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BLOCKED REGULAR FRACTIONAL FACTORIAL DESIGNS WITH MAXIMUM ESTIMATION CAPACITY

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In this paper, the problem of constructing optimal blocked regular fractional factorial designs is considered. The concept of minimum aberration due to Fries and Hunter is a well-accepted criterion for selecting good unblocked fractional factorial designs. Cheng, Steinberg and Sun showed that a minimum aberration design of resolution three or higher maximizes the number of two-factor interactions which are not aliases of main effects and also tends to distribute these interactions over the alias sets very uniformly. We extend this to construct block designs in which (i) no main effect is aliased with any other main effect not confounded with blocks, (ii) the number of two-factor interactions that are neither aliased with main effects nor confounded with blocks is as large as possible and (iii) these interactions are distributed over the alias sets as uniformly as possible. Such designs perform well under the criterion of maximum estimation capacity, a criterion of model robustness which has a direct statistical meaning. Some general results on the construction of blocked regular fractional factorial designs with maximum estimation capacity are obtained by using a finite projective geometric approach.

1. Introduction. Fractional factorial designs are commonly used in industrial experiments especially when a large number of factors have to be studied but the experimental runs are expensive. Blocking, on the other hand, is an effective method of improving the efficiency of an experiment by reducing the heterogeneity of the experimental units. How to choose a good fractional factorial design and a good blocking scheme at the same time is an important issue. When the experimenter knows the possible importance of the factorial effects, an algorithm such as that described in Franklin (1985) can be used to select a fraction and a blocking scheme to estimate certain required effects; see also Bailey's (1977) construction of blocked fractions from Abelian groups. In situations where little information about the relative sizes of the effects is available, it is desirable to have a design with good all-around (or *model robust*) properties.

Although criteria for selecting good unblocked fractional factorial designs such as resolution [Box and Hunter (1961)] and minimum aberration [Fries and Hunter (1980)] are available and widely accepted, only recently have attempts to formulate suitable criteria for "optimal" blocking been made. Let

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$s \geq 2$ be a prime or prime power, n, k and r be positive integers satisfying $k+r < n$, and consider a regular s^{n-k} fractional factorial design d of resolution three or higher, arranged in s^r equal-sized blocks, in which the main effects are not confounded with blocks. Such a design will be called an (s^{n-k}, s^r) regular main effect (RME) design. As usual, an $n \times 1$ nonnull vector $\mathbf{b}$ over $GF(s)$, the finite field with s elements, determines a pencil, which is associated with a k-factor interaction if $\mathbf{b}$ has k nonzero elements. For $\lambda(\neq 0) \in GF(s)$, $\mathbf{b}$ and $\lambda\mathbf{b}$ represent the same pencil carrying $s-1$ degrees of freedom.

For $i \geq 3$, let $A_i(d)$ be the number of distinct i-factor interaction pencils appearing in the defining relation of an RME design d, also, for $i \geq 2$ let $B_i(d)$ be the number of distinct i-factor interaction pencils which are confounded with the blocks of d without appearing in the defining relation. The two sequences $\{A_3(d), A_4(d), \ldots\}$ and $\{B_2(d), B_3(d), \ldots\}$ are called the wordlength patterns of d with respect to the defining equation and blocks, respectively. On the basis of these two wordlength patterns, Sun, Wu and Chen (1997) and Mukerjee and Wu (1999) considered the notion of admissibility, while Chen and Cheng (1997) combined them to extend the notion of minimum aberration to the case of block designs; see also Sitter, Chen and Feder (1997) in this connection.

Motivated by the ideas of Cheng, Steinberg and Sun (1999) and Cheng and Mukerjee (1998) for the unblocked case, in this article, instead of considering wordlength patterns, we propose and study a criterion based on the alias pattern of the interactions. We consider designs which maximize the number of two-factor interaction pencils that are neither aliased with main effects *nor confounded with blocks* and also distribute these interactions over the alias sets as uniformly as possible. As demonstrated in Cheng, Steinberg and Sun (1999) for the unblocked case, this criterion can be tied to some criteria of model robustness, including maximum estimation capacity.

Section 2 defines our optimality criteria and discusses how they relate to those based on wordlength patterns. Section 3 presents a projective geometric formulation of the problem. Section 4 contains some general results on optimal blocked designs under our criteria. Further results on two-level designs are given in Section 5. Section 6 contains tables of 16- and 27-run optimal blocked designs. Most of the proofs are postponed to the Appendix.

2. Optimality criteria.

For any nonnegative integer u, let $L_u = (s^u - 1)/(s-1)$. Then it is easy to see that in an (s^{n-k}, s^r) RME design d, there are $L_{n-k} - L_r - n$ $(= f$, say) alias sets which neither contain a main effect pencil nor are confounded with blocks. Let $m_i(d)$ be the number of two-factor interaction pencils in the ith one of these f alias sets and define the $f \times 1$ vector $\mathbf{m}(d) = (m_1(d), \ldots, m_f(d))^T$. Let d_1 and d_2 be two (s^{n-k}, s^r) RME designs. The d_1 is said to dominate d_2 with respect to the alias pattern of two-factor interactions (written $d_1 \succ_a d_2$) if $\mathbf{m}(d_1)$ is upper weakly majorized by $\mathbf{m}(d_2)$ and not obtainable from $\mathbf{m}(d_2)$ by permuting its elements. [Recall that a vector $\mathbf{x} = (x_1, \ldots, x_t)$ is upper weakly majorized by another vector $\mathbf{y} = (y_1, \ldots, y_t)$ if and only if $\sum_{i=1}^k x_{[i]} \geq \sum_{i=1}^k y_{[i]}$ for all $1 \leq k \leq t$, where

$x_{[1]} \le x_{[2]} \le \cdots \le x_{[t]}$ and $y_{[1]} \le y_{[2]} \le \cdots \le y_{[t]}$ are the ordered components of $\mathbf{x}$ and $\mathbf{y}$, respectively.] An (s^{n-k}, s^r) RME design d is said to be admissible with respect to the alias pattern of two-factor interactions if it is not dominated by any other design. This criterion based on $\mathbf{m}(d)$ seeks to find designs under which the number of two-factor interaction pencils neither aliased with main effects nor confounded with blocks [i.e., $\sum_{i=1}^{f} m_i(d)$] is as large as possible and these interaction pencils are distributed over the alias sets as uniformly as possible [i.e., the $m_i(d)$'s are as equal as possible.] One purpose of this article is to study the construction of such designs.

EXAMPLE 2.1. Let $s = 2$, $n = 6$, $k = 2$, $r = 2$. Then $f = 6$. Denoting the six two-level factors by $A, B, \dots, F$, consider RME designs d_1 and d_2 both of which are given by the defining relation $I = ABCD = CDEF = ABEF$. Let the alias sets containing ACE, ACF and EF be confounded with blocks in d_1 and the alias sets containing AC, AE and CE be confounded with blocks in d_2. Then one can check that

$$\mathbf{m}(d_1) = (2, 2, 2, 2, 2, 2)^T \quad \text{and} \quad \mathbf{m}(d_2) = (3, 2, 2, 2, 0, 0)^T.$$

So $\mathbf{m}(d_1)$ is upper weakly majorized by $\mathbf{m}(d_2)$, and d_1 dominates d_2 with respect to the alias pattern of two-factor interactions. From Lemma 2.1 below it will follow that d_1 also dominates d_2 with respect to the criterion of estimation capacity that we now introduce.

The two criteria of model robustness considered in Cheng, Steinberg and Sun (1999) can be extended to the blocked case. One of them, estimation capacity, will be studied here in detail. Let interactions involving three or more factors be negligible and suppose the main effects are of primary interest. Furthermore, suppose interest lies in having as much information about the two-factor interactions as possible. Now, there are $\binom{n}{2}(s-1)$ distinct two-factor interaction pencils and, for $1 \le u \le \binom{n}{2}(s-1)$, let $E_u(d)$ be the number of models containing all the main effects and u two-factor interaction pencils which can be estimated by d. Then, following Cheng, Steinberg and Sun (1999),

$$E_u(d) = \begin{cases} \displaystyle\sum_{1 \le i_1 < \cdots < i_u \le f} \prod_{j=1}^{u} m_{i_j}(d), & \text{if } u \le f, \\ 0, & \text{otherwise}. \end{cases} \tag{2.1}$$

Under the criterion of *estimation capacity*, it is desirable to choose d so as to make the quantities $E_u(d)$ as large as possible. Let d_1 and d_2 be two (s^{n-k}, s^r) RME designs. Then d_1 is said to dominate d_2 with respect to estimation capacity (written $d_1 \succ_e d_2$) if $E_u(d_1) \ge E_u(d_2)$ for each u, with strict inequality for some u. In particular, if there exists an (s^{n-k}, s^r) RME design that maximizes $E_u(\cdot)$ for each u; then it is said to have maximum estimation capacity. When such a design does not exist, we say that a design d is admissible with respect to estimation capacity if it is not dominated by any other design. Following Cheng, Steinberg and Sun's (1999) Theorem 2, we have the following result.

LEMMA 2.1. *Let d_1 and d_2 be (s^{n-k}, s^r) RME designs. If $d_1 \succ_a d_2$, then $d_1 \succ_e d_2$.*

In addition to having a direct statistical meaning, it is our experience that these criteria often yield substantially fewer admissible designs than that based on the two wordlength patterns as considered in Mukerjee and Wu (1999). From the tables presented in Section 6, it can be seen that in most of the 16-run cases and all the 27-run cases, there is a single design which has maximum estimation capacity.

