbb{S}^{d-1}_+$ is also $d-1$ we obtain that $\Omega_0 = \{x_d > 0\}$ and as in the previous case $u_0(x) = c(x_d)_+$ and $c \leq A$.

     We have therefore shown that $|\nabla u_1| = c$ is satisfied in the sense of viscosity, which as mentioned above, concludes the proof.

□

Theorem 3.9 allows us to prove the equivalence between 4 optimization problems.

Corollary 3.13 *Let $p > P(B)$. The following problems are equivalent:*

(I) $\min\{\lambda_1(\Omega) \mid \Omega \in \mathcal{K}_1^2 \text{ and } P(\Omega) = p\}$ *(III)* $\max\{P(\Omega) \mid \Omega \in \mathcal{K}_1^2 \text{ and } \lambda_1(\Omega) = f(p)\}$

(II) $\min\{\lambda_1(\Omega) \mid \Omega \in \mathcal{K}_1^2 \text{ and } P(\Omega) \geq p\}$ *(IV)* $\max\{P(\Omega) \mid \Omega \in \mathcal{K}_1^2 \text{ and } \lambda_1(\Omega) \leq f(p)\}$.

in the sense that any solution to one of the problem also solves the other ones. Moreover, any solution to these problems is a polygon.

Similarly the following problems are equivalent :

(I') $\max\{\lambda_1(\Omega) \mid \Omega \in \mathcal{K}_1^2 \text{ et } P(\Omega) = p\}$ *(III')* $\min\{P(\Omega) \mid \Omega \in \mathcal{K}_1^2 \text{ et } \lambda_1(\Omega) = g(p)\}$

(II') $\max\{\lambda_1(\Omega) \mid \Omega \in \mathcal{K}_1^2 \text{ et } P(\Omega) \leq p\}$ *(IV')* $\min\{P(\Omega) \mid \Omega \in \mathcal{K}_1^2 \text{ et } \lambda_1(\Omega) \geq g(p)\}$.

and any solution is $C^{1,1}$.

Proof. Let us prove the equivalence between the first four problems.

- We first show that any solution of (I) solves (II): let Ω_p be a solution to (I). Then for every $\Omega \in \mathcal{K}_1^2$ such that $P(\Omega) \geq p$, one has:
$$\lambda_1(\Omega) \geq f\big(P(\Omega)\big) \geq f(p) = \lambda_1(\Omega_p),$$
where we used the monotonicity of f given by Theorem 3.9: therefore Ω_p solves (II).

- Reciprocally, let now Ω^p be a solution of (II): we want to show that Ω^p must be of perimeter p. We notice that
$$f(p) \geq \lambda_1(\Omega^p) \geq f\big(P(\Omega^p)\big) \geq f(p),$$
where the first inequality follows as problem (II) allows more candidates than in the definition of f, and the last inequality uses again the monotonicity of f. Therefore $f(p) = f\big(P(\Omega^p)\big)$, and since f is strictly increasing, we obtain $P(\Omega^p) = p$, which implies that Ω^p solves (I).

We proved the equivalence between problems (I) and (II); equivalence between problems (III) and (IV) is shown by similar manipulations.

It remains to prove the equivalence between (I) and (III).

- Let Ω_p be a solution of (I), which means that $\Omega_p \in \mathcal{K}_1^2$, $P(\Omega_p) = p$ and $\lambda_1(\Omega_p) = f(p)$. Then for every $\Omega \in \mathcal{K}_1^2$ such that $\lambda_1(\Omega) = f(p)$ we have:
$$f(p) = \lambda_1(\Omega) \geq f\big(P(\Omega)\big),$$
thus, since f is increasing, we get $p = P(\Omega_p) \geq P(\Omega)$, which means Ω_p solves (III).

- Let now Ω'_p be a solution of (III), then we have:
$$f(p) = \lambda_1(\Omega'_p) \geq f\big(P(\Omega'_p)\big),$$
thus, by monotonicity of f we get $p \geq P(\Omega'_p)$. On the other hand, since Ω'_p solves (III) and that there exists Ω_p solution to (I), we have $P(\Omega'_p) \geq p$ which finally gives $P(\Omega'_p) = p$ and shows that Ω'_p solves (I).

The same approach can be applied to prove the equivalence between the second four problems. It remains finally to show the geometrical properties of optimal shapes:

- Let Ω be a solution of one of the first four problems. Thanks to the previous equivalence, it is necessarily a solution to (III), which enters the category of "reverse isoperimetric problems". We want to apply [40, Theorem 4] (see also [40, Example 8] for a similar problem, even though here λ_1 appears in the constraint of the problem): to that end one needs to see that the constraints in (III), that is to say $m(\Omega) = (\lambda_1(\Omega), |\Omega|) = (f(p), 1)$ have a first order derivative which is onto. As in the end of the proof of Theorem 3.9, this follows from Lemma 3.11. We deduce that [40, Theorem 4] applies and therefore Ω is a polygon.

- Let Ω be a solution of one of the last four problems: thanks to the equivalence, it is necessarily a solution of (III') so again as in the end of the proof of Theorem 3.9 we apply [40, Theorem 2, Corollary 2] (as in [40, Example 8], we also use [40, Propositions 5-6]) which shows that Ω is $C^{1,1}$.

□

3.2.5 Simple-connectedness of the diagram

In order to complete the proof of Theorem 1.2, we now need the following result:

Theorem 3.14 *We have:*

$$\mathcal{D}_{\mathcal{K}^2} = \Big\{(x,y) \in \mathbb{R}^2,\ \ x \geq P(B) \text{ and } f(x) \leq y \leq g(x)\Big\},$$

thus, $\mathcal{D}_{\mathcal{K}^2}$ is simply connected.

Proof. We consider a coordinate system $(O, \vec{i}, \vec{j})$. Since the involved functionals are invariant by rotations and translations, we preliminarily remark that one may assume if needed that every domain contains the origin O and that its diameter is colinear to the axis $(O, \vec{i})$.

For a given convex body K, we denote by $\mathrm{diam}(K)$ its diameter and by h_K and ρ_K respectively the support and radial functions of K defined by

$$\forall \theta \in \mathbb{S}^1,\ \ h_K(\theta) = \sup\{\langle x, \theta\rangle, x \in K\}, \qquad \rho_K(\theta) = \sup\{\lambda \geq 0,\ \lambda\theta \in K\}. \tag{20}$$

<u>**Step 1: Minkowski sum and continuous paths:**</u>

Let $K_0, K_1 \in \mathcal{K}_1^2$ such that $P(K_0) = P(K_1) = p$. Define:

$$\forall t \in [0,1], \quad K_t := \frac{(1-t)K_0 + tK_1}{\sqrt{|(1-t)K_0 + tK_1|}}. \tag{21}$$

- Since $t \in [0,1] \longmapsto (1-t)K_0 + tK_1 \in (\mathcal{K}^2, d^H)$ and the functionals $(|\cdot|, P, \lambda_1)$ are continuous for the Hausdorff distance, we have by composition that $t \in [0,1] \longmapsto \big(P(K_t), \lambda_1(K_t)\big) \in \mathbb{R}^2$ is also continuous.
- We also notice that thanks to the linearity of the perimeter for the Minkowski sum, as well as the Brunn-Minkowski inequality (see for example [53, Theorem 7.1.1]), one has:

  $$\forall t \in [0,1],\ P\big((1-t)K_0 + tK_1\big) = (1-t)P(K_0) + tP(K_1) = p,$$
  $$\text{and} \quad |(1-t)K_0 + tK_1|^{\frac{1}{2}} \geq (1-t)|K_0|^{\frac{1}{2}} + t|K_1|^{\frac{1}{2}} = 1,$$

  which implies

  $$\forall t \in [0,1], \quad P\big(K_t\big) \leq p. \tag{22}$$

This shows that given two convex domains with same perimeter, (21) defines a continuous path linking them, which "stays on the left" as we can see on Figure 6.

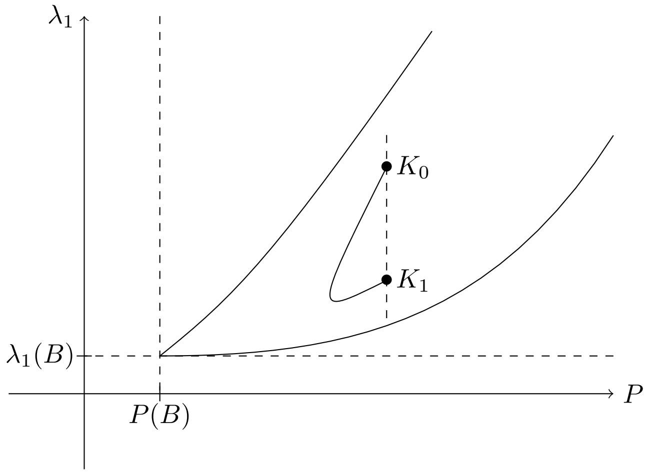

Figure 6: The path goes on the left

For $p \geq P(B)$ and for each $K_0, K_1 \in \mathcal{K}_{1,p}^2$, we therefore denote Γ_{K_0,K_1} the following closed path:

$$\begin{array}{rcccl} \Gamma_{K_0,K_1} & : & [0,1] & \longrightarrow & \mathbb{R}^2 \\ & & t & \longmapsto & \begin{cases} \big(P(K_{2t}), \lambda_1(K_{2t})\big) & \text{if } t \in [0, \frac{1}{2}], \\ \big(P(K_0),\ (2-2t)\lambda_1(K_1) + (2t-1)\lambda_1(K_0)\big) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$

We note that as defined above, the path Γ_{K_0,K_1} contains two components:

- the first one corresponding to $t \in \big[0, \frac{1}{2}\big]$, which is included in the diagram $\mathcal{D}_{\mathcal{K}^2}$,

- and the second one corresponding to $t \in \left[\frac{1}{2}, 1\right]$, which is just "fictional" (not necessarily included in $\mathcal{D}_{\mathcal{K}^2}$) and is introduced in order to obtain a closed path so as we can use the index theory.

