Block diagonalisation of four-dimensional metrics

J D E Grant[1] **and J A Vickers**[2]

[1] Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria

[2] School of Mathematics, University of Southampton, Highfield, Southampton SO17 1BJ, United Kingdom

E-mail: `james.grant@univie.ac.at` and `J.A.Vickers@maths.soton.ac.uk`

Abstract. It is shown that, in 4-dimensions, it is possible to introduce coordinates so that an analytic metric locally takes block diagonal form. i.e. one can find coordinates such that $g_{\alpha\beta} = 0$ for $(\alpha, \beta) \in S$ where $S = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$. We call a coordinate system in which the metric takes this form a 'doubly biorthogonal coordinate system'. We show that all such coordinate systems are determined by a pair of coupled second-order partial differential equations.

AMS classification scheme numbers: Primary: 58A15, secondary: 53B99

Submitted to: *Class. Quantum Grav.*

1. Introduction

This paper is concerned with making coordinate choices to put general metrics into simplified or canonical forms. A metric in 2-dimensions depends upon $\frac{1}{2} \times 2(2+1) = 3$ arbitrary functions g_{11}, g_{12} and g_{22}. On the other hand, the diffeomorphism freedom

$$\begin{aligned} f \colon \mathbb{R}^2 &\to \mathbb{R}^2 \\ (x,y) &\mapsto (f_1(x,y), f_2(x,y)) \end{aligned}$$

contains 2 arbitrary functions. Given any 2-dimensional metric, one would therefore expect to be able to introduce local coordinates such that the metric depended on only $3-2=1$ function. Indeed, it is a classical result that, in two dimensions, every metric is (locally) conformally flat, i.e. there exist coordinates so that

$$ds^2 = \Omega^2(x,y)\left(dx^2+dy^2\right).$$

The proof for analytic metrics goes back to Gauss [1], while the proof for smooth metrics is more recent (see for example [2] for details).

In 3-dimensions the metric depends upon $\frac{1}{2} \times 3(3+1) = 6$ arbitrary functions, while the diffeomorphism freedom $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ involves 3 functions. One would therefore expect to be able to introduce coordinates such that a 3-dimensional metric was specified by $6-3=3$ functions. In fact in 3-dimensions one can introduce coordinates that locally diagonalise the metric. i.e. there exist coordinates such that

$$ds^2 = A(x,y,z)dx^2 + B(x,y,z)dy^2 + C(x,y,z)dz^2.$$

Again the proof of this result in the analytic case goes back a long way [3]. The proof in the smooth case was, again, much more recent [4] and uses the theory of the characteristic variety of an exterior differential system.

In 4-dimensions the metric depends upon $\frac{1}{2} \times 4(4+1) = 10$ arbitrary functions, while the diffeomorphism freedom $f \colon \mathbb{R}^4 \to \mathbb{R}^4$ gives 4 functions. One would therefore expect to be able to write a 4-dimensional metric in a canonical form that depended upon $10-4=6$ arbitrary functions. Thus, in general, one cannot expect to be able to diagonalise a metric in 4-dimensions, although of course in special cases this is possible (this problem was considered in [5]). However it was suggested to one of us by David Robinson that an appropriate local canonical form for 4-dimensional metrics was the 'block diagonal' form

$$g_{\alpha\beta} = \begin{pmatrix} A & B & 0 & 0 \\ B & C & 0 & 0 \\ 0 & 0 & D & E \\ 0 & 0 & E & F \end{pmatrix}.$$

In this paper we use Cartan's theory of exterior differential systems to show that it is indeed possible to write an analytic 4-dimensional metric in this form, at least locally. We show that the problem of finding local coordinates that block-diagonalise a metric may be reformulated as a condition on an orthonormal tetrad (see equation (3.4)).

From this reformulation, we construct an exterior differential system on the orthonormal frame bundle of our manifold, the integral manifolds give rise to solutions of our block-diagonalisation problem. This exterior differential system is not involutive, however, so we must go to the first prolongation. At this point, we discover a consistency condition for our system, (4.10), that must be satisfied. Imposing this constraint on our exterior differential system gives rise to an involutive Pfaffian system, to which the Cartan–Kähler theorem may be applied to show existence of solutions. Note that the consistency condition mentioned above may be interpreted on our manifold as a relation between a curvature component and various components of the connection (cf. equation (4.17) for the Riemannian version of this constraint and equation (4.20) for the Lorentzian version in Newman–Penrose formalism). At the level of our four-dimensional manifold, this constraint may be deduced directly as being a consequence of the conditions (3.4) imposed on the orthonormal tetrad. The constraint involves the extrinsic curvature of the two surfaces and does not impose any additional geometrical restrictions on our manifold. Indeed the fact that we have a Pfaffian system on the first prolongation which satisfies the conditions for the Cartan–Kähler theorem shows that the block-diagonalisation of *any* four-dimensional metric may be carried out locally.

Although our results for local canonical forms have assumed that the metric is Riemannian, they remain true in the Lorentzian case (with obvious modifications). Similarly, we will assume that our metric is Riemannian, although the proof may easily be adapted for metrics of Lorentzian or $(-,-,+,+)$ signature. The Lorentzian version of the 4-dimensional result is, in particular, useful in establishing certain results in general relativity. For example, it can be used to establish some results concerning the geometry of generalised cosmic strings [6] and can also be used to make a gauge choice within the $2+2$ formalism [7] in which all the shifts β^i_α vanish.

Given a local canonical form for a metric one can ask what transformations preserve that form. For the case of a metric in 2-dimensions a conformal (in the sense of complex analytic) transformation of the flat metric will map isothermal coordinates into isothermal coordinates. Similarly in the 3-dimensional case the problem is essentially the same as finding all 'triply orthogonal coordinate systems' which are coordinates in which the flat metric is diagonal. The problem of finding all such coordinate systems was solved by Darboux [8], who showed that it required the solution of a certain third-order partial differential equation. Similarly in 4-dimensions the problem is essentially the same as finding all 'doubly biorthogonal coordinate systems' which are coordinates in which the flat metric is in block diagonal form. We will show in Section 5 that all such coordinates are determined by the solution of a pair of coupled second-order equations.

The plan of this paper is as follows. In Section 2 we briefly review the proofs that 3-dimensional metrics may be diagonalised in both the analytic and smooth case. In Section 3 we explain why these methods fail to give a direct proof of the block diagonalisation of a 4-dimensional metric. However, we reduce the problem of block diagonalising a metric to the problem of constructing an orthonormal tetrad that satisfies a particular set of identities (3.4). In Section 4 we show, using the theory of exterior differential systems, that, in the case where the metric is analytic, such an orthonormal tetrad can always be constructed. As such, we deduce that a four-dimensional analytic metric can be block-diagonalised. In Section 5 we discuss triply orthogonal systems in 3-dimensions to motivate the discussion of doubly biorthogonal systems of coordinates in 4-dimensions. In order to make the paper reasonably self-contained, we have collected together the main background material that we require from the theory of exterior differential systems in Appendix A.

Notation: In the earlier sections of this paper, we will often have cause to refer to a single diagonal component $g_{\alpha\alpha}$ of a metric. Also, when working with exterior differential systems, it is sometimes convenient to explicitly write out the terms in a sum individually, rather than use the summation convention. Therefore, we will generally ***not*** use the Einstein summation convention in this paper, with the exception of Section 5, where the above issues do not arise.

Note also that we will use Greek letters for coordinate indices and Latin letters for frame indices.

2. Diagonalising metrics in 3-dimensions

In this section, we review the methods of proving that a 3-dimensional smooth metric can be diagonalised.

In the analytic case, rather than working with the covariant metric $g_{\alpha\beta}$ it is more convenient to consider the equivalent problem of diagonalising the contravariant metric $g^{\alpha\beta}$. Given $g^{\alpha\beta}(x^1, x^2, x^3)$ we wish to find new coordinates $\{x^{\alpha'}(x^1, x^2, x^3) : \alpha = 1, 2, 3\}$ such that

$$g^{\alpha'\beta'} = \sum_{\gamma,\delta} \frac{\partial x^{\alpha'}}{\partial x^\gamma} \frac{\partial x^{\beta'}}{\partial x^\delta} g^{\gamma\delta} = 0 \qquad \text{for } \alpha' \neq \beta'.$$

This is a non-linear system of 3 equations (taking (α', β') to be $(1, 2)$, $(1, 3)$ and $(2, 3)$) for three unknowns $x^{1'}$, $x^{2'}$ and $x^{3'}$. In the analytic case one can show that solutions to these equations exist but the solutions are not unique (there are trivial transformations given by replacing $x^{1'}$, $x^{2'}$ and $x^{3'}$ with $h^1(x^{1'})$, $h^2(x^{2'})$ and $h^3(x^{3'})$) and the strongly non-linear nature of the equations makes it hard to utilise this method in the smooth case. Instead, DeTurck and Yang [4] seek an orthonormal coframe $\boldsymbol{\epsilon}^1$, $\boldsymbol{\epsilon}^2$, $\boldsymbol{\epsilon}^3$, and a coordinate system x^1, x^2, x^3 such that

$$\boldsymbol{\epsilon}^i = f^i dx^i, \qquad i = 1, 2, 3. \tag{2.1}$$

(Recall **no summation**.) Clearly such a frame would imply that $g_{\mu\nu}$ is diagonal in the coordinate system of the x^μ.

The advantage of condition (2.1) is that, by the Frobenius theorem, it is (locally) equivalent to the existence of a coframe such that

$$\boldsymbol{\epsilon}^i \wedge d\boldsymbol{\epsilon}^i = 0, \qquad i = 1, 2, 3 \tag{2.2}$$

and this is a problem that may be solved without having to consider coordinate transformations. Furthermore, one would expect the $\boldsymbol{\epsilon}^i$ to be unique (up to relabelling) since the lack of uniqueness in the coordinates noted above is absorbed into the f^i.

Let $\{\overline{\boldsymbol{\epsilon}}^i\}$ be some fixed orthonormal frame for $g_{\alpha\beta}$ in some open set. Then, since $\boldsymbol{\epsilon}^i$ and $\overline{\boldsymbol{\epsilon}}^i$ are both orthonormal, they are related by some SO(3) transformation $a^i{}_j$

$$\boldsymbol{\epsilon}^i(x) = \sum_j a^i{}_j(x)\, \overline{\boldsymbol{\epsilon}}^j(x). \tag{2.3}$$

We now substitute (2.3) into (2.2) to obtain

$$\sum_{j,k} a^i{}_j\, \overline{\boldsymbol{\epsilon}}^j \wedge d\left(a^i{}_k\, \overline{\boldsymbol{\epsilon}}^k\right) = 0, \qquad i = 1, 2, 3. \tag{2.4}$$

Note that this gives 3 equations for 3 unknowns (such as the Euler angles) which parameterise elements of SO(3). To show that there exist solutions to (2.4), DeTurck and Yang write the second term as

$$d\left(a^i{}_k\, \overline{\boldsymbol{\epsilon}}^k\right) = \sum_l \left(a^i{}_{k|l}\, \overline{\boldsymbol{\epsilon}}^l \wedge \overline{\boldsymbol{\epsilon}}^k + a^i{}_k\, d\overline{\boldsymbol{\epsilon}}^k\right)$$

(where $f_{|i} = \overline{\mathbf{e}}_i\,(f) = \sum_\mu \overline{e}_i{}^\mu \frac{\partial f}{\partial x^\mu}$, with $\overline{\mathbf{e}}_i$ the dual basis to $\overline{\boldsymbol{\epsilon}}^i$). They then use Cartan's first structure equation to write

$$d\overline{\boldsymbol{\epsilon}}^k = \sum_{l,m} \overline{\gamma}^k{}_{lm}\, \overline{\boldsymbol{\epsilon}}^l \wedge \overline{\boldsymbol{\epsilon}}^m,$$

where $\overline{\gamma}^k{}_{ml}$ are the connection coefficients with respect to the frame $\overline{\boldsymbol{\epsilon}}^i$. (Our conventions are that $d\boldsymbol{\epsilon}^i = -\Gamma^i{}_j \wedge \boldsymbol{\epsilon}^j$ with $\Gamma^i{}_j = \sum_k \gamma^i{}_{jk} \boldsymbol{\epsilon}^k$.)

Substituting in (2.4) gives

$$\sum_{\sigma \in \Sigma_3} \sum_{j,k,l,m} (\operatorname{sign} \sigma)\, a^i{}_{\sigma(j)} \left(a^i{}_{\sigma(k)|\sigma(l)} + a^i{}_m \overline{\gamma}^m{}_{\sigma(k)\sigma(l)}\right) = 0, \qquad i = 1, 2, 3.$$

One can then solve for $a^i{}_{k|l}$ and show that the resulting system is diagonal hyperbolic (a special case of symmetric hyperbolic). In the smooth case one has existence and uniqueness theorems for such systems of equations (see e.g. [9]), so that one can show the existence of a unique (up to relabelling) orthonormal frame satisfying (2.1) and hence a diagonal metric. Note however that as remarked earlier the coordinate expression (2.1) is not unique, but one is free to replace x^1 by $h(x^1)$ etc, so the actual diagonal entries of the metric are not unique.

3. Block diagonalisation of 4-dimensional metrics

In this section and the next, we shall show that it is possible, in the analytic case, to introduce coordinates that block-diagonalise a 4-dimensional metric. The proof will eventually be by an application of the Cartan–Kähler theorem, a generalisation of the Cauchy–Kovalevskya theorem [10]. However, we shall begin by trying to repeat the methods for diagonalising analytic metrics in 3-dimensions.

Given $g^{\alpha\beta}(x^1, x^2, x^3, x^4)$, we want to find new coordinates $\{x^{\alpha'}(x^1, x^2, x^3, x^4) : \alpha = 1, \ldots, 4\}$ such that

$$g^{\alpha'\beta'} = \sum_{\gamma,\delta} \frac{\partial x^{\alpha'}}{\partial x^{\gamma}} \frac{\partial x^{\beta'}}{\partial x^{\delta}} g^{\gamma\delta} = 0 \qquad \text{for } (\alpha', \beta') \in S,$$

where $S = \{(1,3), (1,4), (2,3), (2,4)\}$. This gives 4 equations for 4 unknowns.