In the rest of the paper, unless otherwise specified, when we say d_1 dominates d_2, we mean that d_1 dominates d_2 with respect to the alias pattern of two-factor interactions, and therefore d_1 also dominates d_2 with respect to estimation capacity.

We end this section with a discussion of how our criteria relate to that of Chen and Cheng (1997). Each (s^{n-k}, s^r) RME design d involves $g(= L_{n-k})$ alias sets altogether. Label the alias sets so that the first $f(= L_{n-k} - L_r - n)$ of them neither contain a main effect pencil nor are confounded with blocks. For $1 \le i \le g$, let $m_i(d)$ be the number of distinct two-factor interaction pencils contained in the ith alias set of d. Then $m_1(d), \dots, m_f(d)$ have the same meaning as before, and it is not hard to see that

$$\sum_{i=1}^{f} m_i(d) = \binom{n}{2}(s-1) - 3A_3(d) - B_2(d).$$

Generalizing (2.2) of Cheng, Steinberg and Sun (1999), one can also show that

$$\sum_{i=1}^{g} \{m_i(d)\}^2 = \binom{n}{2}(s-1) + 6\{A_4(d) + (s-2)A_3(d)\}.$$

Under Chen and Cheng's (1997) definition, a minimum aberration $(2^{n-k}, 2^r)$ RME design minimizes $3A_3(d) + B_2(d)$, and then minimizes $A_4(d)$. Therefore it maximizes $\sum_{i=1}^{f} m_i(d)$ and then minimizes $\sum_{i=1}^{g}\{m_i(d)\}^2$. Consequently, $\sum_{i=1}^{f} m_i(d)$ is large and $m_1(d), \dots, m_f(d)$ are expected to be close to one another. By Lemma 2.1, for two-level designs, we expect Chen and Cheng's (1997) minimum aberration criterion to be a good surrogate for our criteria. It must, however, be noted that this line of argument is heuristic and one should not anticipate any neat result connecting the two approaches in general.

3. A projective geometric formulation. Let P be the set of the L_{n-k} distinct points of the finite projective geometry PG$(n-k-1, s)$. A pair of subsets (C_0, C) of P is called an eligible (L_r, n)-pair if (1) C_0 and C are disjoint, (2) C_0 is an $(r-1)$-flat (i.e., it has cardinality L_r and is closed, up to proportionality, under the formation of nonnull linear combinations), and (3) C has cardinality n. Also, let $\mathbf{V}(C)$ be an $(n-k) \times n$ matrix with columns given by the points in C. Then the existence of an (s^{n-k}, s^r) RME design d is equivalent to that of an eligible (L_r, n)-pair of subsets (C_0, C) of P, with $\mathbf{V}(C)$ having full row rank; see Mukerjee and Wu (1999) or Chen and Cheng (1997). Specifically, the s^{n-k} row vectors spanned by the $n-k$ rows of $\mathbf{V}(C)$ give the s^{n-k} level combinations in the fraction, and the $(r-1)$-flat C_0 is used for blocking. Let $\mathbf{V}_0$ be an $(n-k)\times r$ matrix of full column rank such that the columns of $\mathbf{V}_0$ span C_0. Then a typical block consists of all level combinations of the form $\mathbf{V}(C)^T\boldsymbol{l}$, with the $(n-k)\times 1$ vector $\boldsymbol{l}$ over $GF(s)$ satisfying $V_0^T\boldsymbol{l}=\boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is a fixed $r\times 1$ vector over $GF(s)$. Different blocks correspond to different choices of $\boldsymbol{\xi}$. Since $\mathbf{V}_0$ has column rank r, there are s^r choices of $\boldsymbol{\xi}$, leading to a division of the s^{n-k} level combinations into s^r blocks.

Therefore, while studying (s^{n-k}, s^r) RME designs, it is enough to consider eligible (L_r, n)-pairs of subsets (C_0, C) of P, with $\mathbf{V}(C)$ having full row rank. The (s^{n-k}, s^r) RME design corresponding to any such eligible pair will be denoted $d(C_0, C)$. Accordingly, we shall also modify our notation slightly to write $m_i(C_0, C) = m_i(d(C_0, C)), 1 \le i \le f$ and $\mathbf{m}(C_0, C) = \mathbf{m}(d(C_0, C))$. Considering the cardinalities of C_0 and C, it is clear that an (s^{n-k}, s^r) RME design exists if and only if $L_r + n \le L_{n-k}$. In fact, if $L_r + n = L_{n-k}$, then all such designs are isomorphic. Hence, to avoid trivialities, hereafter we assume that $L_r + n < L_{n-k}$. Then $f(= L_{n-k} - L_r - n) > 0$ and the issue of estimation capacity or alias pattern becomes meaningful [cf. (2.1)].

Under the equivalence of an (s^{n-k}, s^r) RME design to an eligible (L_r, n)-pair described in the above, a pencil $\mathbf{b}$ appears in the defining relation if and only if $\mathbf{V}(C)\mathbf{b} = \mathbf{0}$, and it does not appear in the defining relation but is confounded with blocks if and only if $\mathbf{V}(C)\mathbf{b}$ is proportional to some point in C_0. As in the unblocked case, two distinct pencils $\mathbf{b}_1$ and $\mathbf{b}_2$, neither of which appears in the defining equation, are aliased with each other if and only if $\mathbf{V}(C)\mathbf{b}_1$ and $\mathbf{V}(C)\mathbf{b}_2$ are proportional to the same point of P. This establishes a one–one correspondence between the set P, of cardinality L_{n-k}, and the class of the L_{n-k} alias sets. In particular, the L_r points in C_0 correspond to the L_r alias sets which are confounded with blocks while the n points in C correspond to the n alias sets each of which contains a main effect pencil. Thus, defining $\overline{C} = P - (C_0 \cup C)$, the $f(= L_{n-k} - L_r - n)$ points in $\overline{C}$ correspond to the f alias sets which neither contain a main effect pencil nor are confounded with blocks.

EXAMPLE 3.1. Consider again the design d_1 introduced in Example 2.1. For this design both C_0 and C are subsets of $PG(3, 2)$ and are given by

$$\begin{aligned} C_0 &= \{(1,0,1,1)^T, (0,1,1,1)^T, (1,1,0,0)^T\},\\ C &= \{(1,0,0,0)^T, (0,1,0,0)^T, (0,0,1,0)^T(1,1,1,0)^T,\\ &\qquad (0,0,0,1)^T, (1,1,0,1)^T\}, \end{aligned}$$

respectively. The six points in the set C correspond to the six factors $A, B, \dots, F$. It is then easy to see that the pencils that appear in the defining relation and that are confounded with blocks are precisely as described in Example 2.1.

Continuing with the set-up of the paragraph preceding Example 3.1, let $\overline{C} = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_f\}$, $C_0 = \{\boldsymbol{\alpha}_{f+1}, \ldots, \boldsymbol{\alpha}_t\}$, where $t = f + L_r$, and $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_t$ are distinct points in P with $\boldsymbol{\alpha}_{f+1}, \ldots, \boldsymbol{\alpha}_t$ forming an $(r-1)$-flat. For $1 \le i \le f$, let $h_i(C_0, C)$ be the number of distinct two-factor interaction pencils $\mathbf{b}$ such that $\mathbf{V}(C)\mathbf{b}$ is nonnull and proportional to $\boldsymbol{\alpha}_i$, and $\boldsymbol{\phi}_i(C_0, C)$ be the number of linearly dependent triplets $\{\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j, \boldsymbol{\alpha}_u\}$ such that $\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j$ and $\boldsymbol{\alpha}_u$ are distinct members of $C_0 \cup \overline{C}$ and $j < u$. Define $\mathbf{h}(C_0, C)$ and $\boldsymbol{\phi}(C_0, C)$ as $f \times 1$ vectors with the ith elements $h_i(C_0, C)$ and $\boldsymbol{\phi}_i(C_0, C)$, respectively. Then, analogous to the findings of Cheng and Mukerjee (1998) in the unblocked case, the following lemma holds.

LEMMA 3.1. (a) *The vector* $\mathbf{m}(C_0, C)$ *can be obtained from* $\mathbf{h}(C_0, C)$ *by permuting the elements of the latter*.

(b) *For* $1 \le i \le f$, $h_i(C_0, C) = \frac{1}{2}(s-1)(L_{n-k} - 2t + 1) + \boldsymbol{\phi}_i(C_0, C)$.

(c) *Consider* (s^{n-k}, s^r) *RME designs* $d_i = d(C_{0i}, C_i)$, $i = 1, 2$. *If* $\boldsymbol{\phi}(C_{01}, C_1)$ *is upper weakly majorized by* $\boldsymbol{\phi}(C_{02}, C_2)$ *and not obtainable from* $\boldsymbol{\phi}(C_{02}, C_2)$ *by permuting its elements*, *then* d_1 *dominates* d_2.

Consideration of the complementary subset $\overline{C}$, as done here, is reminiscent of Chen and Hedayat (1996) [see also Tang and Wu (1996) and Suen, Chen and Wu (1997)], who explored the issue of minimum aberration in the unblocked case. Our problem is much different from theirs. Unlike them, we consider block designs and have to take care of the aliasing pattern directly. Chen and Cheng (1997) also used complementary sets to construct blocked designs under their definition of minimum aberration.

