

## BLOW-UP OF SEMILINEAR PDE'S AT THE CRITICAL DIMENSION. A PROBABILISTIC APPROACH

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ABSTRACT. We present a probabilistic approach which proves blow-up of solutions of the Fujita equation  $\partial w/\partial t = -(-\Delta)^{\alpha/2}w + w^{1+\beta}$  in the critical dimension  $d = \alpha/\beta$ . By using the Feynman-Kac representation twice, we construct a subsolution which locally grows to infinity as  $t \rightarrow \infty$ . In this way, we cover results proved earlier by analytic methods. Our method also applies to extend a blow-up result for systems proved for the Laplacian case by Escobedo and Levine [2] to the case of  $\alpha$ -Laplacians with possibly different parameters  $\alpha$ .

### 1. INTRODUCTION AND OVERVIEW

Consider the semilinear equation

$$(1.1) \quad \begin{aligned} \frac{\partial w_t}{\partial t} &= \Delta_\alpha w_t + \gamma w_t^{1+\beta}, \\ w_0 &= \varphi, \end{aligned}$$

in  $\mathbb{R}^d$ , where  $\Delta_\alpha := -(-\Delta)^{\alpha/2}$ ,  $0 < \alpha \leq 2$ , denotes the  $\alpha$ -Laplacian,  $\beta$  and  $\gamma$  are positive numbers and the initial condition  $\varphi$  is a nonnegative function on  $\mathbb{R}^d$ .

In Fujita's pioneering work [4] it was shown (originally for the case  $\alpha = 2$ ) that  $d = \alpha/\beta$  is the critical dimension for blow-up of (1.1): if  $d > \alpha/\beta$  then (1.1) admits a global solution for all sufficiently small initial conditions, whereas if  $d < \alpha/\beta$ , then for any non-vanishing initial condition the solution is infinite for suitably large  $t$ .

For the case  $d = \alpha/\beta$  it was proved by Sugitani [12] by subtle analytic arguments that (1.1) blows up. Using different, partly probabilistic methods, this was also proved by Portnoy ([9, 10]) for the special case  $\alpha = 2$ ,  $\beta = 1$ . Related results on systems where the space variable is restricted to a bounded domain in  $\mathbb{R}^d$  can be found in the recent paper of Wang [13] and the references therein.

In this note we give a short probabilistic proof for blow-up at the critical dimension, using the Feynman-Kac representation. Here is an outline.

Recall that the solution  $w$  of the initial value problem on  $[0, T) \times \mathbb{R}^d$

$$(1.2) \quad \begin{aligned} \frac{\partial w_t}{\partial t} &= \Delta_\alpha w_t + w_t v_t, \\ w_0 &= \varphi, \end{aligned}$$

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with  $v : [0, T) \times \mathbb{R}^d \mapsto \mathbb{R}_+$  locally bounded has by the Feynman-Kac formula (cf. Stroock [11], §4.3, Freidlin [3], Thm. 2.2, or Dynkin [1], Thm. 9.7) a probabilistic interpretation as the density (with respect to Lebesgue measure on  $\mathbb{R}^d$ ) of the measure

$$(1.3) \quad \int \mathbb{E}_x \left[ \mathbf{1}(W_t \in dy) \exp \int_0^t v_s(W_s) ds \right] \varphi(x) dx = w_t(y) dy$$

where  $\mathbb{E}_x$  denotes expectation with respect to the symmetric  $\alpha$ -stable process  $(W_t)$  started at  $W_0 = x$ . This shows in particular that any solution  $\tilde{w}$  of (1.2) with  $v$  replaced by  $\tilde{v} \leq v$  and  $\tilde{w}_0 = w_0$  fulfills  $\tilde{w} \leq w$ .

Consider for  $i = 0, 1, 2$  the initial value problems

$$(1.4) \quad \begin{aligned} \frac{\partial w_{t,i}}{\partial t} &= \Delta_\alpha w_{t,i} + \gamma w_{t,i} w_{t,i-1}^\beta \\ w_{0,i} &= \varphi \end{aligned}$$

