

# Blow-up of regular submanifolds in Heisenberg groups and applications

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## Abstract

We obtain a blow-up theorem for regular submanifolds in the Heisenberg group, where intrinsic dilations are used. Main consequence of this result is an explicit formula for the density of  $(p+1)$ -dimensional spherical Hausdorff measure restricted to a  $p$ -dimensional submanifold with respect to the Riemannian surface measure. We explicitly compute this formula in some simple examples and we present a lower semicontinuity result for the spherical Hausdorff measure with respect to the weak convergence of currents. Another application is the proof of an intrinsic coarea formula for vector-valued mappings on the Heisenberg group.

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# 1 Introduction

In recent years, several efforts have been devoted to the project of developing Analysis and Geometry in stratified groups and more general Carnot-Carathéodory spaces with several monographs and surveys on this subject. Among them we mention [3], [6], [13], [16], [23], [28], but this list could be surely enlarged.

Our study fits into the recent project of developing Geometric Measure Theory in these spaces. Ambient of our investigations is the  $(2n+1)$ -dimensional Heisenberg group  $\mathbb{H}^n$ , which represents the simplest model of non-Abelian stratified group, [6], [27]. Aim of this paper is to present an intrinsic blow-up theorem for  $C^1$  submanifolds in the geometry of the Heisenberg group along with its applications. The main feature of this procedure is the use of natural dilations of the group, namely, a one-parameter family of group homomorphisms that are homogeneous with respect to the distance of the group. Recall that dilations in  $\mathbb{H}^n$  are anisotropic, hence they differently act on different directions of the submanifold. The foremost directions are the so-called horizontal directions, that determine the “sub-Riemannian geometry” of the Heisenberg group: at any point  $x \in \mathbb{H}^n$  a  $2n$ -dimensional subspace  $H_x\mathbb{H}^n \subset T_x\mathbb{H}^{2n+1}$  is given and the family of all horizontal spaces  $H_x\mathbb{H}^n$  forms the so-called horizontal subbundle  $H\mathbb{H}^n$ . We will defer full definitions to Section 2.

The blow-up procedure consists in enlarging the submanifold  $\Sigma$  at some point  $x \in \Sigma$  by intrinsic dilations and taking the intersection of the magnified submanifold with a bounded set centered at  $x$ . We are interested in studying the case when  $T_x\Sigma \not\subset H_x\mathbb{H}^n$ , namely,  $x$  is a transverse point. The effect of rescaling the submanifold at a transverse point  $x$  can be obtained by considering the behavior of  $\text{vol}_p(B_{x,r} \cap \Sigma)/r^{p+1}$  as  $r \rightarrow 0^+$ , that heuristically is

$$\frac{\text{vol}_p(B_{x,r} \cap \Sigma)}{r^{p+1}} = \frac{\text{vol}_p(l_x \delta_r(B_1 \cap \Sigma_{x,r}))}{r^{p+1}} = \frac{\text{vol}_p(\delta_r(B_1 \cap \Sigma_{x,r}))}{r^{p+1}} \approx \alpha(x) \text{vol}_p(B_1 \cap \Sigma_{x,r}).$$

Here  $\text{vol}_p$  denotes the  $p$ -dimensional Riemannian measure restricted to  $\Sigma$ , the left translation  $l_x : \mathbb{H}^n \rightarrow \mathbb{H}^n$  is given by  $l_x(y) = x \cdot y$ , the dilation of factor  $r > 0$  is  $\delta_r : \mathbb{H}^n \rightarrow \mathbb{H}^n$ , the dilated submanifold at  $x$  is  $\Sigma_{x,r} = \delta_{1/r}(l_{x^{-1}}\Sigma)$  and  $B_{x,r}$  is the open ball of center  $x$  and radius  $r$  with respect to a fixed homogeneous distance. The meaning of  $\alpha(x)$  will be clear in the following theorem, that makes rigorous our previous consideration and represents our first main result.

**Theorem 1.1 (Blow-up)** *Let  $\Sigma$  be a  $p$ -dimensional  $C^1$  submanifold of  $\Omega$ , where  $\Omega$  is an open subset of  $\mathbb{H}^n$  and let  $x$  be a transverse point. Then the following limit holds*

$$\lim_{r \rightarrow 0^+} \frac{\text{vol}_p(\Sigma \cap B_{x,r})}{r^{p+1}} = \frac{\theta_p^p(\tau_{\Sigma, \nu}(x))}{|\tau_{\Sigma, \nu}(x)|}. \quad (1)$$

A novel object appearing in this limit is the vertical tangent  $p$ -vector  $\tau_{\Sigma, \nu}(x)$ , introduced in Definition 2.13. Its associated  $p$ -dimensional subspace of  $\mathfrak{h}^n$  is a subalgebra

whose image through the exponential map represents the blow-up limit of the rescaled submanifold  $\Sigma_{x,r}$  as  $r \rightarrow 0^+$ . The  $p$ -vector  $\tau_{\Sigma,\nu}(x)$  in higher codimension plays the same role that the well known horizontal normal  $\nu_H$  plays in codimension one (compare for instance with [19]). The metric factor  $\theta(\tau_{\Sigma,\nu}(x))$ , introduced in [19], corresponds to the measure of the intersection of  $B_1$  with the vertical subspace associated to the vertical tangent  $p$ -vector  $\tau_{\Sigma,\nu}(x)$ . A first consequence of Theorem 1.1 is an explicit formula to compute the  $(p+1)$ -dimensional spherical Hausdorff measure of  $p$ -dimensional  $C^1$  submanifolds in the Heisenberg group. In fact, thanks to  $\mathcal{S}^{p+1}$ -negligibility of characteristic points proved in [22], Theorem 1.1 along with standard theorems on differentiation of measures, immediately give the following result.

**Theorem 1.2** *Let  $\rho$  be a homogeneous distance with constant metric factor  $\alpha > 0$  and let  $\mathcal{S}_{\mathbb{H}^n}^{p+1} = \alpha \mathcal{S}_{\rho}^{p+1}$ . Then we have*

$$\mathcal{S}_{\mathbb{H}^n}^{p+1}(\Sigma) = \int_{\Sigma} |\tau_{\Sigma,\nu}(x)| d\text{vol}_p(x). \quad (2)$$

Note that in codimension one, the integral formula (2) fits into the results of [19] in stratified groups. The connection between these results is shown in Proposition 4.15. There are several examples of homogeneous distances satisfying hypothesis of Theorem 1.2, as we show in Example 4.6. Proposition 4.5 shows a class of homogeneous distances having constant metric factor. Proposition 4.7 shows how the computation of the  $(p+1)$ -dimensional spherical Hausdorff measure of a submanifold can be easily performed in several examples, that will appear in Section 4.

Another consequence of Theorem 1.1 is the validity of an intrinsic coarea formula for vector-valued Lipschitz mappings defined on the Heisenberg group. By Sard theorem and the classical Whitney approximation theorem we can assume that a.e. level set is a submanifold of class  $C^1$ , then we apply representation formula (2). The core of the proof stands in the key relation

$$|\tau_{\Sigma,\nu}(x)| = \frac{J_H f(x)}{J_g f(x)}, \quad (3)$$

which surprisingly connects vertical tangent  $p$ -vector with horizontal jacobian  $J_H f$ . The proof of (3) is given in Theorem 3.3. Thus, we can establish the following result.

**Theorem 1.3 (Coarea formula)** *Let  $f : A \rightarrow \mathbb{R}^k$  be a Riemannian Lipschitz map, where  $A \subset \mathbb{H}^n$  is a measurable subset and  $1 \leq k < 2n + 1$ . Let  $\rho$  be a homogeneous distance with constant metric factor  $\alpha > 0$ . Then for every measurable function  $u : A \rightarrow [0, +\infty]$  the formula*

$$\int_A u(x) J_H f(x) dx = \int_{\mathbb{R}^k} \left( \int_{f^{-1}(t) \cap A} u(y) d\mathcal{S}_{\mathbb{H}^n}^{p+1}(y) \right) dt \quad (4)$$

*holds, where  $p = 2n + 1 - k$  and  $\mathcal{S}_{\mathbb{H}^n}^{p+1} = \alpha \mathcal{S}_{\rho}^{p+1}$ .*

This coarea formula along with that of [21], which is a particular case, represent first examples of intrinsic coarea formulae for vector valued mappings defined on non-Abelian Carnot groups. It remains an interesting open question the extension of coarea formula to Lipschitz mappings with respect to a homogeneous distance. Only in the case of real-valued mappings this problem has been settled in [20]. This question is intimately related to a blow-up theorem of “intrinsically regular” submanifolds. In this connection, we mention a recent work by Franchi, Serapioni and Serra Cassano [10], where a notion of intrinsic submanifold in  $\mathbb{H}^n$  has been introduced in arbitrary codimension. According to their terminology, a  $k$ -codimensional  $\mathbb{H}$ -regular submanifold for algebraic reasons must satisfy  $1 \leq k \leq n$ . With this restriction it might be highly irregular, even unrectifiable in the Euclidean sense, [14]. Nevertheless they show that an area-type formula for its  $(p+1)$ -dimensional spherical Hausdorff measure still holds. Here we wish to emphasize the difference in our approach, where we consider  $C^1$  submanifolds, but with no restriction on their codimension.

Let us summarize the contents of the present paper. Section 2 recalls some notions. Section 3 is devoted to the proof of Theorem 1.1. In Section 4 we show the validity of Theorem 1.2, along with its applications. In Proposition 4.5, we single out a privileged class of homogeneous distances having constant metric factor. We present several explicit computations of  $(p+1)$ -dimensional spherical Hausdorff measure in concrete examples. As another application of Theorem 1.2, we show a lower semicontinuity result for the spherical Hausdorff measure with respect to weak convergence of regular currents. Section 5 establishes an intrinsic coarea formula for vector-valued Riemannian Lipschitz mappings on the Heisenberg group.

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## 2 Some basic notions

The  $(2n+1)$ -dimensional Heisenberg group  $\mathbb{H}^n$  is a simply connected Lie group whose Lie algebra  $\mathfrak{h}^n$  is equipped with a basis  $(X_1, \dots, X_{2n}, Z)$  satisfying the bracket relations

$$[X_k, X_{k+n}] = 2Z \tag{5}$$

for every  $k = 1, \dots, n$ . We will identify the Lie algebra  $\mathfrak{h}^n$  with the isomorphic Lie algebra of left invariant vector fields on  $\mathbb{H}^n$ , so that any  $X_j$  also denotes a left invariant vector field of  $\mathbb{H}^n$ . In the terminology of Differential Geometry, the basis  $(X_1, \dots, X_{2n}, Z)$  forms a moving frame in  $\mathbb{H}^n$ . We will say that  $(X_1, \dots, X_{2n}, Z)$  is our *standard frame*. In particular,  $(X_1, \dots, X_{2n})$  is a *horizontal frame* and it spans a smooth distribution of  $2n$ -dimensional hyperplanes, called *horizontal hyperplanes* and denoted by  $H_x\mathbb{H}^n$  for every  $x \in \mathbb{H}^n$ . The collection of all horizontal hyperplanes forms the so called *horizontal subbundle*, denoted by  $H\mathbb{H}^n$ . In the sequel, we

will fix the unique left invariant Riemannian metric  $g$  such that the standard frame  $(X_1, X_2, \dots, X_{2n}, Z)$  forms an orthonormal basis at each point.