REMARK 3.1. Before concluding this section, we introduce a simple implication of Lemma 3.1. If $f = 1$, then recalling that C_0 is a flat, $\boldsymbol{\phi}(C_0, C)$ must equal the scalar 0 for every (s^{n-k}, s^r) RME design $d(C_0, C)$; that is, all such designs are equivalent with respect to estimation capacity as well as the alias pattern of two-factor interactions. Hence, hereafter we shall consider only the situation $f \ge 2$.

REMARK 3.2. Unlike the unblocked case, however, the above equivalence does not hold for $f = 2$ in block designs. As an illustration, let $s = 3, n = 10, k = 7, r = 1$ (so $f = 2$) and consider two $(3^{10-7}, 3)$ RME designs $d_i = d(C_0, C_i)$, $i = 1, 2$, where $C_0 = \{(1, 0, 0)\}^T$, $\overline{C}_1 = \{(0, 1, 0)^T, (1, 1, 0)^T\}$ and $\overline{C}_2 = \{(0, 1, 0)^T, (0, 0, 1)^T\}$, with $\overline{C}_i = P - (C_0 \cup C_i)$, $i = 1, 2$. Then both $\mathbf{V}(C_1)$ and $\mathbf{V}(C_2)$ have full row rank, and $\boldsymbol{\phi}(C_0, C_1) = (1, 1)^T$, $\boldsymbol{\phi}(C_0, C_2) = (0, 0)^T$, so that by Lemma 3.1, d_1 dominates d_2.

4. Blocked s^{n-k} designs with maximum estimation capacity. We continue with the notational system introduced in the last section and first present a lemma whose proof can be found in the Appendix.

LEMMA 4.1. *Consider an* (s^{n-k}, s^r) *RME design* $d(C_0, C)$. *Then for* $1 \leq i \leq f$,

$$\phi_i(C_0, C) \leq \tfrac{1}{2}\min\{f(f-1), s(f-1)(s-1)(f+L_r-1)\}. \tag{4.1}$$

(a) *Let* $2 \leq f \leq s$. *Then equality holds in* (4.1) *for all* $i(1 \leq i \leq f)$ *if and only if*

$$\overline{C} \subset \{\boldsymbol{\beta}_0 + \boldsymbol{\lambda}\boldsymbol{\beta}_1 \colon \boldsymbol{\lambda} \in GF(s)\}, \tag{4.2}$$

for some $\boldsymbol{\beta}_0, \boldsymbol{\beta}_1$ *such that* $\boldsymbol{\beta}_0 \in P - C_0$ *and* $\boldsymbol{\beta}_1 \in C_0$.

(b) *Let* $f > s^r$. *Then equality holds in* (4.1) *for all* $i(1 \leq i \leq f)$ *if and only if* $f + L_r = L_u$ *for some* $u(r+2 \leq u < n-k)$ *and* $C_0 \cup \overline{C}$ *is a* $(u-1)$*-flat*.

(c) *Finally, let* $s < f \leq s^r$, *a situation which can arise only when* $r \geq 2$.

(c1) *For* $s \geq 3$, *equality holds in* (4.1) *for all* $i(1 \leq i \leq f)$ *if and only if* $f = s^u$ *for some* $u(2 \leq u \leq r)$ *and*

$$\overline{C} = \left\{\boldsymbol{\beta}_0 + \sum_{i=1}^{u} \boldsymbol{\lambda}_i \boldsymbol{\beta}_i \colon \boldsymbol{\lambda}_i \in GF(s)\right\}, \tag{4.3}$$

for some $\boldsymbol{\beta}_0 \in P - C_0$ *and some* u *linearly independent points* $\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_u$ *of* C_0.

(c2) *For* $s = 2$, *equality holds in* (4.1) *for all* $i(1 \leq i \leq f)$ *if and only if*

$$\overline{C} = \{\boldsymbol{\beta}_0, \boldsymbol{\beta}_0 + \boldsymbol{\alpha}^{(1)}, \dots, \boldsymbol{\beta}_0 + \boldsymbol{\alpha}^{(f-1)}\}, \tag{4.4}$$

for some $\boldsymbol{\beta}_0 \in P - C_0$ *and some distinct* $\boldsymbol{\alpha}^{(1)}, \dots, \boldsymbol{\alpha}^{(f-1)} \in C_0$.

REMARK 4.1. (a) In particular, if $n - k = 2$, then $r = 1$ and for every (s^{n-k}, s^r) RME design $d(C_0, C)$, $\overline{C}$ is of the form (4.2). By Lemma 3.1 and Lemma 4.1(a), all such designs are equivalent with respect to estimation capacity and the alias pattern of two-factor interactions. Hence, hereafter, we shall consider only the situation $n - k \geq 3$.

(b) Also, if $s = 2$ and $r = n - k - 1$, then for every $(2^{n-k}, 2^r)$ RME design $d(C_0, C)$, $\overline{C}$ is of the form (4.4). Hence by Lemma 3.1 and Lemma 4.1(c), all such designs are also equivalent with regard to estimation capacity and the alias pattern of two-factor interactions. Thus hereafter, for $s = 2$, we shall consider only the situation $r \leq n - k - 2$.

REMARK 4.2. Recall that for an (s^{n-k}, s^r) RME design $d(C_0, C)$, the matrix $\mathbf{V}(C)$ has full row rank. This requirement regarding rank is clearly satisfied $C_0 \cup \overline{C}$ is contained in an $(n-k-2)$-flat of P. Thus, $\mathbf{V}(C)$ has full row rank under (4.2), (4.3) or (4.4) provided $r \leq n - k - 2$ or, if, as needed in Lemma 4.1(b), $C_0 \cup \overline{C}$ is itself a $(u-1)$-flat $(u < n-k)$. Also, for $r = n-k-1$, it can be shown that under (4.2) or (4.3) this rank condition is automatically satisfied.

In consideration of Lemma 3.1(c), Lemma 4.1 and Remark 4.2, we have the following result.

THEOREM 4.1. *Suppose* $f \geq 2$ *and* $n - k \geq 3$.

(a) *Let* $2 \le f \le s$. *Then an* (s^{n-k}, s^r) *RME design* $d(C_0, C)$ *has maximum estimation capacity if and only if* $\overline{C}$ *is as given by* (4.2).

(b) *Let* $f + L_r = L_u$ *for some* $u(r+2 \le u < n-k)$. *Then an* (s^{n-k}, s^r) *RME design* $d(C_0, C)$ *has maximum estimation capacity if and only if* $C_0 \cup \overline{C}$ *is a* $(u-1)$-*flat*.

(c) *Let* $s \ge 3$ *and* $f = s^u$ *where* $2 \le u \le r$ *and* $(u, r) \ne (n-k-1, n-k-1)$. *Then an* (s^{n-k}, s^r) *RME design* $d(C_0, C)$ *has maximum estimation capacity if and only if* $\overline{C}$ *is as given by* (4.3).

(d) *Let* $s = 2, r \le n-k-2$ *and* $2 < f \le 2^r$. *Then a* $(2^{n-k}, 2^r)$ *RME design* $d(C_0, C)$ *has maximum estimation capacity if and only if* $\overline{C}$ *is as given by* (4.4).

REMARK 4.3. All the designs in Theorem 4.1 are also optimal with respect to the alias pattern of two-factor interactions since they maximize $\sum_{i=1}^{f} m_i(d)$ and have the property that all the $m_i(d)$'s are equal.

In view of Theorem 4.1(a), we note that between the two designs d_1 and d_2 consider in Remark 3.2, only d_1 has maximum estimation capacity. Hence it is natural that d_1 dominates d_2. Some more examples follow.

EXAMPLE 4.1. Let $s = 3, n = 27, k = 23, r = 1$. Then $f = 12$ and $f + L_r = L_3$. Let $C_0 = \{(1,0,0,0)^T\}$ and $\overline{C}$ be obtained by deleting the point $(1,0,0,0)^T$ from the 2-flat spanned by $\{(1,0,0,0)^T, (0,1,0,0)^T, (0,0,1,0)^T\}$. Then $C_0 \cup \overline{C}$ is a 2-flat and, by Theorem 4.1(b), the resulting $(3^{27-23}, 3)$ RME design $d(C_0, C)$ has maximum estimation capacity and is optimal with respect to the alias pattern of two-factor interactions.

EXAMPLE 4.2. Let $s = 3, n = 18, k = 14, r = 3$. Then $f = 9$. Let C_0 be the 2-flat spanned by $\{(0,1,0,0)^T, (0,0,1,0)^T, (0,0,0,1)^T\}$ and $\overline{C} = \{(1,i,j,0)^T: i, j = 0,1,2\}$. Then $\overline{C}$ is as given by (4.3) and by Theorem 4.1(c), the resulting $(3^{18-14}, 3^3)$ RME design $d(C_0, C)$ has maximum estimation capacity and is optimal with respect to the alias pattern of two-factor interactions.