where  $w_{t,-1} = 0$ . Then  $f_t := w_{t,0}$ ,  $g_t := w_{t,1}$  and  $h_t := w_{t,2}$  are all subsolutions of (1.1). Since  $f_t(y) = \mathbb{E}_y[\varphi(W_t)]$ , where  $(W_t)$  is a symmetric  $\alpha$ -stable process,  $f_t(y)$  decays like  $\text{const} \cdot t^{-d/\alpha}$  (see Section 2). Since “typically”  $f_s(W_s)$  should be bounded from below by  $\text{const} \cdot s^{-d/\alpha}$ , and also  $\mathbb{P}_x\{W_t \in dy\} \geq \text{const} \cdot t^{-d/\alpha} dy$  as long as  $\|y - x\| \leq t^{1/\alpha}$ , one should expect (using (1.3) with  $v_s = f_s^\beta$  to express the solution of (1.4) for  $i = 1$ ) that

$$(1.5) \quad \begin{aligned} g_t(y) &= \int \mathbb{E}_x \left[ \exp \int_0^t f_s(W_s)^\beta ds \mid W_t = y \right] \varphi(x) dx \\ &\geq ct^{-d/\alpha} \exp \left( \text{const} \int_1^t s^{-d\beta/\alpha} ds \right) \\ &= ct^{-d/\alpha} \exp(\text{const} \cdot \log t) \geq ct^{-d/\alpha + \varepsilon} \end{aligned}$$

as long as  $\|y\| \leq t^{1/\alpha}$ . This intuition can be turned into a proof basically by applying Jensen’s inequality and scaling arguments.

After dealing in this way in Proposition 2.1 with the case  $i = 1$ , we then turn to the case  $i = 2$  in (1.4). Like  $g_t$ , also  $h_t = w_{t,2}$  has a Feynman-Kac representation, but now with  $f_s^\beta$  replaced by  $g_s^\beta$  in the exponent. By (1.5), the integrand  $g_s(W_s)^\beta$  in this exponent should “typically” remain bounded from below by  $\text{const} \cdot s^{-1+\varepsilon\beta}$ . Thus we expect that

$$h_t(y) \geq \text{const} \cdot t^{-d/\alpha} \exp \left( -c \int_0^t s^{-1+\varepsilon\beta} ds \right),$$

and in fact we will prove this in Proposition 2.3. In particular,  $h_t$  is a subsolution of (1.1) which locally grows to infinity. This fact suffices to show blow up, as we will recall in Section 3.

Section 4 comments briefly on the case of subcritical dimensions, and Section 5 on Portnoy’s method. In Section 6 we give some extensions. Apart from re-proving Sugitani’s result, we show that blow-up of (1.1) with a certain *time-dependent* nonlinearity, which was recently proved by Gedda and Kirane [5], arises as an easy corollary of our probabilistic approach.

In Section 7 we obtain conditions for blow-up of a class of semilinear *systems*. We are able to extend a blow-up result of Escobedo and Levine [2] and show blow-up at the critical dimensions of a system which we were able to analyze before only in the case of sub- and supercritical dimensions [7, 8].

## 2. CONSTRUCTING SUBSOLUTIONS BY THE FEYNMAN-KAC FORMULA

In this and the following section we consider  $d = \alpha/\beta$  and prove that (1.1) blows up in this case. Furthermore assume without loss of generality that the initial condition  $\varphi$  of (1.1) does not vanish a.s. on the unit ball. Let  $p_t(x)$  denote the transition density of the symmetric  $\alpha$ -stable process, and write

$$(2.1) \quad f_t(y) := \int p_t(y-x)\varphi(x) dx = \mathbb{E}_y[\varphi(W_t)].$$

For all  $t \geq 1$  we have the inequality

$$(2.2) \quad f_t(y) \geq c_0 t^{-d/\alpha} \mathbf{1}_{B_1}(t^{-1/\alpha}y) \int_{B_1} \varphi(x) dx$$

for some  $c_0 > 0$ , where  $B_r$  denotes the ball in  $\mathbb{R}^d$  with radius  $r$  centered at the origin. Indeed, let  $y \in B_{t^{1/\alpha}}$ . Then we have by the scaling property of  $W_t$

$$\begin{aligned} f_t(y) &= \mathbb{E}_0[\varphi(W_t + y)] = \mathbb{E}_0\left[\varphi\left(t^{1/\alpha}(W_1 + t^{-1/\alpha}y)\right)\right] \\ &\geq \int_{B_1} p_1(x - t^{-1/\alpha}y)\varphi(t^{1/\alpha}x) dx \geq c_0 \int_{B_1} \varphi(t^{1/\alpha}x) dx = c_0 t^{-d/\alpha} \int_{B_{t^{1/\alpha}}} \varphi(x) dx. \end{aligned}$$

This argument also shows that, for sufficiently large  $t$

$$(2.3) \quad f_t(y) \geq c'_0 t^{-d/\alpha} \mathbf{1}_{B_1}(t^{-1/\alpha}y).$$

for some  $c'_0 > 0$ .