Step 2: Continuity of the paths Γ_{K_0,K_1} with respect to (K_0, K_1):

Let $p_0 > P(B)$. Take $K_0, K_1 \in \mathcal{K}^2_{1,p_0}$ and (K_0^n) and (K_1^n) two sequences of $\mathcal{K}^2_1$ converging respectively to K_0 and K_1 for the Hausdorff distance and such that $P(K_0^n) = P(K_1^n)$ for all $n \in \mathbb{N}^*$. Let $\varepsilon > 0$: we will prove that:

$$\exists N_\varepsilon, \forall n \geq N_\varepsilon, \forall t \in [0,1], \quad \left\| \Gamma_{K_0,K_1}(t) - \Gamma_{K_0^n,K_1^n}(t) \right\| < \varepsilon.$$

We have for every $t \in [\frac{1}{2}, 1]$:

$$\begin{aligned}\left\| \Gamma_{K_0,K_1}(t) - \Gamma_{K_0^n,K_1^n}(t) \right\| &\leq |P(K_0) - P(K_0^n)| + (2-2t)|\lambda_1(K_1) - \lambda_1(K_1^n)| + (2t-1)|\lambda_1(K_0) - \lambda_1(K_0^n)| \\ &\leq |P(K_0) - P(K_0^n)| + |\lambda_1(K_1) - \lambda_1(K_1^n)|,\end{aligned}$$

so the estimate is easy to obtain thanks to the convergence of $(K_0^n), (K_1^n)$ and the continuity of λ_1 and P.

For every $t \in [0, \frac{1}{2}]$, we have

$$\left\| \Gamma_{K_0,K_1}(t) - \Gamma_{K_0^n,K_1^n}(t) \right\| \leq \left|P(K_{2t}) - P(K_{2t}^n)\right| + \left|\lambda_1(K_{2t}) - \lambda_1(K_{2t}^n)\right|. \tag{23}$$

We want to control $\left|P(K_{2t}) - P(K_{2t}^n)\right|$ and $\left|\lambda_1(K_{2t}) - \lambda_1(K_{2t}^n)\right|$ independently of t. For the perimeter this will easily follow from the behavior of perimeter and volume with respect to Minkowski sums; for the eigenvalue the situation is more involved and we will use a quantitative version of its continuity with respect to the Hausdorff distance:

- We first notice that for all $t \in [0, 1/2]$ and $n \in \mathbb{N}$:

  $$|(1-2t)K_0 + 2tK_1| \geq 1, \qquad |(1-2t)K_0^n + 2tK_1^n| \geq 1 \quad \text{and} \quad P(K_{2t}), P(K_{2t}^n) \geq P(B).$$

  Therefore using Proposition 3.4

  $$\begin{aligned}\left|P(K_{2t}) - P(K_{2t}^n)\right| &= \frac{|P^2(K_{2t}) - P^2(K_{2t}^n)|}{P(K_{2t}) + P(K_{2t}^n)} \\ &\leq \frac{1}{2P(B)}\left|\frac{P(K_0)^2}{|(1-2t)K_0 + 2tK_1|} - \frac{P(K_0^n)^2}{|(1-2t)K_0^n + 2tK_1^n|}\right| \\ &\leq \Big|P(K_0)^2\big((1-2t)^2W_0(K_0^n,K_1^n) + 4t(1-2t)W_1(K_0^n,K_1^n) + 4t^2W_2(K_0^n,K_1^n)\big) \\ &\quad -P(K_0^n)^2\big((1-2t)^2W_0(K_0,K_1) + 4t(1-2t)W_1(K_0,K_1) + 4t^2W_2(K_0,K_1)\big)\Big| \\ &\leq \underbrace{\sum_{k=0}^{2}\Big(P(K_0)^2|W_k(K_0^n,K_1^n) - W_k(K_0,K_1)| + |W_k(K_0,K_1)| \times |P(K_0^n)^2 - P(K_0)^2|\Big)}_{H^n_{K_0,K_1}}.\end{aligned} \tag{24}$$

  By continuity of the perimeter, P, W_0,W_1 and W_2 for the Hausdorff distance, we have $\lim\limits_{n\to+\infty} H^n_{K_0,K_1} = 0$ while $H^n_{K_0,K_1}$ does not depend on t.

- The result [17, Lemma 2.1] states that if Ω_1 and Ω_2 are starlike planar domains with radial functions ρ_{Ω_1} and ρ_{Ω_2} for which there exists $r_0 > 0$ such that $\rho_{\Omega_1}, \rho_{\Omega_2} \geq r_0$ and $\|\rho_{\Omega_1} - \rho_{\Omega_2}\|_\infty \leq r_0$, then:

  $$\left|\lambda_1(\Omega_1) - \lambda_1(\Omega_2)\right| \leq \frac{3}{r_0^3}\lambda_1(B_1)\|\rho_{\Omega_1} - \rho_{\Omega_2}\|_\infty. \tag{25}$$

  We want to apply this result to (K_{2t}, K_{2t}^n) for $t \in [0, \frac{1}{2}]$ and n large enough. We therefore seek for a suitable r_0 such that the conditions of [17, Lemma 2.1] are satisfied.

  Let $t \in \left[0, \frac{1}{2}\right]$ and $n \in \mathbb{N}^*$ sufficiently large so that $P(K_0^n), P(K_1^n) \leq p_0 + 1$. This implies by (22) that $P(K_{2t}^n) \leq p_0 + 1$ for every $t \in \left[0, \frac{1}{2}\right]$. We now use the classical inequality (see for example [6]) that asserts that for any convex body $\Omega \in \mathcal{K}^d$, one has:

  $$r(\Omega) \geq \frac{|\Omega|}{P(\Omega)},$$

  where $r(\Omega)$ denotes the inradius of Ω. In particular if $\Omega \in \mathcal{K}^2_1$ and $P(\Omega) \leq p_0 + 1$, we have:

  $$r(\Omega) \geq r_0 := \frac{1}{p_0 + 1} > 0. \tag{26}$$

  One can apply this result to K_{2t} and K_{2t}^n, and this implies that one can assume without loss of generality that K_{2t} and K_{2t}^n contain the ball of center O and radius r_0, and this gives $\rho_{K_{2t}} \geq r_0$ and $\rho_{K_{2t}^n} \geq r_0$.

We moreover have:

$$\begin{aligned}\left\|\rho_{K_{2t}}-\rho_{K_{2t}^n}\right\|_\infty &\leq \frac{\|\rho_{K_{2t}}\|_\infty\,\|\rho_{K_{2t}^n}\|_\infty}{r_0^2}d^H(K_{2t},K_{2t}^n) \quad \text{(see [7, Proposition 2])}\\ &\leq \frac{(p_0+1)^2}{r_0^2}d^H(K_{2t},K_{2t}^n) \quad \text{(we used } \|\rho_\Omega\|_\infty\leq diam(\Omega)\leq P(\Omega)\leq p_0+1).\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}d^H(K_{2t},K_{2t}^n) &= \left\|h_{K_{2t}}-h_{K_{2t}^n}\right\|_\infty\\ &= \left\|\frac{(1-2t)h_{K_0}+2th_{K_1}}{\sqrt{|(1-2t)K_0+2tK_1|}}-\frac{(1-2t)h_{K_0^n}+2th_{K_1^n}}{\sqrt{|(1-2t)K_0^n+2tK_1^n|}}\right\|_\infty\\ &\leq (1-2t)\left\|\frac{h_{K_0}}{\sqrt{|(1-2t)K_0+2tK_1|}}-\frac{h_{K_0^n}}{\sqrt{|(1-2t)K_0^n+2tK_1^n|}}\right\|_\infty\\ &\quad +2t\left\|\frac{h_{K_1}}{\sqrt{|(1-2t)K_0+2tK_1|}}-\frac{h_{K_1^n}}{\sqrt{|(1-2t)K_0^n+2tK_1^n|}}\right\|_\infty\\ &\leq \frac{1}{\sqrt{|(1-2t)K_0^n+2tK_1^n|}}\left(\left\|h_{K_0}-h_{K_0^n}\right\|_\infty+\left\|h_{K_1}-h_{K_1^n}\right\|_\infty\right)\\ &\quad +(\|h_{K_0}\|_\infty+\|h_{K_1}\|_\infty)\left|\frac{1}{\sqrt{|(1-2t)K_0^n+2tK_1^n|}}-\frac{1}{\sqrt{|(1-2t)K_0+2tK_1|}}\right|\\ &\leq \left(d^H(K_0,K_0^n)+d^H(K_1,K_1^n)\right)\\ &\quad +(\|h_{K_0}\|_\infty+\|h_{K_1}\|_\infty)\times\Big|\,|(1-2t)K_0^n+2tK_1^n|-|(1-2t)K_0+2tK_1|\,\Big|\\ &\leq \underbrace{\left(d^H(K_0,K_0^n)+d^H(K_1,K_1^n)\right) +(\|h_{K_0}\|_\infty+\|h_{K_1}\|_\infty)\times\sum_{k=0}^{2}|W_k(K_0^n,K_1^n)-W_k(K_0,K_1)|}_{G^n_{K_0,K_1}}.\end{aligned}$$

We then obtain the following estimate:

$$\forall t\in\left[0,\frac{1}{2}\right],\quad \left\|\rho_{K_{2t}}-\rho_{K_{2t}^n}\right\|_\infty\leq\frac{(p_0+1)^2}{r_0^2}\times G^n_{K_0,K_1}. \tag{27}$$

As for $H^n_{K_0,K_1}$, by continuity argument we have $\lim\limits_{n\to+\infty}G^n_{K_0,K_1}=0$. Then, we for n sufficiently large (independently on t), we have $\left\|\rho_{K_{2t}}-\rho_{K_{2t}^n}\right\|_\infty\leq r_0$.
We are finally able to apply (25) on K_{2t} and K_{2t}^n. We get that for n sufficiently large, we have

$$\forall t\in\left[0,\frac{1}{2}\right],\quad \left|\lambda_1(K_{2t})-\lambda_1(K_{2t}^n)\right|\leq\frac{3}{r_0^3}\lambda_1(B_1)\|\rho_{K_{2t}^n}-\rho_{K_{2t}}\|_\infty\leq\frac{3\lambda_1(B_1)}{r_0^5}(p_0+1)^2G^n_{K_0,K_1}. \tag{28}$$

By (23), (24), (28) and the fact that $\lim\limits_{n\to+\infty}G^n_{K_0,K_1}=\lim\limits_{n\to+\infty}H^n_{K_0,K_1}=0$, we conclude that there exists $N_\varepsilon\in\mathbb{N}^*$ such that:

$$\forall n\geq N_\varepsilon,\quad \sup_{t\in[0,1]}\left\|\,\Gamma_{K_0,K_1}(t)-\Gamma_{K_0^n,K_1^n}(t)\,\right\|<\varepsilon.$$

Step 3: The arcs go infinitely to the right when the perimeter increases:

Let $p\geq P(B)$ and (K_0,K_1) two elements of $\mathcal{K}^2_{1,p}$; taking advantage of the invariance with translation and rotation, we choose to align the diameters of K_0 and K_1 with the same axe (say $(O,\vec{i})$). We prove here that this implies:

$$\forall t\in[0,1]\quad P\big(K_t\big)\geq\frac{p}{2},$$

where $(K_t)_{t\in[0,1]}$ is defined in (21).