EXAMPLE 4.3. Let $s = 2, n = 20, k = 15, r = 3$. Then $f = 4$. Let C_0 be the 2-flat spanned by $\{(0,1,0,0,0)^T, (0,0,1,0,0)^T, (0,0,0,1,0)^T\}$ and $\overline{C} = \{(1,0,0,0,0)^T, (1,1,0,0,0)^T, (1,0,1,0,0)^T, (1,0,0,1,0)^T\}$. Then $\overline{C}$ is as given by (4.4) and by Theorem 4.1(d), the resulting $(2^{20-15}, 2^3)$ RME design $d(C_0, C)$ has maximum estimation capacity and is optimal with respect to the alias pattern of two-factor interactions.

REMARK 4.4. The requirement $u \ge r+2$ in Theorem 4.1(b) is not restrictive since evidently $u > r$, and if $u = r+1$ then $f = L_{r+1} - L_r = s^r$, a situation covered in parts (c) and (d) of the theorem. Similarly, the stipulation $(u, r) \ne (n-k-1, n-k-1)$ in Theorem 4.1(c) is not restrictive, for otherwise $f + L_r = s^{n-k-1} + L_{n-k-1} = L_{n-k}$ which is impossible. Parts (a) and (b) of Theorem 4.1 extend some of the findings in Cheng and Mukerjee (1998) in the unblocked case to block designs. Parts (c) and (d), however, do not have any counterparts in the unblocked case. Interestingly, parts (a) and (d) can be particularly helpful in studying estimation capacity for relatively small values of f; this situation corresponds to the nearly saturated case which is of practical importance.

5. Two-level designs. Throughout this section, we consider the case $s = 2$. Then any three distinct points of $P[\equiv \mathrm{PG}(n-k-1, 2)]$ which are linearly dependent form a line. Let $G = \{\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_p\}$ be any nonempty p-subset of P and, for $1 \le i \le p$, let $a_i(G)$ be the number of lines passing through $\boldsymbol{\pi}_i$, and two other distinct points of G. Since any two distinct lines can have at most one point in common,

$$a_i(G) \le \tfrac{1}{2}(p-1), \qquad 1 \le i \le p. \tag{5.1}$$

In the above set-up, we have the following lemma which will be proved in the Appendix.

LEMMA 5.1. (*a*) *Let* $p = 2^u - 3$; $3 \le u < n-k$. *Then*

$$a_i(G) \le 2^{u-1} - 2, \qquad 1 \le i \le p, \tag{5.2}$$

and equality holds in (5.2) *for at most one choice of* i.

(b) *Let* $p = 2^u - 4$, $3 \le u < n-k$. *Then*

$$a_i(G) \le 2^{u-1} - 3, \qquad 1 \le i \le p, \tag{5.3}$$

and equality holds in (5.3) *for at most three distinct choices of* i.

THEOREM 5.1. *Let* $s = 2$ *and* n, k, r *be such that* $f + L_r = 2^u - w$, *where* $w = 2, 3$ *or* 4 *and* $r + 2 \le u < n - k$. *Let* $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_u$ *be linearly independent points of* P, *and* C_0^* *and* $\widetilde{C}$ *be* $(r-1)$- *and* $(u-1)$-*flats spanned by* $\{\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_r\}$ *and* $\{\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_u\}$, *respectively. Let* $\overline{C}^* = \widetilde{C} - C_0^* - T_w$, *where*

$$T_w = \{\boldsymbol{\beta}_{r+1}\} \quad \text{if } w = 2, \tag{5.4}$$

$$T_w = \{\boldsymbol{\beta}_{r+1}, \boldsymbol{\beta}_{r+2}\} \quad \text{if } w = 3, \tag{5.5}$$

$$T_w = \{\boldsymbol{\beta}_1 + \boldsymbol{\beta}_{r+1}, \boldsymbol{\beta}_1 + \boldsymbol{\beta}_{r+2}, \boldsymbol{\beta}_1 + \boldsymbol{\beta}_{r+1} + \boldsymbol{\beta}_{r+2}\} \quad \text{if } w = 4. \tag{5.6}$$

Define $C^* = P - (C_0^* \cup \overline{C}^*)$. *Then* $\mathbf{V}(C^*)$ *has full row rank and the* $(2^{n-k}, 2^r)$ *RME design* $d^* = d(C_0^*, C^*)$ *has maximum estimation capacity and is optimal with respect to the alias pattern of two-factor interactions.*

PROOF. As $u < n-k$, there is an $(n-k-2)$-flat containing $(C_0^* \cup \overline{C}^*)(\subset \widetilde{C})$. Hence $\mathbf{V}(C^*)$ has full row rank (cf. Remark 4.2). For any $(2^{n-k}, 2^r)$ RME design $d(C_0, C)$, as before, let $\overline{C} = P - (C_0 \cup C)$ and write $C_0 \cup \overline{C} = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_t\}$, where $t = f + L_r$, and $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_f \in \overline{C}$. Then, recalling the definition of $\boldsymbol{\phi}_i(C_0, C)$, we have

$$\boldsymbol{\phi}_i(C_0, C) = a_i(C_0 \cup \overline{C}), \qquad 1 \le i \le f. \tag{5.7}$$

(a) Let $w = 2$, that is, $f + L_r = 2^u - 2$. Since $\boldsymbol{\phi}_i(C_0, C)$ is an integer, (5.1) and (5.7) yield $\boldsymbol{\phi}_i(C_0, C) \leq 2^{u-1} - 2, 1 \leq i \leq f$. By (5.4), it is easily seen that the design d^* attains the upper bound for each i. The result follows from Lemma 3.1(c).

(b) and (c) For $w = 3$, let $\boldsymbol{\phi}^*$ be an $f \times 1$ vector with one element $2^{u-1} - 2$ and the remaining $f - 1$ elements $2^{u-1} - 3$, and for $w = 4$, let $\boldsymbol{\phi}^*$ be an $f \times 1$ vector with three elements $2^{u-1} - 3$ and the remaining elements $2^{u-1} - 4$. The latter definition is possible since $u \geq r + 2 \Rightarrow f \geq 3$. Then by (5.7) and Lemma 5.1(a), (b), $\boldsymbol{\phi}(C_0, C)$ upper weakly majorizes $\boldsymbol{\phi}^*$. By (5.5) and (5.6), the vector $\boldsymbol{\phi}(C_0^*, C^*)$, associated with d^*, is obtainable from $\boldsymbol{\phi}^*$ by permuting its coordinates. The result follows from Lemma 3.1(c). □

REMARK 5.1. We do not consider the situation $w = 1$ or $u = r + 1$ in Theorem 5.1 since these are already covered by parts (b) and (d) of Theorem 4.1. However, it will be of interest to extend Theorem 5.1 beyond $w = 4$. Even for $w = 2, 3, 4$, one may wish to completely characterize the designs with maximum estimation capacity. These problems appear to be quite involved at this stage; see also Remark 5.2 below in this context.

REMARK 5.2. From our proof of Theorem 5.1, it is clear that for $w = 2, 3, 4$, the design d^* (i) maximizes $\sum_{i=1}^{f} \boldsymbol{\phi}_i(C_0, C)$ and (ii) at the same time makes the individual $\boldsymbol{\phi}_i(C_0, C)$'s as nearly equal as possible. This enables the use of Lemma 3.1(c) in proving Theorem 5.1. One may wonder if the recent findings in Chen and Hedayat (1996) can help in proving (i) and hence simplifying the proof of Theorem 5.1 [see Cheng and Mukerjee (1998) for a similar approach in the unblocked case]. To that effect, with $s = 2$, note that for any design $d(C_0, C)$,

$$\sum_{i=1}^{f} \boldsymbol{\phi}_i(C_0, C) = 3N_1(\overline{C}) + 2N_2(C_0, \overline{C}), \tag{5.8}$$

where $N_2(C_0, \overline{C})$ is the number of lines passing through two distinct points of $\overline{C}$ and one point of C_0 and, for any nonempty set $G(\subset P)$, $N_1(G)$ is the number of lines contained in G. It is also not hard to see

$$N_1(C_0 \cup \overline{C}) = \text{ constant } + N_1(\overline{C}) + N_2(C_0, \overline{C}), \tag{5.9}$$

where the constant does not depend on the design. By (5.8) and (5.9),

$$\sum_{i=1}^{f} \boldsymbol{\phi}_i(C_0, C) = \text{ constant } + N_1(\overline{C}) + 2N_1(C_0 \cup \overline{C}).$$

While the result in Chen and Hedayat (1996) can be used to maximize $N_1(\overline{C})$ and $N_1(C_0 \cup \overline{C})$ separately, it is not useful for the maximization of $N_1(\overline{C}) + 2N_1(C_0 \cup \overline{C})$.

As another application of Lemma 5.1, we have the following result which will be proved in the Appendix.

THEOREM 5.2. *Let* $r = 1$ *and* $f = 2^u - 1 (2 \le u < n - k)$. *Then* $\mathbf{V}(C)$ *has full row rank and a* $(2^{n-k}, 2^r)$ *RME design* $d(C_0, C)$ *has maximum estimation capacity* (*or is an optimal design with respect to the alias pattern of two-factor interactions*) *if and only if* $\overline{C}$ *is a* $(u-1)$-*flat*.

We now present some examples. In these examples, $\boldsymbol{\beta}_i (1 \le i \le 4)$ represents the 5×1 vector on $GF(2)$ having 1 in the ith position and zeros elsewhere.

