BLOCKING SETS IN DESARGUESIAN PROJECTIVE PLANES

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Abstract

Using theorems of Redéi, and of Brouwer and Schrijver, and Jamison, it is proved that a non-trivial blocking set in a desarguesian projective plane of order q has at least $q+\surd(2q)+1$ points, if q is at least 7, odd and not a square and $q \neq 27$. Further one can show that non-trivial blocking sets in the desarguesian planes PG(2, 11) and PG(2, 13) have at least 18 resp. 21 points, and this is best possible. In addition a nice description of a blocking set of size $q^t+q^{t-1}+1$ in the desarguesian plane $PG(2, q^t)$ is given, where q is some prime power.

Introduction

A blocking set in a linear space is a set S of points, such that each line intersects S in at least one point. S is called *non-trivial*, if no line is completely contained in S, in the case of a projective plane. In this note we want to derive lower bounds for the cardinality of S.

Two useful theorems

The following construction yields interesting blocking sets in the desarguesian plane $PG(2,q)$:

Let $f: GF(q) \to GF(q)$ be any non-linear function. Form a blocking set consisting of

(i) the q points forming the graph of f in $AG(2,q)$,

(ii) the directions determined by f on the line at infinity, say m points.

EXAMPLE 1. Let $q = p$ be a prime, $f(x) = x^{\frac{1}{2}(p+1)}$. This yields a blocking set of $\frac{3}{2}(p+1)$ points, which is conjectured to be best possible ([7], see also [6]).

2. Let $q = q_1^t$ $(t > 1)$, then $GF(q_1)$ is a subfield of $GF(q)$. Let f be the trace map from $GF(q)$ to $GF(q_1)$. Then $S = q+q/q_1+1$. This is also the best known, if q_1 is chosen maximal (compare [**2**, **4**]).

The following theorem gives lower bounds for m, where $q = p^n$, p prime.

THEOREM. ([**8**, p. 237], see also [**6**]).

$$m \geqslant \frac{q-1}{p^{\frac{1}{2}n}+1}+1, \quad \textit{and} \quad m \geqslant \frac{p+3}{2} \quad \textit{if } n=1.$$

COROLLARY. *Let X be a set of q points in the desarguesian affine plane of order q, determining m directions. Then m satisfies the above inequalities.*

Proof. Either X determines all directions, or there is a parallel class all of whose lines contain exactly 1 point of X.

Received 1 June 1984.

1980 *Mathematics Subject Classification* 05B25.

Let S be a minimal blocking set; then each point of S is on at least one tangent. Let $p \in S$ be a point on t tangents, call one tangent l, and form a blocking set of $\mathrm{AG}(2,q) = \mathrm{PG}(2,q) \backslash l$ with $|S|-1+t-1$ points in the obvious way.

THEOREM ([1, 5]). *A blocking set of a desarguesian affine plane* $\mathrm{AG}(2,q)$ *has at least* $2q-1$ *points.*

As a consequence of this, one has that $t \geqslant 2q+1-|S|$ for each point in S. Using these two results it is now a trivial exercise to show that a blocking set in the desarguesian plane PG(2, 11) has at last 18 points, and a rather tedious one to prove $|S| \geqslant 21$ for PG(2, 13).

Blocking sets in the desarguesian plane PG(2, *q*)

It is well known, and due to Bruen [2], that $|S| \geqslant q+\sqrt{q}+1$, with equality if and only if q is a square and S a Baer-subplane. When q is not a square this bound can be improved.

Let S be a blocking set of size $|S| = q+m$. If S contains an m-secant the corollary gives a lower bound for the cardinality of S. The next theorem treats the remaining case.

THEOREM. *Let S be a blocking set of size* $q+m$ *without an m-secant. Then*

$$|S| \geqslant q+\sqrt{(2q)}+1.$$

Proof. Since each line contains at most $m-1$ points of S, it follows that each point is on at most $q-1$ tangents. Counting incident pairs (tangent, point not in S) in two ways, using the second theorem, one gets

$$q(q+m)(q-m+1) \leqslant (q^2-m+1)(q-1)$$

or, rewriting,

$$2q \leqslant (m-1)^2+(m-1)/q, \quad \text{whence } m \geqslant \sqrt{(2q)}+1.$$

COROLLARY. *Suppose q is odd, not a square, at least 7 and not 27. Let S be an arbitrary non-trivial blocking set of the desarguesian plane* PG(2, *q*). *Then* $|S| \leqslant q+\sqrt{(2q)}+1$.

Final remarks

If $q < 7$ everything is known; if $q = 27$ we only get $|S| \geqslant 35$; if $q = 2^{2t+1}$ one obtains $|S| \geqslant 2^{2t+1}+2^{t+1}$.

The first paper relating Redéi's theorem to blocking sets seems to be [3]. We wonder whether Redéi's theorem can be improved to $m \geqslant 1+q/q_1$, where q_1 is the order of maximal subfield of GF(q).

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