As mentioned in the beginning of the proof, we can assume that the diameter of every involved convex $K\in\mathcal{K}_1^2$ is aligned with $(O,\vec{i})$, thus the diameter of K is given by

$$\text{diam}(K)=h_K(0)+h_K(\pi),$$

where h_K is the support functional of K, defined in (20). On the other hand we denote ε_K the width in the direction orthogonal to $(O,\vec{i})$:

$$\varepsilon_K := h_K(\pi/2) + h_K(-\pi/2).$$

We easily get the following estimates from Figure 7:

$$2\times \text{diam}(K) \le P(K) \le 4\times \text{diam}(K) \quad \text{and} \quad |K| \le \varepsilon_K \times \text{diam}(K) \le 2|K|,$$

In particular if $K\in\mathcal{K}^2_{1,p}$, then

$$\text{diam}(K) \le \frac{p}{2} \qquad \text{and} \qquad \varepsilon_K \le \frac{2}{\text{diam}(K)} \le \frac{8}{p}.$$

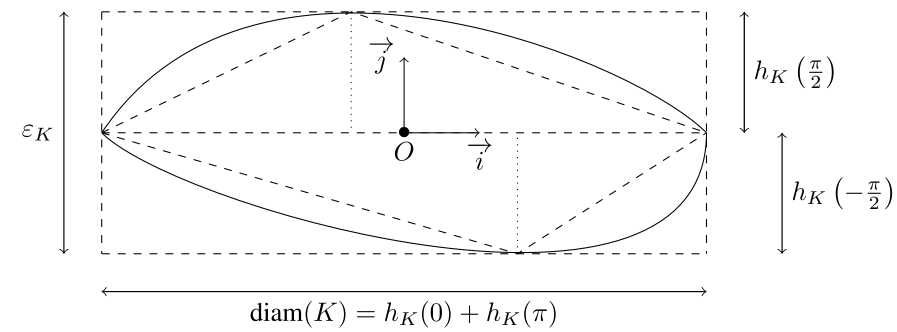

Figure 7: The convex contains a quadrilateral and is contained in a rectangle

We denote by d_t and ε_t the diameter and width in the direction orthogonal to $(O,\vec{i})$ of $(1-t)K_0 + tK_1$, where $t\in[0,1]$. We have:

$$\begin{aligned}
d_t &:= \max_{\theta\in[0,2\pi]} \big(h_{(1-t)K_0+tK_1}(\theta) + h_{(1-t)K_0+tK_1}(\pi+\theta)\big)\\
&= \max_{\theta\in[0,2\pi]} \big((1-t)\times(h_{K_0}(\theta)+h_{K_0}(\pi+\theta)) + t\times(h_{K_1}(\theta)+h_{K_1}(\pi+\theta))\big)\\
&\le (1-t)\times \max_{\theta\in[0,2\pi]}(h_{K_0}(\theta)+h_{K_0}(\pi+\theta)) + t\times \max_{\theta\in[0,2\pi]}(h_{K_1}(\theta)+h_{K_1}(\pi+\theta))\\
&= (1-t)\times\big(\underbrace{h_{K_0}(0)+h_{K_0}(\pi)}_{d_0=\text{diam}(K_0)}\big) + t\times\big(\underbrace{h_{K_1}(0)+h_{K_1}(\pi)}_{d_1=d(K_1)}\big) \quad \text{(because the diameters of } K_0 \text{ and } K_1 \text{ are colinear to } (O,\vec{i}))\\
&= h_{(1-t)K_0+tK_1}(0) + h_{(1-t)K_0+tK_1}(\pi)\\
&\le \max_{\theta\in[0,2\pi]} \big(h_{(1-t)K_0+tK_1}(\theta) + h_{(1-t)K_0+tK_1}(\pi+\theta)\big) = d_t.
\end{aligned}$$

Thus, we have the equalities:

$$\begin{cases} d_t = h_{(1-t)K_0+tK_1}(0) + h_{(1-t)K_0+tK_1}(\pi) = (1-t)d_0 + td_1.\\ \varepsilon_t = h_{(1-t)K_0+tK_1}(\pi/2) + h_{(1-t)K_0+tK_1}(-\pi/2) = (1-t)\varepsilon_0 + t\varepsilon_1.\end{cases}$$

This implies:

$$\begin{aligned}
\forall t\in[0,1],\quad |(1-t)K_0+tK_1| &\le d_t\times\varepsilon_t = \big((1-t)d_0+td_1\big)\times\big((1-t)\varepsilon_0+t\varepsilon_1\big)\\
&\le \left((1-t)\frac{p}{2}+t\frac{p}{2}\right)\times\left((1-t)\frac{8}{p}+t\frac{8}{p}\right) = 4
\end{aligned}$$

Finally, we get:

$$\forall t\in[0,1],\quad P(K_t) = P\Big(\frac{(1-t)K_0+tK_1}{\sqrt{|(1-t)K_0+tK_1|}}\Big) = \frac{(1-t)P(K_0)+tP(K_1)}{\sqrt{|(1-t)K_0+tK_1|}} \ge \frac{p}{2}.$$

Step 4: Relevant paths and conclusion

We denote $\mathcal{E} := \Big\{(x,y)\in\mathbb{R}^2 \mid x\ge P(B) \text{ and } f(x)\le y\le g(x)\Big\}$. We already noticed that $\mathcal{D}_{\mathcal{K}^2}\subset\mathcal{E}$. Assume by contradiction that there exists $A(x_A,y_A)\in\mathcal{E}\setminus\mathcal{D}_{\mathcal{K}^2}$. From Proposition 3.2, there exists $r>0$ such that $B(A,r)\subset\mathcal{E}\backslash\mathcal{D}_{\mathcal{K}^2}$. We are interested in analyzing if A is inside a judiciously chosen closed curve: to that end, let us introduce the set:

$$I = \Big\{\, p\ge x_A+\frac{r}{2} \;\; / \;\exists K_1,K_2\in\mathcal{K}^2_{1,p} \text{ such that } A \text{ is in the interior of } \Gamma_{K_1,K_2}\Big\}.$$

We note that for every $p\ge x_A+\frac{r}{2}$ and $K_1,K_2\in\mathcal{K}^2_{1,p}$, the path Γ_{K_1,K_2} does not cross the point A. Indeed:

- $A \notin \{\Gamma_{K_1,K_2}(t) \mid t \in [0, \frac{1}{2}]\}$, because $\{\Gamma_{K_1,K_2}(t) \mid t \in [0, \frac{1}{2}]\}$ is contained in $\mathcal{D}_{\mathcal{K}^2}$, which is not the case for the point A as assumed above.
- $A \notin \{\Gamma_{K_1,K_2}(t) \mid t \in [\frac{1}{2}, 1]\} = \{(P(K_1), (2-2t)\lambda_1(K_2) + (2t-1)\lambda_1(K_1)) \mid t \in [\frac{1}{2}, 1]\}$, because $P(K_1) = P(K_2) = p \geq x_A + \frac{r}{2} > x_A$.

Moreover, as we do not know whether Γ_{K_1,K_2} is a simple closed curve, we define the interior of Γ_{K_1,K_2} as the set of points A such that the index (also called winding number) of A with respect to the closed curve Γ_{K_1,K_2} is not zero, that is to say $\text{ind}(\Gamma_{K_1,K_2}, A) \neq 0$. We will also say that A is exterior to Γ_{K_1,K_2} if this index is zero.

Using the first step, we note that $x_A + r/2 \in I$: indeed using Corollary 3.3 we know that there exist K_1 and K_2 respectively solutions of the problems

$$\min\left\{\lambda_1(\Omega),\ \ \Omega \in \mathcal{K}^2_{1,x_A+r/2}\right\} \quad \text{and} \quad \max\left\{\lambda_1(\Omega),\ \ \Omega \in \mathcal{K}^2_{1,x_A+r/2}\right\},$$

and as the path Γ_{K_1,K_2} stays on the left of $x_A + r/2$ and its vertical arc is on the right of A, using that $B(A,r) \cap \mathcal{D}_{\mathcal{K}^2} = \emptyset$ we deduce that A is in the interior of Γ_{K_1,K_2}, thus $x_A + r/2 \in I$ and in particular I is not empty. It is also bounded from above, as using Step 3, when the perimeter of two domains K_1, K_2 is sufficiently large, A cannot be in the interior of Γ_{K_1,K_2}. As a consequence, we can define $p_0 = \sup I \in [x_A + r/2, +\infty)$. We analyze the two following cases:

- **<u>Case 1</u>: $p_0 \notin I$, i.e. for every $K_1, K_2 \in \mathcal{K}^2_{1,p_0}$, A is in the exterior of Γ_{K_1,K_2}.**

  As p_0 is defined as the supremum of I, there exists $(p_n)_{n\geq 1}$ converging to p_0 and $(K_1^n, K_2^n)_{n\geq 1}$ two sequences of elements of $\mathcal{K}^2_{1,p_n}$ such that A is in the interior of $\Gamma_{K_1^n,K_2^n}$.