EXAMPLE 5.1. This example demonstrates that Theorem 5.2 cannot be extended to the case $r > 1$. Let $n = 25$, $k = 20$, $r = 2$ (i.e., $f = 3$) and consider two $(2^{25-20}, 2^2)$ RME designs $d_i = d(C_0, C_i)$, $i = 1, 2$, where $C_0 = \{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2\}$, $\overline{C}_1 = \{\boldsymbol{\beta}_3, \boldsymbol{\beta}_1 + \boldsymbol{\beta}_3, \boldsymbol{\beta}_2 + \boldsymbol{\beta}_3\}$ and $\overline{C}_2 = \{\boldsymbol{\beta}_3, \boldsymbol{\beta}_4, \boldsymbol{\beta}_3 + \boldsymbol{\beta}_4\}$, with $\overline{C}_i = P - (C_0 \cup C_i)$, $i = 1, 2$. Then both $\mathbf{V}(C_1)$ and $\mathbf{V}(C_2)$ have full row rank and $\boldsymbol{\phi}(C_0, C_1) = (2, 2, 2)^T$, $\boldsymbol{\phi}(C_0, C_2) = (1, 1, 1)^T$ so that by Lemma 3.1, d_1 dominates d_2 although $\overline{C}_2$ is a 1-flat. Incidentally, by Theorem 4.1(d), the design d_1 has maximum estimation capacity in this set-up.

EXAMPLE 5.2. Let $n = 19, k = 14$, $r = 2$. Then $f = 9$ and $f + L_r = 2^4 - 4$. Let C_0^* and $\widetilde{C}$ be 1- and 3-flats spanned by $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2\}$ and $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3, \boldsymbol{\beta}_4\}$, respectively. Define

$$\overline{C}^* = \widetilde{C} - C_0^* - \{\boldsymbol{\beta}_1 + \boldsymbol{\beta}_3, \boldsymbol{\beta}_1 + \boldsymbol{\beta}_4, \boldsymbol{\beta}_1 + \boldsymbol{\beta}_3 + \boldsymbol{\beta}_4\}, C^* = P - (C_0^* \cup \overline{C}^*).$$

Then by Theorem 5.1, the $(2^{19-14}, 2^2)$ RME design $d(C_0^*, C^*)$ has maximum estimation capacity and is optimal with respect to the alias pattern of two-factor interactions.

EXAMPLE 5.3. Let $n = 23$, $k = 18$, $r = 1$. Then $f = 2^3 - 1$. Let $C_0 = \{\boldsymbol{\beta}_1\}$ and $\overline{C}$ be the 2-flat spanned by $\{\boldsymbol{\beta}_2, \boldsymbol{\beta}_3, \boldsymbol{\beta}_4\}$. Then by Theorem 5.2, the $(2^{23-18}, 2)$ RME design $d(C_0, C)$ has maximum estimation capacity and is optimal with respect to the alias pattern of two-factor interactions.

6. Tables. We now consider in some details the 8- and 16-run two-level designs and 27-run three-level designs. For an 8-run design with $s = 2$, the only possibilities regarding (n, k, r) are (1) (4, 1, 1), (2) (4, 1, 2), (3) (5, 2, 1) and (4) (6, 3, 1). Under (2) (3) or (4), f equals 0, 1, or 0, respectively, and hence all designs are equivalent. Under (1), up to isomorphism, the unique design with maximum estimation capacity is given by $d(C_0, C)$ where $C_0 = \{(1, 1, 0)^T\}, C = \{(1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T, (1, 1, 1)^T\}$ [cf. Theorem 4.1(a)]. This design is also admissible with respect to wordlength patterns and has minimum aberration under Chen and Cheng's (1997) criterion.

Table 1 shown 16-run $(2^{n-k}, 2^r)$ RME designs which are the best in terms of estimation capacity and the alias pattern of two-factor interactions. To construct this table, we use the catalogue of all nonisomorphic 16-run blocked designs compiled by Sun (1993). The triplets $(n, k, r) = (7, 3, 3)$, (8, 4, 3), (11, 7, 2), (12, 8, 2), (13, 9, 1) and (14, 10, 1) are not shown in Table 1 since these yield $f = 0$ or 1 and hence lead to equivalence of all designs. Also, for

TABLE 1
16-run admissible $(2^{n-k}, 2^r)$ RME *designs with respect to the alias pattern of two-factor interactions. These designs are also the best in terms of estimation capacity*

n	k	r	$\{i: \pi_i \in C\}$ (in addition to 1, 2, 4, 8)	Linearly independent members of $\{i: \pi_i \in C_0\}$	$m_1(\mathbf{d}), \ldots, m_f(\mathbf{d})$	$E_1(d), \ldots, E_f(d)$
5	1	1	15	3	1 1 1 1 1 1 1 1 1	9 36 84 126 126 84 36 9 1
			7	11	2 2 2 1 1 1 1 0 0	10 42 96 129 102 44 8 0 0
5	1	2	15	3 5	1 1 1 1 1 1 1	7 21 35 35 21 7 1
			7	3 13	2 2 1 1 1 1 0	8 26 44 41 20 4 0
6	2	1	7 11	13	3 2 2 2 2 2 2 2 0	15 96 340 720 912 640 192 0
			3 13	6	2 2 2 1 1 1 1 1	11 52 138 225 231 146 52 8
6	2	2	7 11	3 13	2 2 2 2 2 2	
7	3	1	7 11 13	14	3 3 3 3 3 3 3	
7	3	2	7 11 13	3 5	3 3 3 3 0	12 54 108 81 0
			3 5 14	6 9	2 2 2 2 2	10 40 80 80 32
8	4	1	7 11 13 14	3	4 4 4 4 4 4	
8	4	2	7 11 13 14	3 5	4 4 4 4	
9	5	1	3 5 9 14 15	6	4 4 4 4 4	
9	5	2	3 5 9 14 15	6 10	4 4 4	
10	6	1	3 5 6 9 14 15	10	5 4 4 4	
10	6	2	3 5 6 9 14 15	7 11	4 4	
11	7	1	3 5 6 9 10 13 14	15	5 5 5	
12	8	1	3 5 6 9 10 13 14 15	7	6 6	

$(n, k, r) = (5, 1, 3)$ or $(6, 2, 3)$, up to isomorphism, there is a unique RME design and hence these triplets are not shown in Table 1. For $(n, k, r) = (5, 1, 1), (5, 1, 2), (6, 2, 1)$ or $(7, 3, 2)$, there are *two* admissible designs with respect to estimation capacity (as well as the alias pattern of two-factor interactions.) For these (n, k, r), we exhibit the two admissible designs and also the corresponding estimation capacity sequences $E_u(d), 1 \leq u \leq f$. For every other (n, k, r), a design with maximum estimation capacity (and optimal with respect to the alias pattern of two-factor interactions) is available and exhibited in Table 1. In the context of Table 1, the points of $PG(3, 2)$ are denoted by $\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_{15}$, and for $1 \leq i \leq 15$, $\boldsymbol{\pi}_i$ is given by the ith column of

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Table 2 shows 27-run $(3^{n-k}, 3^r)$ RME designs with maximum estimation capacity. In preparing this table, Theorem 4.1 is often useful. For any triplet (n, k, r) not covered by this theorem, we start with the catalogue of Chen, Sun and Wu (1993) showing all nonisomorphic unblocked designs, then consider all possible blocking schemes for each such design and finally use Lemma 3.1(c) or, if necessary, (2.1). As before we do not show the triplets $(n, k, r) = (8, 5, 2)$,

TABLE 2
27-*run* $(3^{n-k}, 3^r)$ RME *designs* $d(C_0, C)$ *with maximum estimation capacity*

n	k	r	Members of $\{i: \pi_i \in C\}$ (in addition to 1, 2, 5)	Members of $\{i: \pi_i \in C_0\}$
4	1	1	8	4
4	1	2	8	4 6 7 12
5	2	1	3 9	13
5	2	2	3 9	4 6 7 12
6	3	1	3 9 13	7
6	3	2	3 9 13	4 6 7 12
7	4	1	3 10 11 13	8
7	4	2	3 10 11 13	4 6 7 12
8	5	1	3 8 9 10 11	13
9	6	1	3 8 9 10 11 13	7
10	7	1	3 6 7 8 10 11 12	4

(9, 6, 2), (11, 8, 1) and (12, 9, 1) which yield $f = 0$ or 1. For each other (n, k, r), a design with maximum estimation capacity is available. In the context of Table 2, the points of $PG(2, 3)$ are denoted by $\boldsymbol{\pi}_1, \dots, \boldsymbol{\pi}_{13}$, and for $1 \le i \le 13$, $\boldsymbol{\pi}_i$ is given by the ith column of

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 & 0 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \end{bmatrix}.$$

REMARK 6.1. All the designs shown in Tables 1 and 2, except the second design for $(n, k, r) = (6, 2, 1)$ in Table 1, are admissible in the sense of Mukerjee and Wu (1999).

APPENDIX

Proofs.