**Definition 2.1** Every set of left invariant vector fields  $(Y_1, \dots, Y_{2n})$  spanning the horizontal hyperplane at the unit element of  $\mathbb{H}^n$  will be called *horizontal frame*.

Recall that the exponential map  $\exp : \mathfrak{h}^n \rightarrow \mathbb{H}^n$  is a diffeomorphism, then it is possible to introduce a system of coordinates in all of  $\mathbb{H}^n$ .

**Definition 2.2 (Graded coordinates)** Let  $(Y_1, \dots, Y_{2n})$  be a horizontal frame and let  $W$  be a non horizontal left invariant vector field. The frame  $(Y_1, \dots, Y_{2n}, W)$  defines a coordinate chart  $F : \mathbb{R}^{2n+1} \rightarrow \mathbb{H}^n$  given by

$$F(y) = \exp \left( y_{2n+1} W + \sum_{j=1}^{2n} y_j Y_j \right). \quad (6)$$

Coordinates defined by (6) are called *graded coordinates* in the case  $W = Z$  and *standard coordinates* in the case the standard frame  $(X_1, \dots, X_{2n}, Z)$  is used. In general we will say that the coordinates are associated to the frame  $(Y_1, \dots, Y_{2n}, W)$

We will assume throughout that a system of standard coordinate is fixed, if not stated otherwise.

**Remark 2.3** Note that the horizontal frame  $(Y_1, \dots, Y_{2n})$  of Definition 2.2 may not satisfy relations (5), where  $X_i$  are replaced by  $Y_i$ .

The standard frame with respect to standard coordinates reads as follows

$$\tilde{X}_k = \partial_{x_k} - x_{k+n} \partial_{x_{2n+1}}, \quad \tilde{X}_{k+n} = \partial_{x_{k+n}} + x_k \partial_{x_{2n+1}} \quad \text{and} \quad \tilde{Z} = \partial_{x_{2n+1}} \quad (7)$$

and the group operation is given by the following formula

$$x \cdot y = \left( x_1 + y_1, \dots, x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + \sum_{j=1}^n (x_j y_{j+n} - x_{j+n} y_j) \right). \quad (8)$$

A natural family of dilations which respects the group operation (8) can be defined as follows

$$\delta_r(x) = (rx_1, rx_2, \dots, rx_{2n}, r^2 x_{2n+1}) \quad (9)$$

for every  $r > 0$ . In fact, the map  $\delta_r : \mathbb{H}^n \rightarrow \mathbb{H}^n$  defined above is a group homomorphism with respect to the operation (8).

In contrast with Analysis in Euclidean spaces, where the Euclidean distance is the most natural choice, in the Heisenberg group several distances have been introduced for different purposes. However, all of them are *homogeneous* in the following sense. If  $\rho : \mathbb{H}^n \times \mathbb{H}^n \rightarrow [0, +\infty + [$  is a homogeneous distance, then

1.  $\rho$  is a continuous with respect to the topology of  $\mathbb{H}^n$ ,
2.  $\rho(xy, xz) = \rho(y, z)$  for every  $x, y, z \in \mathbb{H}^n$ ,
3.  $\rho(\delta_r y, \delta_r z) = r \rho(y, z)$  for every  $y, z \in \mathbb{H}^n$  and every  $r > 0$ .

To simplify notations we write  $\rho(x, 0) = \rho(x)$ , where  $0$  denotes either the origin of  $\mathbb{R}^{2n+1}$  or the unit element of  $\mathbb{H}^n$ . The open ball of center  $x$  and radius  $r > 0$  with respect to a homogeneous distance is denoted by  $B_{x,r}$ . The Carnot-Carathéodory distance is an important example of homogeneous distance, [11]. However, all of our computations hold for a general homogeneous distance, therefore in the sequel  $\rho$  will denote a homogeneous distance, if not stated otherwise. Note that the Hausdorff dimension of  $\mathbb{H}^n$  with respect to any homogeneous distance is  $2n + 2$ . Next, we recall the notion of Riemannian jacobian.

**Definition 2.4 (Riemannian jacobian)** Let  $f : M \rightarrow N$  be a  $C^1$  smooth mapping of Riemannian manifolds and let  $x \in M$ , where  $M$  and  $N$  have dimension  $d$  and  $k$ , respectively. The Riemannian jacobian of  $f$  at  $x$  is given by

$$J_g f(x) = \|\Lambda_k(df(x))\|, \quad (10)$$

where  $\Lambda_k(df(x)) : \Lambda_d(T_x M) \rightarrow \Lambda_k(T_{f(x)} N)$  is the canonical linear map associated to  $df(x) : T_x M \rightarrow T_{f(x)} N$ . The norm of  $\Lambda_k(df(x))$  is understood with respect to the induced scalar products on  $\Lambda_d(T_x M)$  and  $\Lambda_k(T_{f(x)} N)$ . We recall scalar products of  $p$ -vectors in (17).

To compute the Riemannian jacobian, we fix two orthonormal bases  $(X_1, \dots, X_d)$  and  $(E_1, \dots, E_k)$  of  $T_x M$  and  $T_{f(x)} N$ , respectively, and we represent  $df(x)$  with respect to these bases by the matrix

$$\nabla_{X,E} f(x) = \begin{bmatrix} \langle E_1, df(x)(X_1) \rangle & \langle E_1, df(x)(X_2) \rangle & \dots & \langle E_1, df(x)(X_d) \rangle \\ \langle E_2, df(x)(X_1) \rangle & \langle E_2, df(x)(X_2) \rangle & \dots & \langle E_2, df(x)(X_d) \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle E_k, df(x)(X_1) \rangle & \langle E_k, df(x)(X_2) \rangle & \dots & \langle E_k, df(x)(X_d) \rangle \end{bmatrix}. \quad (11)$$

Then the jacobian of the matrix  $\nabla_{X,E} f(x)$  coincides with  $J_g f(x)$ . In the sequel, it will be useful to fix the following notation to indicate minors of a matrix.

**Definition 2.5** Let  $G$  be an  $m \times n$  matrix with  $m \leq n$ . We denote by  $G_{i_1 i_2 \dots i_m}$  the  $m \times m$  submatrix with columns  $(i_1, i_2, \dots, i_m)$ . We define the minor

$$M_{i_1 i_2 \dots i_m}(G) = \det(G_{i_1 i_2 \dots i_m}). \quad (12)$$

**Definition 2.6 (Horizontal jacobian)** Let  $\Omega$  be an open subset of  $\mathbb{H}^n$  and let  $x \in \Omega$ . The *horizontal jacobian* of a  $C^1$  mapping  $f : \Omega \rightarrow \mathbb{R}^k$  at  $x$  is given by

$$J_H f(x) = \|\Lambda_k(df(x)|_{H_x \mathbb{H}^n})\|, \quad (13)$$

where  $\Lambda_k(df(x)|_{H_x \mathbb{H}^n}) : \Lambda_k(H_x \mathbb{H}^n) \rightarrow \Lambda_k(\mathbb{R}^k)$ .

From definition of horizontal jacobian, it follows that it only depends on the restriction of  $g$  to the horizontal subbundle, namely, from the “sub-Riemannian metric”. Let us consider a horizontal frame  $(Y_1, Y_2, \dots, Y_{2n})$ , hence  $J_H f(x)$  is given by the jacobian of

$$\nabla_Y f(x) = \begin{bmatrix} Y_1 f^1(x) & Y_2 f^1(x) & \dots & Y_{2n} f^1(x) \\ Y_1 f^2(x) & Y_2 f^2(x) & \dots & Y_{2n} f^2(x) \\ \vdots & \vdots & \vdots & \vdots \\ Y_1 f^k(x) & Y_2 f^k(x) & \dots & Y_{2n} f^k(x) \end{bmatrix}. \quad (14)$$

As a consequence, we have the formula

$$J_H f(x) = \sqrt{\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq 2n} [M_{i_1 i_2 \dots i_k}(\nabla_Y f(x))]^2}. \quad (15)$$

**Proposition 2.7** *Let  $(Y_1, Y_2, \dots, Y_{2n}, W)$  be an orthonormal frame with respect to a left invariant metric  $h$  and let  $F : \mathbb{R}^{2n+1} \rightarrow \mathbb{H}^n$  define coordinates with respect to this frame. Then we have  $F_{\sharp} \mathcal{L}^{2n+1} = \text{vol}_{2n+1}$ , where  $\text{vol}_{2n+1}$  denotes the Riemannian volume measure with respect to the metric  $h$ .*

PROOF. Let  $A$  be a measurable set of  $\mathbb{R}^{2n+1}$ . By classical area formula and taking into account the left invariance of both  $\text{vol}_p$  and  $F_{\sharp} \mathcal{L}^{2n+1}$  we have

$$c \mathcal{L}^{2n+1}(A) = \text{vol}_{2n+1}(F(A)) = \int_A J_h F(x) dx$$

for some constant  $c > 0$ . Then  $\int_A J_h F = c$  for any measurable  $A$ . By continuity of  $x \rightarrow J_h F(x)$  we obtain that  $J_h F(x) = c$  for any  $x \in \mathbb{R}^q$ . We have  $F = \exp \circ L$ , with

$$L(y) = y_{2n+1} W + \sum_{j=1}^{2n} y_j Y_j$$

and  $(Y_1, Y_2, \dots, Y_{2n}, W)$  is orthonormal. Since the map  $dF(0) = d \exp(0) \circ L = L$  has jacobian equal to one, then  $c = 1$  and the thesis follows.  $\square$

**Remark 2.8** By previous proposition, the volume measure of a measurable subset  $A$  of  $\mathbb{H}^n$  corresponds to the  $(2n + 1)$ -dimensional Lebesgue measure of the same subset read with respect to coordinates associated to an orthonormal frame. Here the volume measure is defined by the same left invariant metric.

Recall that  $(F_{\sharp}\mu)(A) = \mu(F^{-1}(A))$ , where  $\mu$  is a measure defined on the domain of  $F$  and  $A$  is a measurable set defined on the codomain of  $F$ . The  $d$ -dimensional spherical Hausdorff measure  $\mathcal{S}^d$  is defined as

$$\mathcal{S}^d(A) = \sup_{\varepsilon > 0} \inf \left\{ \sum_{j=1}^{\infty} \text{diam}(E_j)^d \mid A \subset \bigcup E_j, \text{diam}(E_j) \leq \varepsilon \right\}, \quad (16)$$

where the diameter is considered with respect to a homogeneous distance  $\rho$  of  $\mathbb{H}^n$  and we do not consider any dimensional factor. The  $k$ -dimensional Hausdorff measure built with respect to the Riemannian distance is denoted by  $\text{vol}_k$  and it corresponds to the classical Riemannian volume measure with respect to the graded metric  $g$ , see for instance 3.2.46 of [5].