For ease of notation, we let $x^{\alpha'}(x^1, x^2, x^3, x^4) = y^{\alpha}(x^1, x^2, x^3, x^4) = y^{\alpha}(x^{\beta})$. We now linearise about $y^{\alpha}_o(x^{\beta})$ and obtain

$$\sum_{\gamma,\delta} \left(y^{\alpha}_{,\gamma} y^{\beta}_{o,\delta} + y^{\beta}_{,\gamma} y^{\alpha}_{o,\delta} \right) g^{\gamma\delta} = - \sum_{\gamma,\delta} y^{\alpha}_{o,\gamma} y^{\beta}_{o,\delta} g^{\gamma\delta} \qquad \text{for } (\alpha, \beta) \in S.$$

This is a system of the form

$$P^{\alpha} \frac{\partial}{\partial x^{\alpha}} \mathbf{y} = \mathbf{c},$$

where

$$P^{\alpha} = \begin{pmatrix} y^{3,\alpha}_o & 0 & y^{1,\alpha}_o & 0 \\ y^{4,\alpha}_o & 0 & 0 & y^{1,\alpha}_o \\ 0 & y^{3,\alpha}_o & y^{2,\alpha}_o & 0 \\ 0 & y^{4,\alpha}_o & 0 & y^{2,\alpha}_o \end{pmatrix},$$

$$\mathbf{c} = \begin{pmatrix} -y^1_{o,\alpha} y^3_{o,\beta} g^{\alpha\beta} \\ -y^1_{o,\alpha} y^4_{o,\beta} g^{\alpha\beta} \\ -y^2_{o,\alpha} y^3_{o,\beta} g^{\alpha\beta} \\ -y^2_{o,\alpha} y^4_{o,\beta} g^{\alpha\beta} \end{pmatrix},$$

and $y^{\beta,\alpha}_o = y^{\beta}_{o,\gamma} g^{\alpha\gamma}$.

Unfortunately, when one attempts to find the characteristic surfaces, one finds

$$\det(P^{\alpha}\xi_{\alpha}) = 0, \qquad \forall \xi_{\alpha} \in \mathbb{R}^4,$$

so that there are no non-characteristic surfaces and the initial data must satisfy some constraint. As a result, one cannot directly apply the Cauchy–Kovalevskya theorem, unlike in the apparently similar problem of diagonalising a metric in 3-dimensions.

We therefore turn to the method of DeTurck and Yang. In this case, this involves finding a coframe $\{\boldsymbol{\epsilon}^i\}$ and a coordinate system such that

$$\boldsymbol{\epsilon}^1 \wedge \boldsymbol{\epsilon}^2 = f dx^1 \wedge dx^2, \tag{3.1a}$$

$$\boldsymbol{\epsilon}^3 \wedge \boldsymbol{\epsilon}^4 = g dx^3 \wedge dx^4. \tag{3.1b}$$

Note that (3.1*a*) implies

$$\boldsymbol{\epsilon}^i = \sum_{\mu=1,2} \boldsymbol{\epsilon}^i_{\mu} dx^{\mu} \qquad i = 1, 2 \tag{3.2}$$

and that (3.1*b*) implies

$$\boldsymbol{\epsilon}^i = \sum_{\mu=3,4} \boldsymbol{\epsilon}^i_{\mu} dx^{\mu} \qquad i = 3, 4 \tag{3.3}$$

and hence $g_{\mu\nu} = \sum_{i,j} \delta_{ij} \boldsymbol{\epsilon}^i_\mu \boldsymbol{\epsilon}^j_\nu$ is block diagonal. Conversely, if $g_{\mu\nu}$ is block diagonal, we can certainly find a coframe that satisfies (3.2) and (3.3) and hence (3.1a) and (3.1b).

This leads to the following characterisation of metrics that can be block-diagonalised:

Proposition 3.1. *A Riemannian metric* $\mathbf{g}$ *can be block-diagonalised if and only if it admits an orthonormal coframe,* $\{\boldsymbol{\epsilon}^a : a = 1, \ldots, 4\}$*, that satisfies the relations*

$$\begin{aligned} \boldsymbol{\epsilon}^1 \wedge \boldsymbol{\epsilon}^2 \wedge d\boldsymbol{\epsilon}^1 &= 0, \\ \boldsymbol{\epsilon}^1 \wedge \boldsymbol{\epsilon}^2 \wedge d\boldsymbol{\epsilon}^2 &= 0, \\ \boldsymbol{\epsilon}^3 \wedge \boldsymbol{\epsilon}^4 \wedge d\boldsymbol{\epsilon}^3 &= 0, \\ \boldsymbol{\epsilon}^3 \wedge \boldsymbol{\epsilon}^4 \wedge d\boldsymbol{\epsilon}^4 &= 0. \end{aligned} \tag{3.4}$$

Proof. Given a coframe that obeys relations (3.4), the Frobenius theorem implies the existence of local coordinates (t, x, y, z) and functions $\alpha, \ldots, \theta$ such that

$$\begin{aligned} \boldsymbol{\epsilon}^1 &= \alpha\, dt + \beta\, dx, & \boldsymbol{\epsilon}^2 &= \gamma\, dt + \delta\, dx, \\ \boldsymbol{\epsilon}^3 &= \epsilon\, dy + \zeta\, dz, & \boldsymbol{\epsilon}^4 &= \eta\, dy + \theta\, dz. \end{aligned} \tag{3.5}$$

The metric $\mathbf{g}$ is then block-diagonal in this coordinate system. Conversely, if the metric $\mathbf{g}$ is block-diagonal with respect to a coordinate system (t, x, y, z), then we can choose a coframe of the form (3.5), which then automatically satisfies (3.4). □

Remark 3.2. Although we have stated the block-diagonalisation problem in terms of Riemannian manifolds, it is clear that the problem of block-diagonalising a metric is conformally invariant, In particular, a coordinate system that block-diagonalises a representative metric in a conformal equivalence class will block-diagonalise all representatives in that conformal equivalence class. We will pursue the Riemannian version of the problem for simplicity, although all of our calculations can be reformulated in a conformally equivariant fashion.

In the next section, the characterisation given in Proposition 3.1 will be used to show that all analytic four-dimensional metrics can be block-diagonalised.

4. Exterior differential systems

In this section, we use the theory of exterior differential systems, in particular the Cartan–Kähler theorem, to show that, for a given analytic metric $\mathbf{g}$, we can find an orthonormal coframe that satisfies the conditions (3.4) of Proposition 3.1. Our notation, generally, follows that of [10]. The methods that we use are similar to those used in the study of orthogonal coordinates for Riemannian metrics in Chapter III, Section 3, Example 3.2, and Chapter VII, Section 3 of [10]. For completeness, however, a summary of the relevant terminology and results from exterior differential systems theory has been included in Appendix A.

Let X be an oriented four-manifold with a Riemannian metric $\mathbf{g}$, and let $\pi\colon \mathcal{F} \to X$ be the bundle of orthonormal coframes of $(X, \mathbf{g})$. We will denote points in $\mathcal{F}$ by either p or, since we are working locally, we will assume a trivialisation $\pi^{-1}(X) \cong X \times \mathrm{SO}(4)$ and denote points in $\mathcal{F}$ by (x, g) where $x \in X$ and $g \in \mathrm{SO}(4)$. The bundle $\mathcal{F}$ comes equipped with a canonical basis of 1-forms consisting of the components, $\{\boldsymbol{\omega}^a\}_{a=1,\dots,4}$, of the tautological 1-form on $\mathcal{F}$ and the components, $\{\boldsymbol{\omega}^a{}_b\}_{a,b=1,\dots,4}$, of the Levi-Civita connection (see, e.g., [11]). These differential forms have the following properties:

- **Reproducing property**: An orthonormal coframe $\{\boldsymbol{\epsilon}^a\}_{a=1}^4$ on M defines a corresponding section $f\colon X \to \mathcal{F}$. Pulling back the tautological 1-forms on $\mathcal{F}$ by this section reproduces the coframe $\{\boldsymbol{\epsilon}^a\}$ i.e. $f^*\boldsymbol{\omega}^a = \boldsymbol{\epsilon}^a$.
- **Canonical coframing**: A canonical coframing of $\mathcal{F}$ consists of the tautological 1-forms $\boldsymbol{\omega}^a$, $a = 1, \dots, 4$ and the connection 1-forms $\boldsymbol{\omega}^a{}_b$, where $a, b = 1, \dots, 4$ with $a < b$. Note that we will often write summations that involve terms of the form $\boldsymbol{\omega}^a{}_b$ with $a > b$. In this case, we identify $\boldsymbol{\omega}^a{}_b$ with $-\sum_{c,d} \delta^{ac}\delta_{bd}\boldsymbol{\omega}^d{}_c$, consistent with the $\mathrm{SO}(4)$ nature of the connection. We adopt similar conventions with quantities such as $\lambda^b{}_{ca}$ introduced later.
- **Cartan structure equations**: The one-forms $\{\boldsymbol{\omega}^a, \boldsymbol{\omega}^a{}_b\}$ obey the Cartan structure equations
$$\begin{aligned} d\boldsymbol{\omega}^a + \sum_b \boldsymbol{\omega}^a{}_b \wedge \boldsymbol{\omega}^b &= 0, \\ d\boldsymbol{\omega}^a{}_b + \sum_c \boldsymbol{\omega}^a{}_c \wedge \boldsymbol{\omega}^c{}_b &= \boldsymbol{\Omega}^a{}_b, \end{aligned}$$
where
$$\boldsymbol{\Omega}^a{}_b = \frac{1}{2}\sum_{c,d} R^a{}_{bcd}\, \boldsymbol{\omega}^c \wedge \boldsymbol{\omega}^d \in \Omega^2(\mathcal{F}, \mathfrak{so}(4))$$
is the curvature form of the connection form $\boldsymbol{\omega}^a{}_b$. (Recall our convention mentioned above for $\boldsymbol{\omega}^a{}_b$ with $a > b$.)

Following Proposition 3.1, let $\mathcal{I} \subset \Omega^*(\mathcal{F})$ be the exterior differential system on $\mathcal{F}$ generated by the 4-forms
$$\begin{aligned} \boldsymbol{\Theta}^1 &:= \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge d\boldsymbol{\omega}^1 = \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^1{}_3 + \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^4 \wedge \boldsymbol{\omega}^1{}_4, \\ \boldsymbol{\Theta}^2 &:= \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge d\boldsymbol{\omega}^2 = \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^2{}_3 + \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^4 \wedge \boldsymbol{\omega}^2{}_4, \\ \boldsymbol{\Theta}^3 &:= \boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^4 \wedge d\boldsymbol{\omega}^3 = -\boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^4 \wedge \boldsymbol{\omega}^1{}_3 - \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^4 \wedge \boldsymbol{\omega}^2{}_3, \\ \boldsymbol{\Theta}^4 &:= \boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^4 \wedge d\boldsymbol{\omega}^4 = -\boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^4 \wedge \boldsymbol{\omega}^1{}_4 - \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^4 \wedge \boldsymbol{\omega}^2{}_4. \end{aligned} \tag{4.1}$$
(Therefore, $\mathcal{I}$ is the ideal in $\Omega^*(\mathcal{F})$ generated, algebraically, by the 4-forms $\boldsymbol{\Theta}^i$ and the 5-forms $d\boldsymbol{\Theta}^i$.) We consider the exterior differential system with independence condition $(\mathcal{I}, \boldsymbol{\Omega})$ on the ten-dimensional manifold $\mathcal{F}$, where the independence condition is defined by the 4-form
$$\boldsymbol{\Omega} := \boldsymbol{\omega}^1 \wedge \ldots \wedge \boldsymbol{\omega}^4 \in \Omega^4(\mathcal{F}).$$

As a result of the previous discussion, we have the following:

Lemma 4.1. *Let $U \subseteq X$ is an open set, and $f: U \to \mathcal{F}$ a section of $\mathcal{F}$ that satisfies $f^*\boldsymbol{\varphi} = 0$, for all $\boldsymbol{\varphi} \in \mathcal{I}$, and $f^*\boldsymbol{\Omega} \neq 0$ on U. Then the 1-forms $\boldsymbol{\epsilon}^a := f^*\boldsymbol{\omega}^a \in \Omega^1(U)$ define an orthonormal coframe on U that satisfies (3.4).*

Let $E_4 \subset T_p\mathcal{F}$ be a 4-dimensional integral element of $(\mathcal{I}, \boldsymbol{\Omega})$ based at point $p \in \mathcal{F}$ (i.e. $\boldsymbol{\varphi}|E_4 = 0$, for all $\boldsymbol{\varphi} \in \mathcal{I}$ and $\boldsymbol{\Omega}|E_4 \neq 0$). The space of such integral elements is denoted by $V_4(\mathcal{I}, \boldsymbol{\Omega})$, and is a subset of $\mathrm{Gr}_4(T\mathcal{F}, \boldsymbol{\Omega})$, which is the subset of the Grassmannian bundle $\mathrm{Gr}_4(T\mathcal{F})$ consisting of 4-planes, E_4, for which $\boldsymbol{\Omega}|E_4 \neq 0$. Let $(\mathbf{v}_1, \ldots, \mathbf{v}_4)$ be a basis for E_4 which, without loss of generality, we may take to be of the form

$$\mathbf{v}_a = \frac{\partial}{\partial \boldsymbol{\omega}^a}(p) + \sum_{b<c} \lambda^b{}_{ca} \frac{\partial}{\partial \boldsymbol{\omega}^b{}_c}(p), \tag{4.2}$$

where $\{\partial/\partial\boldsymbol{\omega}^a, \partial/\partial\boldsymbol{\omega}^a{}_b\}$ denotes the basis of $T\mathcal{F}$ dual to $\{\boldsymbol{\omega}^a, \boldsymbol{\omega}^a{}_b\}$ (see, e.g., [12, pp. 253] for a discussion of this notation.). Note that the coordinates (x, g) on $\mathcal{F}$ along with the parameters $\left\{\lambda^b{}_{ca} : a, b, c = 1, \ldots, 4;\ b < c\right\}$, give a local coordinate system on $\mathrm{Gr}_4(T\mathcal{F}, \boldsymbol{\Omega})$. The condition that E_4 be an integral element of $\mathcal{I}$ is that $\boldsymbol{\Theta}^i(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = 0$ for $i = 1, \ldots, 4$. Substituting (4.2) into (4.1), we find that this is equivalent to the conditions

$$\begin{aligned} &\lambda^2{}_{31} = \lambda^1{}_{32}, \qquad \lambda^2{}_{41} = \lambda^1{}_{42}, \\ &\lambda^1{}_{43} = \lambda^1{}_{34}, \qquad \lambda^2{}_{43} = \lambda^2{}_{34}. \end{aligned} \tag{4.3}$$

At each point $p \in \mathcal{F}$, these equations impose 4 linear constraints on the coordinates $\lambda^b{}_{ca}$. It therefore follows that $V_4(\mathcal{I}, \boldsymbol{\Omega})$ is a smooth submanifold of $Gr_4(T\mathcal{F})$ of codimension 4.