PROOF OF LEMMA 4.1. For any distinct $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in P$, let $\boldsymbol{\sigma}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ denote the set of the $s-1$ distinct points spanned by $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ (excluding themselves) and note that

$$\boldsymbol{\gamma} \in \boldsymbol{\sigma}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \text{ if and only if } \boldsymbol{\beta} \in \boldsymbol{\sigma}(\boldsymbol{\alpha}, \boldsymbol{\gamma}). \tag{A.1}$$

Write $\overline{C} = \{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_f\}$ and for $1 \le i,\ j \le f,\ i \ne j$, let $\boldsymbol{\psi}_{ij}$ and $\boldsymbol{\theta}_{ij}$ be the cardinalities of $\overline{C} \cap \boldsymbol{\sigma}(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j)$ and $C_0 \cap \boldsymbol{\sigma}(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j)$, respectively. Clearly,

$$\boldsymbol{\psi}_{ij} + \boldsymbol{\theta}_{ij} \le s - 1, \qquad \boldsymbol{\psi}_{ij} \le f - 2, \qquad \boldsymbol{\theta}_{ij} \le 1, \tag{A.2}$$

the last inequality being a consequence of the fact that C_0 is a flat.

Since C_0 is a flat, any linearly dependent triplet containing $\boldsymbol{\alpha}_i (1 \le i \le f)$ and two other distinct members of $C_0 \cup \overline{C}$ must contain at least one member of $\overline{C}$ other than $\boldsymbol{\alpha}_i$. Hence by (A.1) and the definition of $\boldsymbol{\phi}_i(C_0, C)$,

$$\boldsymbol{\phi}_i(C_0, C) = \sum_{j=1,\ j\neq i}^{f} \left(\tfrac{1}{2}\boldsymbol{\psi}_{ij} + \boldsymbol{\theta}_{ij}\right), \qquad 1 \le i \le f. \tag{A.3}$$

We now prove inequality (4.1) and study the situation of equality there, considering separately the three exclusive and exhaustive cases $2 \le f \le s$, $f > s^r$ and $s < f \le s^r$.

(a) Let $2 \le f \le s$. Then (4.1) reduces to $\boldsymbol{\phi};(C_0, C) \le \frac{1}{2} f(f-1)$, the validity of which follows from (A.3) and the last two inequalities in (A.2). Thus equality holds in (4.1) for all i if and only if

$$\boldsymbol{\psi}_{ij} = f - 2, \qquad \boldsymbol{\theta}_{ij} = 1 \quad \text{for all } 1 \le i,\ j \le f,\ i \ne j. \tag{A.4}$$

If (A.4) holds, then in particular,

$$\boldsymbol{\sigma}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) \supset [\overline{C} - \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\}] \cup \{\boldsymbol{\alpha}\},$$

for some $\boldsymbol{\alpha} \in C_0$, so that using (A.1) and recalling that two nontrivial $(n-k)\times 1$ vectors with proportional coordinates represent the same point in P, it follows that $\overline{C}$ must be given by (4.2) with $\boldsymbol{\beta}_0 = \boldsymbol{\alpha}_1$ and $\boldsymbol{\beta}_1 = \boldsymbol{\alpha}$. Conversely, it is easy to see that (4.2) implies (A.4). Thus equality is attained in (4.1) for all i if and only if (4.2) holds.

(b) Let $f > s^r$. Then for $1 \le i \le f$, the inequality in (4.1) reduces to

$$\boldsymbol{\phi}_i(C_0, C) \le \tfrac{1}{2}(s-1)(f + L_r - 1). \tag{A.5}$$

Now given any $\boldsymbol{\alpha}_i \in \overline{C}$ and $\boldsymbol{\alpha}(\ne \boldsymbol{\alpha}_i) \in C_0 \cup \overline{C}$, the set $\boldsymbol{\sigma}(\boldsymbol{\alpha}_i, \boldsymbol{\alpha})$ contains $s-1$ distinct points. Hence by (A.1), the validity of (A.5) is evident. Next suppose equality holds in (4.1) or, equivalently, in (A.5) for every i. Then for each $\boldsymbol{\alpha}_i \in \overline{C}$ and $\boldsymbol{\alpha}(\ne \boldsymbol{\alpha}_i) \in C_0 \cup \overline{C}$, we have $\boldsymbol{\sigma}(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}) \subset C_0 \cup \overline{C}$. Since C_0 itself is an $(r-1)$-flat, it follows that $C_0 \cup \overline{C}$ is a $(u-1)$-flat (so that $f + L_r = L_u$), where $r+2 \le u < n-k$; note that $u < n-k$ as $C_0 \cup \overline{C}$ is a proper subset of P, while $u > r+1$ as $f > s^r$. Conversely, if $C_0 \cup \overline{C}$ is a flat, then trivially equality holds in (A.5) and hence in (4.1) for each i.

(c) Now suppose $r \ge 2$ and $s < f \le s^r$. Then (4.1) reduces to $\boldsymbol{\phi}_i(C_0, C) \le \frac{1}{2}s(f-1)$, the validity of which follows from (A.3) and the first and third inequalities in (A.2), noting that

$$\boldsymbol{\phi}_i(C_0, C) = \tfrac{1}{2} \sum_{j=1,\ j\neq i}^{f} \{(\boldsymbol{\psi}_{ij} + \boldsymbol{\theta}_{ij}) + \boldsymbol{\theta}_{ij}\}.$$

Thus equality holds in (4.1) for each i if only if $\boldsymbol{\psi}_{ij} + \boldsymbol{\theta}_{ij} = s-1$ and $\boldsymbol{\theta}_{ij} = 1$, that is, if and only if

$$\boldsymbol{\psi}_{ij} = s - 2, \qquad \boldsymbol{\theta}_{ij} = 1 \quad \text{for all } 1 \le i,\ j \le f,\ i \ne j. \tag{A.6}$$

Suppose (A.6) holds. Then for any distinct $\boldsymbol{\alpha}_i,\ \boldsymbol{\alpha}_j \in \overline{C}$,

$$\boldsymbol{\alpha} \in \boldsymbol{\sigma}(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j), \qquad \boldsymbol{\alpha} \notin C_0 \Rightarrow \boldsymbol{\alpha} \in \overline{C}, \tag{A.7}$$

and $\boldsymbol{\sigma}(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j)$ contains some point of C_0. The latter fact implies that rank $[\mathbf{V}(C_0 \cup \overline{C})] = r+1$, for otherwise, there will exist linearly independent points $\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_r,\ \boldsymbol{\alpha}_i,\ \boldsymbol{\alpha}_j$ such that $\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_r$ span C_0 and $\boldsymbol{\alpha}_i,\ \boldsymbol{\alpha}_j \in \overline{C}$; then $\boldsymbol{\sigma}(\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j)$ will not contain any point of C_0. Hence $\overline{C}$ must be of the form

$$\overline{C} = \{\boldsymbol{\beta}_0, \boldsymbol{\beta}_0 + \boldsymbol{\rho}_1 \boldsymbol{\alpha}^{(1)}, \dots, \boldsymbol{\beta}_0 + \boldsymbol{\rho}_{f-1} \boldsymbol{\alpha}^{(f-1)}\}, \tag{A.8}$$

where $\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_{f-1}$ are nonzero elements of $GF(s)$, $\boldsymbol{\beta}_0 \in P - C_0$ and $\boldsymbol{\alpha}^{(1)}, \dots, \boldsymbol{\alpha}^{(f-1)} \in C_0$. We now consider the cases $s \geq 3$ and $s = 2$ separately.

(c1) Let $s \geq 3$. Suppose rank$(\boldsymbol{\alpha}^{(1)}, \dots, \boldsymbol{\alpha}^{(f-1)} = u$, where $2 \leq u \leq r$ (note that $u \geq 2$ as $f > s$). Without loss of generality, let the points $\boldsymbol{\alpha}^{(1)}, \dots, \boldsymbol{\alpha}^{(u)}$ be linearly independent, forming a basis of $\{\boldsymbol{\alpha}^{(1)}, \dots, \boldsymbol{\alpha}^{(f-1)}\}$. Then writing $\boldsymbol{\beta}_i = \boldsymbol{\alpha}^{(i)}$, $1 \leq i \leq u$, by (A.8),

$$\overline{C} \supset \{\boldsymbol{\beta}_0, \boldsymbol{\beta}_0 + \boldsymbol{\rho}_1 \boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_0 + \boldsymbol{\rho}_u \boldsymbol{\beta}_u\}, \tag{A.9}$$

$$\overline{C} \subset \left\{\boldsymbol{\beta}_0 + \sum_{i=1}^{u} \boldsymbol{\lambda}_i \boldsymbol{\beta}_i \colon \boldsymbol{\lambda}_i \in GF(s)\right\} \tag{A.10}$$

We note that for each i and each $\boldsymbol{\lambda}_i \in GF(s)$,

$$\boldsymbol{\beta}_0 + \boldsymbol{\lambda}_i \boldsymbol{\beta}_i \in \overline{C}. \tag{A.11}$$

If $\boldsymbol{\lambda}_i \in \{0, \boldsymbol{\rho}_i\}$, then (A.11) follows from (A.9). Even otherwise, (A.11) follows from (A.7) and (A.9), nothing that $\boldsymbol{\beta}_0 + \boldsymbol{\lambda}_i \boldsymbol{\beta}_i \in \boldsymbol{\sigma}(\boldsymbol{\beta}_0, \boldsymbol{\beta}_0 + \boldsymbol{\rho}_i \boldsymbol{\beta}_i)$ and $\boldsymbol{\beta}_0 + \boldsymbol{\lambda}_i \boldsymbol{\beta}_i \notin C_0$.