**2.1. The first iteration: a subsolution with a slow decay.** We are going to obtain a lower bound for the solution  $g_t$  of

$$(2.4) \quad \begin{aligned} \frac{\partial g_t}{\partial t} &= \Delta_\alpha g_t + \gamma g_t f_t^\beta, \\ g_0 &= \varphi, \end{aligned}$$

where  $f_t$  is defined in (2.1). Since  $f_t$  is a subsolution of (1.1),  $g_t$  is a subsolution of (1.1) as well.

**Proposition 2.1.** *There exist  $\varepsilon, c > 0$  such that, for all  $t \geq 2$  and all  $y \in \mathbb{R}^d$  obeying  $\|y\| \leq t^{1/\alpha}$ ,*

$$(2.5) \quad g_t(y) \geq c t^{-d/\alpha + \varepsilon}.$$

*Proof.* By the Feynman-Kac formula,  $g_t$  arises as the density of the measure defined in (1.3) (with  $v_s$  replaced by  $f_s^\beta$ ). We therefore have, using (2.2) and Jensen's inequality,

$$\begin{aligned} g_t(y) &= \int \varphi(x) p_t(y-x) \mathbb{E}_x \left[ \exp \int_0^t \gamma f_s(W_s)^\beta ds \mid W_t = y \right] dx \\ &\geq \int \varphi(x) p_t(y-x) \mathbb{E}_x \left[ \exp \int_1^{t/2} c_2 s^{-\beta d/\alpha} \mathbf{1}_{B_{s^{1/\alpha}}}(W_s) ds \mid W_t = y \right] dx \\ &\geq \int_{B_1} \varphi(x) p_t(y-x) \exp \left( c_2 \int_1^{t/2} s^{-\beta d/\alpha} \mathbb{P}_x \{ W_s \in B_{s^{1/\alpha}} \mid W_t = y \} ds \right) dx \\ (2.6) \quad &\geq c_3 t^{-d/\alpha} \exp \left( c_4 \int_1^{t/2} s^{-\beta d/\alpha} ds \right) \end{aligned}$$

where the last estimate relies on Lemma 2.2 below. (Here and below  $c_i$ ,  $i = 1, 2, \dots$  denote “locally defined” positive constants). The assertion now follows from our assumption  $d = \alpha/\beta$ .  $\square$

The intuition behind the following assertion is clear: conditioning on some “typical” state at time  $t$  does not much affect the behavior of  $(W_t)$  between times 0 and  $t/2$ .

**Lemma 2.2.** *There exists a  $c > 0$  such that for all  $t \geq 2$ ,  $y \in B_{t^{1/\alpha}}$ ,  $x \in B_1$  and  $s \in [1, t/2]$ ,*

$$(2.7) \quad \mathbb{P}_x \{W_s \in B_{s^{1/\alpha}} | W_t = y\} \geq c.$$

*Proof.* First note that (2.7) is equivalent to

$$(2.8) \quad \int_{B_{s^{1/\alpha}}} p_s(z-x)p_{t-s}(y-z) dz \geq c_5 p_t(y-x).$$

Next, let us state the following facts, which are easy consequences of the scaling property of  $(W_t)$ :

(i) For all  $z \in B_{s^{1/\alpha}}$  and  $r := t-s$

$$\begin{aligned} p_r(y-z) dz &= \mathbb{P}_0 \left\{ r^{1/\alpha} W_1 + y \in dz \right\} \geq \inf_{a \in B_{2^{1/\alpha}}} \mathbb{P}_0 \left\{ W_1 \in 2^{1/\alpha} t^{-1/\alpha} dz - a \right\} \\ &\geq c_5 t^{-d/\alpha} dz. \end{aligned}$$

(ii) Similarly, for all  $z \in B_{s^{1/\alpha}}$ ,  $p_s(z-x) \geq c_6 s^{-d/\alpha}$ .

Combining (i) and (ii) we see that the LHS of (2.8) is bounded from below by  $c_7 t^{-d/\alpha}$ . Since  $p_t(\cdot)$  is bounded above by  $\text{const} \cdot t^{-d/\alpha}$  the claim is proved.  $\square$

**2.2. The second iteration: a subsolution growing to infinity.** We are now aiming at a lower estimate for the solution  $h_t$  of

$$(2.9) \quad \begin{aligned} \frac{\partial h_t}{\partial t} &= \Delta_\alpha h_t + h_t g_t^\beta, \\ h_0 &= \varphi \end{aligned}$$

where  $g_t$  is the subsolution of (1.1) constructed in the previous subsection. Clearly, also  $h_t$  is a subsolution of (1.1).