We now consider an integral flag $(0)_p \subset E_1 \subset E_2 \subset E_3 \subset E_4 \subset T_p(\mathcal{F})$, and wish to calculate the integers $c_k, k = 0, \ldots, 4$ (see Definition A.4 in Appendix A). Since $\mathcal{I}$ contains no non-zero 1-forms, 2-forms or 3-forms, it follows that

$$c_0 = c_1 = c_2 = 0$$

and, from its definition, we have

$$c_4 = \dim \mathcal{F} - 4 = 6.$$

Therefore, it only remains to calculate c_3. To do this, we first define the one-forms

$$\boldsymbol{\pi}^a{}_b := \boldsymbol{\omega}^a{}_b(p) - \sum_c \lambda^a{}_{bc}\, \boldsymbol{\omega}^c(p) \in T^*_p\mathcal{F}.$$

Note that the $\boldsymbol{\pi}^a{}_b$, with $a < b$, span the subspace of $T^*_p(\mathcal{F})$ that annihilate the vectors $\mathbf{v}_a$. It then follows that E_4 may be described as

$$E_4 = \left\{\mathbf{v} \in T_p\mathcal{F} : \boldsymbol{\pi}^a{}_b(\mathbf{v}) = 0,\ \text{for } a, b = 1, \ldots, 4;\ a < b\right\}.$$

We now note that, by (4.3), we may write

$$\begin{aligned} \boldsymbol{\Theta}^1 &= \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3 \wedge \boldsymbol{\pi}^1{}_3 + \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^4 \wedge \boldsymbol{\pi}^1{}_4, \\ \boldsymbol{\Theta}^2 &= \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3 \wedge \boldsymbol{\pi}^2{}_3 + \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^4 \wedge \boldsymbol{\pi}^2{}_4, \\ \boldsymbol{\Theta}^3 &= -\boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^4 \wedge \boldsymbol{\pi}^1{}_3 - \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^4 \wedge \boldsymbol{\pi}^2{}_3, \\ \boldsymbol{\Theta}^4 &= -\boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^4 \wedge \boldsymbol{\pi}^1{}_4 - \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^4 \wedge \boldsymbol{\pi}^2{}_4. \end{aligned}$$

We let $E_3 := \operatorname{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \subset E_4$, where

$$\mathbf{e}_i = \sum_{a=1}^{4} e_i^a \mathbf{v}_a, \qquad i = 1, 2, 3,$$

and define the quantities

$$\begin{aligned} A &:= \left(\boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3\right)(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3), \qquad & B &:= \left(\boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^4\right)(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3), \\ C &:= \left(\boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^4\right)(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3), \qquad & D &:= \left(\boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^4\right)(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3). \end{aligned}$$

We then wish to consider the polar space

$$H(E_3) := \left\{\mathbf{v} \in T_p\mathcal{F} : \boldsymbol{\varphi}(\mathbf{v}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = 0, \forall \boldsymbol{\varphi} \in \mathcal{I}\right\}$$

(see Definition A.3 in Appendix A). It follows that $\mathbf{v} \in T_p\mathcal{F}$ lies in $H(E_3)$ if and only if

$$\begin{aligned} \boldsymbol{\Theta}^1(\mathbf{v}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) &= -A\boldsymbol{\pi}^1{}_3(\mathbf{v}) - B\boldsymbol{\pi}^1{}_4(\mathbf{v}) = 0, \\ \boldsymbol{\Theta}^2(\mathbf{v}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) &= -A\boldsymbol{\pi}^2{}_3(\mathbf{v}) - B\boldsymbol{\pi}^2{}_4(\mathbf{v}) = 0, \\ \boldsymbol{\Theta}^3(\mathbf{v}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) &= D\boldsymbol{\pi}^1{}_3(\mathbf{v}) + C\boldsymbol{\pi}^2{}_3(\mathbf{v}) = 0, \\ \boldsymbol{\Theta}^4(\mathbf{v}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) &= D\boldsymbol{\pi}^1{}_4(\mathbf{v}) + C\boldsymbol{\pi}^2{}_4(\mathbf{v}) = 0. \end{aligned} \tag{4.4}$$

Since $\boldsymbol{\pi}^1{}_3, \boldsymbol{\pi}^1{}_4, \boldsymbol{\pi}^2{}_3, \boldsymbol{\pi}^2{}_4$ are linearly-independent 1-forms on $\mathcal{F}$, it follows that the number of linearly-independent constraints imposed on a vector $\mathbf{v} \in T_p\mathcal{F}$ by equations (4.4) is equal to the rank of the matrix

$$\alpha := \begin{pmatrix} -A & -B & 0 & 0 \\ 0 & 0 & -A & -B \\ D & 0 & C & 0 \\ 0 & D & 0 & C \end{pmatrix}.$$

Since $\det\alpha = 0$, it follows that $c_3 \leq 3$. Any flag $(0)_p \subset E_1 \subset E_2 \subset E_3 \subset E_4$ such that $\operatorname{rank}\alpha = 3$ (e.g. $A = C = 1$, $B = D = 0$) will give rise to 3 linearly-independent 1-forms, $(\boldsymbol{\pi}^1, \boldsymbol{\pi}^2, \boldsymbol{\pi}^3)$, such that $H(E_3) = \left\{\mathbf{v} \in T_p\mathcal{F} : \boldsymbol{\pi}^1(\mathbf{v}) = \boldsymbol{\pi}^2(\mathbf{v}) = \boldsymbol{\pi}^3(\mathbf{v}) = 0\right\}$. Hence $c_3 = 3$ for such an integral flag.

Corollary 4.2. *The exterior differential system with independence condition* $(\mathcal{I}, \boldsymbol{\Omega})$ *contains no integral elements of dimension* 4 *that pass Cartan's test.*

Proof. The codimension of $V_4(\mathcal{I}, \boldsymbol{\Omega})$ at any integral element is equal to 4. Any four-dimensional integral flag has $c_0 = c_1 = c_2 = 0$ and $c_3 \leq 3$. Therefore $c_0 + c_1 + c_2 + c_3 \leq 3 \neq 4$, so no such integral element passes Cartan's test. □

Note that the non-maximality of the rank of α is essentially the same algebraic condition that led to the non-existence of non-characteristic surfaces when we studied the linearisation of the block-diagonalisation problem in Section 3.

4.1. Prolongation

Since the system $(\mathcal{I}, \boldsymbol{\Omega})$ is not involutive, we cannot directly apply the Cartan–Kähler theorem. There is a standard technique for dealing with such non-involutive exterior differential systems, namely *prolongation* (see, e.g., [10, 11, 12]). In the current context, the (first) prolongation of the system $(\mathcal{I}, \boldsymbol{\Omega})$ is a Pfaffian system defined on the manifold of four-dimensional integral elements of the system $(\mathcal{I}, \boldsymbol{\Omega})$, $V_4(\mathcal{I}, \boldsymbol{\Omega})$. In particular, recall that $(x, g, \lambda^b{}_{ca})$ define a local coordinate system on the Grassmannian bundle $Gr_4(T\mathcal{F})$ of four-planes in the tangent bundle of $\mathcal{F}$. Moreover, the space $M^{(1)} := V_4(\mathcal{I}, \boldsymbol{\Omega})$ is a thirty-dimensional manifold of the form $\mathcal{F} \times \mathbb{R}^{20}$, with the parameters $\lambda^b{}_{ca}$ subject to the symmetry conditions (4.3) as coordinates in the $\mathbb{R}^{20}$ direction. (In particular, the conditions imposed by the exterior differential system $(\mathcal{I}, \boldsymbol{\Omega})$ have already been imposed.) As such $M^{(1)}$ may be viewed as a subspace of the bundle $Gr_4(T\mathcal{F})$. The bundle $Gr_4(T\mathcal{F})$ comes equipped with a natural set of contact forms, and the Pfaffian system that we consider on $M^{(1)}$ is generated by the restriction of these differential forms.

More explicitly, we now consider the exterior differential system with independence condition, $(\mathcal{I}^{(1)}, \boldsymbol{\Omega})$, on the space $M^{(1)}$ generated by the 1-forms

$$\boldsymbol{\theta}^a{}_b := \boldsymbol{\omega}^a{}_b - \sum_c \lambda^a{}_{bc}\, \boldsymbol{\omega}^c, \qquad a, b = 1, \ldots, 4; \qquad a < b, \tag{4.5}$$

where $\boldsymbol{\omega}^a{}_b$ and $\boldsymbol{\omega}^a$ now denote the pull-backs to $M^{(1)}$ of the corresponding forms on $\mathcal{F}$, with the independence condition defined by the 4-form $\boldsymbol{\Omega} := \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^4$.

We now look for four-dimensional integral elements, $E_4 \in V_4(\mathcal{I}^{(1)}, \boldsymbol{\Omega})$, of this system. The point is that if U is an open subset of X and $f\colon U \to M^{(1)}$ a local section of the bundle $M^{(1)}$ with the property that $f^*\boldsymbol{\theta}^a{}_b = 0$, $f^*\boldsymbol{\Omega} \neq 0$, then $\boldsymbol{\epsilon}^i := f^*\boldsymbol{\omega}^i$ define an orthonormal coframe on U that obeys (3.4). As such, integral manifolds of $(\mathcal{I}^{(1)}, \boldsymbol{\Omega})$ define solutions of our block-diagonalisation problem. As a first step in showing the existence of such integral manifolds we show that the system $(\mathcal{I}^{(1)}, \boldsymbol{\Omega})$ on $M^{(1)}$ is involutive. Applying the Cartan–Kähler theorem then gives the solution to our block-diagonalisation problem. Our method here follows that of [10], Chapter VII, §3.

A short calculation shows that

$$d\boldsymbol{\theta}^a{}_b \equiv -\sum_c d\lambda^a{}_{bc} \wedge \boldsymbol{\omega}^c + \frac{1}{2} \sum_{c,d} T^a{}_{bcd}\, \boldsymbol{\omega}^c \wedge \boldsymbol{\omega}^d \mod \boldsymbol{\theta}, \tag{4.6}$$

where we have defined

$$T^a{}_{bcd} := R^a{}_{bcd} + \sum_e \left[\lambda^a{}_{be}\left(\lambda^e{}_{dc} - \lambda^e{}_{cd}\right) - \lambda^a{}_{ec}\lambda^e{}_{bd} + \lambda^a{}_{ed}\lambda^e{}_{bc}\right].$$

The second term in Equation (4.6) implies that there is torsion in the Pfaffian system. We would like to absorb the torsion terms by writing (4.6) in the form $d\boldsymbol{\theta}^a{}_b \equiv -\sum_c \boldsymbol{\pi}^a{}_{bc} \wedge \boldsymbol{\omega}^c \mod \boldsymbol{\theta}$, where $\boldsymbol{\pi}^a{}_{bc} \equiv d\lambda^a{}_{bc} \mod \boldsymbol{\omega}^i$ and the 1-forms $\boldsymbol{\pi}_a{}^b{}_c$, $a, b, c = 1, \ldots, 4$, $b < c$ obey symmetry relations analogous to (4.3) (e.g. $\boldsymbol{\pi}^2{}_{31} = \boldsymbol{\pi}^1{}_{32}$).

However, in the present case, there is an obstruction to the existence of such 1-forms $\boldsymbol{\pi}^b{}_{ca}$, which lies in the quantity, $T(x,g,\lambda)$, defined by the relation

$$\begin{aligned}\boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^3 \wedge d\boldsymbol{\theta}^1{}_3 + \boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^4 \wedge d\boldsymbol{\theta}^1{}_4 + \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3 \wedge d\boldsymbol{\theta}^2{}_3 + \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^4 \wedge d\boldsymbol{\theta}^2{}_4 \\ \equiv -2T(x,g,\lambda)\,\boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^2 \wedge \boldsymbol{\omega}^3 \wedge \boldsymbol{\omega}^4 \mod \boldsymbol{\theta}.\end{aligned} \tag{4.7}$$

$T(x,g,\lambda)$ is then given in terms of the curvature by the expression

$$\begin{aligned}T(x,g,\lambda) := \ & R_{1234}(x,g) + \lambda^2{}_{31}\left(\lambda^2{}_{42} - \lambda^1{}_{41}\right) + \lambda^2{}_{41}\left(\lambda^1{}_{31} - \lambda^2{}_{32}\right) \\ & + \lambda^4{}_{13}\left(\lambda^4{}_{24} - \lambda^3{}_{23}\right) + \lambda^4{}_{23}\left(\lambda^3{}_{13} - \lambda^4{}_{14}\right).\end{aligned}$$

In particular, an explicit calculation (for details, see B.1) shows that it is possible to absorb most of the torsion terms in (4.7) and there exist 1-forms, $\boldsymbol{\pi}^b{}_{ca}$, on $M^{(1)}$ satisfying $\boldsymbol{\pi}^b{}_{ca} \equiv d\lambda^b{}_{ca} \mod \boldsymbol{\omega}^i$ in terms of which equations (4.6) take the form

$$\begin{aligned}
d\boldsymbol{\theta}^1{}_2 &\equiv -\sum_a \boldsymbol{\pi}^1{}_{2a} \wedge \boldsymbol{\omega}^a \mod \boldsymbol{\theta}, \\
d\boldsymbol{\theta}^1{}_3 &\equiv -\sum_a \boldsymbol{\pi}^1{}_{3a} \wedge \boldsymbol{\omega}^a \mod \boldsymbol{\theta}, \\
d\boldsymbol{\theta}^1{}_4 &\equiv -\sum_a \boldsymbol{\pi}^1{}_{4a} \wedge \boldsymbol{\omega}^a \mod \boldsymbol{\theta}, \\
d\boldsymbol{\theta}^2{}_3 &\equiv -\sum_a \boldsymbol{\pi}^2{}_{3a} \wedge \boldsymbol{\omega}^a \mod \boldsymbol{\theta}, \\
d\boldsymbol{\theta}^2{}_4 &\equiv -\sum_a \boldsymbol{\pi}^2{}_{4a} \wedge \boldsymbol{\omega}^a + 2T\boldsymbol{\omega}^1 \wedge \boldsymbol{\omega}^3 \mod \boldsymbol{\theta}, \\
d\boldsymbol{\theta}^3{}_4 &\equiv -\sum_a \boldsymbol{\pi}^3{}_{4a} \wedge \boldsymbol{\omega}^a \mod \boldsymbol{\theta}.
\end{aligned} \tag{4.8}$$

Equation (4.7) implies, however, that it is not possible to absorb the remaining torsion by a redefinition of the forms $\boldsymbol{\pi}^b{}_{ca}$. In particular, it implies that there is essential torsion in the system characterised by the function T. The existence of such essential torsion implies that a necessary condition for the existence of an integral element $E_4 \subset T_pM^{(1)}$ based at a point $p \in M^{(1)}$ is that p satisfies the compatibility condition $T(p) = 0$. We define the non-singular part of the subspace where this condition holds,

$$S^{(1)} := \left\{p \in M^{(1)} : T(p) = 0, dT(p) \neq 0\right\},$$

which (by the implicit function theorem) is a codimension-one submanifold, $i\colon S \to M^{(1)}$, of $M^{(1)}$.

Remark 4.3. Note that an explicit calculation of dT shows that, given $(x,g) \in \mathcal{F}$, for generic λ we have $dT(x,g,\lambda) \neq 0$.