**Definition 2.9 (Horizontal  $p$ -vectors)** For each  $x \in \mathbb{H}^n$ , we say that any linear combination of wedge products  $X_{j_1}(x) \wedge X_{j_2}(x) \wedge \cdots \wedge X_{j_p}(x)$ , where  $1 \leq j_s \leq 2n$  and  $j = 1, \dots, 2n$ , is a *horizontal  $p$ -vector*. The space of horizontal  $p$ -vectors is denoted by  $\Lambda_p(H_x\mathbb{H}^n)$ .

**Definition 2.10 (Vertical  $p$ -vectors)** For each  $x \in \mathbb{H}^n$ , we say that any linear combination of wedge products  $X_{j_1}(x) \wedge X_{j_2}(x) \wedge \cdots \wedge X_{j_{p-1}}(x) \wedge Z(x)$ , where  $1 \leq j_s \leq 2n$  and  $j = 1, \dots, 2n$ , is a *vertical  $p$ -vector*. The space of vertical  $p$ -vectors is denoted by  $\mathcal{V}_p(H_x\mathbb{H}^n)$ .

For every couple of simple  $p$ -vectors  $v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p \in \Lambda_p(T_x\mathbb{H}^n)$ , we define the scalar product induced by the left invariant Riemannian metric  $g$  on  $T_x\mathbb{H}^n$  as

$$\langle v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p \rangle = \det((g(x)(v_i, w_j))), \quad (17)$$

see for instance 1.7.5 of [5] for more details. This allows us to regard the space of vertical  $p$ -vectors  $\mathcal{V}_p(T_x\mathbb{H}^n)$  as the orthogonal complement of the horizontal subspace  $\Lambda_p(H_x\mathbb{H}^n)$ . We have the orthogonal decomposition

$$\Lambda_p(T_x\mathbb{H}^n) = \Lambda_p(H_x\mathbb{H}^n) \oplus \mathcal{V}_p(T_x\mathbb{H}^n), \quad (18)$$

which generalizes the case  $p = 1$ , corresponding to  $T_x\mathbb{H}^n = H_x\mathbb{H}^n \oplus \langle Z(x) \rangle$ .

**Definition 2.11 (Vertical projection)** Let  $x \in \mathbb{H}^n$  and let  $\xi \in \Lambda_p(T_x\mathbb{H}^n)$ . The orthogonal decomposition  $\xi = \xi_H + \xi_V$  associated to (18) uniquely defines the vertical  $p$ -vector  $\xi_V \in \mathcal{V}_p(T_x\mathbb{H}^n)$ . We say that  $\xi_V$  is the *vertical projection* of  $\xi$  and that the mapping  $\pi_V : \Lambda_p(T_x\mathbb{H}^n) \rightarrow \mathcal{V}_p(T_x\mathbb{H}^n)$ , which associates  $\xi_V$  to  $\xi$ , is the *vertical projection*.

We have omitted  $x$  in the definition of vertical projection  $\pi_V$ .



**Definition 2.12 (Characteristic points and transverse points)** Let  $\Sigma \subset \Omega$  be a  $C^1$  submanifold and let  $x \in \Sigma$ . We say that  $x \in \Sigma$  is a *characteristic point* if  $T_x \Sigma \subset H_x \mathbb{H}^n$  and that it is a *transverse point* otherwise. The *characteristic set* of  $\Sigma$  is the subset of all characteristic points and it is denoted by  $C(\Sigma)$ .

Recall that a *tangent  $p$ -vector* to a  $p$ -dimensional submanifold  $\Sigma$  of class  $C^1$  at  $x \in \Sigma$  is defined by the wedge product  $t_1 \wedge t_2 \wedge \cdots \wedge t_p$ , where  $(t_1, \dots, t_p)$  is an orthonormal basis of  $T_x \Sigma$ . We denote this simple  $p$ -vector by  $\tau_\Sigma(x)$ . Notice that the tangent  $p$ -vector (which belongs to a one-dimensional space) cannot be continuously defined on all of  $\Sigma$ , unless the submanifold is oriented.

**Definition 2.13 (Vertical tangent  $p$ -vector)** Let  $\Sigma \subset \Omega$  be a  $p$ -dimensional submanifold of class  $C^1$  and let  $x \in \Sigma$ . A *vertical tangent  $p$ -vector* to  $\Sigma$  at  $x$  is defined by  $\pi_\mathcal{V}(\tau_\Sigma)$ , where  $\tau_\Sigma$  is a tangent  $p$ -vector and  $\pi_\mathcal{V}$  is the vertical projection. The vertical tangent  $p$ -vector will be denoted by  $\tau_{\Sigma, \mathcal{V}}(x)$ .

### 3 Blow-up at transverse points

This section is devoted to the proof of Theorem 1.1. In the following proposition, we give a simple characterization of characteristic points using vertical tangent  $p$ -vectors.

**Proposition 3.1** *Let  $\Sigma \subset \Omega$  be a submanifold of class  $C^1$  and let  $x \in \Sigma$ . Then  $x \in C(\Sigma)$  if and only if  $\tau_{\Sigma, \mathcal{V}}(x) = 0$ .*

PROOF. Let  $x \in \Sigma$  and let  $(t_1, t_2, \dots, t_p)$  be an orthonormal basis of  $T_x \Sigma$ . We have the unique decomposition  $t_j = V_j + \gamma_j Z$ , where  $V_j \in H_x \mathbb{H}^n$  for every  $j = 1, \dots, p$ . It follows that

$$\begin{aligned} \tau &= t_1 \wedge t_2 \wedge \cdots \wedge t_p \\ &= (V_1 + \gamma_1 Z) \wedge (V_2 + \gamma_2 Z) \wedge \cdots \wedge (V_p + \gamma_p Z) \\ &= V_1 \wedge V_2 \wedge \cdots \wedge V_p + \sum_{j=1}^p \gamma_j V_1 \wedge V_2 \wedge \cdots \wedge V_{j-1} \wedge Z \wedge V_{j+1} \wedge \cdots \wedge V_p. \end{aligned}$$

Assume that  $x \notin C(\Sigma)$ . If  $V_1, V_2, \dots, V_p$  are linearly dependent, then we get

$$t_1 \wedge t_2 \wedge \cdots \wedge t_p = \sum_{j=1}^p \gamma_j V_1 \wedge V_2 \wedge \cdots \wedge V_{j-1} \wedge Z \wedge V_{j+1} \wedge \cdots \wedge V_p.$$

As a result,  $\pi_\mathcal{V}(\tau) = \tau$  hence it is not vanishing. If  $V_1, \dots, V_p$  are linearly independent, then all wedge products of the form

$$V_1 \wedge V_2 \wedge \cdots \wedge V_{j-1} \wedge Z \wedge V_{j+1} \wedge \cdots \wedge V_p \tag{19}$$

are non-vanishing for every  $j = 1, \dots, p$ . The fact that  $x$  is transverse implies that there exists  $\gamma_{j_0} \neq 0$ , then the projection

$$\pi_{\mathcal{V}}(\tau) = \sum_{j=1}^p \gamma_j V_1 \wedge V_2 \wedge \dots \wedge V_{j-1} \wedge Z \wedge V_{j+1} \wedge \dots \wedge V_p \quad (20)$$

is non-vanishing. Conversely, if  $\pi_{\mathcal{V}}(\tau) \neq 0$ , then (20) yields some  $\gamma_{j_1} \neq 0$ , therefore  $t_{j_1} \notin H_x \mathbb{H}^n$ .  $\square$

**Proposition 3.2** *Let  $f : \Omega \rightarrow \mathbb{R}^k$  be of class  $C^1$ , with surjective differential at each point of  $\Omega$ . Let  $\Sigma$  denote the submanifold  $f^{-1}(0)$  of  $\Omega$  and let  $x \in \Sigma$ . Then  $x \in C(\Sigma)$  if and only if  $df(x)|_{H_x \mathbb{H}^n}$  is not surjective.*

PROOF. We first notice that  $\text{Ker} \left( df(x)|_{H_x \mathbb{H}^n} \right) = T_x \Sigma \cap H_x \mathbb{H}^n$ , then we have

$$\dim(H_x \mathbb{H}^n \cap T_x \Sigma) = 2n - \dim \left( \text{Im} \left( df(x)|_{H_x \mathbb{H}^n} \right) \right). \quad (21)$$

This last formula allows us to get our claim as follows. Assume that  $x \in C(\Sigma)$ . Then  $T_x \Sigma \subset H_x \mathbb{H}^n$  and (21) gives

$$2n + 1 - k = 2n - \dim \left( \text{Im} \left( df(x)|_{H_x \mathbb{H}^n} \right) \right).$$

From this equation we conclude that  $df(x)|_{H_x \mathbb{H}^n}$  is not surjective. Conversely, if  $df(x)|_{H_x \mathbb{H}^n}$  is not surjective, then (21) implies

$$\dim(H_x \mathbb{H}^n \cap T_x \Sigma) \geq 2n - k + 1 = \dim(T_x \Sigma)$$

therefore  $T_x \Sigma \subset H_x \mathbb{H}^n$ , namely,  $x \in C(\Sigma)$ .  $\square$

**Theorem 3.3** *Let  $f : \Omega \rightarrow \mathbb{R}^k$  be of class  $C^1$ , with surjective differential at each point of  $\Omega$ . Let  $\Sigma$  denote the submanifold  $f^{-1}(0)$  of  $\Omega$  and let  $x \in \Sigma$ . Then we have*

$$|\tau_{\Sigma, \mathcal{V}}(x)| = \frac{J_H f(x)}{J_g f(x)}. \quad (22)$$

PROOF. Left invariance of Riemannian metric allows us to consider the left translated submanifold  $l_{x^{-1}} \Sigma$ . Replacing  $f$  with  $f \circ l_x$  and  $\Omega$  with  $l_{x^{-1}} \Omega$  we can assume that  $x$  is the unit element 0 of  $\mathbb{H}^n$ . Recall that  $l_x : \mathbb{H}^n \rightarrow \mathbb{H}^n$  is the left translation  $l_x(y) = x \cdot y$ . If  $x \in C(\Sigma)$ , then Proposition 3.1 and Proposition 3.2 make (22) the trivial identity  $0 = 0$ . Assume that  $x \in \Sigma \setminus C(\Sigma)$ . Then Proposition 3.2 implies that the horizontal gradients

$$\nabla_H f^i = (X_1 f^i(0), X_2 f^i(0), \dots, X_{2n} f^i(0)) \quad \text{for } i = 1, 2, \dots, k$$

span a  $k$ -dimensional space of  $\mathbb{R}^{2n}$ . Let  $c_1, c_2, \dots, c_k \in \mathbb{R}^{2n}$  be orthogonal unit vectors generating this vector space and choose  $c_{k+1}, \dots, c_{2n} \in \mathbb{R}^{2n}$  such that  $(c_1, c_2, \dots, c_{2n})$  is an orthonormal basis of  $\mathbb{R}^{2n}$ . These vectors allow us to define a new horizontal frame

$$Y_j = \sum_{k=1}^{2n} c_j^k X_k \quad \text{for every } j = 1, \dots, 2n. \quad (23)$$