**Proposition 2.3.**  *$\inf \{h_t(y) | \|y\| \leq 1\} \rightarrow \infty$  as  $t \rightarrow \infty$ , more specifically there exist constants  $\varepsilon, c', c'' > 0$  such that*

$$h_t(y) \geq c' t^{-d/\alpha} \exp(c'' t^{\varepsilon\beta}) 1_{B_1}(y).$$

*Proof.* We proceed as in the proof of Proposition 2.1. First we note that the Feynman-Kac formula gives

$$(2.10) \quad h_t(y) = \int \varphi(x) p_t(y-x) \mathbb{E}_x \left[ \exp \int_0^t \gamma g_s(W_s)^\beta ds \middle| W_t = y \right] dx.$$

Using Jensen's inequality and (2.5), we see that the RHS of (2.10) is bounded from below by

$$(2.11) \quad \int \varphi(x) p_t(y-x) \exp \left( \gamma \int_2^{t/2} \mathbb{E}_x [g_s(W_s)^\beta | W_t = y] ds \right) dx \\ \geq \int_{B_1} \varphi(x) p_t(y-x) \\ \cdot \exp \left( \gamma \int_2^{t/2} c_8 s^{-\beta d/\alpha + \varepsilon \beta} \mathbb{P}_x \{W_s \in B_{s^{1/\alpha}} | W_t = y\} ds \right) dx$$

$$(2.12) \quad \geq c_8 t^{-d/\alpha} \exp(c_9 t^{\varepsilon \beta}).$$

Here, we used Lemma 2.2 and the assumption  $d = \alpha/\beta$  in the last inequality.  $\square$

### 3. COMPLETION OF THE PROOF OF BLOW-UP

From Proposition 2.3 we know that

$$(3.1) \quad K(t) := \inf_{x \in B_1} w_t(x) \rightarrow \infty \text{ as } t \rightarrow \infty$$

where  $B_1$  denotes the unit ball. In fact this is enough to guarantee blow-up. Here is an easy argument which is borrowed from [6] §4, and which we include for convenience.

We are going to re-start (1.1) with the initial condition  $w_{t_0}$ , with a suitable choice of  $t_0$  given below. Writing  $u_t := w_{t_0+t}$  we first recall the integral form of (1.1)

$$(3.2) \quad u_t(x) = \int p_t(y-x) u_0(y) dy + \int_0^t \gamma ds \int p_{t-s}(y-x) u_s(y)^{1+\beta} dy.$$

Noting that  $\zeta := \min_{x \in B_1} \min_{0 \leq s \leq 1} \mathbb{P}_x \{W_s \in B_1\}$  is strictly positive, we obtain for all  $t \in [0, 1]$  from (3.1) the estimate

$$(3.3) \quad \min_{x \in B_1} u_t(x) \geq \zeta K(t_0) + \gamma \zeta \int_0^t \left( \min_{y \in B_1} u_s(y) \right)^{1+\beta} ds.$$

Now choose  $t_0$  so big that the blow-up time of the equation

$$(3.4) \quad v(t) = \zeta K(t_0) + \gamma \zeta \int_0^t v(s)^{1+\beta} ds$$

is smaller than 1. Then, *a fortiori*,  $\min_{x \in B_1} u_1(x) = \infty$ , which shows blow-up of  $w$ .

### 4. SUBCRITICAL DIMENSIONS: ONE ITERATION SUFFICES

In the case  $d < \alpha/\beta$ , (2.6) shows that already the first subsolution  $g_t$  (constructed in Section 2.1) grows to infinity on the unit ball  $B_1$  in the sense that  $\inf \{g_t(y) | \|y\| \leq 1\} \rightarrow \infty$  as  $t \rightarrow \infty$ . Thus, in view of the previous section, for subcritical dimensions a single application of the Feynman-Kac formula suffices to show blow-up of (1.1).