We define the 1-forms $\widetilde{\boldsymbol{\theta}}^a{}_b := i^*\boldsymbol{\theta}^a{}_b$, $\widetilde{\boldsymbol{\omega}}^a := i^*\boldsymbol{\omega}^a$ on S, and consider the Pfaffian system $(\widetilde{\mathcal{I}}, \widetilde{\Omega})$ on S generated by $\{\widetilde{\boldsymbol{\theta}}^a{}_b\}$ with independence condition $\widetilde{\Omega} := i^*\Omega = \widetilde{\boldsymbol{\omega}}^1 \wedge \widetilde{\boldsymbol{\omega}}^2 \wedge \widetilde{\boldsymbol{\omega}}^3 \wedge \widetilde{\boldsymbol{\omega}}^4$. We then have the following:

Proposition 4.4. *There exist* 1*-forms,* $\widetilde{\boldsymbol{\pi}}^b{}_{ca} \in \Omega^1(S)$*, for* $a,b,c = 1,\dots,4$ *with* $b < c$*, that satisfy*

(i) $\widetilde{\boldsymbol{\pi}}^b{}_{ca} \equiv i^*\left(d\lambda^b{}_{ca}\right) \quad \bmod \widetilde{\boldsymbol{\omega}}^i$,
(ii) $\widetilde{\boldsymbol{\pi}}^2{}_{31} = \widetilde{\boldsymbol{\pi}}^1{}_{32}, \quad \widetilde{\boldsymbol{\pi}}^2{}_{41} = \widetilde{\boldsymbol{\pi}}^1{}_{42}, \quad \widetilde{\boldsymbol{\pi}}^1{}_{43} = \widetilde{\boldsymbol{\pi}}^1{}_{34}, \quad \widetilde{\boldsymbol{\pi}}^2{}_{43} = \widetilde{\boldsymbol{\pi}}^2{}_{34}$,

with the property that

$$d\widetilde{\boldsymbol{\theta}}^a{}_b \equiv -\sum_c \widetilde{\boldsymbol{\pi}}^a{}_{bc} \wedge \widetilde{\boldsymbol{\omega}}^c \quad \bmod \widetilde{\boldsymbol{\theta}}. \tag{4.9}$$

Proof. Taking the pull-back of equations (4.8) to S, and using the fact that $T \circ i = 0$, we deduce that the 1-forms $\widetilde{\boldsymbol{\pi}}^b{}_{ca} := i^*\left(\boldsymbol{\pi}^b{}_{ca}\right)$ on S have the required properties. □

Rather than using $\lambda^b{}_{ca}$ as coordinates it will be useful to introduce new coordinates $y^1, \ldots, y^8$ and $z^1, \ldots, z^4$ on $M^{(1)}$ defined by

$$\begin{aligned} &y^1 := \lambda^2{}_{31}, && y^2 := \tfrac{1}{2}\left(\lambda^2{}_{42} - \lambda^1{}_{41}\right), && y^3 := \lambda^2{}_{41}, && y^4 := \tfrac{1}{2}\left(\lambda^2{}_{32} - \lambda^1{}_{31}\right), \\ &y^5 := \lambda^4{}_{13}, && y^6 := \tfrac{1}{2}\left(\lambda^4{}_{24} - \lambda^3{}_{23}\right), && y^7 := \lambda^4{}_{23}, && y^8 := \tfrac{1}{2}\left(\lambda^4{}_{14} - \lambda^3{}_{13}\right) \end{aligned}$$

and

$$\begin{aligned} &z^1 := \tfrac{1}{2}\left(\lambda^2{}_{42} + \lambda^1{}_{41}\right), && z^2 := \tfrac{1}{2}\left(\lambda^2{}_{32} + \lambda^1{}_{31}\right), \\ &z^3 := \tfrac{1}{2}\left(\lambda^4{}_{24} + \lambda^3{}_{23}\right), && z^4 := \tfrac{1}{2}\left(\lambda^4{}_{14} + \lambda^3{}_{13}\right). \end{aligned}$$

In terms of these coordinates our constraint equation takes the form

$$T(x, g, y, z) = R_{1234}(x, g) + 2\left(y^1 y^2 - y^3 y^4 + y^5 y^6 - y^7 y^8\right) = 0, \tag{4.10}$$

so that the constraint does not depend upon the z coordinates.

We now write the structure equations (4.9) in the form

$$d\begin{pmatrix} \widetilde{\boldsymbol{\theta}}^1{}_2 \\ \widetilde{\boldsymbol{\theta}}^1{}_3 \\ \widetilde{\boldsymbol{\theta}}^1{}_4 \\ \widetilde{\boldsymbol{\theta}}^2{}_3 \\ \widetilde{\boldsymbol{\theta}}^2{}_4 \\ \widetilde{\boldsymbol{\theta}}^3{}_4 \end{pmatrix} \equiv \pi \wedge \begin{pmatrix} \widetilde{\boldsymbol{\omega}}^1 \\ \widetilde{\boldsymbol{\omega}}^2 \\ \widetilde{\boldsymbol{\omega}}^3 \\ \widetilde{\boldsymbol{\omega}}^4 \end{pmatrix} \quad \bmod \widetilde{\boldsymbol{\theta}}. \tag{4.11}$$

Here, the matrix of 1-forms π (which, modulo $\{\widetilde{\boldsymbol{\theta}}, \widetilde{\boldsymbol{\omega}}\}$, is the tableau matrix of $(\widetilde{\mathcal{I}}, \widetilde{\boldsymbol{\Omega}})$ at x) is given by

$$\pi = -\begin{pmatrix} \widetilde{\boldsymbol{\pi}}^1{}_{21} & \widetilde{\boldsymbol{\pi}}^1{}_{22} & \widetilde{\boldsymbol{\pi}}^1{}_{23} & \widetilde{\boldsymbol{\pi}}^1{}_{24} \\ \widetilde{\boldsymbol{\pi}}^1{}_{31} & \widetilde{\boldsymbol{\pi}}^1{}_{32} & \widetilde{\boldsymbol{\pi}}^1{}_{33} & \widetilde{\boldsymbol{\pi}}^1{}_{34} \\ \widetilde{\boldsymbol{\pi}}^1{}_{41} & \widetilde{\boldsymbol{\pi}}^1{}_{42} & \widetilde{\boldsymbol{\pi}}^1{}_{34} & \widetilde{\boldsymbol{\pi}}^1{}_{44} \\ \widetilde{\boldsymbol{\pi}}^1{}_{32} & \widetilde{\boldsymbol{\pi}}^2{}_{32} & \widetilde{\boldsymbol{\pi}}^2{}_{33} & \widetilde{\boldsymbol{\pi}}^2{}_{34} \\ \widetilde{\boldsymbol{\pi}}^1{}_{42} & \widetilde{\boldsymbol{\pi}}^2{}_{42} & \widetilde{\boldsymbol{\pi}}^2{}_{34} & \widetilde{\boldsymbol{\pi}}^2{}_{44} \\ \widetilde{\boldsymbol{\pi}}^3{}_{41} & \widetilde{\boldsymbol{\pi}}^3{}_{42} & \widetilde{\boldsymbol{\pi}}^3{}_{43} & \widetilde{\boldsymbol{\pi}}^3{}_{44} \end{pmatrix} \tag{4.12}$$

In order to simplify notation, we define the 1-forms $\widetilde{\boldsymbol{\pi}}^\alpha$, $\alpha = 1, \ldots, 8$, by

$$\begin{aligned} &\widetilde{\boldsymbol{\pi}}^1 := \widetilde{\boldsymbol{\pi}}^2{}_{31}, && \widetilde{\boldsymbol{\pi}}^2 := \tfrac{1}{2}\left(\widetilde{\boldsymbol{\pi}}^2{}_{42} - \widetilde{\boldsymbol{\pi}}^1{}_{41}\right), \\ &\widetilde{\boldsymbol{\pi}}^3 := \widetilde{\boldsymbol{\pi}}^2{}_{41}, && \widetilde{\boldsymbol{\pi}}^4 := \tfrac{1}{2}\left(\widetilde{\boldsymbol{\pi}}^2{}_{32} - \widetilde{\boldsymbol{\pi}}^1{}_{31}\right), \\ &\widetilde{\boldsymbol{\pi}}^5 := \widetilde{\boldsymbol{\pi}}^4{}_{13}, && \widetilde{\boldsymbol{\pi}}^6 := \tfrac{1}{2}\left(\widetilde{\boldsymbol{\pi}}^4{}_{24} - \widetilde{\boldsymbol{\pi}}^3{}_{23}\right), \\ &\widetilde{\boldsymbol{\pi}}^7 := \widetilde{\boldsymbol{\pi}}^4{}_{23}, && \widetilde{\boldsymbol{\pi}}^8 := \tfrac{1}{2}\left(\widetilde{\boldsymbol{\pi}}^4{}_{14} - \widetilde{\boldsymbol{\pi}}^3{}_{13}\right), \end{aligned}$$

which have the property that $\widetilde{\boldsymbol{\pi}}^\alpha \equiv i^* dy^\alpha \mod \widetilde{\boldsymbol{\omega}}^a$ for $\alpha = 1, \ldots, 8$. We also define the 1-forms $\widetilde{\boldsymbol{\rho}^a}$, $a = 1, \ldots, 4$ by

$$\widetilde{\boldsymbol{\rho}}^1 := \tfrac{1}{2}\left(\widetilde{\boldsymbol{\pi}}^2{}_{42} + \widetilde{\boldsymbol{\pi}}^1{}_{41}\right), \qquad \widetilde{\boldsymbol{\rho}}^2 := \tfrac{1}{2}\left(\widetilde{\boldsymbol{\pi}}^2{}_{32} + \widetilde{\boldsymbol{\pi}}^1{}_{31}\right),$$
$$\widetilde{\boldsymbol{\rho}}^3 := \tfrac{1}{2}\left(\widetilde{\boldsymbol{\pi}}^4{}_{24} + \widetilde{\boldsymbol{\pi}}^3{}_{23}\right), \qquad \widetilde{\boldsymbol{\rho}}^4 := \tfrac{1}{2}\left(\widetilde{\boldsymbol{\pi}}^4{}_{14} + \widetilde{\boldsymbol{\pi}}^3{}_{13}\right),$$

which have the property that that $\widetilde{\boldsymbol{\rho}}^a \equiv i^* dz^a \pmod{\widetilde{\boldsymbol{\omega}}^1, \ldots, \widetilde{\boldsymbol{\omega}}^4}$ for $a = 1, \ldots, 4$. In addition, we define 1-forms $\{\widetilde{\boldsymbol{\mu}}^a\}_{a=1}^4$ and $\{\widetilde{\boldsymbol{\nu}}^a\}_{a=1}^4$ by

$$\left(\widetilde{\boldsymbol{\mu}}^1, \ldots, \widetilde{\boldsymbol{\mu}}^4\right) := \left(\widetilde{\boldsymbol{\pi}}^1{}_{21}, \widetilde{\boldsymbol{\pi}}^1{}_{22}, \widetilde{\boldsymbol{\pi}}^1{}_{23}, \widetilde{\boldsymbol{\pi}}^1{}_{24}\right),$$
$$\left(\widetilde{\boldsymbol{\nu}}^1, \ldots, \widetilde{\boldsymbol{\nu}}^4\right) := \left(\widetilde{\boldsymbol{\pi}}^3{}_{41}, \widetilde{\boldsymbol{\pi}}^3{}_{42}, \widetilde{\boldsymbol{\pi}}^3{}_{43}, \widetilde{\boldsymbol{\pi}}^3{}_{44}\right).$$

In this notation we have

$$\pi = -\begin{pmatrix} \widetilde{\boldsymbol{\mu}}^1 & \widetilde{\boldsymbol{\mu}}^2 & \widetilde{\boldsymbol{\mu}}^3 & \widetilde{\boldsymbol{\mu}}^4 \\ \widetilde{\boldsymbol{\rho}}^2 - \widetilde{\boldsymbol{\pi}}^4 & \widetilde{\boldsymbol{\pi}}^1 & -\widetilde{\boldsymbol{\rho}}^4 + \widetilde{\boldsymbol{\pi}}^8 & \widetilde{\boldsymbol{\pi}}^5 \\ \widetilde{\boldsymbol{\rho}}^1 - \widetilde{\boldsymbol{\pi}}^2 & \widetilde{\boldsymbol{\pi}}^3 & \widetilde{\boldsymbol{\pi}}^5 & -\widetilde{\boldsymbol{\rho}}^4 - \widetilde{\boldsymbol{\pi}}^8 \\ \widetilde{\boldsymbol{\pi}}^1 & \widetilde{\boldsymbol{\rho}}^2 + \widetilde{\boldsymbol{\pi}}^4 & -\widetilde{\boldsymbol{\rho}}^3 + \widetilde{\boldsymbol{\pi}}^6 & \widetilde{\boldsymbol{\pi}}^7 \\ \widetilde{\boldsymbol{\pi}}^3 & \widetilde{\boldsymbol{\rho}}^1 + \widetilde{\boldsymbol{\pi}}^2 & \widetilde{\boldsymbol{\pi}}^7 & -\widetilde{\boldsymbol{\rho}}^3 - \widetilde{\boldsymbol{\pi}}^6 \\ \widetilde{\boldsymbol{\nu}}^1 & \widetilde{\boldsymbol{\nu}}^2 & \widetilde{\boldsymbol{\nu}}^3 & \widetilde{\boldsymbol{\nu}}^4 \end{pmatrix}. \tag{4.13}$$

We now note that, since the functions (x, g, y) obey equation (4.10) on $S^{(1)}$, they will not be functionally independent when pulled back to S. In particular, we require that $i^*(dT) = 0$, which translates into the condition that

$$2\left(\widetilde{y}^1 d\widetilde{y}^2 + \widetilde{y}^2 d\widetilde{y}^1 - \widetilde{y}^3 d\widetilde{y}^4 - \widetilde{y}^4 d\widetilde{y}^3 + \widetilde{y}^5 d\widetilde{y}^6 + \widetilde{y}^6 d\widetilde{y}^5 - \widetilde{y}^7 d\widetilde{y}^8 - \widetilde{y}^8 d\widetilde{y}^7\right) + \sum_a \Phi_a \widetilde{\boldsymbol{\omega}}^a \equiv 0 \mod \widetilde{\boldsymbol{\theta}} \tag{4.14}$$

on S, where

$$\Phi_a = i^*\left(\frac{\partial}{\partial \boldsymbol{\omega}^a} R_{1234}\right) + \sum_b \left[\widetilde{\lambda}^b{}_{1a} \widetilde{R}_{b234} + \widetilde{\lambda}^b{}_{2a} \widetilde{R}_{1b34} + \widetilde{\lambda}^b{}_{3a} \widetilde{R}_{12b4} + \widetilde{\lambda}^b{}_{4a} \widetilde{R}_{123b}\right],$$

and $\widetilde{y}^\alpha := i^* y^\alpha = y^\alpha \circ i$, etc, denote the pull-backs to S of the corresponding functions on $M^{(1)}$. In particular, since $\widetilde{\boldsymbol{\pi}}^\alpha \equiv i^* dy^\alpha \mod \widetilde{\boldsymbol{\omega}}^a$ for $\alpha = 1, \ldots, 8$, there exist functions Ψ_a on S such that

$$i^*(dT) = \widetilde{y}^1 \widetilde{\boldsymbol{\pi}}^2 + \widetilde{y}^2 \widetilde{\boldsymbol{\pi}}^1 - \widetilde{y}^3 \widetilde{\boldsymbol{\pi}}^4 - \widetilde{y}^4 \widetilde{\boldsymbol{\pi}}^3 + \widetilde{y}^5 \widetilde{\boldsymbol{\pi}}^6 + \widetilde{y}^6 \widetilde{\boldsymbol{\pi}}^5 - \widetilde{y}^7 \widetilde{\boldsymbol{\pi}}^8 - \widetilde{y}^8 \widetilde{\boldsymbol{\pi}}^7 + \sum_a \Psi_a \widetilde{\boldsymbol{\omega}}^a \equiv 0 \mod \widetilde{\boldsymbol{\theta}}. \tag{4.15}$$