  By Blaschke selection Theorem, there exist $(K_1^{p_0}, K_2^{p_0})$ such that up to a subsequence (that we do not denote) $K_1^n \underset{n\to\infty}{\longrightarrow} K_1^{p_0}$ and $K_2^n \underset{n\to\infty}{\longrightarrow} K_2^{p_0}$ for the Hausdorff distance. Using the result of Step 2 we get that:

  $$\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}^*, \forall t \in [0,1], \quad \left\| \Gamma_{K_1^{p_0},K_2^{p_0}}(t) - \Gamma_{K_1^{n_\varepsilon},K_2^{n_\varepsilon}}(t) \right\| < \varepsilon,$$

  so for a sufficiently small value of $\varepsilon > 0$, by continuity of the index under this uniform estimate, we have:

  $$\text{ind}\big(\Gamma_{K_1^{p_0},K_2^{p_0}}, A\big) = \text{ind}\big(\Gamma_{K_1^{n_\varepsilon},K_2^{n_\varepsilon}}, A\big).$$

  This is a contradiction (see Figure 8) since A is in the interior of $\Gamma_{K_1^{p_{n_\varepsilon}},K_2^{p_{n_\varepsilon}}}$ (ie. $\text{ind}\big(\Gamma_{K_1^{p_{n_\varepsilon}},K_2^{p_{n_\varepsilon}}}, A\big) \neq 0$) while as $K_1^{p_0}, K_2^{p_0} \in \mathcal{K}^2_{1,p_0}$, it must also be in the exterior of $\Gamma_{K_1^{p_0},K_2^{p_0}}$ (ie. $\text{ind}\big(\Gamma_{K_1^{p_0},K_2^{p_0}}, A\big) = 0$).

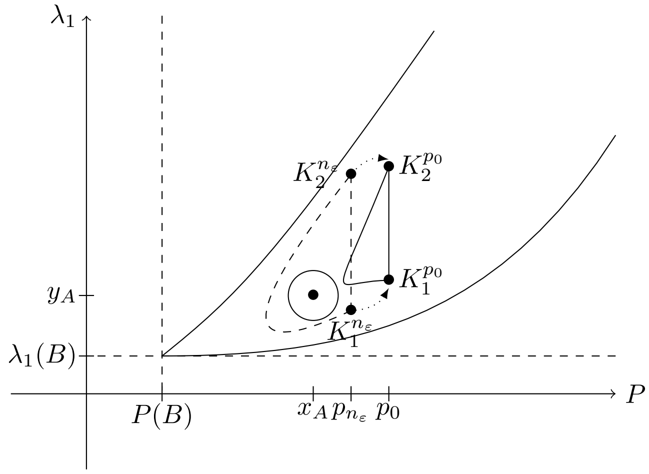

Figure 8: Using compactness to find sets $K_1^{p_0}$ and $K_2^{p_0}$

- **<u>Case 2</u>: $p_0 \in I$, i.e. there exist $K_1^{p_0}, K_2^{p_0} \in \mathcal{K}^2_{1,p_0}$ such that A is in the interior of Γ_{K_1,K_2}.**

  Consider $p_n = p_0 + 1/n$ for $n \geq 1$. By Corollary 3.6, there exist (K_1^n, K_2^n) two sequences in $\mathcal{K}^2_{1,p_n}$ such that $K_1^n \underset{n\to\infty}{\longrightarrow} K_1^{p_0}$ and $K_2^n \underset{n\to\infty}{\longrightarrow} K_2^{p_0}$ for the Hausdorff distance, see Figure 9.

  Similarly to the first case, using Step 3 and the continuity of the index (with respect to the curve) we have for n large enough

  $$\text{ind}\big(\Gamma_{K_1^{p_0},K_2^{p_0}}, A\big) = \text{ind}\big(\Gamma_{K_1^n,K_2^n}, A\big).$$

  This is also a contradiction since A is in the interior of $\Gamma_{K_1^{p_0},K_2^{p_0}}$ while as $p_n > \sup I$, it must be in the exterior of $\Gamma_{K_1^n,K_2^n}$.

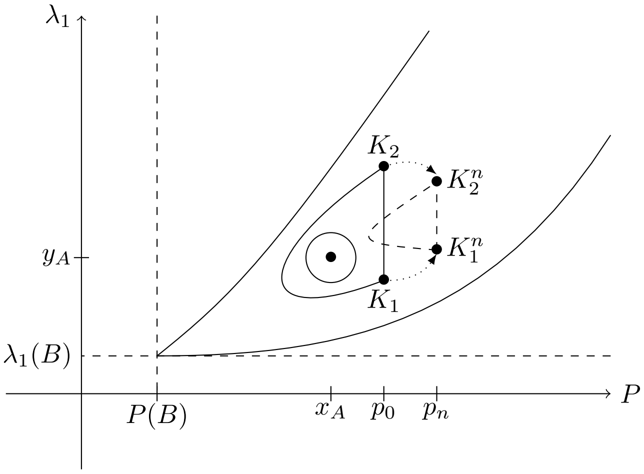

Figure 9: Using Corollary 3.6 to increase the perimeter

We obtained a contradiction in both cases, which proves that $\mathcal{D}_{\mathcal{K}^2} = \mathcal{E}$. Thus $\mathcal{D}_{\mathcal{K}^2} \subset \mathbb{R}^2$ does not contain any hole and so is simply connected. This concludes the proof. $\square$

The following result is a direct consequence of Theorem 3.14.

Corollary 3.15 *The diagram $\mathcal{D}_{\mathcal{K}^2}$ is vertically and horizontally convex.*

3.3 Asymptotics of the diagram

Upper behavior: It has been proven in [46] and [18, Proposition 5.5] that

Proposition 3.16 *Let B_1 be a ball of radius 1 in $\mathbb{R}^d$ with $d \geq 2$, and $p > d$.*

1. *If $\gamma < \dfrac{d(d+1)P(B_1)}{4\lambda_1(B_1)(\lambda_1(B_1)-d)}$, then B_1 is a local minimizer of $P - \gamma\lambda_1$ in a $W^{2,p}$-neighborhood with volume constraint, in the sense that there exists $\eta = \eta(\gamma) > 0$ such that*

$$P(B_1^\varphi) - P(B_1) \geq \gamma\left[\lambda_1(B_1^\varphi) - \lambda_1(B_1)\right]$$

for every B_1^φ such that $|B_1^\varphi| = |B_1|$ and being nearly spherical in the sense that

$$B_1^\varphi := \left\{tx(1+\varphi(x)), t \in [0,1[, x \in \mathbb{S}^{d-1}\right\} \quad \textit{with } \varphi : \mathbb{S}^{d-1} \to \mathbb{R} \textit{ satisfying } \|\varphi\|_{W^{2,p}(\mathbb{S}^{d-1})} \leq \eta.$$

2. *If $\gamma > \dfrac{d(d+1)P(B_1)}{4\lambda_1(B_1)(\lambda_1(B_1)-d)}$, then B_1 is not a local minimizer $P - \gamma\lambda_1$ among domains with given volume; more precisely, for $\eta > 0$, there exists $\varphi : \mathbb{S}^{d-1} \to \mathbb{R}$ such that*

$$|B_1^\varphi| = |B_1|, \quad \|\varphi\|_{W^{2,p}(\mathbb{S}^{d-1})} \leq \eta, \qquad \textit{and} \quad P(B_1^\varphi) - P(B_1) < \gamma\left[\lambda_1(B_1^\varphi) - \lambda_1(B_1)\right].$$

Corollary 3.17 *Let $d \geq 2$, $x_0 = \frac{P(B_1)}{|B_1|^{\frac{d-1}{d}}}$ and g the function defined in Section 3.2.4. Then*

$$\limsup_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} \geq \frac{4|B_1|^{\frac{d+1}{d}}\lambda_1(B_1)\big(\lambda_1(B_1) - d\big)}{d(d+1)P(B_1)}. \tag{29}$$

Remark 3.18 *When $d = 2$, inequality (29) becomes*

$$\limsup_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} \geq \frac{\sqrt{\pi}}{3}\lambda_1(B_1) \times \big(\lambda_1(B_1) - 2\big) \geq 12.9264,$$

where the numerical lower bound is obtained by using a lower numerical value of $\lambda_1(B_1) = j_{0,1}^2$ (where $j_{0,1}$ denotes the first zero of the Bessel function J_0).

Proof. Given $\gamma > \dfrac{d(d+1)P(B_1)}{4\lambda_1(B_1)(\lambda_1(B_1)-d)}$ and $n \in \mathbb{N}^*$, from Proposition 3.16, there exists $\varphi_n : \mathbb{S}^{d-1} \to \mathbb{R}$ such that

$$|B_1| = |B_1^{\varphi_n}|, \quad \|\varphi_n\|_{W^{2,p}(\mathbb{S}^{d-1})} \leq \frac{1}{n}, \qquad \text{and} \quad P(B_1^{\varphi_n}) - P(B_1) < \gamma\left[\lambda_1(B_1^{\varphi_n}) - \lambda_1(B_1)\right].$$

Defining $\Omega_n = \frac{B_1^{\varphi_n}}{|B_1^{\varphi_n}|^{1/d}}$ and $B = \frac{B_1}{|B_1|^{1/d}}$ having unit area, we get

$$P(\Omega_n) - P(B) = \frac{P(B_1^{\varphi_n}) - P(B_1)}{|B_1|^{\frac{d-1}{d}}} < \frac{\gamma}{|B_1|^{\frac{d-1}{d}}} [\lambda_1(B_1^{\varphi_n}) - \lambda_1(B_1)] = \frac{\gamma}{|B_1|^{\frac{d+1}{d}}} [\lambda_1(\Omega_n) - \lambda_1(B)] .$$

Defining $x_n = P(\Omega_n)$, we get, as g is defined as a maximum:

$$x_n - x_0 < \frac{\gamma}{|B_1|^{\frac{d+1}{d}}} (g(x_n) - g(x_0)).$$

When n diverges to $+\infty$, x_n goes to x_0, and therefore

$$\limsup_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} \geq \lim_{n \to +\infty} \frac{g(x_n) - g(x_0)}{x_n - x_0} \geq \frac{|B_1|^{\frac{d+1}{d}}}{\gamma},$$

where γ is arbitrary chosen in $\left(\frac{d(d+1)P(B_1)}{4\lambda_1(B_1)(\lambda_1(B_1) - d)}, +\infty \right)$.