We denote by  $C$  the  $2n \times 2n$  orthogonal matrix whose  $i$ -th column corresponds to the vector  $c_i$ , then by our choice of vectors  $c_j$ , we obtain  $\nabla_Y f(x) = \nabla_X f(x) C$  and

$$\nabla_Y f(x) = \begin{bmatrix} \langle \nabla_H f^1, c_1 \rangle & \langle \nabla_H f^1, c_2 \rangle & \cdots & \langle \nabla_H f^1, c_k \rangle & 0 & \cdots & 0 \\ \langle \nabla_H f^2, c_1 \rangle & \langle \nabla_H f^2, c_2 \rangle & \cdots & \langle \nabla_H f^2, c_k \rangle & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & 0 & \cdots & 0 \\ \langle \nabla_H f^k, c_1 \rangle & \langle \nabla_H f^k, c_2 \rangle & \cdots & \langle \nabla_H f^k, c_k \rangle & 0 & \cdots & 0 \end{bmatrix}, \quad (24)$$

where the symbol  $\langle, \rangle$  denotes the standard scalar product of  $\mathbb{R}^{2n}$ . Let us consider  $F : \mathbb{R}^{2n+1} \rightarrow \mathbb{H}^n$ , defining graded coordinates  $(y_1, \dots, y_{2n+1})$  associated to the frame  $(Y_1, \dots, Y_{2n}, Z)$ , according to Definition 2.2. Then the differential of  $f$  at 0 with respect to  $(y_1, \dots, y_{2n+1})$  can be represented by the matrix

$$\nabla_y f(0) = \begin{bmatrix} f_{y_1}^1(0) & f_{y_2}^1(0) & \cdots & f_{y_k}^1(0) & 0 & \cdots & 0 & f_{y_{2n+1}}^1(0) \\ f_{y_1}^2(0) & f_{y_2}^2(0) & \cdots & f_{y_k}^2(0) & 0 & \cdots & 0 & f_{y_{2n+1}}^2(0) \\ \vdots & \vdots & \cdots & \vdots & 0 & \cdots & 0 & \vdots \\ f_{y_1}^k(0) & f_{y_2}^k(0) & \cdots & f_{y_k}^k(0) & 0 & \cdots & 0 & f_{y_{2n+1}}^k(0) \end{bmatrix}. \quad (25)$$

It follows that

$$f_{y_j}^i(0) = \langle \nabla_H f^i, c_j \rangle$$

for every  $i, j = 1, \dots, 2n$ . The implicit function theorem gives us a  $C^1$  map  $\varphi : A \rightarrow \mathbb{R}^k$  such that  $A \subset \mathbb{R}^p$  is an open neighbourhood of the origin and

$$f(\varphi^1(\tilde{y}), \dots, \varphi^k(\tilde{y}), y_{k+1}, \dots, y_{2n+1}) = 0 \quad (26)$$

for every  $\tilde{y} = (y_{k+1}, \dots, y_{2n+1}) \in A$ . Then we define the mapping  $\phi : A \rightarrow \mathbb{R}^{2n+1}$  as

$$\phi(\tilde{y}) = (\varphi^1(\tilde{y}), \dots, \varphi^k(\tilde{y}), y_{k+1}, \dots, y_{2n+1}), \quad (27)$$

so that differentiating (26) we get

$$0 = \partial_{y_j} (f^i \circ \phi) = \sum_{l=1}^k f_{y_l}^i \varphi_{y_j}^l + f_{y_j}^i \quad (28)$$

for every  $i = 1, \dots, k$  and  $j = k + 1, \dots, 2n + 1$ . Equations (28) can be more concisely written in matrix form as follows

$$\nabla_z f \cdot \varphi_{y_j} = -f_{y_j}, \quad (29)$$

where  $z = (y_1, \dots, y_k)$ , the  $k \times k$  matrix  $\nabla_z f$  has coefficients  $f_{y_l}^i$ , where  $i, l = 1, \dots, k$  and  $j = k + 1, \dots, 2n + 1$ . In order to achieve a more explicit formula for the differential of the implicit map, we explicitly write the inverse matrix of  $\nabla_z f$  as

$$(\nabla_z f)^{-1} = \frac{1}{M_{12\dots k}(\nabla_z f)} \begin{bmatrix} C_{11}(\nabla_z f) & C_{21}(\nabla_z f) & \cdots & C_{k1}(\nabla_z f) \\ C_{12}(\nabla_z f) & C_{22}(\nabla_z f) & \cdots & C_{k2}(\nabla_z f) \\ \vdots & \vdots & \cdots & \vdots \\ C_{1k}(\nabla_z f) & C_{2k}(\nabla_z f) & \cdots & C_{kk}(\nabla_z f) \end{bmatrix},$$

where  $C_{ij}(\nabla_z f)$  denotes the cofactor of  $\nabla_z f$ , which is equal to  $(-1)^{i+j} \det(\hat{D}_{ij} f)$  and  $\hat{D}_{ij} f$  is the  $(k-1) \times (k-1)$  square matrix obtained by removing the  $i$ -th row and the  $j$ -th column from  $\nabla_z f$ . In view of (29) we have

$$\varphi_{y_j} = -(\nabla_z f)^{-1} f_{y_j} = -\frac{1}{M_{12\dots k}(\nabla_z f)} \begin{bmatrix} \sum_{i=1}^k C_{i1}(\nabla_z f) f_{y_j}^i \\ \sum_{i=1}^k C_{i2}(\nabla_z f) f_{y_j}^i \\ \vdots \\ \sum_{i=1}^k C_{ik}(\nabla_z f) f_{y_j}^i \end{bmatrix}.$$

An elementary formula for computing the determinant of a matrix implies

$$\sum_{i=1}^k C_{is}(\nabla_z f) f_{y_j}^i = M_{12\dots s-1 j s+1\dots k}(\nabla_y f)$$

for every  $j = k + 1, \dots, 2n + 1$ . As a consequence, we get

$$\varphi_{y_j}^s = -\frac{M_{12\dots s-1 j s+1\dots k}(\nabla_y f)}{M_{1\dots k}(\nabla_z f)}. \quad (30)$$

Note that  $M_{1\dots k}(\nabla_z f)$  corresponds to the determinant of the matrix  $\nabla_z f$ . As a consequence of (30) and of (25), we conclude that

$$\varphi_{y_j}^s(0) = 0$$

for every  $j = k + 1, \dots, 2n$ . Previous considerations and expression (27) lead us to the formula

$$\nabla_{\tilde{y}}\phi(0) = \begin{bmatrix} 0 & 0 & \cdots & 0 & \varphi_{y_{2n+1}}^1(0) \\ 0 & 0 & \cdots & 0 & \varphi_{y_{2n+1}}^2(0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \varphi_{y_{2n+1}}^k(0) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & 1 & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad (31)$$

where  $\nabla_{\tilde{y}}\phi(0)$  is a  $(2n + 1) \times p$  matrix whose  $p \times p$  lower block is the identity matrix. Notice that columns of (31) represent a basis of the tangent space  $T_0\Sigma$  with respect to coordinates  $(y_{k+1}, \dots, y_{2n+1})$ . More precisely, the set of vectors

$$\left( Y_{k+1}(0), Y_{k+2}(0), \dots, Y_{2n}(0), \frac{Z(0) + \sum_{j=1}^k v_j Y_j(0)}{\left(1 + \sum_{j=1}^k v_j^2\right)^{1/2}} \right)$$

form an orthonormal basis of  $T_0\Sigma$ , where we have defined  $v_j = \varphi_{y_{2n+1}}^j(0)$ . Then the tangent  $p$ -vector  $\tau_\Sigma$  to  $\Sigma$  at 0 is given by the wedge product

$$\tau_\Sigma(0) = \frac{Y_{k+1}(0) \wedge Y_{k+2}(0) \wedge \cdots \wedge Y_{2n}(0) \wedge \left( Z(0) + \sum_{j=1}^k v_j Y_j(0) \right)}{\left(1 + \sum_{j=1}^k v_j^2\right)^{1/2}}.$$

Obviously,  $p$ -vectors  $Y_{k+1}(0) \wedge Y_{k+2}(0) \wedge \cdots \wedge Y_{2n}(0) \wedge Y_j(0)$  are horizontal, hence they disappear in the vertical projection. It follows that

$$\tau_{\Sigma, \nu}(0) = \pi_\nu(\tau_\Sigma, \nu(0)) = \frac{Y_{k+1}(0) \wedge Y_{k+2}(0) \wedge \cdots \wedge Y_{2n}(0) \wedge Z(0)}{\left(1 + \sum_{j=1}^k v_j^2\right)^{1/2}},$$

therefore we clearly obtain

$$|\tau_{\Sigma, \nu}(0)| = \left(1 + \sum_{l=1}^k v_l^2\right)^{-1/2}. \quad (32)$$

Due to formula (30) in the case  $j = 2n + 1$  and to (25), we obtain

$$1 + \sum_{l=1}^k v_l^2 = \frac{(M_{1\dots k}(\nabla_z f))^2 + \sum_{s=1}^k (M_{12\dots s-1 2n+1 s+1 \dots k}(\nabla_y f))^2}{(M_{1\dots k}(\nabla_z f))^2} = \left( \frac{J_g f(0)}{J_H f(0)} \right)^2,$$

then (32) shows the validity of (22) in the case  $x = 0$ . Left invariance of the Riemannian metric  $g$  leads us to the conclusion.  $\square$

**Definition 3.4 (Metric factor)** Let  $\tau$  be a vertical simple  $p$ -vector of  $\Lambda_p(\mathfrak{h}^n)$  and let  $\mathcal{L}(\tau)$  be the unique associated subspace, with  $L = \exp \mathcal{L}(\tau)$ . The *metric factor* of a homogeneous distance  $\rho$  with respect to  $\tau$  is defined by

$$\theta_p^\rho(\tau) = \mathcal{H}_{|\cdot|}^p(F^{-1}(L \cap B_1)),$$

where  $F : \mathbb{R}^{2n+1} \rightarrow \mathbb{H}^n$  defines a system of graded coordinates,  $\mathcal{H}_{|\cdot|}^p$  denotes the  $p$ -dimensional Hausdorff measure with respect to the Euclidean distance of  $\mathbb{R}^{2n+1}$  and  $B_1$  is the unit ball of  $\mathbb{H}^n$  with respect to the distance  $\rho$ . Recall that the subspace associated to a simple  $p$ -vector  $\tau$  is defined as  $\{v \in \mathfrak{h}^n \mid v \wedge \tau = 0\}$ .

**Remark 3.5** In the case of subspaces  $\mathcal{L}$  of codimension one, the notion of metric factor fits into the one introduced in [19]. It is easy to observe that the notion of metric factor does not depend on the system of coordinates we are using. In fact,  $F_1^{-1} \circ F_2 : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$  is an Euclidean isometry whenever  $F_1, F_2 : \mathbb{R}^{2n+1} \rightarrow \mathbb{H}^n$  represent systems of graded coordinates with respect to the same left invariant Riemannian metric.