Since $s \geq 3$, there exist $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 (\neq 0) \in GF(s)$ such that $\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 = 1$. Hence for each $i,\ j (1 \leq i,\ j \leq u;\ i \neq j)$ and each $\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_j \in GF(s)$,

$$\boldsymbol{\beta}_0 + \boldsymbol{\lambda}_i \boldsymbol{\beta}_i + \boldsymbol{\lambda}_j \boldsymbol{\beta}_j = \boldsymbol{\mu}_1(\boldsymbol{\beta}_0 + \boldsymbol{\lambda}_i \boldsymbol{\mu}_1^{-1} \boldsymbol{\beta}_i) + \boldsymbol{\mu}_2(\boldsymbol{\beta}_0 + \boldsymbol{\lambda}_j \boldsymbol{\mu}_2^{-1} \boldsymbol{\beta}_j),$$

so that by (A.7) and (A.11),

$$\boldsymbol{\beta}_0 + \boldsymbol{\lambda}_i \boldsymbol{\beta}_i + \boldsymbol{\lambda}_j \boldsymbol{\beta}_j \in \overline{C}. \tag{A.12}$$

Similarly, for each distinct $i,\ j,\ w (1 \leq i,\ j, w \leq u)$ and each $\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_j, \boldsymbol{\lambda}_w \in GF(s)$,

$$\begin{aligned} \boldsymbol{\beta}_0 + \boldsymbol{\lambda}_i \boldsymbol{\beta}_i + \boldsymbol{\lambda}_j \boldsymbol{\beta}_j + \boldsymbol{\lambda}_w \boldsymbol{\beta}_w = {} & \boldsymbol{\mu}_1(\boldsymbol{\beta}_0 + \boldsymbol{\lambda}_i \boldsymbol{\mu}_1^{-1} \boldsymbol{\beta}_i + \boldsymbol{\lambda}_j \boldsymbol{\mu}_1^{-1} \boldsymbol{\beta}_j) \\ & + \boldsymbol{\mu}_2(\boldsymbol{\beta}_0 + \boldsymbol{\lambda}_w \boldsymbol{\mu}_2^{-1} \boldsymbol{\beta}_w), \end{aligned}$$

and by (A.7), (A.11) and (A.12), $\boldsymbol{\beta}_0 + \boldsymbol{\lambda}_i \boldsymbol{\beta}_i + \boldsymbol{\lambda}_j \boldsymbol{\beta}_j + \boldsymbol{\lambda}_w \boldsymbol{\beta}_w \in \overline{C}$. Proceeding in this manner, it is seen that the reverse set inequality also holds in (A.10); that is, $\overline{C}$ is given by (4.3) in which case $f = s^u$. Conversely, (4.3) implies (A.6) and hence leads to the attainment of equality in (4.1) for each i.

(c2) Let $s = 2$. Then in (A.8), $\boldsymbol{\rho}_1 = \cdots = \boldsymbol{\rho}_{f-1} = 1$ and hence $\boldsymbol{\alpha}^{(1)}, \dots, \boldsymbol{\alpha}^{(f-1)}$ are distinct so that $\overline{C}$ is as given by (4.4). Conversely, if (4.4) holds, then (A.6) also holds and hence equality is attained in (4.1) for every i. □

PROOF OF LEMMA 5.1. (a) Inequality (5.2) follows trivially from (5.1). If possible, suppose equality is attained in (5.2) for two distinct choices of i, say $i = 1$ and 2, without loss of generality. Then there are $2^{u-1} - 2$ lines which pass through $\boldsymbol{\pi}_1$ (or $\boldsymbol{\pi}_2$) and jointly cover all the remaining points of G. Hence

$$\boldsymbol{\pi}_1 + \boldsymbol{\pi}_i \in G \quad \text{for each } \boldsymbol{\pi}_i (\neq \boldsymbol{\pi}_1) \in G, \tag{A.13}$$

$$\boldsymbol{\pi}_2 + \boldsymbol{\pi}_i \in G \quad \text{for each } \boldsymbol{\pi}_i (\neq \boldsymbol{\pi}_2) \in G. \tag{A.14}$$

By (A.13) and (A.14); $\boldsymbol{\pi}_1 + \boldsymbol{\pi}_2 \in G$ and

$$\boldsymbol{\pi}_1 + \boldsymbol{\pi}_2 + \boldsymbol{\pi}_i \in G \quad \text{for each } \boldsymbol{\pi}_i (\neq \boldsymbol{\pi}_1 + \boldsymbol{\pi}_2) \in G. \tag{A.15}$$

Let $G^* = G - \{\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \boldsymbol{\pi}_1 + \boldsymbol{\pi}_2\}$. Any two points of G^*, say $\boldsymbol{\pi}_i$ and $\boldsymbol{\pi}_{i'}$, which are not necessarily distinct, will be said to be associated with each other (written $\boldsymbol{\pi}_i \wedge \boldsymbol{\pi}_{i'}$) if $\boldsymbol{\pi}_i + \boldsymbol{\pi}_{i'} \in \{0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \boldsymbol{\pi}_1 + \boldsymbol{\pi}_2\}$. By (A.13), (A.14) and (A.15), each point of G^* is associated with four distinct points of G^* including itself. Also, it is easy to recognize that $\wedge$ is an equivalence relation. Thus it partitions G^* into equivalence classes of size 4. This is, however, impossible as the cardinality of G^*, namely $2^u - 6 (= p - 3)$ is not an integral multiple of 4. Hence part (a) is proved.

(b) Since $a_i(G)$ is an integer for each i, the validity of (5.3) is obvious from (5.1). Let $G_0 = \{\boldsymbol{\pi}_w \colon 1 \le w \le p,\ a_w(G) = 2^{u-1} - 3\}$. If G_0 is empty or singleton, then there is nothing more to prove. Consider, therefore, the situation where the cardinality of G_0 is at least two. Without loss of generality, let $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2 \in G_0$. If possible, suppose $\boldsymbol{\pi}_1 + \boldsymbol{\pi}_2 \notin G$. Define $G_1 = G \cup \{\boldsymbol{\pi}_1 + \boldsymbol{\pi}_2\}$. Then

$$a_1(G_1) = a_2(G_1) = 2^{u-1} - 2, \tag{A.16}$$

since, in addition to the $2^{u-1} - 3$ lines of $G (\subset G_1)$ that pass through $\boldsymbol{\pi}_1$(or $\boldsymbol{\pi}_2$), we now have the extra line $\{\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \boldsymbol{\pi}_1 + \boldsymbol{\pi}_2\}$ of G_1. However, (A.16) is impossible by part (a) of this lemma since G_1 has cardinality $2^u - 3$. Hence

$$\boldsymbol{\pi}_1 + \boldsymbol{\pi}_2 \in G. \tag{A.17}$$

For ease in presentation, we split the rest of the proof into three steps.

STEP 1. Since $\boldsymbol{\pi}_1$(or $\boldsymbol{\pi}_2$) $\in G_0$, there are $2^{u-1} - 3$ lines which pass through $\boldsymbol{\pi}_1$(or $\boldsymbol{\pi}_2$) and jointly cover all but one of the remaining points of G. Hence there exist $\boldsymbol{\pi}_{j_1}, \boldsymbol{\pi}_{j_2} \in G$ such that

$$\boldsymbol{\pi}_1 \neq \boldsymbol{\pi}_{j1}, \boldsymbol{\pi}_2 \neq \boldsymbol{\pi}_{j2},\ \boldsymbol{\pi}_1 + \boldsymbol{\pi}_{j1} \notin G,\ \boldsymbol{\pi}_2 + \boldsymbol{\pi}_{j2} \notin G, \tag{A.18}$$

$$\boldsymbol{\pi}_1 + \boldsymbol{\pi}_i \in G \quad \text{for each } \boldsymbol{\pi}_i (\neq \boldsymbol{\pi}_1, \boldsymbol{\pi}_{j1}) \in G, \tag{A.19}$$

$$\boldsymbol{\pi}_2 + \boldsymbol{\pi}_i \in G \quad \text{for each } \boldsymbol{\pi}_i (\neq \boldsymbol{\pi}_2, \boldsymbol{\pi}_{j2}) \in G. \tag{A.20}$$

If possible, suppose $\boldsymbol{\pi}_{j1} \neq \boldsymbol{\pi}_{j2}$. Since $\boldsymbol{\pi}_{j1} \neq \boldsymbol{\pi}_2$ [by (A.17) and (A.18)], by (A.20), $\boldsymbol{\pi}_2 + \boldsymbol{\pi}_{j1} \in G$. From (A.18), $\boldsymbol{\pi}_2 + \boldsymbol{\pi}_{j1} \neq \boldsymbol{\pi}_1, \boldsymbol{\pi}_{j1}$. Hence, by (A.19), $\boldsymbol{\pi}_1 + \boldsymbol{\pi}_2 + \boldsymbol{\pi}_{j1} (= \boldsymbol{\pi}, \text{ say}) \in G$. Now by (A.18), $\boldsymbol{\pi} \neq \boldsymbol{\pi}_2$ and $\boldsymbol{\pi}_2 + \boldsymbol{\pi} \notin G$.