This ends the proof. □

Lower behavior: In the next result, we study the stability of the ball for the minimality of $\lambda_1 - c[P - P(B)]^\alpha$ in order to have information about the behavior of the lower part of the diagram near the ball:

Theorem 3.19 *Let B_1 be the ball of radius 1 in $\mathbb{R}^d$ with $d \geq 2$, and $p > d$.*

1. *Then there exists $c > 0$ and $\eta > 0$ such that*

$$\lambda_1(B_1^\varphi) - \lambda_1(B_1) \geq c\Big[P(B_1^\varphi) - P(B_1)\Big]^{3/2}$$

for every B_1^φ such that $|B_1^\varphi| = |B_1|$ and being nearly spherical with $\|\varphi\|_{W^{2,p}(\mathbb{S}^{d-1})} \leq \eta$.

2. *If however $\alpha \in (0, 3/2)$, then for any $c > 0$ and $\eta > 0$, there exists $\varphi : \mathbb{S}^{d-1} \to \mathbb{R}$ such that*

$$|B_1^\varphi| = |B_1|, \quad \|\varphi\|_{W^{2,p}(\mathbb{S}^{d-1})} \leq \eta, \quad \textit{and} \quad \lambda_1(B_1^\varphi) - \lambda_1(B_1) < c\Big[P(B_1^\varphi) - P(B_1)\Big]^\alpha.$$

Remark 3 *Such nearly spherical sets were considered by Fuglede in [27] where he was studying the stability of the ball for the usual isoperimetric problem. See also [11] where the authors use nearly spherical sets when studying the quantitative Faber-Krahn inequality, and [18] for a more general approach about stability among smooth deformations of a given set.*

Proof.

1. Let $\varphi : \mathbb{S}^{d-1} \to \mathbb{R}$ such that $|B_1^\varphi| = |B_1|$ and the barycenter of B_1^φ is 0. From [27], there exists $\eta_1 > 0$ and $C_1 > 0$ such that

$$P(B_1^\varphi) - P(B_1) \leq C_1 \|\varphi\|^2_{H^1(\mathbb{S}^{d-1})} \quad \text{if } \|\varphi\|_{W^{1,\infty}(\mathbb{S}^{d-1})} \leq \eta_1.$$

Moreover, using [18, Theorem 1.3 and Section 5.2] (see also [11, Theorem 3.3] in the context of $C^{2,\alpha}$-perturbations), there exists $\eta_2 > 0$ and $c_2 > 0$ such that

$$\lambda_1(B_1^\varphi) - \lambda_1(B_1) \geq c_2 \|\varphi\|^2_{H^{1/2}(\mathbb{S}^{d-1})} \quad \text{if } \|\varphi\|_{W^{2,p}(\mathbb{S}^{d-1})} \leq \eta_2.$$

Therefore, setting $\eta = \min\{\eta_1, \eta_2\}$ and assuming $\|\varphi\|_{W^{2,p}(\mathbb{S}^{d-1})} \leq \eta$, we get for $c > 0$:

$$\lambda_1(B_1^\varphi) - \lambda_1(B_1) - c\Big[P(B_1^\varphi) - P(B_1)\Big]^{3/2} \geq c_2 \|\varphi\|^2_{H^{1/2}(\mathbb{S}^{d-1})} - cC_1^{3/2} \|\varphi\|^3_{H^1(\mathbb{S}^{d-1})},$$

but from a Gagliardo-Nirenberg type inequality (see for example [12]), we have that there exists $C_3, C_4 > 0$ such that

$$\|\varphi\|_{H^1(\mathbb{S}^{d-1})} \leq C_3 \|\varphi\|^{2/3}_{H^{1/2}(\mathbb{S}^{d-1})} \|\varphi\|^{1/3}_{H^2(\mathbb{S}^{d-1})} \leq C_4 \|\varphi\|^{2/3}_{H^{1/2}(\mathbb{S}^{d-1})} \|\varphi\|^{1/3}_{W^{2,p}(\mathbb{S}^{d-1})},$$

(we used $p > d \geq 2$) therefore

$$\lambda_1(B_1^\varphi) - \lambda_1(B_1) - c\Big[P(B_1^\varphi) - P(B_1)\Big]^{3/2} \geq \|\varphi\|^2_{H^{1/2}(\mathbb{S}^{d-1})} \Big[c_2 - cC_1^{3/2} C_4^3 \|\varphi\|_{W^{2,p}(\mathbb{S}^{d-1})}\Big],$$

which is positive if $\|\varphi\|_{W^{2,p}(\mathbb{S}^{d-1})} \leq \eta$ is small enough.

2. Assume to the contrary that for every $\varphi : \mathbb{S}^{d-1} \to \mathbb{R}$ such that $|B_1^\varphi| = |B_1|$ and $\|\varphi\|_{W^{2,p}(\mathbb{S}^{d-1})} \le \eta$, we have

$$\lambda_1(B_1^\varphi) - \lambda_1(B_1) \ge c\Big[P(B_1^\varphi) - P(B_1)\Big]^\alpha,$$

where $\alpha \in [0, 3/2)$ and $c > 0$. We choose the origin as the center of B_1 so that $\mathrm{Bar}(B_1) = 0$ where Bar denotes the barycenter of a given shape. We also denote $\mathrm{Vol} : \Omega \mapsto |\Omega|$.

We now use the framework from [18], and in particular, if J denotes a shape functional, then $J'(B_1), J''(B_1)$ denote respectively the first and second order derivatives of $\varphi \mapsto J(B_1^\varphi)$. From [18, Proposition 4.5] the perimeter functional satisfies $(\mathbf{IT}_{H^1,W^{1,\infty}})$ which means there exists ω_1 a modulus of continuity such that

$$P(B_1^\varphi) - P(B_1) = P'(B_1).\varphi + \frac{1}{2}P''(B_1).(\varphi,\varphi) + \omega_1(\|\varphi\|_{W^{1,\infty}(\mathbb{S}^{d-1})})\|\varphi\|^2_{H^1(\mathbb{S}^{d-1})}.$$

Moreover, B_1 is a stable critical shape of P under volume constraint and up to translations (see [18, Section 5.1]), which means that there exists $\mu \in \mathbb{R}$ a Lagrange multiplier such that

$$P'(B_1).\varphi = \mu \mathrm{Vol}'(B_1).\varphi, \quad \forall \varphi \in W^{1,\infty}(\mathbb{S}^{d-1})$$

and there exists $c_1 > 0$ such that

$$(P - \mu\mathrm{Vol})''(B_1).(\varphi,\varphi) \ge c_1\|\varphi\|^2_{H^1(\mathbb{S}^{d-1})}, \qquad \forall \varphi \in W^{1,\infty}(\mathbb{S}^{d-1}) \text{ such that } \mathrm{Vol}'(B_1).\varphi = 0 \text{ and } \mathrm{Bar}'(B_1).\varphi = 0.$$

Therefore, one gets that there exists $\eta_1 > 0$ such that for any $\varphi \in W^{1,\infty}(\mathbb{S}^{d-1})$ satisfying

$$\mathrm{Vol}'(B_1).\varphi = 0, \qquad \mathrm{Bar}'(B_1).\varphi = 0 \qquad \text{and} \qquad \|\varphi\|_{W^{1,\infty}(\mathbb{S}^{d-1})} \le \eta_1,$$

one has

$$\begin{aligned} P(B_1^\varphi) - P(B_1) &\ge \frac{1}{2}(P - \mu\mathrm{Vol})''(B_1).(\varphi,\varphi) + \frac{\mu}{2}\mathrm{Vol}''(B_1).(\varphi,\varphi) - \frac{c_1}{4}\|\varphi\|^2_{H^1(\mathbb{S}^{d-1})} \\ &\ge \frac{c_1}{2}\|\varphi\|^2_{H^1(\mathbb{S}^{d-1})} - C_1\|\varphi\|^2_{L^2(\mathbb{S}^{d-1})}, \end{aligned}$$

for some $C_1 \in \mathbb{R}$ (coming from the fact that $\mathrm{Vol}''(B)$ is a continuous quadratic form on $L^2(\mathbb{S}^{d-1})$, see [18, Section 2.2]).

Similarly, from [18, Theorem 1.4] λ_1 satisfies $(\mathbf{IT}_{H^{1/2},W^{2,p}})$ and moreover B_1 is a critical point of λ_1 under volume constraint, and $\lambda_1''(B_1)$ is a continuous quadratic form on $H^{1/2}(\mathbb{S}^{d-1})$, so there exists $\eta_2 > 0$ such that for any $\varphi \in W^{2,p}(\mathbb{S}^{d-1})$ satisfying

$$\mathrm{Vol}'(B_1).\varphi = 0, \qquad \mathrm{Bar}'(B_1).\varphi = 0 \qquad \text{and} \qquad \|\varphi\|_{W^{2,p}(\mathbb{S}^{d-1})} \le \eta_2,$$

one has

$$\lambda_1(B_1^\varphi) - \lambda_1(B_1) \le C_2\|\varphi\|^2_{H^{1/2}(\mathbb{S}^{d-1})}.$$

Therefore we get as above, setting $\eta = \min\{\eta_1, \eta_2\}$:

$$\forall \varphi \text{ such that } \|\varphi\|_{W^{2,p}(\mathbb{S}^{d-1})} \le \eta, \quad \mathrm{Vol}'(B_1).\varphi = 0, \quad \text{and} \quad \mathrm{Bar}'(B_1).\varphi = 0,$$
$$\|\varphi\|^2_{H^1(\mathbb{S}^{d-1})} \le C\|\varphi\|^{2/\alpha}_{H^{1/2}(\mathbb{S}^{d-1})} + \widetilde{C}\|\varphi\|^2_{L^2(\mathbb{S}^{d-1})},$$

for some $(C, \widetilde{C}) \in \mathbb{R}_+^2$. Using scaling, and looking at the expressions of $\mathrm{Vol}'(B_1), \mathrm{Bar}'(B_1)$, we finally obtain:

$$\forall \varphi \text{ such that } \int_{\mathbb{S}^{d-1}} \varphi = 0, \quad \text{and} \quad \forall i \in [\![1, d]\!], \int_{\mathbb{S}^{d-1}} x_i\varphi(x) = 0,$$
$$\|\varphi\|^2_{H^1(\mathbb{S}^{d-1})} \le C\eta^{2/\alpha-2}\|\varphi\|^{2-2/\alpha}_{W^{2,p}(\mathbb{S}^{d-1})}\|\varphi\|^{2/\alpha}_{H^{1/2}(\mathbb{S}^{d-1})} + \widetilde{C}\|\varphi\|^2_{L^2(\mathbb{S}^{d-1})}. \tag{30}$$