*Proof of Theorem 1.1.* As in the proof of Theorem 3.3, left invariance of the Riemannian metric  $g$  allows us to assume that  $x = 0$ . For  $r_0 > 0$  sufficiently small, we can suppose the existence of a function  $f : B_{r_0} \rightarrow \mathbb{R}^k$  such that  $\Sigma \cap B_{r_0} = f^{-1}(0)$  and whose differential is surjective at every point of  $B_{r_0}$ . By Proposition 3.2, the horizontal gradients

$$\nabla_H f^i = (X_1 f^i(0), X_2 f^i(0), \dots, X_{2n} f^i(0)) \quad \text{for } i = 1, 2, \dots, k$$

span a  $k$ -dimensional space of  $\mathbb{R}^{2n}$ . Now, repeating the argument in the proof of Theorem 3.3, we define the system of graded coordinates  $(y_1, \dots, y_{2n+1})$  associated to the frame  $(Y_1, \dots, Y_{2n}, Z)$ , where  $Y_j$  are given by (23). The differential of  $f$  at 0 can be represented by the matrix

$$\nabla_y f(0) = \begin{bmatrix} f_{y_1}^1(0) & f_{y_2}^1(0) & \cdots & f_{y_k}^1(0) & 0 & \cdots & 0 & f_{y_{2n+1}}^1(0) \\ f_{y_1}^2(0) & f_{y_2}^2(0) & \cdots & f_{y_k}^2(0) & 0 & \cdots & 0 & f_{y_{2n+1}}^2(0) \\ \vdots & \vdots & \cdots & \vdots & 0 & \cdots & 0 & \vdots \\ f_{y_1}^k(0) & f_{y_2}^k(0) & \cdots & f_{y_k}^k(0) & 0 & \cdots & 0 & f_{y_{2n+1}}^k(0) \end{bmatrix}, \quad (33)$$

whose first  $k$  columns are linearly independent. By the implicit function theorem there exists a  $C^1$  mapping  $\varphi : A \rightarrow \mathbb{R}^k$  such that  $A \subset \mathbb{R}^p$  is an open neighbourhood of the origin and

$$f(\varphi^1(\tilde{y}), \dots, \varphi^k(\tilde{y}), y_{k+1}, \dots, y_{2n+1}) = 0 \quad (34)$$

for every  $\tilde{y} = (y_{k+1}, \dots, y_{2n+1}) \in A$ . Proceeding as in the proof of Theorem 3.3, we define the mapping  $\phi : A \rightarrow \mathbb{R}^{2n+1}$  as

$$\phi(\tilde{y}) = (\varphi^1(\tilde{y}), \dots, \varphi^k(\tilde{y}), y_{k+1}, \dots, y_{2n+1}), \quad (35)$$

and by the same computations, differentiating (34) we obtain

$$\nabla_{\tilde{y}}\phi(0) = \begin{bmatrix} 0 & 0 & \cdots & 0 & \varphi^1_{y_{2n+1}}(0) \\ 0 & 0 & \cdots & 0 & \varphi^2_{y_{2n+1}}(0) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \varphi^k_{y_{2n+1}}(0) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & 1 & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad (36)$$

where  $\nabla_{\tilde{y}}\phi(0)$  is a  $(2n+1) \times p$  matrix whose  $p \times p$  lower block is the identity matrix. For each  $r < r_0$ , write the ball  $B_r$  in terms of graded coordinates defining  $\tilde{B}_r = F^{-1}(B_r) \subset \mathbb{R}^{2n+1}$ . The surface  $\Sigma$  read in graded coordinates can be seen as the image of  $\phi$ . Then we have established

$$\frac{\text{vol}_p(\Sigma \cap B_r)}{r^{p+1}} = r^{-1-p} \int_{\phi^{-1}(\tilde{B}_r)} J_g\phi(\tilde{y}) d\tilde{y}. \quad (37)$$

The dilation  $\delta_r$  restricted to coordinates  $(y_1, \dots, y_{2n+1})$  gives

$$\delta_r\tilde{y} = \delta_r((y_{k+1}, \dots, y_{2n+1})) = (ry_{k+1}, ry_{k+2}, \dots, ry_{2n}, r^2y_{2n+1}), \quad (38)$$

therefore, performing a change of variable in (37) we get

$$\frac{\text{vol}_p(\Sigma \cap B_{x,r})}{r^{p+1}} = \int_{\delta_{1/r}(\phi^{-1}(\tilde{B}_r))} J_g\phi(\delta_r\tilde{y}) d\tilde{y}. \quad (39)$$

The set  $\delta_{1/r}(\phi^{-1}(\tilde{B}_r))$  can be written as follows

$$(\delta_{1/r} \circ \phi \circ \delta_r)^{-1}(\tilde{B}_1) = \left\{ \tilde{y} \in \mathbb{R}^p \left| \left( \frac{\varphi^1(\delta_r\tilde{y})}{r}, \dots, \frac{\varphi^k(\delta_r\tilde{y})}{r}, y_{k+1}, \dots, y_{2n+1} \right) \in \tilde{B}_1 \right. \right\}. \quad (40)$$

From expressions (36) and (38) one easily gets that

$$\lim_{r \rightarrow 0^+} \frac{\varphi^j(\delta_r\tilde{y})}{r} = 0 \quad (41)$$

for every  $j = 1, \dots, k$ . As a result, the limit

$$\mathbf{1}_{\delta_{1/r}(\phi^{-1}(\tilde{B}_r))} \longrightarrow \mathbf{1}_{\tilde{B}_1 \cap \Pi} \quad \text{as } r \rightarrow 0^+ \quad (42)$$

holds a.e. in  $\mathbb{R}^p$ , where we have defined

$$\Pi = \{(0, \dots, 0, y_{k+1}, \dots, y_{2n+1}) \in \mathbb{R}^{2n+1} \mid y_j \in \mathbb{R}, j = k+1, \dots, 2n+1\}.$$

From (39), we conclude that

$$\lim_{r \rightarrow 0^+} \frac{\text{vol}_p(\Sigma \cap B_{x,r})}{r^{p+1}} = J_g \phi(0) \mathcal{H}^p(\Pi \cap \tilde{B}_1). \quad (43)$$

To compute  $J_g \phi(0)$ , we use both the canonical form of the tangent space  $T_0 \Sigma$  given by (36) and the fact that our frame  $(Y_1, \dots, Y_{2n}, Z)$  is orthonormal. Thus, according to Definition 2.4 the Riemannian jacobian of  $\phi$  at zero is given by

$$J_g \phi(0) = \left(1 + \sum_{j=1}^k v_j^2\right)^{1/2}, \quad (44)$$

where we have defined  $v_j = \varphi_{y_{2n+1}}^j(0)$  for every  $j = 1, \dots, 2n$ . Again, following the same steps of the proof of Theorem 3.3, we get

$$|\tau_{\Sigma, \nu}(0)| = \left(1 + \sum_{l=1}^k v_l^2\right)^{-1/2} = (J_g \phi(0))^{-1}.$$

Then (43) yields

$$\lim_{r \rightarrow 0^+} \frac{\text{vol}_p(\Sigma \cap B_r)}{r^{p+1}} = \frac{\mathcal{H}^p(\Pi \cap \tilde{B}_1)}{|\tau_{\Sigma, \nu}(0)|}. \quad (45)$$

The subspace  $\mathcal{L}(\tau_{\Sigma, \nu}(x))$  associated to the  $p$ -vector  $\tau_{\Sigma, \nu}(x)$  satisfies the relation

$$\exp(\mathcal{L}(\tau_{\Sigma, \nu}(x))) = F(\Pi)$$

therefore the metric factor of  $\rho$  with respect to  $\tau_{\Sigma, \nu}(x)$  is  $\mathcal{H}^p(\Pi \cap \tilde{B}_1)$ . This fact along with (45) implies the validity of (1) and ends the proof.  $\square$

## 4 Spherical Hausdorff measure of submanifolds

This section deals with various applications of Theorem 1.2. A key result to obtain this theorem is the  $\mathcal{S}^{Q-k}$ -negligibility of characteristic points of a  $k$ -codimensional submanifold of a Carnot group of Hausdorff dimension  $Q$ , see [22]. This result in the case of Heisenberg groups reads as follows.



**Theorem 4.1** *Let  $\Sigma \subset \Omega$  be a  $C^1$  submanifold of dimension  $p$ . Then the set of characteristic points  $C(\Sigma)$  is  $S^{p+1}$ -negligible.*

**Remark 4.2** In order to apply the negligibility result of [22] one has to check that the notion of characteristic point in arbitrary stratified groups coincides with our definition stated in the Heisenberg group. According to [22] a point  $x \in \Sigma$  is characteristic if

$$\dim(H_x \mathbb{H}^n) - \dim(T_x \Sigma \cap H_x \mathbb{H}^n) \leq k - 1. \quad (46)$$

If  $x$  is characteristic according to Definition 2.12, then

$$\dim(T_x \Sigma \cap H_x \mathbb{H}^n) = p = 2n + 1 - k$$

and (46) holds. Conversely, if (46) holds, then

$$p = \dim(T_x \Sigma) = 2n - k + 1 \leq \dim(T_x \Sigma \cap H_x \mathbb{H}^n),$$

hence  $T_x \Sigma \subset H_x \mathbb{H}^n$ .

**Corollary 4.3** *Let  $\Sigma \subset \Omega$  be a  $C^1$  submanifold of dimension  $p$ . Then we have*

$$\int_{\Sigma} \theta(\tau_{\Sigma, \nu}(x)) dS^{p+1}(x) = \int_{\Sigma} |\tau_{\Sigma, \nu}(x)| d\text{vol}_p(x) \quad (47)$$

PROOF. We apply Theorem 2.10.17(2) and Theorem 2.10.18(1) of [5], hence from limit (1) and Theorem 4.1 the proof follows by a standard argument.  $\square$

**Remark 4.4** Proof of Theorem 1.2 immediately follows from (47).

Next, we present a class of homogeneous distances in the Heisenberg group which possess constant metric factor. The standard system of graded coordinates  $F : \mathbb{R}^{2n+1} \rightarrow \mathbb{H}^n$  induced by  $(X_1, \dots, X_{2n}, Z)$  will be understood in the sequel. To simplify notation we will write  $x = F(\tilde{x}, x_{2n+1}) \in \mathbb{H}^n$ , with  $\tilde{x} = (x_1, \dots, x_{2n}) \in \mathbb{R}^{2n}$ .