Recall, however, that the 1-forms, $\widetilde{\boldsymbol{\pi}}^b{}_{ca}$, are not uniquely determined, and that we may add to them any linear combination of the 1-forms $\{\widetilde{\boldsymbol{\omega}}^a\}$ consistent with equations (4.11) and (4.12). At a generic point $p \in S$ at which $\widetilde{y}^1(p), \ldots, \widetilde{y}^8(p)$ are all non-zero, it is shown in Appendix B.2 that all the functions Ψ_a in equation (4.15) may be absorbed into a redefinition of the 1-forms $\widetilde{\boldsymbol{\pi}}^1, \ldots, \widetilde{\boldsymbol{\pi}}^8$ and $\widetilde{\boldsymbol{\rho}}^1, \ldots, \widetilde{\boldsymbol{\rho}}^4$. Noting that the non-vanishing of $\widetilde{y}^1, \ldots, \widetilde{y}^8$ is an open condition, we deduce that we may take the 1-forms $\widetilde{\boldsymbol{\pi}}^1, \ldots, \widetilde{\boldsymbol{\pi}}^8$ to obey the linear-dependence condition

$$\widetilde{y}^1 \widetilde{\boldsymbol{\pi}}^2 + \widetilde{y}^2 \widetilde{\boldsymbol{\pi}}^1 - \widetilde{y}^3 \widetilde{\boldsymbol{\pi}}^4 - \widetilde{y}^4 \widetilde{\boldsymbol{\pi}}^3 + \widetilde{y}^5 \widetilde{\boldsymbol{\pi}}^6 + \widetilde{y}^6 \widetilde{\boldsymbol{\pi}}^5 - \widetilde{y}^7 \widetilde{\boldsymbol{\pi}}^8 - \widetilde{y}^8 \widetilde{\boldsymbol{\pi}}^7 \equiv 0 \mod \widetilde{\boldsymbol{\theta}} \tag{4.16}$$

on an open neighbourhood, U, of the point p in S. This relationship implies (via the structure equations (4.11)) that the essential torsion of the system $(\widetilde{\mathcal{I}}, \widetilde{\boldsymbol{\Omega}})$ is zero on the open set U. It follows from Proposition A.11 that the system $(\widetilde{\mathcal{I}}, \widetilde{\boldsymbol{\Omega}})$ is involutive at p if and only if the tableau A_p is involutive.

To show that this is the case, we need to know the reduced Cartan characters of the tableau A_p, and the dimension of the first prolongation, $A_p^{(1)}$, of A_p.

Proposition 4.5. *The first prolongation of the tableau A_p is an affine-linear space of dimension* 41.

Proof. See Appendix B.2. □

Proposition 4.6. *The system $(\widetilde{\mathcal{I}}, \widetilde{\Omega})$ has reduced Cartan characters*

$$s_1' = 6, \quad s_2' = 6, \quad s_3' = 5, \quad s_4' = 2.$$

Proof. Let $p \in S$ with $\widetilde{y}^1(p), \ldots, \widetilde{y}^8(p)$ all non-zero. Equation (4.15) may then be looked on as defining one of the 1-forms, say $\widetilde{\boldsymbol{\pi}}^8$, in terms of the other seven. Note that, since the thirty 1-forms $\{\widetilde{\boldsymbol{\omega}}^i, \widetilde{\boldsymbol{\theta}}^a{}_b, \widetilde{\boldsymbol{\rho}}^a, \widetilde{\boldsymbol{\mu}}^a, \widetilde{\boldsymbol{\nu}}^a\}$ must span the cotangent space at each point of the twenty-nine-dimensional manifold S, it follows that the linear relation (4.15) is the only relation obeyed by these 1-forms on S. As such, once we have substituted for $\widetilde{\boldsymbol{\pi}}^8$, say, the remaining differential forms $\{\widetilde{\boldsymbol{\pi}}^1, \ldots, \widetilde{\boldsymbol{\pi}}^7, \widetilde{\boldsymbol{\rho}}^a, \widetilde{\boldsymbol{\mu}}^a, \widetilde{\boldsymbol{\nu}}^a\}$ that appear in the matrix π are linearly-independent on S.

We then consider the tableau matrix, $\overline{\pi} := \pi \mod \widetilde{\boldsymbol{\theta}}, \widetilde{\boldsymbol{\omega}}$, and we wish to calculate the reduced Cartan characters. This should be computed with respect to a generic basis of 1-forms $\{\boldsymbol{\omega}^i\}$, so we note that the tableau relative to a different basis, $\underline{\widetilde{\boldsymbol{\omega}}}^a := \sum_b \left(\sigma^{-1}\right)^a{}_b \widetilde{\boldsymbol{\omega}}^b$ where $\sigma \in \mathrm{GL}(4, \mathbb{R})$, is given by $\overline{\pi}_\sigma := \overline{\pi}\sigma$. Substituting for $\widetilde{\boldsymbol{\pi}}^8$ into the tableau matrix and noting that this is the only relationship that our differential forms obey, we see that π then has six linearly-independent 1-forms in its first column:

$$\widetilde{\boldsymbol{\mu}}^1, \quad \widetilde{\boldsymbol{\rho}}^2 - \widetilde{\boldsymbol{\pi}}^4, \quad \widetilde{\boldsymbol{\rho}}^1 - \widetilde{\boldsymbol{\pi}}^2, \quad \widetilde{\boldsymbol{\pi}}^1, \quad \widetilde{\boldsymbol{\pi}}^3, \quad \widetilde{\boldsymbol{\nu}}^1.$$

Therefore $s_1' = 6$. The 1-forms in column three:

$$\widetilde{\boldsymbol{\mu}}^3, \quad -\widetilde{\boldsymbol{\rho}}^4 + \widetilde{\boldsymbol{\pi}}^8, \quad \widetilde{\boldsymbol{\pi}}^5, \quad -\widetilde{\boldsymbol{\rho}}^3 + \quad \widetilde{\boldsymbol{\pi}}^6, \quad \widetilde{\boldsymbol{\pi}}^7, \quad \widetilde{\boldsymbol{\nu}}^3$$

are then linearly-independent, and independent of those in column one. (In the preceding equation, we substitute for $\widetilde{\boldsymbol{\pi}}^8$ using equation (4.16).) Therefore $s_2' = 6$. If we then consider the linear combination of α times column two and β times column four of (4.12), we gain the 1-forms

$$\alpha\widetilde{\boldsymbol{\mu}}^2 + \beta\widetilde{\boldsymbol{\mu}}^4, \quad \alpha\widetilde{\boldsymbol{\pi}}^3 - \beta(\widetilde{\boldsymbol{\rho}}^4 + \widetilde{\boldsymbol{\pi}}^8), \quad \alpha(\widetilde{\boldsymbol{\rho}}^2 + \widetilde{\boldsymbol{\pi}}^4) + \beta\widetilde{\boldsymbol{\pi}}^7,$$
$$\alpha(\widetilde{\boldsymbol{\rho}}^1 + \widetilde{\boldsymbol{\pi}}^2) - \beta(\widetilde{\boldsymbol{\rho}}^3 + \widetilde{\boldsymbol{\pi}}^6), \quad \alpha\widetilde{\boldsymbol{\nu}}^2 + \beta\widetilde{\boldsymbol{\nu}}^4.$$

If we then take α, β both non-zero, this gives five more linearly-independent 1-forms. Therefore $s_3' = 5$. Finally, $s_1' + s_2' + s_3' + s_4' = 19$, the number of linearly-independent 1-forms in π, which fixes $s_4' = 2$.

Note that the above is equivalent to taking

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 0 & \alpha & * \\ 0 & 1 & 0 & * \\ 0 & 0 & \beta & * \end{pmatrix},$$

where the last column is only constrained by the requirement that σ be non-singular. □

Proposition 4.7. *The Pfaffian differential system $(\widetilde{\mathcal{I}}, \widetilde{\mathbf{\Omega}})$ is involutive at p.*

Proof.
$$s'_1 + 2s'_2 + 3s'_3 + 4s'_4 = 6 + 12 + 15 + 8 = 41 = \dim A_p^{(1)}.$$
□

Theorem 4.8. *Let X be an analytic manifold, and $\mathbf{g}$ an analytic Riemannian metric on X. For each $x \in X$, there exists a neighbourhood of x on which there exists an analytic coordinate system in terms of which the metric $\mathbf{g}$ takes block-diagonal form.*

Proof. Given any point $x \in M$, choose a generic point $p \in \pi^{-1}(x) \in S$. By the previous Proposition, the system $(\widetilde{\mathcal{I}}, \widetilde{\mathbf{\Omega}})$ is involutive. Applying the Cartan–Kähler theorem (cf. Remark A.12), we deduce that there exists an integral manifold of the exterior differential system with independence condition $(\widetilde{\mathcal{I}}, \widetilde{\Omega})$ through p. This integral manifold corresponds to a section $f : X \to S$ and hence to an orthonormal coframe $\{\boldsymbol{\epsilon}^i\}$ on a neighbourhood of x that obeys equation (3.4). □

Remark 4.9. The solution to (3.4) is not unique but one has the freedom to independently make rotations in the $(\boldsymbol{\epsilon}^1, \boldsymbol{\epsilon}^2)$ and $(\boldsymbol{\epsilon}^3, \boldsymbol{\epsilon}^4)$ planes (equivalently, in the (t, x) and (y, z) planes of the proof of Proposition 3.1). This corresponds to the freedom to make rotations in the $(\boldsymbol{\omega}^1, \boldsymbol{\omega}^2)$ and $(\boldsymbol{\omega}^3, \boldsymbol{\omega}^4)$ planes without changing $(\mathcal{I}, \mathbf{\Omega})$. As a result the characteristic manifold is parameterised by two functions of four variables, consistent with the result that $s'_4 = 2$.

Remark 4.10. The coordinate functions $\lambda^b{}_{ca}$ pull back to define functions on X that give the components, $\{\mathbf{\Gamma}^a{}_b\}$, of the Levi-Civita connection of the coframe $\{\boldsymbol{\epsilon}^a\}$. The curvature of $\mathbf{\Gamma}$, $\mathbf{R}^{\mathbf{\Gamma}}$, then automatically obeys the condition that

$$\begin{aligned} R^{\mathbf{\Gamma}}_{1234} + \Gamma^2{}_{31}\left(\Gamma^2{}_{42} - \Gamma^1{}_{41}\right) + \Gamma^2{}_{41}\left(\Gamma^1{}_{31} - \Gamma^2{}_{32}\right) \\ + \Gamma^4{}_{13}\left(\Gamma^4{}_{24} - \Gamma^3{}_{23}\right) + \Gamma^4{}_{23}\left(\Gamma^3{}_{13} - \Gamma^4{}_{14}\right) = 0. \end{aligned} \tag{4.17}$$

In the present context, this condition is derived from pulling back the condition $T(p) = 0$ that was required for our Pfaffian system on S to have integral elements. However, it can also be shown that this condition arises directly from the symmetry requirements on the Levi-Civita connection (analogous to (4.3)) that follow from imposing (3.4).

It turns out that (4.17) has a simple geometrical interpretation. Let $R^{\perp}_{1234}$ denote the curvature of the connection of the bundle normal to the $\boldsymbol{\epsilon}^1 \wedge \boldsymbol{\epsilon}^2$ plane. This is related to the full curvature and the associated fundamental form $A_{\mathbf{U}}$ by the Ricci equation

$$\mathbf{g}\left(\mathbf{R}^{\perp}(\mathbf{X}, \mathbf{Y})\mathbf{V}, \mathbf{U}\right) = \mathbf{g}\left(\mathbf{R}(\mathbf{X}, \mathbf{Y})\mathbf{V}, \mathbf{U}\right) - \mathbf{g}\left(\left[A_{\mathbf{U}}, A_{\mathbf{V}}\right]\mathbf{X}, \mathbf{Y}\right).$$

In the same way one can use the Ricci equation to obtain an expression for the curvature $\tilde{R}^{\perp}_{3412}$ of the connection of the bundle normal to the $\boldsymbol{\epsilon}^3 \wedge \boldsymbol{\epsilon}^4$ plane. Then by adding the expressions for the two normal curvatures together one may write the curvature condition (4.17) in the alternative form

$$R^{\perp}_{1234} + \tilde{R}^{\perp}_{3412} = R_{1234}. \tag{4.18}$$

So that the full curvature is just the sum of the two normal curvatures.

4.2. The Lorentzian case

Although we have carried out all of our calculations for the case of a Riemannian four-manifold, the calculations carry through, essentially unchanged, if the metric has Lorentzian signature. We can easily obtain the geometric condition corresponding to (4.18) by using the Newman–Penrose null formalism (see e.g. [13]). We start by introducing a (complex) basis of null 1-forms $(\boldsymbol{\ell}, \mathbf{n}, \mathbf{m}, \overline{\mathbf{m}})$. Then in terms of this basis the condition (3.4) that the metric can be block diagonalised is given by

$$\begin{aligned}
\boldsymbol{\ell} \wedge \mathbf{n} \wedge d\boldsymbol{\ell} \quad &= 0, \\
\boldsymbol{\ell} \wedge \mathbf{n} \wedge d\mathbf{n} \quad &= 0, \\
\mathbf{m} \wedge \overline{\mathbf{m}} \wedge d\mathbf{m} &= 0, \\
\mathbf{m} \wedge \overline{\mathbf{m}} \wedge d\overline{\mathbf{m}} &= 0.
\end{aligned}$$

From equation (4.13.44) in [13], the above conditions result in reality constraints on the spin coefficients given by

$$\rho = \overline{\rho}, \qquad \rho' = \overline{\rho'}, \qquad \overline{\tau'} = \tau, \qquad \tau' = \overline{\tau}. \tag{4.19}$$

We now make use of the Newman–Penrose equations (4.11.12) in [13] to obtain the equation

$$\begin{aligned}
D'\rho - \delta'\tau + D\rho' - \delta\tau' = 2\rho\rho' - \left(\tau\overline{\tau} + \tau'\overline{\tau'}\right) + \rho(\gamma + \overline{\gamma}) + \rho'(\gamma' + \overline{\gamma'}) \\
- \left(\tau(\alpha + \overline{\alpha'}) + \overline{\tau}(\alpha' + \overline{\alpha})\right) - 4\Lambda - 2\left(\Psi_2 + \kappa\kappa' - \sigma\sigma'\right).
\end{aligned}$$

Because of the reality conditions on the spin coefficients (4.19), we see that the imaginary part of the left hand side of this equation must vanish. Similarly all the terms but the final one on the right hand side are real and have vanishing imaginary part. It must therefore be the case that the final term also has vanishing imaginary part so that

$$\operatorname{Im}\left(\Psi_2 + \kappa\kappa' - \sigma\sigma'\right) = 0. \tag{4.20}$$

Therefore, our block-diagonalisation condition necessarily implies that this constraint must be satisfied. Note that both (4.19) and (4.20) are invariant under spin and boost transformations which reflects the fact that the 2-forms $\boldsymbol{\ell} \wedge \mathbf{n}$ and $\mathbf{m} \wedge \overline{\mathbf{m}}$ are invariant under such transformations.