We want to get a contradiction by testing such interpolation inequality for an oscillating function φ. To that end (see [55, pages 139-141] for more details), we denote $\mathcal{H}_k$ the space of spherical harmonics of degree $k \in \mathbb{N}$ (that is, the restriction to $\mathbb{S}^{d-1}$ of homogeneous polynomials in $\mathbb{R}^d$, of degree k) and $(Y^{k,l})_{1\le l\le d_k}$ an orthonormal basis of $\mathcal{H}_k$ with respect to the $L^2(\mathbb{S}^{d-1})$ scalar product. The family $(Y^{k,l})_{k\in\mathbb{N},1\le l\le d_k}$ is a Hilbert basis of $L^2(\mathbb{S}^{d-1})$, so any function φ in $L^2(\mathbb{S}^{d-1})$ can be decomposed:

$$\varphi(x) = \sum_{k=0}^{\infty}\sum_{l=1}^{d_k}\alpha_{k,l}(\varphi)Y^{k,l}(x), \quad \text{for } x \in \mathbb{S}^{d-1}, \quad \text{with} \quad \alpha_{k,l}(\varphi) = \int_{\mathbb{S}^{d-1}} \overline{Y_{k,l}}\varphi.$$

Moreover, for $s \in \mathbb{R}_+$, $\|\varphi\|_{H^s(\mathbb{S}^{d-1})}$ is equivalent to $\big(\sum_{k\in\mathbb{N}}(1+k)^{2s}\sum_l |\alpha_{k,l}|^2\big)^{1/2}$.

We therefore choose $\varphi_k = Y_{k,1}$ for $k \geq 2$. As $\mathcal{H}_0$ is made of constant functions and $\mathcal{H}_1 = \text{span}(x_i)_{i\in[\![1,d]\!]}$, we have

$$\forall k \geq 2, \quad \int_{\mathbb{S}^{d-1}} \varphi_k = 0 \quad \text{and} \quad \forall i \in [\![1,d]\!], \int_{\mathbb{S}^{d-1}} x_i\varphi_k(x) = 0$$

so that $(\varphi_k)_{k\geq 2}$ are admissible for (30). Moreover

$$\forall s \geq 0, \ \|\varphi_k\|_{H^s(\mathbb{S}^{d-1})} \underset{k\to\infty}{\sim} k^s, \quad \text{and} \quad \|\varphi_k\|_{W^{2,p}(\mathbb{S}^{d-1})} \geq \|\varphi_k\|_{H^2(\mathbb{S}^{d-1})}$$

so that (30) gives $k^2 = O(k^{4-4/\alpha}k^{1/\alpha} + 1)$ which contradicts the assumption $\alpha < 3/2$ and concludes the proof.

□

Corollary 3.20 *Take $\alpha \in (0, 3/2)$, $d \geq 2$ and $x_0 = \frac{P(B_1)}{|B_1|^{\frac{d-1}{d}}}$, we have*

$$\liminf_{x\to x_0} \frac{f(x) - f(x_0)}{(x - x_0)^\alpha} = 0.$$

Proof. Take $c > 0$. By the second assertion of Theorem 3.19, for every $n \in \mathbb{N}^*$ there exists $\varphi_n : \mathbb{S}^{d-1} \to \mathbb{R}$ such that

$$|B_1^{\varphi_n}| = |B_1|, \quad \|\varphi_n\|_{W^{2,p}(\mathbb{S}^{d-1})} \leq \frac{1}{n}, \quad \text{and} \quad 0 \leq \lambda_1(B_1^{\varphi_n}) - \lambda_1(B_1) < c \times \Big[P(B_1^{\varphi_n}) - P(B_1)\Big]^\alpha.$$

The last inequality is equivalent to

$$0 \leq |B_1^{\varphi_n}|^{\frac{2}{d}}\lambda_1(B_1^{\varphi_n}) - |B_1|^{\frac{2}{d}}\lambda_1(B_1) < c \times |B_1|^{\frac{d+1}{d}} \left(\frac{P(B_1^{\varphi_n})}{|B_1^{\varphi_n}|^{\frac{d-1}{d}}} - \frac{P(B_1)}{|B_1|^{\frac{d-1}{d}}}\right)^\alpha,$$

so, we get

$$0 \leq f(x_n) - f(x_0) \leq c \times |B_1|^{1+\frac{1}{d}}(x_n - x_0)^\alpha,$$

where $x_n := P(B_1^{\varphi_n})/|B_1^{\varphi_n}|^{\frac{d-1}{d}} \underset{n\to+\infty}{\longrightarrow} P(B_1)/|B_1|^{\frac{d-1}{d}} = x_0$, because $\|\varphi_n\|_{W^{2,p}(\mathbb{S}^{d-1})} \leq \frac{1}{n} \underset{n\to+\infty}{\longrightarrow} 0$.

Thus, we can write

$$\forall c > 0, \quad 0 \leq \liminf_{x\to x_0} \frac{f(x) - f(x_0)}{(x - x_0)^\alpha} \leq \liminf_{n\to+\infty} \frac{f(x_n) - f(x_0)}{(x_n - x_0)^\alpha} \leq c|B_1|^{1+\frac{1}{d}}.$$

Finally, we get the result $\liminf\limits_{x\to x_0} \frac{f(x)-f(x_0)}{(x-x_0)^\alpha} = 0$.

□

The most interesting part of Theorem 3.19 and Corollary 3.20 is that the exponent $3/2$ was apparently unknown and seems to be optimal (see Section 4); in the planar case $d = 2$, we actually improve the result of Corollary 3.20 and retrieve the same exponent in a completely different (and independent) way, by studying the asymptotics of λ_1 and P for regular polygons:

Proposition 3.21 *Let $d = 2$ and for $n \geq 3$, denote S_n the regular $n-$gon with unit area (and again B denotes the disk of unit area). We have:*

$$\big(\lambda_1(S_n) - \lambda_1(B)\big) \underset{n\to\infty}{\sim} \beta_0 \times \big(P(S_n) - P(B)\big)^{3/2},$$

with

$$\beta_0 := \frac{4 \times 3^{\frac{3}{2}}\zeta(3)\lambda_1(B)}{\pi^{\frac{15}{4}}},$$

where $\zeta : x \in (1, +\infty) \longmapsto \sum\limits_{n=1}^{\infty} \frac{1}{n^x}$ is the Riemann zeta function.

Proof. We take the asymptotic expansion of the fundamental frequency of regular polygons found in [45] :

$$\lambda_1(S_n) - \lambda_1(B) = \frac{4\zeta(3)\lambda_1(B)}{n^3} + o\left(\frac{1}{n^3}\right) \tag{31}$$

and

$$P(S_n) - P(B) = 2\sqrt{n}\sqrt{\tan\frac{\pi}{n}} - 2\sqrt{\pi} = \frac{\pi^{\frac{5}{2}}}{3} \times \frac{1}{n^2} + o\left(\frac{1}{n^3}\right). \tag{32}$$

Then we can write:

$$\frac{\lambda_1(S_n) - \lambda_1(B)}{\big(P(S_n) - P(B)\big)^{3/2}} \underset{n\to+\infty}{\sim} \frac{4 \times 3^{\frac{3}{2}}\zeta(3)\lambda_1(B)}{\pi^{\frac{15}{4}}} = \beta_0.$$

□

This proposition allows us to get the following asymptotic property on f.

Corollary 3.22 *Let $d=2$, x_0 and f defined in Section 3.2.4. Then*

1. $\forall \alpha \in (0,3/2), \quad f(x)-f(x_0) = o\big((x-x_0)^\alpha\big)$,
   in particular by taking $\alpha=1$, we get that f is differentiable at x_0 and $f'(x_0)=0$.

2. $0 \le \liminf\limits_{x\to x_0} \frac{f(x)-f(x_0)}{(x-x_0)^{3/2}} \le \limsup\limits_{x\to x_0} \frac{f(x)-f(x_0)}{(x-x_0)^{3/2}} \le \beta_0$.

Remark 3.23 *Enlightened with the results of Theorem 3.19 and Corollary 3.20, which are stated for arbitrary dimensions, we believe that the first assertion holds for $d\ge 2$. Unfortunately, we did not manage to prove it because of the lack of information on the asymptotic behaviour of $x_n := P(B_1^{\varphi_n})/|B_1^{\varphi_n}|^{\frac{d-1}{d}}$ introduced in the proof of Corollary 3.20.*

Proof. Take $\alpha \in (0,3/2]$, we introduce the integer valued function which associates to $x\in \big(x_0, P(S_3)\big)$ the integer $n_x := \max\{n\ge 3,\ \ P(S_n)\ge x\}$, note that $\lim\limits_{x\to x_0} n_x = +\infty$.