## 5. A REMARK ON PORTNOY'S METHOD

Portnoy [9] studies the iteration scheme

$$(5.1) \quad \begin{aligned} v_{n+1}(x) &= (\Pi_1 v_n)(x) + (\Pi_1 v_n)^2(x) \\ v_0 &= \varphi \geq 0 \end{aligned}$$

where  $\Pi_1$  is a transition probability on  $\mathbb{R}^d$ . He shows that under suitable assumptions on  $\Pi_1$  (which include the case of a standard Brownian transition probability), (5.1) admits no bounded solution for  $d = 1$  and  $d = 2$  provided  $\varphi$  does not a.s. vanish.

A closer look on his proofs shows that he achieves this by analyzing subsolutions  $v_n^{(i)}$  of (5.1) which are given by the scheme

$$(5.2) \quad \begin{aligned} v_{n+1}^{(0)} &= \Pi_1 v_n^{(0)} = \Pi_{n+1} \varphi \\ v_{n+1}^{(i)} &= \Pi_1 v_n^{(i)} + \left( \Pi_1 v_n^{(i)} \right) \left( \Pi_1 v_n^{(i-1)} \right), \quad i = 1, 2. \end{aligned}$$

The analysis of (5.2) is carried through probabilistically in terms of random walks, which is much in the spirit of a discrete time Feynman-Kac approach.

It can be extracted from Portnoy's arguments that, for the Brownian case, say,

$$(5.3) \quad v_n^{(1)} \text{ grows to infinity for } d = 1,$$

and

$$(5.4) \quad v_n^{(2)} \text{ grows to infinity for } d = 2.$$

An easy application of Jensen's inequality plus induction shows that  $w_n$  is bounded from below by  $v_n$  (where  $w_t$  is the solution of (1.1) with  $\beta = 1$ ). Indeed,

$$\begin{aligned} w_n &= \Pi_1 w_{n-1} + \int_0^1 \Pi_s w_{n-s}^2 ds \geq \Pi_1 w_{n-1} + \left( \int_0^1 \Pi_s w_{n-s} ds \right)^2 \\ &\geq \Pi_1 w_{n-1} + \left( \int_0^1 \Pi_s \Pi_{1-s} w_{n-1} ds \right)^2 \geq \Pi_1 v_{n-1} + (\Pi_1 v_{n-1})^2 = v_n. \end{aligned}$$

Together with the argument in Section 3 above, (5.3) and (5.4) thus imply blow-up of  $w$  for  $\beta = 1$  and  $\alpha = 2$  in one and two dimensions. (In [10], a more complicated argument is used to show  $w_n \geq v_n$  and the blow-up of  $w$ .)

## 6. EXTENSIONS

**6.1. Sugitani's condition.** Sugitani [12] considers instead of (1.1) the slightly more general equation

$$(6.1) \quad \begin{aligned} \frac{\partial w_t}{\partial t} &= \Delta_\alpha w_t + F(w_t) \\ w_0 &= \varphi, \end{aligned}$$

where  $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is increasing and convex, and  $F(u) \sim \gamma u^{1+\beta}$  as  $u \rightarrow 0$ . This requires only slight modifications in Section 2:

In (2.4) and below,  $f_t(u)^\beta$  has to be replaced by  $F(f_t(u))/f_t(u)$ , which by assumption can be bounded from below by  $c f_t(u)^\beta$ .

Similarly, in (2.9) and below,  $g_t(u)^\beta$  has to be replaced by  $F(g_t(u))/g_t(u)$ .

**6.2. A time dependent nonlinearity.** Recently, Guedda and Kirane [5] showed by analytic methods blow-up of the equation

$$(6.2) \quad \frac{\partial w_t}{\partial t} = \Delta_\alpha w_t + \gamma t^\sigma w_t^{1+\beta}, \quad w_0 = \varphi \quad (\geq 0, \neq 0)$$

for  $\sigma \geq \beta d/\alpha - 1$ . This result also follows quickly from our probabilistic approach. In fact, it suffices to consider the case  $\sigma = \beta d/\alpha - 1$ .

**Lemma 6.1.** *The solution of*

$$(6.3) \quad \begin{aligned} \frac{\partial w_t}{\partial t} &= \Delta_\alpha w_t + v_t w_t^{1+\beta}, \\ w_0 &= \varphi \quad (\geq 0, \neq 0) \end{aligned}$$

with  $v : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}_+$ ,  $v_t(x) \geq \text{const} \cdot t^{\beta d/\alpha - 1} 1_{B_1}(t^{-1/\alpha}x)$  for  $t \geq 1$  blows up in finite time.

We briefly indicate the changes required in the arguments presented in sections 2 and 3 in order to prove Lemma 6.1.