**Proposition 4.5** *Let  $F : \mathbb{R}^{2n+1} \rightarrow \mathbb{H}^n$  define standard coordinates and let  $\rho$  be a homogeneous distance of  $\mathbb{H}^n$  such that  $\rho(0, F(\cdot)) : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}$  only depends on  $(|\tilde{x}|, x_{2n+1})$ . Then  $\theta_p^\rho(\tau) = \theta_p^\rho(\tilde{\tau})$  whenever  $\tau, \tilde{\tau}$  are vertical simple  $p$ -vectors.*

PROOF. Let  $\tau = U_1 \wedge \dots \wedge U_{p-1} \wedge Z$  and  $\tilde{\tau} = W_1 \wedge \dots \wedge W_{p-1} \wedge Z$  be vertical simple  $p$ -vectors, where it is not restrictive assuming that both  $(U_1, \dots, U_{p-1}, Z)$  and  $(W_1, \dots, W_{p-1}, Z)$  are orthonormal systems of  $\mathfrak{h}^{2n+1}$ . Then we easily find an isometry  $J : \mathfrak{h}^{2n+1} \rightarrow \mathfrak{h}^{2n+1}$  such that  $J(\mathcal{L}(\tau)) = \mathcal{L}(\tilde{\tau})$  and  $J(Z) = Z$ . Recall that our graded coordinates are defined by  $F = \exp \circ \mathcal{I}$ , where  $\mathcal{I} : \mathbb{R}^{2n+1} \rightarrow \mathfrak{h}^{2n+1}$  is an isometry such that

$$\mathcal{I}(x_1, \dots, x_{2n+1}) = x_{2n+1} Z + \sum_{j=1}^{2n} x_j X_j$$

for every  $(x_1, \dots, x_{2n+1}) \in \mathbb{R}^{2n+1}$ . Thus, defining  $\tilde{B}_1 = F^{-1}(B_1) \subset \mathbb{R}^{2n+1}$ , we have

$$F^{-1}(\exp \mathcal{L}(\tilde{\tau}) \cap B_1) = \mathcal{I}^{-1} \circ J(\mathcal{L}(\tau)) \cap \tilde{B}_1 = \varphi(\mathcal{I}^{-1}(\mathcal{L}(\tau))) \cap \tilde{B}_1, \quad (48)$$

where  $\varphi = \mathcal{I}^{-1} \circ J \circ \mathcal{I} : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$  is an Euclidean isometry such that  $\varphi(e_{2n+1}) = e_{2n+1}$  and  $e_{2n+1}$  is the  $(2n+1)$ -th vector of the canonical basis of  $\mathbb{R}^{2n+1}$ . Then  $|\tilde{x}| = |\tilde{y}|$  whenever  $\varphi(\tilde{x}, t) = (\tilde{y}, t)$ . As a result, the fact that  $\rho(0, F(\tilde{x}, t))$  only depends on  $(|\tilde{x}|, t)$  easily implies that  $\varphi(\tilde{B}_1) = \tilde{B}_1$ . Thus, due to (48), it follows that

$$\theta_p^\rho(\tau) = \mathcal{H}_{|\cdot|}^p(\mathcal{I}^{-1}(\mathcal{L}(\tau)) \cap \tilde{B}_1) = \mathcal{H}_{|\cdot|}^p(\varphi(\mathcal{I}^{-1}(\mathcal{L}(\tau))) \cap \tilde{B}_1) = \theta_p^\rho(\tilde{\tau}). \quad (49)$$

This ends the proof.  $\square$

**Example 4.6** An example of homogeneous distance satisfying hypotheses of Proposition 4.5 is the gauge distance, also called Korányi distance, [15]. The gauge distance from  $x$  to the origin is given by

$$d(x, 0) = (|\tilde{x}|^4 + 16 x_{2n+1}^2)^{1/4},$$

where  $x = (\tilde{x}, x_{2n+1})$ . Then we define  $d(x, y) = d(0, x^{-1}y)$ , for any  $x, y \in \mathbb{H}^n$ . Another example of homogeneous distance with this property is the “maximum distance”, defined by

$$d_\infty(x, 0) = \max \left\{ |\tilde{x}|, |x_{2n+1}|^{1/2} \right\}.$$

Due to Proposition 4.5, both of these distances have constant metric factor.

Next, we apply (2) to compute the spherical Hausdorff measure of some submanifolds. We will use the following proposition.

**Proposition 4.7** *Let  $\phi : U \rightarrow \mathbb{R}^{2n+1}$  be a  $C^1$  embedding, where  $U \subset \mathbb{R}^p$  is a bounded open set. Let  $F : \mathbb{R}^{2n+1} \rightarrow \mathbb{H}^n$  define standard coordinates and set  $\Phi = F \circ \phi : U \rightarrow \mathbb{H}^n$ , where  $\Sigma = \Phi(U)$ . Let  $\rho$  be a homogeneous distance with constant metric factor  $\alpha > 0$ . Then we have*

$$\mathcal{S}_{\mathbb{H}^n}^{p+1}(\Sigma) = \int_U |\pi_{\mathcal{V}}(\Phi_{u_1}(u) \wedge \Phi_{u_2}(u) \wedge \dots \wedge \Phi_{u_p}(u))| du, \quad (50)$$

for every measurable set  $A \subset \mathbb{H}^n$ , where the norm  $|\cdot|$  is induced by the scalar product (17) on  $p$ -vectors.

PROOF. By definition of Riemannian volume, formula (2) can be written with respect to  $\phi$  as

$$\mathcal{S}_{\mathbb{H}^n}^{p+1}(\Sigma) = \int_U |\tau_{\Sigma, \mathcal{V}}(\phi(u))| \sqrt{\det [g(\phi(u))(\Phi_{u_i}(u), \Phi_{u_j}(u))]} du, \quad (51)$$

where we have

$$|\Phi_{u_1}(u) \wedge \Phi_{u_2}(u) \wedge \cdots \wedge \Phi_{u_p}(u)| = \sqrt{\det [g(\Phi(u))(\Phi_{u_i}(u), \Phi_{u_j}(u))]} . \quad (52)$$

Therefore, taking into account the formula

$$\tau_\Sigma(\Phi(u)) = \frac{\Phi_{u_1}(u) \wedge \Phi_{u_2}(u) \wedge \cdots \wedge \Phi_{u_p}(u)}{|\Phi_{u_1}(u) \wedge \Phi_{u_2}(u) \wedge \cdots \wedge \Phi_{u_p}(u)|} , \quad (53)$$

the definition of vertical tangent  $p$ -vector  $\pi_\mathcal{V}(\tau_\Sigma) = \tau_{\Sigma, \mathcal{V}}$  and joining (51), (52) and (53), formula (50) follows.  $\square$

**Example 4.8** Let  $\phi : \mathbb{R}^3 \rightarrow \mathbb{R}^5$ , defined by  $\phi(u) = (u_1, u_2, u_3, 0, \frac{u_1^2 + u_2^2 + u_3^2}{2})$ . The mapping  $\phi$  parametrizes a 3-dimensional paraboloid  $\Sigma = \Phi(U)$  of  $\mathbb{R}^5$ , where  $U$  is an open bounded set of  $\mathbb{R}^3$ ,  $\Phi = F \circ \phi$  and  $F : \mathbb{R}^3 \rightarrow \mathbb{H}^2$  represents standard coordinates, according to Definition 2.2. Using expressions (7), we have

$$\begin{aligned} \phi_{u_1}(u) &= \tilde{X}_1(\phi(u)) + (\phi_3(u) + u_1)\tilde{T}(\phi(u)), \\ \phi_{u_2}(u) &= \tilde{X}_2(\phi(u)) + (\phi_4(u) + u_2)\tilde{T}(\phi(u)), \\ \phi_{u_3}(u) &= \tilde{X}_3(\phi(u)) + (u_3 - \phi_1(u))\tilde{T}(\phi(u)). \end{aligned}$$

Observing that for every  $j = 1, \dots, 2n$ , we have

$$\begin{aligned} dF(\phi(u))\tilde{X}_j(\phi(u)) &= X_j(\Phi(u)) \in H_{\Phi(u)}\mathbb{H}^n \quad \text{and} \\ dF(\phi(u))\tilde{Z}(\phi(u)) &= Z(\Phi(u)) \in T_{\Phi(u)}\mathbb{H}^n, \end{aligned}$$

hence we obtain

$$\begin{aligned} \Phi_{u_1}(u) &= X_1(\Phi(u)) + (u_3 + u_1)T(\Phi(u)), \quad \Phi_{u_2}(u) = X_2(\Phi(u)) + u_2T(\Phi(u)), \\ \Phi_{u_3}(u) &= X_3(\Phi(u)) + (u_3 - u_1)T(\Phi(u)). \end{aligned}$$

Thus, we can compute

$$\begin{aligned} \pi_\mathcal{V}(\Phi_{u_1} \wedge \Phi_{u_2} \wedge \Phi_{u_3}) \\ = (u_3 - u_1) X_1 \wedge X_2 \wedge T - u_2 X_1 \wedge X_3 \wedge T + (u_3 + u_1) X_2 \wedge X_3 \wedge T, \end{aligned}$$

hence formula (50) yields

$$\mathcal{S}_{\mathbb{H}^2}^4(\Sigma) = \int_U \sqrt{u_2^2 + 2(u_3^2 + u_1^2)} \, du.$$

**Example 4.9** Let  $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ ,  $\phi(u_1, u_2) = (a_1u_1, a_2u_2, bu_1 + cu_2)$ , define a hyperplane in  $\mathbb{R}^3$ , where  $a_1, a_2, b, c \in \mathbb{R}$  and  $(J\phi)^2 = a_1^2a_2^2 + a_1^2c^2 + a_2^2b^2 > 0$ . Embedding the hyperplane in  $\mathbb{H}^1$  through standard coordinates  $F : \mathbb{R}^3 \rightarrow \mathbb{H}^1$ , we obtain

$$\begin{aligned}\Phi_{u_1}(u) &= a_1X_1(\Phi(u)) + (a_1a_2u_2 + b)T(\Phi(u)), \\ \Phi_{u_2}(u) &= a_2X_2(\Phi(u)) + (c - a_1a_2u_1)T(\Phi(u)),\end{aligned}$$

where  $\Phi = F \circ \phi$ . Then we get

$$\pi_{\mathcal{V}}(\Phi_{u_1}(u) \wedge \Phi_{u_2}(u)) = a_1(c - a_1a_2u_1)X_1 \wedge T - (a_1a_2u_2 + b)a_2X_2 \wedge T$$

and formula (50) yields

$$\mathcal{S}_{\mathbb{H}^1}^3(\Pi) = \int_U \sqrt{a_1^2(c - a_1a_2u_1)^2 + a_2^2(a_1a_2u_2 + b)^2} du, \quad (54)$$

where  $\Pi = \Phi(U)$  and  $U$  is an open bounded set of  $\mathbb{R}^2$ .