To relate this condition to equation (4.18) above we introduce the complex curvature curvature of the surface spanned by $\mathbf{m} \wedge \overline{\mathbf{m}}$ which is given by the formula

$$K = \sigma\sigma' - \Psi_2 - \rho\rho' + \Phi_{11} + \Lambda.$$

Twice the real part of this gives the Gaussian curvature while twice the imaginary part gives the curvature of the connection of the normal bundle, which in view of the reality conditions on the spin coefficients is given by

$$\mathrm{Im}\, K = \mathrm{Im}\, (\sigma\sigma' - \Psi_2)$$

The corresponding curvature of the connection of the normal bundle to $\boldsymbol{\ell} \wedge \mathbf{n}$ is obtained by applying the Sachs $*$-operation (which has the effect of swapping $\mathbf{m} \wedge \overline{\mathbf{m}}$ with $\boldsymbol{\ell} \wedge \mathbf{n}$). Under this operation we have

$$\sigma^* = -\kappa, \qquad \sigma'^* = \kappa', \qquad \Psi_2^* = \Psi_2,$$

so that the normal curvature is this time given by

$$\mathrm{Im}\, K^* = \mathrm{Im}\, (-\kappa\kappa' - \Psi_2)$$

Finally we note that the full curvature for the orthonormal frame corresponding to the Newman–Penrose null tetrad is given by $R_{TXYZ} = -2\, \mathrm{Im}\, \Psi_2$. Hence condition (4.18) becomes

$$\mathrm{Im}\, K + \mathrm{Im}\, K^* = \mathrm{Im}\, \Psi_2.$$

Substituting for $\mathrm{Im}\, K$ and $\mathrm{Im}\, K^*$ we again obtain equation (4.20). So that the constraint obtained from the Newman–Penrose equations agrees with that obtained from the prolongation process.

Finally, with reference to Remark 3.2, it should be noted that the constraints (4.17) and (4.20) that have arisen via the prolongation procedure are both preserved under conformal transformations of the metric, $\mathbf{g}$. This is, again, a manifestation of the fact that our problem is actually a problem in conformal, rather than Riemannian/Lorentzian, geometry.

5. Doubly biorthogonal coordinates

The problem of diagonalising a metric in 3-dimensions is equivalent to that of finding three families of 2-surfaces

$$f^i(x^1, x^2, x^3) = c^i, \qquad i = 1, 2, 3$$

that are mutually orthogonal. Given such 'triply orthogonal' surfaces the change of coordinates

$$x^{i'} = f^i(x^1, x^2, x^3)$$

brings the metric to diagonal form. Darboux [8] (see also Eisenhart [14])) was able to find all triply orthogonal systems for the flat metric by first giving a condition on two families of 2-surfaces that guaranteed the existence of a third family orthogonal to both.

Let

$$f(x, y, z) = a = \text{constant},$$
$$g(x, y, z) = b = \text{constant}$$

be two 1-parameter families of 2-surfaces S^1_a and S^2_b. The normal 1-form to S^1_a is df and the normal 1-form to S^2_b is dg. We require these to be orthogonal so that

$$\mathbf{g}(df, dg) = 0. \tag{5.1}$$

We now construct a 1-form $\boldsymbol{\omega}$ orthogonal to both S^1_a and S^2_b

$$\boldsymbol{\omega} = \star\,(df \wedge dg)\,. \tag{5.2}$$

In order for there to be a 2-surface mutually orthogonal to both S^1_a and S^2_b we require $\boldsymbol{\omega}$ to be surface forming and hence

$$d\boldsymbol{\omega} \wedge \boldsymbol{\omega} = 0. \tag{5.3}$$

Substituting for (5.2) into (5.3) gives the condition

$$d\,(\star\,(df \wedge dg)) \wedge (df \wedge dg) = 0. \tag{5.4}$$

In components (5.4) takes the form

$$\epsilon^{cab}\epsilon_{cde}\{(\nabla_b\nabla^d f)(\nabla^e g) + (\nabla^d f)(\nabla_b\nabla^e g)\}\epsilon_{akl}(\nabla^k f)(\nabla^l g) = 0,$$

which can be simplified to read

$$\epsilon^{abc}\nabla_b f\nabla_c g\left[(\nabla^d g)(\nabla_d\nabla_a f) - (\nabla^d f)(\nabla_d\nabla_a g)\right] = 0. \tag{5.5}$$

On the other hand differentiating (5.1) gives

$$(\nabla_b\nabla^a f)(\nabla_b g) + (\nabla_a f)(\nabla_b\nabla^a g) = 0. \tag{5.6}$$

We can can now use (5.6) to replace the second derivatives of g in (5.5) by second derivatives of f to obtain:

$$\epsilon^{abc}(\nabla_b f)(\nabla_c g)(\nabla^d g)(\nabla_d\nabla_a f) = 0.$$

Now since $\nabla^d g$ is normal to S^2_b, it is tangent to S^1_a. Hence if we are given some function f that defines a family of surfaces S^1_a, any surface S^2_b that intersects it orthogonally with the mutually orthogonal direction surface forming, must intersect S^1_a in a line with tangent direction X^a that satisfies

$$\epsilon^{abc}(\nabla_b f)X_c X^d(\nabla_d\nabla_a f) = 0. \tag{5.7}$$

This is just the classical result that the surfaces intersect in lines of curvature [8, 14].

The significant point about this is that given f we can solve (5.7) to give X^a algebraically in terms of first and second derivatives of f. Since X^a is tangent to both S^1_a and S^2_b it is normal to the third surface and must satisfy the surface orthogonal condition

$$\epsilon^{abc}(\nabla_a X_b)X_c = 0.$$

Substituting for X^a we obtain a third-order partial differential equation for f; the Darboux equation [8], see also Eisenhart [14] for details.

We see from the above that the coordinate surface of a triply orthogonal system must satisfy Darboux's equation. Conversely, given a solution $f(x, y, z)$ of the Darboux equation one can calculate the lines of curvature of the surfaces S^1_a given by $f(x, y, z) =$

a, and then find an orthogonal family of surfaces S^2_b which intersects S^1_a orthogonally along these lines. One then knows that the direction orthogonal to both normals is surface orthogonal and hence one has a triply orthogonal system of surfaces. (Note in practice it is often simpler to perform the last two steps in the opposite order.) Hence all triply orthogonal surface are determined by solutions to the third-order Darboux partial differential equation.

In the case of 'doubly biorthogonal' coordinate systems we proceed in a similar manner. We first ask when there exists a family of two surfaces orthogonal to a given two-parameter family of 2-surfaces.

Let the given two-parameter family of two surfaces $S_{a,b}$ be given by

$$f(x, y, z, w) = a, \qquad g(x, y, z, w) = b.$$

Since df and dg are both co-normals to S we require $\boldsymbol{\omega} = \star(df \wedge dg)$ to be surface-orthogonal. By the Frobenius theorem this is the condition

$$(\star d\boldsymbol{\omega}) \wedge \star\boldsymbol{\omega} = 0,$$

which, in components, takes the form

$$\epsilon^{ijkl}(\nabla_j f)(\nabla_k g)\left\{(\nabla_m f)(\nabla^m \nabla_l g) - (\nabla_m g)(\nabla^m \nabla_l f)\right\} = 0. \tag{5.8}$$

If one contracts (5.8) with $\nabla_i f$ or $\nabla_i g$ then the expression vanishes whatever the value of the final term. On the other hand if one contracts it with an element μ_i that is not in the linear span of $\nabla_i f$ and $\nabla_i g$ then $Y^i = \epsilon^{ijkl}\mu_i \nabla_j f \nabla_k g$ is a non-zero vector orthogonal to $\nabla_i f$ and $\nabla_i g$. Furthermore any vector Y^i orthogonal to $\nabla_i f$ and $\nabla_i g$ can be obtained in this way by choosing μ_i suitably. Hence we require

$$Y^i \left\{(\nabla_j f)(\nabla^j \nabla_i g) - (\nabla_j g)(\nabla^j \nabla_i f)\right\} = 0 \text{ for all } Y^i \text{ such that } Y^i \nabla_i f = Y^i \nabla_i g = 0. \tag{5.9}$$

This gives a pair of coupled second-order equations for f and g. Note that, unlike the case of triply orthogonal systems, $g^{ij}\nabla_i f \nabla_j g \neq 0$ in general since we cannot be expected to diagonalise one of the 2×2 blocks as well as obtain block diagonal form (this would involve setting five terms in the metric to zero). Hence there is no possibility of eliminating the second derivative of g in favour of derivatives of f as was done in three dimensions. Indeed (5.9) implies (5.8) and hence that $\boldsymbol{\omega} = \star(df \wedge dg)$ is surface orthogonal. Thus (5.9) is a necessary and sufficient condition for the existence of a doubly biorthogonal coordinate system.

Proposition 5.1. *All doubly biorthogonal systems are determined by solutions to the pair of coupled second-order partial differential equations*

$$Y^i \left\{(\nabla_j f)(\nabla^j \nabla_i g) - (\nabla_j g)(\nabla^j \nabla_i f)\right\} = 0 \textit{ for all } Y^i \textit{ such that } Y^i \nabla_i f = Y^i \nabla_i g = 0.$$

Acknowledgments

This work was partially supported by START-project Y237–N13 of the Austrian Science Fund.

Appendix A. Results from the theory of exterior differential systems

We now recall some standard definitions and results from the theory of exterior differential systems. For more information, see [10], the terminology and notation of which we will generally follow.

Throughout this section, let M be an arbitrary smooth manifold of dimension n. Let $\Omega^p(M)$ denote the space of C^∞ sections of $\bigwedge^p T^*M$ and $\Omega^*(M) := \bigoplus_{p=0}^n \Omega^p(M)$.

An *exterior differential system*, $\mathcal{I}$, on M consists of a two-sided, homogeneous differential ideal, $\mathcal{I} \subset \Omega^*(M)$. In particular, we have

- Given $\boldsymbol{\alpha} \in \mathcal{I}$, then $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \in \mathcal{I}$ and $\boldsymbol{\beta} \wedge \boldsymbol{\alpha} \in \mathcal{I}$ for all $\boldsymbol{\beta} \in \Omega^*(M)$.
- $\mathcal{I} = \bigoplus \mathcal{I}^q$ where $\mathcal{I}^q := \mathcal{I} \cap \Omega^q(M)$ and, for any $\boldsymbol{\alpha} \in \mathcal{I}$, the part of $\boldsymbol{\alpha} \in \mathcal{I}$ lying in $\mathcal{I}^q$ also lies in $\mathcal{I}$, for $q = 0, \ldots, n$.
- For all $\boldsymbol{\alpha} \in \mathcal{I}$ we have $d\boldsymbol{\alpha} \in \mathcal{I}$.

Given a point $x \in M$, a k-dimensional linear subspace $E_k \subseteq T_xM$ (where $k \in \{1, \ldots, n\}$) is an *integral element of* $(\mathcal{I}, \boldsymbol{\Omega})$ *(of dimension* k*) based at* x if $\boldsymbol{\varphi}|E_k = 0$ for all $\boldsymbol{\varphi} \in \mathcal{I}$, where $\boldsymbol{\alpha}|E_k$ denotes the restriction of a form $\boldsymbol{\alpha}$ to E_k. The set of integral elements of $\mathcal{I}$ of dimension k is denoted $V_k(\mathcal{I})$.

An *exterior differential system with independence condition*, $(\mathcal{I}, \boldsymbol{\Omega})$, on M consists of an exterior differential system $\mathcal{I} \subset \Omega^*(M)$, and a non-vanishing differential form $\boldsymbol{\Omega} \in \Omega^p(M)$. Given a point $x \in M$, an p-dimensional linear subspace $E_p \subseteq T_xM$ is an *integral element of* $(\mathcal{I}, \boldsymbol{\Omega})$ *based at* x if $\boldsymbol{\varphi}|E_p = 0$ for all $\boldsymbol{\varphi} \in \mathcal{I}$ and $\boldsymbol{\Omega}|E_p \neq 0$. The set of integral elements of $(\mathcal{I}, \boldsymbol{\Omega})$ is denoted $V_p(\mathcal{I}, \boldsymbol{\Omega})$.

Definition A.1. An *integral manifold* of $(\mathcal{I}, \boldsymbol{\Omega})$ is an immersed sub-manifold $i \colon N \to M$ with the property that $i^*\boldsymbol{\varphi} = 0$, for all $\boldsymbol{\varphi} \in \mathcal{I}$, and $i^*\boldsymbol{\Omega} \neq 0$. Equivalently, $i_*(T_xN) \subset T_{i(x)}M$ should be an integral element of $(\mathcal{I}, \boldsymbol{\Omega})$, for each $x \in N$.

Definition A.2. An *integral flag of* $(\mathcal{I}, \boldsymbol{\Omega})$ *based at* x is a nested sequence of subspaces $(0)_x \subset E_1 \subset E_2 \subset \ldots \subset E_p \subseteq T_xM$, with the properties that

- E_k is of dimension k, for $k = 0, \ldots, p-1$;
- E_p is an integral element of $(\mathcal{I}, \boldsymbol{\Omega})$.