**1.** Concerning the subsolution  $g_t$ , all what happens is that a factor  $s^\sigma 1_{B_{s^{1/\alpha}}}(\cdot)$  enters into the exponentials in the Feynman-Kac representation in the RHS of (2.6). Since  $s^{-\beta d/\alpha}$  in the RHS of (2.6) cancels against  $s^\sigma$ , the lower bound (2.6) remains unchanged, and so does the estimate (2.5).

**2.** Concerning the subsolution  $h_t$ , again a factor  $s^\sigma$  enters into the exponentials in (2.10) and (2.11). Since again  $(s^{-d/\alpha})^\beta$  cancels against  $s^\sigma$ , the lower bound (2.12) remains unchanged, and so does the assertion in Proposition 2.3.

**3.** Concerning the argument in Section 3, from the space-time-inhomogeneity in (6.3) a factor  $(t_0 + t)^\sigma$  enters in front of the integral in (3.3) (Observe that by our assumption  $v_t \geq \text{const} \cdot t^\sigma$  uniformly on  $B_1$  for  $t \geq 1$ ). Still, since (2.12) guarantees a super-algebraic growth of  $K(t)$ , we can choose  $t_0$  so big that the blow-up time of the equation

$$v(t) = \zeta K(t_0) + \gamma \zeta (t_0 + 1)^\sigma \int_0^t v(s)^{1+\beta} ds$$

is smaller than 1, so that the argument of Section 3 remains valid.

## 7. BLOW-UP OF SYSTEMS

In this section we apply our probabilistic approach to extend a blow-up result of Escobedo and Levine [2] (Theorem 7.1 and Remark 7.2). In Theorem 7.3 we show that a system which we investigated in [8] in high dimensions blows up at the critical dimension.

**Theorem 7.1.** *Assume that  $(u, v)$  solves*

$$(7.1) \quad \begin{aligned} \frac{\partial u_t}{\partial t} &= \Delta_{\alpha_1} u_t + u_t^{1+\beta_1} v_t^{\beta_2} \\ \frac{\partial v_t}{\partial t} &= \Delta_{\alpha_2} v_t + F(u_t, v_t) \\ u_0 &= \varphi_1, \quad v_0 = \varphi_2, \end{aligned}$$

where  $\alpha_1, \alpha_2 \in (0, 2]$ ,  $\beta_1 > 0$ ,  $\beta_2 \geq 0$ ,  $F \geq 0$ ,  $\varphi_1 \geq 0$ ,  $\varphi_2 \geq 0$  and both  $\varphi_1$  and  $\varphi_2$  do not a.s. vanish. Then  $u$  blows up if

$$(7.2) \quad \alpha_2 \leq \alpha_1 \text{ and } d \leq \left( \frac{\beta_1}{\alpha_1} + \frac{\beta_2}{\alpha_2} \right)^{-1}.$$

*Remark 7.2.* For  $\alpha_1 = \alpha_2 =: \alpha$ , (7.2) turns into the condition  $d \leq \alpha/(\beta_1 + \beta_2)$ , which is also the condition for blow-up of the partial differential equation

$$\frac{\partial u}{\partial t} = \Delta_\alpha u + u^{1+\beta_1+\beta_2}.$$

For  $\alpha = 2$ , this specializes to one of the main results in Escobedo and Levine's paper [2]. They investigate by analytic tools the system

$$\frac{\partial u}{\partial t} = \Delta u + u^{1+\beta_1}v^{\beta_2}, \quad \frac{\partial v}{\partial t} = \Delta v + u^{\theta_1}v^{\theta_2}$$

and prove blow-up under the condition  $d \leq 2/(\beta_1 + \beta_2)$ .

*Proof of Theorem 7.1.* Let  $f_{t,j}(y) := \int \varphi_j(x)p_{t,j}(y-x)dx$ ,  $j = 1, 2$ , where  $p_{t,j}$  denotes the symmetric  $\alpha_j$ -stable transition density. Obviously,  $(f_{t,1}, f_{t,2})$  is a sub-solution of (7.1), and from (2.2) we have for  $t \geq 1$

$$(7.3) \quad f_{t,1}(y) \geq Ct^{-d/\alpha_1} \mathbf{1}_{B_1}(t^{-1/\alpha_1}y)$$

and

$$(7.4) \quad f_{t,2}(y) \geq Ct^{-d/\alpha_2} \mathbf{1}_{B_1}(t^{-1/\alpha_1}y),$$

where we used the assumption  $\alpha_2 \leq \alpha_1$  to obtain (7.4). Consequently for  $t \geq 1$  and  $\|y\| \leq t^{1/\alpha_1}$

$$v_t(y)^{\beta_2} \geq C't^{-d\beta_2/\alpha_2} \geq C't^{d\beta_1/\alpha_1-1}$$

where we used the assumption (7.2) in the last inequality. Now we infer blow-up of  $u$  using Lemma 6.1.  $\square$