**Example 4.10** Let  $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ ,  $\phi(u_1, u_2) = (u_1, u_2, \frac{u_1^2 + u_2^2}{2})$ , define a paraboloid in  $\mathbb{R}^3$ . By standard coordinates  $F : \mathbb{R}^3 \rightarrow \mathbb{H}^1$  and arguing as in the previous examples, we have

$$\begin{aligned}\Phi_{u_1}(u) &= X_1(\Phi(u)) + (u_2 + u_1)T(\Phi(u)) \quad \text{and} \\ \Phi_{u_2}(u) &= X_2(\Phi(u)) + (u_2 - u_1)T(\Phi(u)),\end{aligned}$$

where  $\Phi = F \circ \phi$ . It follows that

$$\pi_{\mathcal{V}}(\Phi_{u_1}(u) \wedge \Phi_{u_2}(u)) = (u_2 - u_1)X_1 \wedge T - (u_2 + u_1)X_2 \wedge T$$

and formula (50) yields

$$\mathcal{S}_{\mathbb{H}^1}^3(\mathcal{P}) = \int_U \sqrt{2u_1^2 + 2u_2^2} du \quad (55)$$

where  $\mathcal{P} = \Phi(U)$  and  $U$  is an open bounded set of  $\mathbb{R}^2$ .

**Remark 4.11** It is curious to notice that the density of  $\mathcal{S}_{\mathbb{H}^1}^3$  restricted to the paraboloid  $\mathcal{P}$ , computed in (55), is proportional to the density of  $\mathcal{S}_{\mathbb{H}^1}^3$  restricted to the horizontal projection of  $\mathcal{P}$  onto the plane  $F(\{(x_1, x_2, x_3) \mid x_3 = 0\}) \subset \mathbb{H}^1$ , whose density is given by (54) in the case  $a_1 = a_2 = 1$  and  $b = c = 0$ .

**Example 4.12** From computations of Example 4.9, one can get the 2-dimensional spherical Hausdorff measure of the line  $\Phi(t) = F(at, 0, bt)$  defined on an interval  $[\alpha, \beta]$ . We have

$$\Phi'(t) = aX_1(\Phi(t)) + bT(\Phi(t)) \quad \text{and} \quad \pi_{\mathcal{V}}(\Phi'(t)) = bT(\Phi(t)), \quad (56)$$

then defining the submanifold  $\mathcal{L} = \Phi([\alpha, \beta])$ , the formula  $\mathcal{S}_{\mathbb{H}^1}^2(\mathcal{L}) = |b|(\beta - \alpha)$  holds.

Another consequence of (2) is the lower semicontinuity of the spherical Hausdorff measure with respect to weak convergence of regular currents. To see this, it suffices to establish the following formula

$$\mathcal{S}_{\mathbb{H}^n}^{p+1}(\Sigma) = \sup_{\omega \in \mathcal{F}_c^p(\Omega)} \int_{\Sigma} \langle \tau_{\Sigma, \nu}, \omega \rangle d\text{vol}_p, \quad (57)$$

where  $\mathcal{F}_c^p(\Omega)$  is the space of smooth  $p$ -forms with compact support in  $\Omega$  with  $|\omega| \leq 1$ . The norm of  $\omega$  is defined making the standard frame of  $p$ -forms  $(dx_1, dx_2, \dots, dx_{2n}, \tilde{\theta})$  orthonormal and extending this scalar product to  $p$ -forms exactly as we have seen in formula (17). The 1-form  $\tilde{\theta}$  is the so called *contact form*

$$\tilde{\theta} = dx_{2n+1} + \sum_{j=1}^n x_{j+n} dx_j - x_j dx_{j+n} \quad (58)$$

written in standard coordinates. Note that  $(dx_1, dx_2, \dots, dx_{2n}, \tilde{\theta})$  is the dual basis of  $(\tilde{X}_1, \dots, \tilde{X}_{2n}, \tilde{Z})$ . Formula (57) follows from (2) observing that

$$\int_{\Sigma} |\tau_{\Sigma, \nu}| d\text{vol}_p = \sup_{\omega \in \mathcal{F}_c^p(\Omega)} \int_{\Sigma} \langle \tau_{\Sigma, \nu}, \omega \rangle d\text{vol}_p, \quad (59)$$

as one can check by standard arguments. As a consequence of these observations, we can establish the following proposition.

**Proposition 4.13** *Let  $(\Sigma_m)$  be a sequence of  $C^1$  submanifolds of  $\Omega$  which weakly converges in the sense of currents to the  $C^1$  submanifold  $\Sigma$ . Then*

$$\liminf_{m \rightarrow \infty} \mathcal{S}_{\mathbb{H}^n}^{p+1}(\Sigma_m) \geq \mathcal{S}_{\mathbb{H}^n}^{p+1}(\Sigma). \quad (60)$$

PROOF. By hypothesis

$$\int_{\Sigma_m} \langle \tau_{\Sigma_m, \nu}, \omega \rangle d\text{vol}_p \longrightarrow \int_{\Sigma} \langle \tau_{\Sigma, \nu}, \omega \rangle d\text{vol}_p, \quad (61)$$

for every  $\omega \in \mathcal{F}_c^p(\Omega)$ . Then (57) ends the proof.  $\square$

**Remark 4.14** It is clear the importance of (60) in studying versions of the Plateau problem with respect to the geometry of Heisenberg groups.

Recall that the horizontal normal is the orthogonal projection of the normal to  $\nu(x)$  to  $T_x \Sigma$  onto the horizontal subspace  $H_x \mathbb{H}^n$ . In the next proposition we show that in codimension one an explicit relationship can be established between vertical tangent  $2n$ -vector and horizontal normal  $\nu_H$ .

**Proposition 4.15** *Let  $\Sigma$  be a  $2n$ -dimensional submanifold of class  $C^1$  and let  $\nu_H(x)$  a horizontal normal at  $x \in \Sigma$ . Then we have*

$$\nu_H^j = (-1)^j \tau_{\Sigma, \nu}^j$$

where  $\nu_H = \sum_{j=1}^{2n} \nu_H^j X_j$  and  $\tau_{\Sigma, \nu} = \sum_{j=1}^{2n} \tau_{\Sigma, \nu}^j X_1 \wedge \cdots \wedge X_{j-1} \wedge X_{j+1} \wedge \cdots \wedge X_{2n} \wedge Z$ . In particular, the equality  $|\tau_{\Sigma, \nu}| = |\nu_H|$  holds.

PROOF Let  $(t_1, t_2, \dots, t_{2n})$  be an orthonormal basis of  $T_x \Sigma$ , where  $x$  is a transverse point. Then

$$t_j = \sum_{i=1}^{2n} c_j^i X_i(x) + c_j^{2n+1} Z(x)$$

where  $C = (c_j^i)$  is a  $(2n+1) \times 2n$  matrix, whose columns are orthonormal vectors of  $\mathbb{R}^{2n+1}$ . Then we have

$$\tau_{\Sigma}(x) = t_1 \wedge t_2 \wedge \cdots \wedge t_{2n} = \sum_{j=1}^{2n+1} \det(\hat{C}^j) X_1 \wedge X_2 \wedge \cdots \wedge X_{j-1} \wedge X_{j+1} \wedge \cdots \wedge Z,$$

where  $\hat{C}^j$  is the  $2n \times 2n$  matrix obtained by removing the  $j$ -th row from  $C$ . The vertical projection yields

$$\tau_{\Sigma, \nu}(x) = \pi_{\nu}(\tau_{\Sigma}(x)) = \sum_{j=1}^{2n} \det(\hat{C}^j) X_1 \wedge X_2 \wedge \cdots \wedge X_{j-1} \wedge X_{j+1} \wedge \cdots \wedge Z \quad (62)$$

and by elementary linear algebra one can deduce that

$$\sum_{j=1}^{2n+1} (-1)^j \det(\hat{C}^j) c_k^j = \det [ C \ c_k ] = 0 \quad (63)$$

for every  $k = 1, \dots, 2n$ . Then the vector

$$\nu = \sum_{j=1}^{2n} (-1)^j \det(\hat{C}^j) X_j + (-1)^{2n+1} \det(\hat{C}^{2n+1}) Z$$

yields a unit normal to  $\Sigma$  at  $x$ . Its horizontal projection is

$$\nu_H = \sum_{j=1}^{2n} (-1)^j \det(\hat{C}^j) X_j. \quad (64)$$

Formulae (62) and (64) yield the thesis.  $\square$

## 5 Coarea formula

This section is devoted to the proof of Theorem 1.3. Next, we recall the Riemannian coarea formula, see Section 13.4 of [4].

**Theorem 5.1** *Let  $f : \mathbb{H}^n \rightarrow \mathbb{R}^k$  be a Riemannian Lipschitz function, with  $1 \leq k < 2n + 1$ . Then for any summable map  $u : \mathbb{H}^n \rightarrow \mathbb{R}$ , the following formula holds*

$$\int_{\mathbb{H}^n} u(x) J_g f(x) d\text{vol}_{2n+1}(x) = \int_{\mathbb{R}^k} \left( \int_{f^{-1}(t)} u(y) d\text{vol}_p(y) \right) dt, \quad (65)$$

where  $p = 2n + 1 - k$

In the previous theorem the Heisenberg group  $\mathbb{H}^n$  is equipped with its left invariant Riemannian metric  $g$ . The terminology ‘‘Riemannian Lipschitz map’’ means that the map is Lipschitz with respect to the Riemannian distance.

*Proof of Theorem 1.3.* We first prove (4) in the case  $f$  is defined on all of  $\mathbb{H}^n$  and is of class  $C^1$ . Let  $\Omega$  be an open subset of  $\mathbb{H}^n$ . In view of Riemannian coarea formula (65), we have

$$\int_{\Omega} u(x) J_g f(x) dx = \int_{\mathbb{R}^k} \left( \int_{f^{-1}(t) \cap \Omega} u(y) d\text{vol}_p(y) \right) dt, \quad (66)$$

where  $u : \Omega \rightarrow [0, +\infty]$  is a measurable function. Note that in the left hand side of (66) we have used the Lebesgue measure in that, by Proposition 2.7, it coincides with the volume measure expressed in terms of standard coordinates, namely  $F_{\sharp}(\mathcal{L}^{2n+1}) = \text{vol}_{2n+1}$ . Now we define

$$u(x) = J_H f(x) \mathbf{1}_{\{Jf \neq 0\} \cap \Omega}(x) / Jf(x)$$

and use (66), obtaining

$$\int_{\Omega} J_H f(x) dx = \int_{\mathbb{R}^k} \left( \int_{f^{-1}(t) \cap \Omega} \frac{J_H f(x) \mathbf{1}_{\{Jf \neq 0\}}(x)}{Jf(x)} d\text{vol}_p(y) \right) dt. \quad (67)$$

The validity of (66) also implies that for a.e.  $t \in \mathbb{R}^k$  the set of points of  $f^{-1}(t)$  where  $J_g f$  vanishes is  $\text{vol}_p$ -negligible, then the previous formula becomes

$$\int_{\Omega} J_H f(x) dx = \int_{\mathbb{R}^k} \left( \int_{f^{-1}(t) \cap \Omega} \frac{J_H f(x)}{Jf(x)} d\text{vol}_p(y) \right) dt. \quad (68)$$