We have:

$$\begin{aligned}
0 \ \le\ \frac{f(x)-f(x_0)}{(x-x_0)^\alpha} &\le \frac{f\big(P(S_{n_x})\big)-f\big(P(B)\big)}{\big(P(S_{n_x+1})-P(B)\big)^\alpha} \quad \text{(because } x\ge P(S_{n_x+1}) \text{ and } f(x)\le f\big(P(S_{n_x})\big)\text{)}\\
&\le \frac{\lambda_1(S_{n_x})-\lambda_1(B)}{(P(S_{n_x+1})-P(B))^\alpha} \quad \text{(by the definition of } f\text{)}\\
&= \frac{\lambda_1(S_{n_x})-\lambda_1(B)}{\big(P(S_{n_x})-P(B)\big)^\alpha} \times \left(\frac{P(S_{n_x})-P(B)}{P(S_{n_x+1})-P(B)}\right)^\alpha\\
&\underset{x\to x_0}{\sim} \frac{\frac{4\zeta(3)\lambda_1(B)}{n_x^3}}{\left(\frac{\pi^{\frac{5}{2}}}{3}\times\frac{1}{n_x^2}\right)^\alpha} \times \left(\frac{\frac{\pi^{\frac{5}{2}}}{3}\times\frac{1}{n_x^2}}{\frac{\pi^{\frac{5}{2}}}{3}\times\frac{1}{(n_x+1)^2}}\right)^\alpha \quad \text{(by (31) and (32))}\\
&\underset{x\to x_0}{\sim} \beta_0 \times \left(\frac{\pi^{5/2}}{3}\frac{1}{n_x^2}\right)^{3/2-\alpha} \quad \text{(because } n_x \underset{x\to x_0}{\longrightarrow} +\infty\text{)},
\end{aligned}$$

thus, if $\alpha\in(0,3/2)$, we have $\lim\limits_{x\to x_0} \frac{f(x)-f(x_0)}{(x-x_0)^\alpha} = 0$, which is equivalent to the first assertion. On the other hand if $\alpha = 3/2$, we get the proof of the second assertion. □

4 Further remarks and Conjectures

4.1 About $\mathcal{D}_{\mathcal{K}^2}$

Our theoretical and numerical studies highlight some remaining open problems about $\mathcal{D}_{\mathcal{K}^2}$:

1. is it true that f and g defined in (14) and (15) are convex?

2. is it true that

$$g'(x_0) = \frac{\sqrt{\pi}}{3}\lambda_1(B_1)\times\big(\lambda_1(B_1)-2\big) \qquad \text{and} \qquad f(x)-f(x_0) \underset{x\to x_0}{\sim} c(x-x_0)^{3/2}$$

   for some $c>0$? These questions are closely related to the following: for $\gamma > \frac{\sqrt{\pi}}{3}\lambda_1(B_1)\times\big(\lambda_1(B_1)-2\big)$, can we find $c>0$ and $\mathcal{V}$ a neighborhood of B in $\mathcal{K}_1^2$ (for the Hausdorff distance) such that

$$\forall \Omega\in\mathcal{V},\quad c(P(\Omega)-P(B))^{3/2} \le \lambda_1(\Omega)-\lambda_1(B) \le \gamma(P(\Omega)-P(B)) \quad ?$$

   Proposition 3.16 and Theorem 3.19 shows that such inequalities are valid in a smooth neighborhood of B, but it is well-known that achieving a similar result in a non-smooth neighborhood requires more work (see for example [1, 28] and [18, Section 6]). We note that numerical evidence that will appear in [26] suggests that the optimal value of the constant c is given by the β_0 introduced in Proposition 3.21. It also supports that the inequality may in fact be global, which means we conjecture the following inequality:

$$\forall\Omega\in\mathcal{K}_1^2,\qquad \lambda_1(\Omega)-\lambda_1(B)\ge \beta_0\times\big(P(\Omega)-P(B)\big)^{3/2}.$$

Remark 4.1 *It is interesting to note that if we combine the conjectured inequality*

$$\lambda_1(\Omega)-\lambda_1(B)\ge c\big(P(\Omega)-P(B)\big)^{3/2}, \tag{33}$$

with the famous quantitative isoperimetric inequality of [29], which affirms the existence of a constant α_d , depending only on the dimension d, such that for every Borel set $\Omega \subset \mathbb{R}^d$, one has

$$P(\Omega) - P(B) \geq \alpha_d \times \mathcal{A}(\Omega)^2,$$

where $\mathcal{A}(\Omega)$ is the so called Fraenkel asymmetry of the set Ω, we get that for every $\Omega \in \mathcal{K}_1^2$

$$\lambda_1(\Omega) - \lambda_1(B) \geq c\alpha_d^{3/2} \times \mathcal{A}(\Omega)^3.$$

The exponent 3 is not optimal, it is higher than the optimal one given in [11], where the authors prove that there exists a dimensional constant σ_d such that for every open set $\Omega \subset \mathbb{R}^d$ with unit measure, one has

$$\lambda_1(\Omega) - \lambda_1(B) \geq \sigma_d \times \mathcal{A}(\Omega)^2. \tag{34}$$

Nevertheless, we note that inequality (33) *is stronger in some cases than* (34)*. Indeed, if we take the regular polygons (S_n) introduced in Proposition 3.21, we have by straightforward computations:*

$$P(S_n) - P(B) \underset{n\to\infty}{\sim} \mu_2 \times \mathcal{A}(S_n),$$

where μ_2 is a positive constant. Thus, for sufficiently large values of n, we have

$$\lambda_1(\Omega) - \lambda_1(B) \geq c\big(P(S_n) - P(B)\big)^{3/2} \geq c' \times \mathcal{A}(S_n)^{3/2},$$

where c' is a positive constant. This shows that (34) *is (in this case) weaker than the conjecture* (33)*.*

One could also wonder if we can improve our understanding of the shapes realizing the boundary of the diagram, that is to say solutions of the optimization problems in Corollary 3.13. For example, one can state:

Conjecture 1 *The regular polygons are on the lower part of $\partial\mathcal{D}_{\mathcal{K}^2}$.*

This result seems to be verified numerically. Using Theorem 3.9, we will observe however (see the proof below) that this statement (regular polygons are on the lower boundary) is actually a stronger statement than Polya's conjecture in the restricted class of convex sets. Recall that this conjecture states that among polygons of fixed measure and whose number of sides is bounded by N, the regular N-gon has the lowest first Dirichlet eigenvalue. This conjecture is expected to be valid for any polygon, but even in the class of convex polygons, the result is not known (for $N \geq 5$) and already expected to be very challenging.

Indeed, let us take $N \geq 3$ and denote Ω_N^* the regular polygon of unit measure and N sides. By the isoperimetric inequality for polygons (see [47, Theorem 5.1]), we have $P(\Omega) \geq P(\Omega_N^*)$, for every convex polygon Ω of unit measure and at most N sides. Now, if we assume that the regular polygon Ω_N^* is on the lower boundary of the diagram, that is to say $\lambda_1(\Omega_N^*) = f\big(P(\Omega_N^*)\big)$, then by monotonicity of f, we conclude that: $\lambda_1(\Omega_N^*) = f\big(P(\Omega_N^*)\big) \leq f\big(P(\Omega)\big) \leq \lambda_1(\Omega)$, with equality if and only if Ω is equal to Ω_N^* up to rigid motions.

4.2 About $\mathcal{D}_{\mathcal{K}^d}$ for $d \geq 3$

As stated in the introduction, a major part of our results for convex sets are restricted to the planar case mainly because some of the assertions of Lemma 3.5 are only given in dimension 2 and seem to be rather challenging to extend to higher dimensions, see Remark 2. Nevertheless, we believe that once a similar result is proven for $d \geq 3$, it would be possible to apply the same strategy developed in the present paper to prove the following conjecture:

Conjecture 2 *We denote $x_0 = P(B)$ where B is a ball of unit volume.*

1. *the diagram $\mathcal{D}_{\mathcal{K}^d}$ is made of all points in $\mathbb{R}^2$ lying between the graphs of f and g, more precisely:*

$$\mathcal{D}_{\mathcal{K}^d} = \Big\{(x, y) \in \mathbb{R}^2,\ x \geq x_0 \ \text{ and } \ f(x) \leq y \leq g(x)\Big\}, \tag{35}$$

   where f and g are defined in (14) *and* (15)*.*

2. *functions f and g are continuous and strictly increasing.*

4.3 About $\mathcal{D}_{\mathcal{S}^d}$ where $\mathcal{S}^d$ is the class of simply connected domains

We decided to focus on two classes of domains, $\mathcal{O}$ the class of open domains in $\mathbb{R}^d$, and $\mathcal{K}^d$ the class of convex domains in $\mathbb{R}^d$. But one could also focus on an intermediate class which is

$$\mathcal{S}^d = \{\Omega \subset \mathbb{R}^d,\ \ \Omega \text{ is open, bounded and simply-connected}\}.$$

We give here some thoughts about the Blaschke-Santaló diagram of $(\lambda_1, P, |\cdot|)$ in this class, denoted $\mathcal{D}_{\mathcal{S}^d}$: note first that as for the class of open domains, there is uncertainty about the definition of the perimeter P. But since we are not giving any specific statement here, we consider part of the investigation to decide in which way a change in the definition of P may affect the shape of $\mathcal{D}_{\mathcal{S}^d}$.

1. Assume first that $d = 2$: since inequalities (8) and (11) also hold for the class of planar simply connected domains, the diagram $\mathcal{D}_{\mathcal{S}^2}$ is bounded from above by a continuous function, and therefore different from the diagram of open sets $\mathcal{D}_{\mathcal{O}}$ described in Theorem 1.1. However, we expect that the lower boundary of the diagram is simply given by the horizontal half line $[P(B), +\infty) \times \{\lambda_1(B)\}$. More precisely we formulate:

   Conjecture 3 *There exists h a continuous and increasing function such that*

$$\mathcal{D}_{\mathcal{S}^2} = \{\big(P(B), \lambda_1(B)\big)\} \cup \{(x, y) \mid x > P(B),\ \ \lambda_1(B) < y \leq h(x)\}.$$

   This is supported by the fact that we can find shapes with a high perimeter but whose first Dirichlet-eigenvalue is close to the one of the ball, for example by adding a thin tail to a ball (see for example [20] for results on tailed domains).

   Finally, notice that we do not know whether we should expect h and g to be equal or not. This is probably also a challenging question.

2. If we now assume $d \geq 3$, the class of simply connected domains behave very differently. Actually, we can introduce an even more restrictive class of domains, namely

$$\widetilde{\mathcal{S}}^d = \{\Omega \subset \mathbb{R}^d,\ \ \Omega \text{ is open and homeomorphic to a ball}\}.$$

   We believe that in this case we have

$$\mathcal{D}_{\mathcal{S}^d} = \mathcal{D}_{\widetilde{\mathcal{S}}^d} = \mathcal{D}_{\mathcal{O}}.$$

   To support this conjecture, we refer to the construction described in [18, Remark 6.2] and inspired by [25].