**Theorem 7.3.** *Assume that  $(u, v)$  solves*

$$(7.5) \quad \begin{aligned} \frac{\partial u_t}{\partial t} &= \Delta_{\alpha_1} u_t + u_t v_t \\ \frac{\partial v_t}{\partial t} &= \Delta_{\alpha_2} v_t + u_t v_t \\ u_0 &= \varphi_1, \quad v_0 = \varphi_2, \end{aligned}$$

where  $\alpha_1, \alpha_2 \in (0, 2]$ ,  $\varphi_1 \geq 0$ ,  $\varphi_2 \geq 0$  and both  $\varphi_1$  and  $\varphi_2$  do not a.s. vanish. Then  $(u, v)$  blows up if  $d \leq \min(\alpha_1, \alpha_2)$ .

*Remark 7.4.* It was shown in [8] that (7.5) admits global solutions if  $d > \min(\alpha_1, \alpha_2)$  and  $\varphi_1$  and  $\varphi_2$  are sufficiently small.

Before proving Theorem 7.3, we prepare with a lemma which is an easy generalization of Lemma 2.2. Here and below,  $(W_t^{(i)})$  denotes the symmetric stable process with index  $\alpha_i$  and  $p_{t,i}(x)$  its transition density,  $i = 1, 2$ .

**Lemma 7.5.** *Assume that  $\alpha := \alpha_2 \leq \alpha_1$ . There exists a  $c > 0$  such that for all  $t \geq 2$ ,  $y \in B_{t^{1/\alpha}}$ ,  $x \in B_1$  and  $s \in [1, t/2]$ ,*

$$\mathbb{P}_x \left\{ W_s^{(2)} \in B_{s^{1/\alpha_1}} \mid W_t^{(2)} = y \right\} \geq cs^{d/\alpha_1-d/\alpha_2}$$

*Proof.* It suffices to show (2.8) with  $cs^{d/\alpha_1-d/\alpha_2}$  instead of  $c_5$  and  $p_{t,2}$  instead of  $p_t$ .

Again we have (i) and (ii) from the proof of Lemma 2.2, now with  $(W_t^{(2)})$  instead of  $(W_t)$ . Integrating the bound  $s^{-d/\alpha_2}$  over  $B_{s^{1/\alpha_1}}$  then gives the factor  $\text{const} \cdot s^{d/\alpha_1-d/\alpha_2}$ .  $\square$



*Proof of Theorem 7.3.* The proof proceeds in three steps. First we prove using the Feynman-Kac representation (see (1.3)) that (at least one component of) the solution  $(u, v)$  locally grows to  $\infty$ . In a second step we show that  $(u, v)$  can be bounded below uniformly in  $B_1 \times B_1$  similarly as in Section 3 but this time by comparison with the solution of a suitable coupled pair of ODEs. Finally, in step 3 we show that this system of ODEs blows up.

1. From (2.3) we have

$$(7.6) \quad u_t \geq c_1 t^{-d/\alpha_1} \mathbf{1}_{B_{t^{1/\alpha_1}}}$$

and

$$(7.7) \quad v_t \geq c_2 t^{-d/\alpha_2} \mathbf{1}_{B_{t^{1/\alpha_2}}}$$

for all  $t \geq t_0$  for some sufficiently large  $t_0$ . Let us now assume without loss of generality that  $\alpha_2 \leq \alpha_1$ . By the Feynman-Kac formula we have

$$u_t(y) = \int \varphi_1(x) p_{t,1}(y-x) \mathbb{E}_x \left[ \exp \int_0^t v_s(W_s^{(1)}) ds \mid W_t^{(1)} = y \right] dx.$$

For  $t \geq 2t_0$ , by Jensen's inequality and (7.7), this can be bounded from below by

$$\int \varphi_1(x) p_{t,1}(y-x) \exp \left( \int_{t_0}^{t/2} c_2 s^{-d/\alpha_2} \mathbb{P}_x \left\{ W_s^{(1)} \in B_{s^{1/\alpha_2}} \mid W_t^{(1)} = y \right\} ds \right) dx.$$

Noting that  $B_{s^{1/\alpha_2}} \supseteq B_{s^{1/\alpha_1}}$  and using Lemma 2.2, we thus arrive at the lower bound

$$(7.8) \quad c_3 t^{-d/\alpha_1} \exp \left( c_4 \int_{t_0}^{t/2} s^{-d/\alpha_2} ds \right).$$

If  $d < \alpha_2$ , then this lower bound grows super-algebraically from which we will infer blow-up in steps 2 and 3.