Definition A.3. Let $\mathbf{e}_1, \ldots, \mathbf{e}_k$ be a basis for $E_k \subseteq T_xM$. The *polar space* of E is the vector space

$$H(E) = \left\{\mathbf{v} \in T_xM : \boldsymbol{\varphi}(\mathbf{v}, \mathbf{e}_1, \ldots, \mathbf{e}_k) = 0 \text{ for all } \boldsymbol{\varphi} \in \mathcal{I}^{k+1}\big|_x\right\}.$$

Definition A.4. Let $(0)_x \subset E_1 \subset E_2 \subset \ldots \subset E_p \subseteq T_xM$ be an integral flag of $(\mathcal{I}, \boldsymbol{\Omega})$ based at $x \in M$. We define the integers $\{c_k : k = -1, 0, \ldots, p\}$ as follows:

$$c_k = \begin{cases} 0 & k = -1, \\ \operatorname{codim} H(E_k) & k = 0, \ldots, p-1 \\ \dim M - p & k = p. \end{cases}$$

We now quote the first half of Theorem 1.11 from Chapter III of [10]:

Proposition A.5. *Let $(\mathcal{I}, \boldsymbol{\Omega})$ be an exterior differential system with independence condition on manifold M, where $\mathcal{I}$ contains no non-zero forms of degree* 0*. Let $(0)_x \subset E_1 \subset E_2 \subset \ldots \subset E_p \subset T_xM$ be an integral flag of $(\mathcal{I}, \boldsymbol{\Omega})$. Then $V_p(\mathcal{I}, \boldsymbol{\Omega}) \subseteq Gr_p(TM)$ is of codimension at least $c_0 + c_1 + \ldots + c_{p-1}$ at E_p.*

If there exists a neighbourhood, U of E_p in $Gr_p(TM)$ such that $V_p(\mathcal{I}, \boldsymbol{\Omega}) \cap U$ is a smooth sub-manifold of codimension $c_0 + c_1 + \ldots + c_{p-1}$ in U at E_p, then we say that the integral flat E_p *passes Cartan's test.*

The key result is the following:

Theorem A.6 (Cartan–Kähler Theorem: [10], Chapter III, Corollary 2.3)**.** *Let $(\mathcal{I}, \boldsymbol{\Omega})$ be an analytic differential ideal on a manifold M. Let $E_p \subset T_xM$ be an integral element of $(\mathcal{I}, \boldsymbol{\Omega})$ that passes Cartan's test. Then there exists an integral manifold of $(\mathcal{I}, \boldsymbol{\Omega})$ through x, the tangent space to which, at x, is E_p.*

A.1. Linear Pfaffian systems

A *Pfaffian system* is an exterior differential system with independence condition, $(\mathcal{I}, \boldsymbol{\Omega})$, on a manifold M such that $\mathcal{I}$ is generated, as a differential ideal, by sections of a sub-bundle $I \subset T^*M$. (It is assumed that I is of constant rank, s_0.) The independence condition, $\boldsymbol{\Omega}$, may be characterised by a sub-bundle $J \subset T^*M$, with $I \subset J \subset T^*M$ and rank $J/I = n$, in which case $\boldsymbol{\Omega}$ corresponds to a non-vanishing section of $\wedge^n (J/I)$. Such a Pfaffian system is *linear* if

$$dI \equiv 0 \mod J.$$

Locally, we may choose a coframe $\{\boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^{s_0}, \boldsymbol{\omega}^1, \ldots, \boldsymbol{\omega}^n, \boldsymbol{\pi}^1, \ldots, \boldsymbol{\pi}^t\}$ on M such that $I_x = \operatorname{span}(\boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^{s_0})$, $J_x = \operatorname{span}(\boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^{s_0}, \boldsymbol{\omega}^1, \ldots, \boldsymbol{\omega}^n)$. In this case, the condition that the Pfaffian system be linear is that there exist functions ${A^a}_{\varepsilon i}$, ${c^a}_{ij}$ on M such that

$$d\boldsymbol{\theta}^a \equiv \sum_{\varepsilon,i} {A^a}_{\varepsilon i} \boldsymbol{\pi}^\varepsilon \wedge \boldsymbol{\omega}^i + \frac{1}{2} \sum_{i,j} {c^a}_{ij} \boldsymbol{\omega}^i \wedge \boldsymbol{\omega}^j \mod \boldsymbol{\theta}. \tag{A.1}$$

Under a change of coframe of the form

$$(\boldsymbol{\theta}^\sigma, \boldsymbol{\omega}^i, \boldsymbol{\pi}^\varepsilon) \mapsto (\boldsymbol{\theta}^\sigma, \boldsymbol{\omega}^i, \boldsymbol{\pi}^\varepsilon + \sum_i {p^\varepsilon}_i \boldsymbol{\omega}^i), \tag{A.2}$$

the coefficients ${c^a}_{ij}$ transform according to the rule

$${c^a}_{ij} \mapsto {c^a}_{ij} + \sum_\varepsilon \left({A^a}_{\varepsilon i}\, {p^\varepsilon}_j - {A^a}_{\varepsilon j}\, {p^\varepsilon}_i\right).$$

We define two collections of coefficients ${c^a}_{ij}$, ${\widetilde{c}^a}_{ij}$ to be equivalent if there exists parameters ${p^\varepsilon}_i$ such that ${\widetilde{c}^a}_{ij} = {c^a}_{ij} + \sum_\varepsilon \left({A^a}_{\varepsilon i}\, {p^\varepsilon}_j - {A^a}_{\varepsilon j}\, {p^\varepsilon}_i\right)$, and denote the corresponding equivalence class of coefficients by $[c]$. $[c]$ is the *essential torsion* of the linear Pfaffian system $(\mathcal{I}, \boldsymbol{\Omega})$. If it is possible to choose the ${p^\varepsilon}_i$ such that ${\widetilde{c}^a}_{ij} = 0$ (i.e.

there is no essential torsion) then we say that the torsion can be *absorbed*. Given a point $x \in M$, there exists an integral element of $(\mathcal{I}, \boldsymbol{\Omega})$ based at x if and only if $[c]\,(x) = 0$.

In the terminology of Olver [12, pp. 351], the *degree of indeterminacy*, $r^{(1)}$, of the above coframe is the number of the number of solutions of the homogeneous problem

$$\sum_{\varepsilon} (A^a{}_{\varepsilon i}\, p^\varepsilon{}_j - A^a{}_{\varepsilon j}\, p^\varepsilon{}_i) = 0.$$

Equivalently, it is the number of transformations of the form (A.2) that leave the structure equations (A.1) unchanged.

If the torsion vanishes on an open neighbourhood, U, of x, then we write (A.1) in the form

$$d\boldsymbol{\theta}^a \equiv \sum_i \boldsymbol{\pi}^a{}_i \wedge \boldsymbol{\omega}^i \mod \boldsymbol{\theta}, \tag{A.3}$$

where $\boldsymbol{\pi}^a{}_i \equiv \sum_{\varepsilon,i} A^a{}_{\varepsilon i} \boldsymbol{\pi}^\varepsilon \mod \{\boldsymbol{\theta}, \boldsymbol{\omega}\}$.

To determine the involutivity of a torsion-free linear Pfaffian system at $x \in M$, we need to consider its *tableau* A_x, which is a linear subspace of $I_x^* \otimes (J_x/I_x)$. For our purposes, however, it is simpler (but equivalent) to consider the corresponding tableau matrix:

Definition A.7. Given a linear Pfaffian system with structure equations as in (A.3) and a point $x \in M$, the *tableau matrix at* x is the $s_0 \times n$ matrix of elements of $T_x^* M / J_x$ given by

$$\pi_x = (\boldsymbol{\pi}^a{}_i(x)) \mod \{\boldsymbol{\theta}(x), \boldsymbol{\omega}(x)\}.$$

The reduced Cartan characters, $s'_1, \ldots, s'_4$, of the tableau A_x are defined by

$$s'_1 + \ldots + s'_k = \text{ the number of linearly-independent 1-forms in the first } k \text{ columns of } \pi_x,$$

for a generic choice of the 1-forms $\{\boldsymbol{\omega}^i\}$.

In order to check for involutivity of the system $(\mathcal{I}, \boldsymbol{\Omega})$ at $x \in M$, we need to know the dimension of the first prolongation, $A^{(1)}$, of the tableau A_x. We do not give a formal definition of $A^{(1)}$, but content ourselves with the following characterisation, which gives us sufficient information to calculate its dimension:

Proposition A.8 ([11], Proposition 5.7.1)**.** *Let* $x \in M$ *and* $\boldsymbol{\pi}^a{}_i \in T_x^* M$ *satisfy* $d\boldsymbol{\theta}^a \equiv \boldsymbol{\pi}^a{}_i \wedge \boldsymbol{\omega}^i \mod \boldsymbol{\theta}$*. Then the first prolongation,* $A^{(1)}$*, of the tableau* A_x *may be identified with the space of* 1*-forms* $\widetilde{\boldsymbol{\pi}}^a{}_i \equiv \boldsymbol{\pi}^a{}_i \mod \boldsymbol{\theta}$ *such that* $d\boldsymbol{\theta}^a \equiv \widetilde{\boldsymbol{\pi}}^a{}_i \wedge \boldsymbol{\omega}^i \mod \boldsymbol{\theta}$*.*

Remark A.9. Proposition A.8 implies that $\dim A^{(1)}$ is equal to the degree of indeterminacy, $r^{(1)}$ of the coframe. Therefore, in this notation, a Pfaffian system is involutive if it satisfies

$$s'_1 + 2s'_2 + \ldots + ns'_n = r^{(1)}.$$

Proposition A.10 ([10], pp. 318)**.** *The first prolongation of the tableau, A_x, and the reduced Cartan characters obey the inequality*

$$\dim A^{(1)} \leq s_1' + 2s_2' + \ldots + ns_n'.$$

The tableau, A_x, is involutive *if equality holds in this equation.*

Proposition A.11 ([10], Chapter IV, Theorem 5.16)**.** *The linear Pfaffian system $(\mathcal{I}, \boldsymbol{\Omega})$ is involutive at $x \in M$ if and only if*

(i) $[c](x) = 0$*;*

(ii) the tableau A is involutive.

Remark A.12. If the system $(\mathcal{I}, \boldsymbol{\Omega})$ is involutive at $x \in M$, then the Cartan–Kähler theorem implies the existence of an integral manifold of the system $(\mathcal{I}, \boldsymbol{\Omega})$ through the point x.

Appendix B. Absorption formulae

B.1. Explicit absorption procedures

The structure equations for the Pfaffian system $(\mathcal{I}, \boldsymbol{\Omega})$ on the manifold $M^{(1)}$ are given in equation (4.6). We can absorb most of the torsion in the original problem by setting

$$\begin{aligned}
\boldsymbol{\pi}^1{}_{21} &= d\lambda^1{}_{21} + T^1{}_{212}\,\boldsymbol{\omega}^2 + T^1{}_{213}\,\boldsymbol{\omega}^3 + T^1{}_{214}\,\boldsymbol{\omega}^4,\\
\boldsymbol{\pi}^1{}_{22} &= d\lambda^1{}_{22} + T^1{}_{223}\,\boldsymbol{\omega}^3 + T^1{}_{224}\,\boldsymbol{\omega}^4,\\
\boldsymbol{\pi}^1{}_{23} &= d\lambda^1{}_{23} + T^1{}_{234}\,\boldsymbol{\omega}^4,\\
\boldsymbol{\pi}^1{}_{24} &= d\lambda^1{}_{24},\\
\boldsymbol{\pi}^1{}_{31} &= d\lambda^1{}_{31} + T^1{}_{312}\,\boldsymbol{\omega}^2 + T^1{}_{313}\,\boldsymbol{\omega}^3 + T^1{}_{314}\,\boldsymbol{\omega}^4,\\
\boldsymbol{\pi}^1{}_{32} &= \boldsymbol{\pi}^2{}_{31} = d\lambda^1{}_{32} + T^1{}_{324}\,\boldsymbol{\omega}^4,\\
\boldsymbol{\pi}^1{}_{33} &= d\lambda^1{}_{33} - T^1{}_{323}\,\boldsymbol{\omega}^2 + T^1{}_{334}\,\boldsymbol{\omega}^4,\\
\boldsymbol{\pi}^1{}_{34} &= \boldsymbol{\pi}_3{}^1{}_4 = d\lambda^1{}_{34},\\
\boldsymbol{\pi}^1{}_{41} &= d\lambda^1{}_{41} + T^1{}_{412}\,\boldsymbol{\omega}^2 + T^1{}_{413}\,\boldsymbol{\omega}^3 + T^1{}_{414}\,\boldsymbol{\omega}^4,\\
\boldsymbol{\pi}^1{}_{42} &= \boldsymbol{\pi}^2{}_{41} = d\lambda^1{}_{42} + T^1{}_{423}\,\boldsymbol{\omega}^3,\\
\boldsymbol{\pi}^1{}_{44} &= d\lambda^1{}_{44} - T^1{}_{424}\,\boldsymbol{\omega}^2 - T^1{}_{434}\,\boldsymbol{\omega}^3,\\
\boldsymbol{\pi}^2{}_{32} &= d\lambda^2{}_{32} - T^2{}_{312}\,\boldsymbol{\omega}^1 + T^2{}_{323}\,\boldsymbol{\omega}^3 + T^2{}_{324}\,\boldsymbol{\omega}^4,\\
\boldsymbol{\pi}^2{}_{33} &= d\lambda^2{}_{33} - T^2{}_{313}\,\boldsymbol{\omega}^1 + T^2{}_{334}\,\boldsymbol{\omega}^4,\\
\boldsymbol{\pi}^2{}_{34} &= \boldsymbol{\pi}^2{}_{43} = d\lambda^2{}_{34} + \left(T^1{}_{324} + T^2{}_{341}\right)\,\boldsymbol{\omega}^1,\\
\boldsymbol{\pi}^2{}_{42} &= d\lambda^2{}_{42} - T^2{}_{412}\,\boldsymbol{\omega}^1 + T^2{}_{423}\,\boldsymbol{\omega}^3 + T^2{}_{424}\,\boldsymbol{\omega}^4,\\
\boldsymbol{\pi}^2{}_{44} &= d\lambda^2{}_{44} - T^2{}_{414}\,\boldsymbol{\omega}^1 - T^2{}_{434}\,\boldsymbol{\omega}^3,\\
\boldsymbol{\pi}^3{}_{41} &= d\lambda^3{}_{41} + T^3{}_{412}\,\boldsymbol{\omega}^2 + T^3{}_{413}\,\boldsymbol{\omega}^3 + T^3{}_{414}\,\boldsymbol{\omega}^4,\\
\boldsymbol{\pi}^3{}_{42} &= d\lambda^3{}_{42} + T^3{}_{423}\,\boldsymbol{\omega}^3 + T^3{}_{424}\,\boldsymbol{\omega}^4,\\
\boldsymbol{\pi}^3{}_{43} &= d\lambda^3{}_{43} + T^3{}_{434}\,\boldsymbol{\omega}^4,\\
\boldsymbol{\pi}^3{}_{44} &= d\lambda^3{}_{44}.
\end{aligned}$$

The structure equations then take the form given in equation (4.9). Note that the quantity on the left-hand-side of equation (4.7) is invariant under transformations of the form $\boldsymbol{\pi}^b{}_{ca} \to \boldsymbol{\pi}^b{}_{ca} + \delta\boldsymbol{\pi}^b{}_{ca}$ with $\delta\boldsymbol{\pi}^b{}_{ca} = \sum_d \Pi_a{}^b{}_{cd}\boldsymbol{\omega}^d$ that preserve the required symmetries of the $\boldsymbol{\pi}^b{}_{ca}$ (i.e. $\boldsymbol{\pi}^2{}_{31} = \boldsymbol{\pi}^1{}_{32}$). As such, it follows that, at points of $M^{(1)}$ at which $T(x, g, \lambda) \neq 0$, there remains essential torsion in the system that cannot be absorbed into a redefinition of the 1-forms $\boldsymbol{\pi}^b{}_{ca}$.