Therefore by (A.20), $\boldsymbol{\pi} = \boldsymbol{\pi}_{j2}$, that is, $\boldsymbol{\pi}_1 + \boldsymbol{\pi}_{j1} = \boldsymbol{\pi}_2 + \boldsymbol{\pi}_{j2} = \boldsymbol{\pi}^*$ (say), where $\boldsymbol{\pi}^* \notin G$. Redefine the set G_1, with cardinality $2^u - 3$, as $G_1 = G \cup \{\boldsymbol{\pi}^*\}$. Then as before, (A.16) holds, which is impossible. Hence $\boldsymbol{\pi}_{j1} = \boldsymbol{\pi}_{j2} = \boldsymbol{\pi}_j$ (say), where $\boldsymbol{\pi}_j \in G$. In fact, $\boldsymbol{\pi}_j \in G - G_0$ since by (A.18), $\boldsymbol{\pi}_j \neq \boldsymbol{\pi}_w$ and $\boldsymbol{\pi}_w + \boldsymbol{\pi}_j \notin G$ for $w = 1, 2$.

Using the same argument for any pair of distinct members of G_0, it follows that there exists unique $\boldsymbol{\pi}_j \in G - G_0$ such that for each $\boldsymbol{\pi}_w \in G_0$,

$$\boldsymbol{\pi}_w + \boldsymbol{\pi}_j \notin G \tag{A.21}$$

and

$$\boldsymbol{\pi}_w + \boldsymbol{\pi}_i \notin G \text{ whenever } \boldsymbol{\pi}_i \in G \quad \text{and} \quad \boldsymbol{\pi}_i \neq \boldsymbol{\pi}_w, \boldsymbol{\pi}_j. \tag{A.22}$$

STEP 2. We now show that the set G_0 is closed under the addition of distinct elements. Without loss of generality, it will be enough to show that $\boldsymbol{\pi}_1 + \boldsymbol{\pi}_2 \in G_0$, that is,

$$\boldsymbol{\pi}_1 + \boldsymbol{\pi}_2 + \boldsymbol{\pi}_i \in G \quad \text{for every } \boldsymbol{\pi}_i (\neq \boldsymbol{\pi}_1 + \boldsymbol{\pi}_2, \boldsymbol{\pi}_j) \in G, \tag{A.23}$$

in view of (A.21) and (A.22). If $\boldsymbol{\pi}_i$ equals $\boldsymbol{\pi}_1$ or $\boldsymbol{\pi}_2$, then (A.23) holds trivially. Suppose $\boldsymbol{\pi}_i$ is different from each of $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \boldsymbol{\pi}_1 + \boldsymbol{\pi}_2$ and $\boldsymbol{\pi}_j$. Then by (A.22), $\boldsymbol{\pi}_1 + \boldsymbol{\pi}_i \in G$. Also, $\boldsymbol{\pi}_1 + \boldsymbol{\pi}_i \neq \boldsymbol{\pi}_2$ and by (A.21), $\boldsymbol{\pi}_1 + \boldsymbol{\pi}_i \neq \boldsymbol{\pi}_j$. Hence by (A.22), $\boldsymbol{\pi}_2 + \boldsymbol{\pi}_1 + \boldsymbol{\pi}_i \in G$; that is, (A.23) holds again.

STEP 3. If possible, let the cardinality of G_0 exceed three. Then, by Step 2, this cardinality equals $2^z - 1$ where $z (\geq 3)$ is an integer and $u \geq z + 1$ as $G_0 \subset G$. Let $G^* = G - [G_0 \cup \{\boldsymbol{\pi}_j\}]$. By (A.21), (A.22) and the closure property of G_0 as noted in Step 2,

$$\boldsymbol{\pi}_w + \boldsymbol{\pi}_i \in G^* \text{ whenever } \boldsymbol{\pi}_i \in G^* \quad \text{and} \quad \boldsymbol{\pi}_w \in G_0. \tag{A.24}$$

For any two points of G^*, say $\boldsymbol{\pi}_i$ and $\boldsymbol{\pi}_i'$, which are not necessarily distinct, we write $\boldsymbol{\pi}_i \wedge \boldsymbol{\pi}_{i'}$ if $\boldsymbol{\pi}_i + \boldsymbol{\pi}_{i'} \in G_0 \cup \{\mathbf{0}\}$. Then, as in the proof of part (a), by (A.24), the relation $\wedge$ partitions G^* into equivalence classes of size 2^z. This is, however, impossible since for $u \geq z + 1$ and $z \geq 3$, the cardinality of G^*, namely $2^u - 2^z - 4$, is not an integral multiple of 2^z. This completes the proof of part (b).

PROOF OF THEOREM 5.2. Here $f + L_r = 2^u$ and for any design $d(C_0, C)$, noting that $\boldsymbol{\phi}_i(C_0, C)$ is an integer, by (5.1) and (5.7),

$$\boldsymbol{\phi}_i(C_0, C) \leq 2^{u-1} - 1, \qquad 1 \leq i \leq f. \tag{A.25}$$

If. Suppose $\overline{C}$ is a $(u-1)$-flat. Then it is not hard to see that $V(C)$ has full row rank and that equality holds in (A.25) for each i. Hence by Lemma 3.1(c), the design $d(C_0, C)$, has maximum estimation capacity.

Only if. this is trivially true for $u = 2$. Consider, therefore, the case $3 \leq u < n - k$ and suppose the design $d(C_0, C)$ has maximum estimation capacity. Then by Lemma 3.1(c) and the "if" part of this theorem, equality holds in

(A.25) for all i. Hence writing $\overline{C} = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_f\}$, $C_0 = \{\boldsymbol{\alpha}_{f+1}\}$ and $G = C_0 \cup \overline{C}$, we have

$$a_i(G) = \boldsymbol{\phi}_i(C_0, C) = 2^{u-1} - 1, \qquad 1 \le i \le f. \tag{A.26}$$

Also, it is easy to see that

$$a_i(G) = a_i(\overline{C}) + \boldsymbol{\delta}_i, \qquad 1 \le i \le f. \tag{A.27}$$

$$a_{f+1}(G) = \tfrac{1}{2} \sum_{i=1}^{f} \boldsymbol{\delta}_i, \tag{A.28}$$

where, for $1 \le i \le f$, the indicator $\boldsymbol{\delta}_i$ equals 1 if there is a line passing through $\boldsymbol{\alpha}_{f+1}, \boldsymbol{\alpha}_i$ and another point of $\overline{C}$, and 0 otherwise.

If possible, suppose $\boldsymbol{\delta}_i = 1$ for some i. Then there is a line passing through $\boldsymbol{\alpha}_{f+1}$ and two distinct points, say $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$, of $\overline{C}$. Consequently,

$$\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_{f+1}. \tag{A.29}$$

By (A.28), $\boldsymbol{\delta}_i = 0$ for some i, otherwise $a_{f+1}(G) = \frac{1}{2}f$, which is not an integer. Since $\boldsymbol{\delta}_1 = \boldsymbol{\delta}_2 = 1$ [see (A.29)], without loss of generality, let $\boldsymbol{\delta}_3 = 0$. Then by (A.26) and (A.27), $a_3(\overline{C}) = 2^{u-1} - 1$ and, as $\overline{C}$ has cardinality $2^u - 1$, it follows that $\boldsymbol{\alpha}_3 + \boldsymbol{\alpha}_i \in \overline{C}$, $1 \le i (\ne 3) \le f$. In particular, the points $\boldsymbol{\alpha}_3 + \boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_3 + \boldsymbol{\alpha}_2$, which are evidently distinct and each different from $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ or $\boldsymbol{\alpha}_3$ [see (A.29)], belong to $\overline{C}$. Without loss of generality, let

$$\boldsymbol{\alpha}_3 + \boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_4, \qquad \boldsymbol{\alpha}_3 + \boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_5. \tag{A.30}$$

By (A.29), (A.30), $\boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_4 = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_5 = \boldsymbol{\alpha}_3 + \boldsymbol{\alpha}_{f+1}$ and, as $\boldsymbol{\delta}_3 = 0$, we have

$$\boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_4 \notin G, \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_5 \notin G. \tag{A.31}$$

Also, by (A.29) and (A.30),

$$\boldsymbol{\alpha}_4 + \boldsymbol{\alpha}_5 = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_{f+1}. \tag{A.32}$$

Now, let $G_1 = G - \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_4, \boldsymbol{\alpha}_{f+1}\}$. Then by (A.31) and (A.32), among the $2^{u-1} - 1$ lines that pass through $\boldsymbol{\alpha}_2$ and are contained in G [see (A.26)], only one, namely $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_{f+1}\}$ ceases to remain in G_1. Thus there are $2^{u-1} - 2$ lines passing through $\boldsymbol{\alpha}_2$ and two other distinct points of G_1. A similar argument is applicable to the point $\boldsymbol{\alpha}_5 (\in G_1)$. Hence the set G_1, of cardinality $2^u - 3$, contains two distinct points $\boldsymbol{\alpha}_2$ and $\boldsymbol{\alpha}_5$ through either of which pass $2^{u-1} - 2$ lines that are contained in G_1. However, this is impossible by Lemma 5.1(a).

Consequently, $\boldsymbol{\delta}_i = 0, 1 \le i \le f$. Therefore, by (A.26) and (A.27), $a_i(\overline{C}) = 2^{u-1} - 1, 1 \le i \le f$. As $\overline{C}$ has cardinality $2^u - 1$, it follows that $\overline{C}$ must be a $(u-1)$-flat. □

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