By classical Sard’s theorem and Theorem 4.1 for a.e.  $t \in \mathbb{R}^k$  the  $C^1$  submanifold  $f^{-1}(t)$  has  $\mathcal{S}_{\mathbb{H}^n}^{p+1}$ -negligible characteristic points, hence Proposition 3.2 implies that

$$C_t = \{y \in f^{-1}(t) \cap \Omega \mid J_H f(y) = 0\}$$

is  $\mathcal{S}_{\mathbb{H}^n}^{p+1}$ -negligible. As a result, from formulae (22) and (2) we have proved that for a.e.  $t \in \mathbb{R}^k$  the equalities

$$\int_{f^{-1}(t) \cap \Omega} \frac{J_H f(x)}{J_g f(x)} d\text{vol}_p(y) = \mathcal{S}_{\mathbb{H}^n}^{p+1}(f^{-1}(t) \cap \Omega \setminus C_t) = \mathcal{S}_{\mathbb{H}^n}^{p+1}(f^{-1}(t) \cap \Omega)$$

hold, therefore (68) yields

$$\int_{\Omega} J_H f(x) dx = \int_{\mathbb{R}^k} \mathcal{S}_{\mathbb{H}^n}^{p+1}(f^{-1}(t) \cap \Omega) dt. \quad (69)$$

The arbitrary choice of  $\Omega$  yields the validity of (69) also for arbitrary closed sets. Then, approximation of measurable sets by closed ones, Borel regularity of  $\mathcal{S}_{\mathbb{H}^n}^{p+1}$  and the coarea estimate 2.10.25 of [5] extend the validity of (69) to the following one

$$\int_A J_H f(x) dx = \int_{\mathbb{R}^k} \mathcal{S}_{\mathbb{H}^n}^{p+1}(f^{-1}(t) \cap A) dt, \quad (70)$$

where  $A$  is a measurable subset of  $\mathbb{H}^n$ . Now we consider the general case, where  $f : A \rightarrow \mathbb{R}^k$  is a Lipschitz map defined on a measurable bounded subset  $A$  of  $\mathbb{H}^3$ . Let  $f_1 : \mathbb{H}^n \rightarrow \mathbb{R}^k$  be a Lipschitz extension of  $f$ , namely,  $f_1|_A = f$  holds. Due to the Whitney extension theorem (see for instance 3.1.15 of [5]) for every arbitrarily fixed  $\varepsilon > 0$  there exists a  $C^1$  function  $f_2 : \mathbb{H}^n \rightarrow \mathbb{R}^k$  such that the open subset  $O = \{z \in \mathbb{H}^n \mid f_1(z) \neq f_2(z)\}$  has Lebesgue measure less than or equal to  $\varepsilon$ . We wish to prove

$$\left| \int_A J_H f(x) dx - \int_{\mathbb{R}^k} \mathcal{S}_{\mathbb{H}^n}^{p+1}(f^{-1}(t) \cap A) dt \right| \leq \int_{A \cap O} J_H f(x) dx + \int_{\mathbb{R}^k} \mathcal{S}_{\mathbb{H}^n}^{p+1}(f^{-1}(t) \cap A \cap O) dt. \quad (71)$$

In fact, due to the validity of (70) for  $C^1$  mappings, we have

$$\int_{A \setminus O} J_H f_2(x) dx = \int_{\mathbb{R}^k} \mathcal{S}_{\mathbb{H}^n}^{p+1}(f_2^{-1}(t) \cap A \setminus O) dt.$$

Note here that the horizontal jacobian  $J_H f$  is well defined on  $A$ , in that  $df$  is well defined at density points of the domain, see for instance Definition 7 and Proposition 2.2 of [17]. The equality  $f_2|_{A \setminus O} = f|_{A \setminus O}$  implies that  $J_H f_2 = J_H f$  a.e. on  $A \setminus O$ , therefore

$$\int_{A \setminus O} J_H f(x) dx = \int_{\mathbb{R}^k} \mathcal{S}_{\mathbb{H}^n}^{p+1}(f^{-1}(t) \cap A \setminus O) dt$$

holds and inequality (71) is proved. Now we observe that for a.e.  $x \in A$ , we have

$$J_H f(x) \leq \prod_{i=1}^k \left( \sum_{j=1}^{2n} (X_j f^i(x))^2 \right)^{1/2} \leq \|df(x)|_{H_x \mathbb{H}^n}\|^k$$



therefore the estimate

$$J_H f(x) \leq \text{Lip}(f)^k \quad (72)$$

holds for a.e.  $x \in A$ . By virtue of the general coarea inequality 2.10.25 of [5] there exists a dimensional constant  $c_1 > 0$  such that

$$\int_{\mathbb{R}^k} \mathcal{S}_{\mathbb{H}^n}^{p+1}(f^{-1}(t) \cap A \cap O) dt \leq c_1 \text{Lip}(f)^k \mathcal{H}^{2n+2}(O). \quad (73)$$

The fact that the  $2n + 2$ -dimensional Hausdorff measure  $\mathcal{H}^{2n+2}$  with respect to the homogeneous distance  $\rho$  is proportional to the Lebesgue measure, gives us a constant  $c_2 > 0$  such that

$$\int_{\mathbb{R}^k} \mathcal{S}_{\mathbb{H}^n}^{p+1}(f^{-1}(t) \cap A \cap O) dt \leq c_2 \text{Lip}(f)^k \mathcal{L}^{2n+1}(O) \leq c_2 \text{Lip}(f)^k \varepsilon. \quad (74)$$

Thus, estimates (72) and (74) joined with inequality (71) yield

$$\left| \int_A J_H f(x) dx - \int_{\mathbb{R}^k} \mathcal{S}_{\mathbb{H}^n}^{p+1}(f^{-1}(t) \cap A) dt \right| \leq (1 + c_2) \text{Lip}(f)^k \varepsilon.$$

Letting  $\varepsilon \rightarrow 0^+$ , we have proved that

$$\int_A J_H f(x) dx = \int_{\mathbb{R}^k} \mathcal{S}_{\mathbb{H}^n}^{p+1}(f^{-1}(t) \cap A) dt. \quad (75)$$

Finally, utilizing increasing sequences of step functions pointwise converging to  $u$  and applying Beppo Levi convergence theorem the proof of (4) is achieved in the case  $A$  is bounded. If  $A$  is not bounded, then one can take the limit of (4) where  $A$  is replaced by  $A_k$  and  $\{A_k\}$  is an increasing sequence of measurable bounded sets whose union yields  $A$ . Then the Beppo Levi convergence theorem concludes the proof.  $\square$

**Remark 5.2** Notice that once  $f : A \rightarrow \mathbb{R}^k$  in the previous theorem is considered with respect to standard coordinates it is easy to check that the locally Lipschitz property with respect to the Euclidean distance of  $\mathbb{R}^{2n+1}$  is equivalent to the locally Lipschitz property with respect to the Riemannian distance.

## References

- [1] L.AMBROSIO, *Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces*, Adv. Math., **159**, 51-67, (2001)
- [2] Z.M.BALOGH, *Size of characteristic sets and functions with prescribed gradients*, J. Reine Angew. Math., **564**, 63-83, (2003)

- [3] A.BELLAÏCHE, J.J.RISLER eds, *Sub-Riemannian geometry*, Progress in Mathematics, **144**, Birkhäuser Verlag, Basel, (1996).
- [4] Y.D.BURAGO, V.A.ZALGALLER, *Geometric inequalities*, Grundlehren Math. Springer, Berlin.
- [5] H.FEDERER, *Geometric Measure Theory*, Springer, (1969).
- [6] G.B. FOLLAND, E.M. STEIN, *Hardy Spaces on Homogeneous groups*, Princeton University Press, 1982
- [7] B.FRANCHI, R.SERAPIONI, F.SERRA CASSANO, *Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields*, Houston Jour. Math., **22**, 859-889, (1996)
- [8] B.FRANCHI, R.SERAPIONI, F.SERRA CASSANO, *Rectifiability and Perimeter in the Heisenberg group*, Math. Ann. **321**, n. 3, (2001).
- [9] B.FRANCHI, R.SERAPIONI, F.SERRA CASSANO, *Regular hypersurfaces, intrinsic perimeter and implicit function theorem in Carnot groups*, Comm. Anal. Geom. **11**, n.5, 909-944, (2003)
- [10] B.FRANCHI, R.SERAPIONI, F.SERRA CASSANO, *Regular submanifolds, graphs and area formula in Heisenberg groups*, preprint (2004)
- [11] M.GROMOV, *Carnot-Carathéodory spaces seen from within*, Subriemannian Geometry, Progress in Mathematics, **144**. ed. by A.Bellaïche and J.Risler, Birkhauser Verlag, Basel, (1996).
- [12] N.GAROFALO, D.M.NHIEU, *Isoperimetric and Sobolev Inequalities for Carnot-Carathéodory Spaces and the Existence of Minimal Surfaces*, Comm. Pure Appl. Math. **49**, 1081-1144 (1996)
- [13] P.HAJLASZ, P.KOSKELA, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145**, (2000)
- [14] B.KIRCHHEIM, F.SERRA CASSANO, *Rectifiability and parametrization of intrinsic regular surfaces in the Heisenberg group*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), **3**, n.4, 871-896, (2004)
- [15] A.KORÁNYI, *Geometric properties of Heisenberg-type groups*, Adv. in Math. **56**, n.1, 28-38, (1985).
- [16] I.KUPKA, *Géométrie sous-riemannienne*, Astérisque, **241**, Exp. No. 817, 5, 351-380,(1997)

- [17] V.MAGNANI, *Differentiability and Area formula on stratified Lie groups*, Houston Jour. Math., **27**, n.2, 297-323, (2001)
- [18] V.MAGNANI, *On a general coarea inequality and applications*, Ann. Acad. Sci. Fenn. Math., **27**, 121-140, (2002)
- [19] V.MAGNANI, *A Blow-up Theorem for regular hypersurfaces on nilpotent groups*, Manuscripta Math., **110**, n.1, 55-76, (2003)
- [20] V.MAGNANI, *The coarea formula for real-valued Lipschitz maps on stratified groups*, Math. Nachr., **278**, n.14, 1-17, (2005)
- [21] V.MAGNANI, *Note on coarea formulae in the Heisenberg group*, Publ. Mat. **48**, n.2, 409-422, (2004)
- [22] V.MAGNANI. *Characteristic points, rectifiability and perimeter measure on stratified groups*, to appear on J. Eur. Math. Soc.
- [23] R.MONTGOMERY, *A Tour of Subriemannian Geometries, Their Geodesics and Applications*, Mathematical Surveys and Monographs, **91**, American Mathematical Society, Providence, (2002).
- [24] P.PANSU, *Geometrie du Group d'Heisenberg*, These de doctorat, Université Paris VII, (1982)
- [25] P.PANSU, *Une inégalité isoperimétrique sur le groupe de Heisenberg*, C.R. Acad. Sc. Paris, **295**, Série I, 127-130, (1982)
- [26] P.PANSU, *Métriques de Carnot-Carathéodory quasimétriques des espaces symétriques de rang un*, Ann. Math., **129**, 1-60, (1989)
- [27] E.M.STEIN, *Harmonic Analysis*, Princeton University Press, (1993)
- [28] N.TH.VAROPOULOS, L.SALOFF-COSTE, T.COULHON, *Analysis and Geometry on Groups*, Cambridge University Press, Cambridge, (1992).