3. It would also be interesting to study the diagram in the class of sharshaped domains. In dimension 2, it is not clear whether we expect the diagram to be the same as the diagram for simply connected sets or not. In dimension higher than 3 however, it would be more natural to expect that the diagram differs from the one of simply connected sets, but we did not investigate this question yet.

Acknowledgements. This work was partially supported by the project ANR-18-CE40-0013 SHAPO financed by the French Agence Nationale de la Recherche (ANR). The authors would also like to thank Dorin Bucur, Antoine Henrot, Ilaria Lucardesi, Bozhidar Velichkov and Davide Zucco for valuable discussions about this paper. Finally, the authors thank the referees for their thorough reading and many helpful suggestions.

References

[1] E. Acerbi, N. Fusco, and M. Morini. Minimality via second variation for a nonlocal isoperimetric problem. *Comm. Math. Phys.*, 322(2):515–557, 2013.

[2] P. Antunes and P. Freitas. New bounds for the principal Dirichlet eigenvalue of planar regions. *Experimental Mathematics*, 15(3):333–342, March 2006.

[3] P. Antunes and A. Henrot. On the range of the first two Dirichlet and Neumann eigenvalues of the laplacian. *Proceedings of the Royal Society of London*, 467(2):1577–1603, 2011.

[4] C. Bianchini, A. Henrot, and T. Takahashi. Elastic energy of a convex body. *Mathematische Nachrichten*, 289(5):546–574, 2016.

[5] W. Blaschke. Eine frage über konvexe Köirper. *Jahresber. Deutsch. Math. Ver.*, 25:121–125, 1916.

[6] T. Bonnesen and W. Fenchel. *Theory of Convex bodies*. B C S Associates, 1987.

[7] A. Boulkhemair. On a shape derivative formula in the Brunn-Minkowski theory. *SIAM J. Control Optim.*, 55(1):156–171, 2017.

[8] B. Brandolini, C. Nitsch, and C. Trombetti. An upper bound for nonlinear eigenvalues on convex domains by means of the isoperimetric deficit. *Arch. Math. (Basel)*, 94(4):391–400, 2010.

[9] L. Brasco. On principal frequencies and inradius in convex sets. *Bruno Pini Math. Anal. Semin.*, page 20, 2018.

[10] L. Brasco. On principal frequencies and isoperimetric ratios in convex sets. *Ann. Fac. Sci. Toulouse Math.*, page 25, 2019.

[11] L. Brasco, G. De Philippis, and B. Velichkov. Faber-Krahn inequalities in sharp quantitative form. *Duke Math. J.*, 164(9):1777–1831, 2015.

[12] H. Brezis and P. Mironescu. Gagliardo-Nirenberg inequalities and non-inequalities: The full story. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 35(5):1355 – 1376, 2018.

[13] F. Brock. Continuous Steiner-symmetrization. *Math. Nachr.*, 172:25–48, 1995.

[14] D. Bucur, G. Buttazzo, and I. Figueiredo. On the attainable eigenvalues of the Laplace operator. *SIAM J. Math. Ana.*, 30:527–536, 1999.

[15] T. Chatelain and A. Henrot. Some results about Schiffer's conjectures. *Inverse Problems*, 15:647–658, March 1999.

[16] D. Cioranescu and F. Murat. A strange term coming from nowhere. In *Topics in the mathematical modelling of composite materials*, volume 31 of *Progr. Nonlinear Differential Equations Appl.*, pages 45–93. Birkhäuser Boston, Boston, MA, 1997.

[17] S. J. Cox and M. Ross. Extremal eigenvalue problems for starlike planar domains. *J. Differential Equations*, 120(1):174–197, 1995.

[18] M. Dambrine and J. Lamboley. Stability in shape optimization with second variation. *Journal of Differential Equations*, 267(5):3009–3045, 2019.

[19] D. De Silva. Free boundary regularity for a problem with right hand side. *Interfaces Free Bound.*, 13(2):223–238, 2011.

[20] A. L. Delitsyn, B. T. Nguyen, and D. S. Grebenkov. Exponential decay of Laplacian eigenfunctions in domains with branches of variable cross-sectional profiles. *European Physical Journal B*, 85(11):371, November 2012.

[21] F. Della Pietra and N. Gavitone. Sharp bounds for the first eigenvalue and the torsional rigidity related to some anisotropic operators. *Math. Nachr.*, 287(2-3):194–209, 2014.

[22] A. Delyon, A. Henrot, and Y. Privat. Non-dispersal and density properties of infinite packings. *SIAM Journal on Control and Optimization*, 57(2):1467–1492, 2019.

[23] A. Delyon, A. Henrot, and Y. Privat. The missing (A, d, r) diagram. *Submitted - https://hal.archives-ouvertes.fr/hal-02334941*, 2020.

[24] L. Esposito, N. Fusco, and C. Trombetti. A quantitative version of the isoperimetric inequality: the anisotropic case. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 4(4):619–651, 2005.

[25] A. Fraser and R. Schoen. Shape optimization for the Steklov problem in higher dimensions. *Adv. Math.*, 348:146–162, 2019.

[26] I. Ftouhi. Numerical study of shape optimization and Blaschke-Santaló diagrams in the class of convex domains, (in preparation).

[27] B. Fuglede. Stability in the isoperimetric problem for convex or nearly spherical domains in $\mathbf{R}^n$. *Trans. Amer. Math. Soc.*, 314(2):619–638, 1989.

[28] N. Fusco, M. S. Gelli, and G. Pisante. On a Bonnesen type inequality involving the spherical deviation. *Journal de Mathématiques Pures et Appliquées*, 98(6):616 – 632, 2012.

[29] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. *Ann. of Math. (2)*, 168(3):941–980, 2008.

[30] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[31] A. Henrot. *Extremum problems for eigenvalues of elliptic operators*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[32] A. Henrot and M. Pierre. *Variation et Optimisation de Formes*, volume 16 of *Springer Series in Computational Mathematics*. Springer, 2005.

[33] M.A. Hernandez Cifre, G. Salinas, and S.S Gomis. Complete systems of inequalities. *Journal of inequalities in pure and applied mathematics*, 2(1), March 2001.

[34] I. Joó and L. L. Stachó. Generalization of an inequality of G. Pólya concerning the eigenfrequences of vibrating bodies. *Publ. Inst. Math. (Beograd) (N.S.)*, 31(45):65–72, 1982.

[35] B. Kawohl. *Rearrangements and convexity of level sets in PDE*, volume 1150 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1985.

[36] D. Kinderlehrer and L. Nirenberg. Regularity in free boundary problems. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 4(2):373–391, 1977.

[37] T. Lachand-Robert and M. A. Peletier. Newton's problem of the body of minimal resistance in the class of convex developable functions. *Math. Nachr.*, 226:153–176, 2001.

[38] J. Lamboley and A. Novruzi. Polygons as optimal shapes with convexity constraint. *SIAM J. Control Optim.*, 48(5):3003–3025, 2009/10.

[39] J. Lamboley and A. Novruzi. Maximizers of Dirichlet eigenvalue among convex domains are polygons, (in preparation).

[40] J. Lamboley, A. Novruzi, and M. Pierre. Regularity and singularities of optimal convex shapes in the plane. *Arch. Ration. Mech. Anal.*, 205(1):311–343, 2012.

[41] J. Lamboley, A. Novruzi, and M. Pierre. Estimates of first and second order shape derivatives in nonsmooth multidimensional domains and applications. *Journal of Functional Analysis*, 270(7):2616–2652, 2016.

[42] J. L. Lewis and A. Vogel. *On Some Almost Everywhere Symmetry Theorems*, pages 347–374. Birkhäuser Boston, Boston, MA, 1992.

[43] I. Lucardesi and D. Zucco. On Blaschke-Santaló diagrams for the torsional rigidity and the first Dirichlet eigenvalue. *Submitted - http://cvgmt.sns.it/paper/4490/*, 2019.

[44] E. Makai. On the principal frequency of a membrane and the torsional rigidity of a beam. In *Studies in mathematical analysis and related topics*, pages 227–231. Stanford Univ. Press, Stanford, Calif., 1962.

[45] L. Molinari. On the ground state of regular polygonal billiards. *J. Phys. A*, 30(18):6517–6524, 1997.

[46] C. Nitsch. An isoperimetric result for the fundamental frequency via domain derivative. *Calc. Var. Partial Differential Equations*, 49(1-2):323–335, 2014.

[47] R. Osserman. The isoperimetric inequality. *Bull. Amer. Math. Soc.*, 84(6):1182–1238, 11 1978.

[48] L. E. Payne and H. F. Weinberger. Some isoperimetric inequalities for membrane frequencies and torsional rigidity. *J. Math. Anal. Appl.*, 2:210–216, 1961.

[49] G. Pólya. Two more inequalities between physical and geometrical quantities. *J. Indian Math. Soc. (N.S.)*, 24:413–419 (1961), 1960.

[50] E. Russ, B. Trey, and B. Velichkov. Existence and regularity of optimal shapes for elliptic operators with drift. *To appear in Calc. Var. PDE*, 2020.

[51] V. Sander. `http://cglab.ca/~sander/misc/ConvexGeneration/convex.html`.

[52] L. Santaló. Sobre los sistemas completos de desigualdades entre treselementos de una figura convexa plana. *Math. Notae*, 17:82–104, March 1961.

[53] R. Schneider. *Convex Bodies: The Brunn-Minkowski Theory*. Cambridge University Press, 2nd expanded edition edition, 2013.

[54] J. Serrin. A symmetry problem in potential theory. *Arch. Rational Mech. Anal.*, 43:304–318, 1971.

[55] E. M. Stein and G. Weiss. *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.

[56] P. Valtr. Probability that n random points are in convex position. *Discrete Comput. Geom.*, 13(3-4):637–643, 1995.

[57] M. van den Berg, G. Buttazzo, and A. Pratelli. On the relations between principal eigenvalue and torsional rigidity. *Submitted - http://cvgmt.sns.it/paper/4514/*, 2019.