B.2. Calculation of degree of indeterminacy

We let $\mathbf{X} := (y^1, \dots, y^8) \in \mathbb{R}^{4,4}$ with the split-signature metric

$$\mathbf{q}(\mathbf{X}, \mathbf{X}) := 2\left(y^1y^2 - y^3y^4 + y^5y^6 - y^7y^8\right).$$

Then our constraint equation (4.10) takes the

$$T(x, g, \mathbf{X}) := \mathbf{q}(\mathbf{X}, \mathbf{X}) + R_{1234}(x, g) = 0. \tag{B.1}$$

We then need to consider the pull-back to S of the exterior derivative of T, and find that

$$i^*(dT) = \widetilde{y}^1\widetilde{\boldsymbol{\pi}}^2 + \widetilde{y}^2\widetilde{\boldsymbol{\pi}}^1 - \widetilde{y}^3\widetilde{\boldsymbol{\pi}}^4 - \widetilde{y}^4\widetilde{\boldsymbol{\pi}}^3 + \widetilde{y}^5\widetilde{\boldsymbol{\pi}}^6 + \widetilde{y}^6\widetilde{\boldsymbol{\pi}}^5 - \widetilde{y}^7\widetilde{\boldsymbol{\pi}}^8 - \widetilde{y}^8\widetilde{\boldsymbol{\pi}}^7 + \sum_a \Psi_a\widetilde{\boldsymbol{\omega}}^a \equiv 0 \mod \widetilde{\boldsymbol{\theta}}. \tag{B.2}$$

Note that the 1-forms $\{\widetilde{\boldsymbol{\pi}}^\alpha, \widetilde{\boldsymbol{\rho}}^a, \widetilde{\boldsymbol{\mu}}^a, \widetilde{\boldsymbol{\nu}}^a\}$ are not uniquely determined by the structure equations (4.11) and (4.13). In particular, we are free to consider variations of the form

$$\widetilde{\boldsymbol{\pi}}^\alpha \mapsto \widetilde{\boldsymbol{\pi}}^\alpha + \delta\widetilde{\boldsymbol{\pi}}^\alpha, \qquad \widetilde{\boldsymbol{\rho}}^i \mapsto \widetilde{\boldsymbol{\rho}}^i + \delta\widetilde{\boldsymbol{\rho}}^i, \tag{B.3}$$

$$\widetilde{\boldsymbol{\mu}}^a \mapsto \widetilde{\boldsymbol{\mu}}^a + \delta\widetilde{\boldsymbol{\mu}}^a, \qquad \widetilde{\boldsymbol{\nu}}^a \mapsto \widetilde{\boldsymbol{\nu}}^a + \delta\widetilde{\boldsymbol{\nu}}^a \tag{B.4}$$

with

$$\delta\widetilde{\boldsymbol{\pi}}^\alpha, \delta\widetilde{\boldsymbol{\rho}}^a, \delta\widetilde{\boldsymbol{\mu}}^a, \delta\widetilde{\boldsymbol{\nu}}^a \equiv 0 \mod \widetilde{\boldsymbol{\omega}}^a, \tag{B.5}$$

as long as they preserve (4.11) and (4.13). We first wish to show that, in the generic case where $\widetilde{y}^1, \dots, \widetilde{y}^8$ are all non-zero, we may use such transformations to absorb the $\sum_a \Psi_a\widetilde{\boldsymbol{\omega}}^a$ term in (B.2) into a redefinition of the 1-forms $\widetilde{\boldsymbol{\pi}}^\alpha$, $\widetilde{\boldsymbol{\rho}}^i$.

Firstly, it is straightforward to show that the most general variation that preserves the structure equations (4.11) and (4.12) is of the form (from now on, we drop tildes on all quantities)

$$\begin{aligned}
\delta\boldsymbol{\pi}^1 &= \alpha\boldsymbol{\omega}^1 + \beta\boldsymbol{\omega}^2 + \gamma\boldsymbol{\omega}^3 + \delta\boldsymbol{\omega}^4,\\
\delta\boldsymbol{\pi}^3 &= \epsilon\boldsymbol{\omega}^1 + \zeta\boldsymbol{\omega}^2 + \delta\boldsymbol{\omega}^3 + \eta\boldsymbol{\omega}^4,\\
\delta\boldsymbol{\pi}^5 &= \theta\boldsymbol{\omega}^1 + \delta\boldsymbol{\omega}^2 + \iota\boldsymbol{\omega}^3 + \kappa\boldsymbol{\omega}^4,\\
\delta\boldsymbol{\pi}^7 &= \delta\boldsymbol{\omega}^1 + \lambda\boldsymbol{\omega}^2 + \mu\boldsymbol{\omega}^3 + \nu\boldsymbol{\omega}^4,
\end{aligned}$$

along with

$$\begin{aligned}
\delta\boldsymbol{\pi}^2 &= \xi\boldsymbol{\omega}^1 + o\boldsymbol{\omega}^2 + \tfrac{1}{2}\left(\lambda - \theta\right)\boldsymbol{\omega}^3 + \tfrac{1}{2}\left(\pi - \rho\right)\boldsymbol{\omega}^4,\\
\delta\boldsymbol{\pi}^4 &= \sigma\boldsymbol{\omega}^1 + \tau\boldsymbol{\omega}^2 + \tfrac{1}{2}\left(\upsilon - \phi\right)\boldsymbol{\omega}^3 + \tfrac{1}{2}\left(\lambda - \theta\right)\boldsymbol{\omega}^4,\\
\delta\boldsymbol{\pi}^6 &= \tfrac{1}{2}\left(\gamma - \eta\right)\boldsymbol{\omega}^1 + \tfrac{1}{2}\left(\upsilon - \pi\right)\boldsymbol{\omega}^2 + \chi\boldsymbol{\omega}^3 + \psi\boldsymbol{\omega}^4,\\
\delta\boldsymbol{\pi}^8 &= \tfrac{1}{2}\left(\phi - \rho\right)\boldsymbol{\omega}^1 + \tfrac{1}{2}\left(\gamma - \eta\right)\boldsymbol{\omega}^2 + \omega\boldsymbol{\omega}^3 + \Omega\boldsymbol{\omega}^4,
\end{aligned}$$

and

$$\begin{aligned}
\delta\boldsymbol{\rho}^1 &= (\zeta-\xi)\boldsymbol{\omega}^1 + (o+\epsilon)\boldsymbol{\omega}^2 + \tfrac{1}{2}(\lambda+\theta)\,\boldsymbol{\omega}^3 + \tfrac{1}{2}(\pi+\rho)\,\boldsymbol{\omega}^4,\\
\delta\boldsymbol{\rho}^2 &= (\beta-\sigma)\boldsymbol{\omega}^1 + (\tau+\alpha) + \tfrac{1}{2}(\upsilon+\phi)\,\boldsymbol{\omega}^3 + \tfrac{1}{2}(\lambda+\theta)\,\boldsymbol{\omega}^4,\\
\delta\boldsymbol{\rho}^3 &= -\tfrac{1}{2}(\gamma+\eta)\,\boldsymbol{\omega}^1 - \tfrac{1}{2}(\upsilon+\pi)\,\boldsymbol{\omega}^2 - (\chi+\nu)\boldsymbol{\omega}^3 - (\phi+\mu)\boldsymbol{\omega}^4,\\
\delta\boldsymbol{\rho}^4 &= -\tfrac{1}{2}(\phi+\rho)\,\boldsymbol{\omega}^1 - \tfrac{1}{2}(\gamma+\eta)\,\boldsymbol{\omega}^2 - (\omega+\kappa)\boldsymbol{\omega}^3 - (\delta-\Omega)\boldsymbol{\omega}^4,
\end{aligned}$$

where $\alpha,\dots,\omega$ and Ω are 25 free parameters. We now wish to find a transformation of the form (B.3) with the property that

$$y^1\delta\boldsymbol{\pi}^2 + y^2\delta\boldsymbol{\pi}^1 - y^3\delta\boldsymbol{\pi}^4 - y^4\delta\boldsymbol{\pi}^3 + y^5\delta\boldsymbol{\pi}^6 + y^6\delta\boldsymbol{\pi}^5 - y^7\delta\boldsymbol{\pi}^8 - y^8\delta\boldsymbol{\pi}^7 = -\sum_a \Psi_a\boldsymbol{\omega}^a.$$

Using the form of $\delta\boldsymbol{\pi}^\alpha$ given above, this implies that we need to find vectors $\mathbf{Y}_1,\dots,\mathbf{Y}_4$ of the form

$$\begin{aligned}
\mathbf{Y}_1 &= \left(\alpha,\xi,\epsilon,\sigma,\theta,\tfrac{1}{2}(\gamma-\eta),\delta,\tfrac{1}{2}(\phi-\rho)\right),\\
\mathbf{Y}_2 &= \left(\beta,o,\zeta,\tau,\delta,\tfrac{1}{2}(\phi-\rho),\lambda,\tfrac{1}{2}(\gamma-\eta)\right),\\
\mathbf{Y}_3 &= \left(\gamma,\tfrac{1}{2}(\lambda-\theta),\delta,\tfrac{1}{2}(\upsilon-\phi),\iota,\chi,\mu,\omega\right),\\
\mathbf{Y}_4 &= \left(\delta,\tfrac{1}{2}(\pi-\rho),\eta,\tfrac{1}{2}(\lambda-\theta),\kappa,\psi,\nu,\Omega\right),
\end{aligned}$$

with the property that

$$\mathbf{q}(\mathbf{X},\mathbf{Y}_i) = -\Psi_i, \qquad i=1,\dots,4. \tag{B.6}$$

In the generic case where $y^1,\dots,y^8$ are all non-zero, these equations may be solved for four of the free parameters in the $\mathbf{Y}_i$, and hence will yield the required transformation (B.3) in terms of the remaining 21 free parameters. Substituting these expressions into $\delta\boldsymbol{\pi}^\alpha$, we therefore generate a 21-parameter family of 1-forms $\boldsymbol{\pi}'^\alpha := \boldsymbol{\pi}^\alpha + \delta\boldsymbol{\pi}^\alpha$, $\boldsymbol{\rho}'^i := \boldsymbol{\rho}^i + \delta\boldsymbol{\rho}^i$ in terms of which the constraint equation (B.2) takes the required form

$$y^1\boldsymbol{\pi}'^2 + y^2\boldsymbol{\pi}'^1 - y^3\boldsymbol{\pi}'^4 - y^4\boldsymbol{\pi}'^3 + y^5\boldsymbol{\pi}'^6 + y^6\boldsymbol{\pi}'^5 - y^7\boldsymbol{\pi}'^8 - y^8\boldsymbol{\pi}'^7 \equiv 0 \mod \boldsymbol{\theta}. \tag{B.7}$$

Finally, based on the the preceding calculations, we deduce Proposition 4.5:

Proof of Proposition 4.5. Since we are dealing with a linear Pfaffian system, the first prolongation of A_p is necessarily an affine-linear space (cf. [10], Chapter IV) the dimension of which, by Proposition A.8, is equal to $r^{(1)}$, the degree of indeterminacy of our coframe. By definition, $r^{(1)}$ is equal to the number of parameters in a change of the 1-forms as in equations (B.3), (B.4) and (B.5) that preserve the form of the structure equations (4.11) and (4.12). Setting $\Psi_a = 0$ in the calculations above, we see that there exists a 21-parameter family of 1-forms, $\delta\widetilde{\boldsymbol{\pi}}^\alpha, \delta\widetilde{\boldsymbol{\rho}}^a$ on S that satisfy these conditions. In addition, we have 10 free parameters in the choice of $\delta\widetilde{\boldsymbol{\mu}}^a$ and 10 free parameters in the choice of $\delta\widetilde{\boldsymbol{\nu}}^a$ consistent with the structure equations. In total, therefore, at a generic point $p \in S$, we have 41 free parameters in choosing the 1-forms in a way that is consistent with the structure equations.

Therefore $\dim A^{(1)} = r^{(1)} = 41$, as required. □

References

[1] Gauss C F 1822 Allgemeine Auflösung der Aufgabe die Theile einer gegebenen Fläche auf einer andern gegebnen Fläche so abzubilden, dass die Abbildung dem Abgebildeten in den kleinsten Theilen ähnlich wird. See Gauss Werke IV, pp. 189–216.

[2] Spivak M 1975 *A comprehensive introduction to differential geometry. Vol. IV* (Publish or Perish Inc., Boston, Mass., 1975).

[3] Cartan E 1945 *Les systèmes différentiels extérieurs et leurs applications géométriques* (Actualités Sci. Ind., no. 994, Hermann et Cie., Paris).

[4] DeTurck D M and Yang D 1984 Existence of elastic deformations with prescribed principal strains and triply orthogonal systems, *Duke Math. J.* **51** 243–60.

[5] Tod K P 1992, On choosing coordinates to diagonalize the metric, *Classical and Quantum Gravity* **9** 1693–1705.

[6] Kini D A 1997 *Weak Singularities in General Relativity*, PhD Thesis, University of Southampton.

[7] d'Inverno R A and Smallwood J 1980 Covariant $2+2$ formulation of the initial value problem in general relativity, *Phys. Rev. D* **22** 1233–47.

[8] Darboux G 1898 *Leçons sur les systèmes orthogonaux et les coordonnées curvilignes* (Gauthier-Villars).

[9] Taylor M E 1981 *Pseudodifferential operators* (Princeton Mathematical Series, Princeton University Press, Princeton, N.J.).

[10] Bryant R L, Chern S S, Gardner R B, Goldschmidt H L, and Griffiths P A 1991 *Exterior differential systems*, (Mathematical Sciences Research Institute Publications, vol. 18, Springer-Verlag, New York).

[11] Ivey T A and Landsberg J M 2003 *Cartan for beginners: differential geometry via moving frames and exterior differential systems*, (Graduate Studies in Mathematics, vol. 61, American Mathematical Society, Providence, RI).

[12] Olver P J 1995 *Equivalence, invariants, and symmetry*, (Cambridge University Press, Cambridge).

[13] Penrose R and Rindler W 1987 *Spinors and space-time. Vol. 1*, (Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge).

[14] Eisenhart L P 1960 *A treatise on the differential geometry of curves and surfaces* (Dover Publications Inc., New York).