Let us now assume  $d = \alpha_2$ . Then (7.8) turns into the lower bound

$$(7.9) \quad u_t(y) \geq c_5 t^{-d/\alpha_1 + \varepsilon}$$

(uniformly in  $y \in B_{t^{1/\alpha_1}}$  for  $t$  sufficiently large). Another application of the Feynman-Kac formula gives

$$(7.10) \quad v_t(y) = \int \varphi_2(x) p_{t,2}(y-x) \mathbb{E}_x \left[ \exp \int_0^t u_s(W_s^{(2)}) ds \mid W_t^{(2)} = y \right] dx.$$

Using Jensen's inequality and (7.9), we can bound this from below by

$$\int \varphi_2(x) p_{t,2}(y-x) \exp \int_{t_0}^{t/2} c_1 s^{-d/\alpha_1 + \varepsilon} \mathbb{P}_x \left\{ W_s^{(2)} \in B_{s^{1/\alpha_1}} \mid W_t^{(2)} = y \right\} ds dx.$$

In view of Lemma 7.5 we thus obtain as a lower bound for  $v_t(y)$  (as long as  $t$  is sufficiently large and  $y \in B_{t^{1/\alpha_2}}$ ):

$$\begin{aligned} c_6 t^{-d/\alpha_2} \exp \int_{t_0}^{t/2} c_7 s^{-d/\alpha_1 + \varepsilon} s^{d/\alpha_1 - d/\alpha_2} ds &= c_6 t^{-d/\alpha_2} \exp \int_{t_0}^{t/2} c_7 s^{-d/\alpha_2 + \varepsilon} ds \\ &= c_6 t^{-d/\alpha_2} \exp(c_8 t^\varepsilon). \end{aligned}$$

Thus in this case  $v$  grows (super-algebraically).

2. Rewriting (7.5) in integral form we obtain for  $t, t_0 \geq 0$

$$\begin{aligned} u_{t+t_0}(x) &= \int dy p_{t,1}(y-x)u_{t_0}(y) + \int_0^t ds \int dy p_{t-s,1}(y-x)u_{t_0+s}(y)v_{t_0+s}(y) \\ v_{t+t_0}(x) &= \int dy p_{t,2}(y-x)v_{t_0}(y) + \int_0^t ds \int dy p_{t-s,2}(y-x)u_{t_0+s}(y)v_{t_0+s}(y). \end{aligned}$$

Let  $\zeta := \min_{x \in B_1} \min_{0 \leq s \leq 1} \left( \mathbb{P}_x(W_s^{(1)} \in B_1) \wedge \mathbb{P}_x(W_s^{(2)} \in B_1) \right) > 0$  and  $\tilde{u}(t) := \min_{x \in B_1} u_t(x)$ ,  $\tilde{v}(t) := \min_{x \in B_1} v_t(x)$ . This allows us to estimate for  $t \in [0, 1]$

$$(7.11) \quad \begin{aligned} \tilde{u}(t_0+t) &\geq \zeta \tilde{u}(t_0) + \zeta \int_0^t ds \tilde{u}(t_0+s) \tilde{v}(t_0+s), \\ \tilde{v}(t_0+t) &\geq \zeta \tilde{v}(t_0) + \zeta \int_0^t ds \tilde{u}(t_0+s) \tilde{v}(t_0+s). \end{aligned}$$

In step 1 we saw that  $(\tilde{u} \vee \tilde{v})(t_0) \rightarrow \infty$  super-algebraically while  $(\tilde{u} \wedge \tilde{v})(t_0)$  decays at most algebraically. Thus,  $t_0$  can be chosen so big that the blow-up time of

$$(7.12) \quad U(t) = \zeta \tilde{u}(t_0) + \zeta \int_0^t ds U(s)V(s), \quad V(t) = \zeta \tilde{v}(t_0) + \zeta \int_0^t ds U(s)V(s)$$

is less than 1 (see step 3). We conclude that  $(u, v)$  blows up.

3. It remains to study (7.12) which in ODE form is

$$U'(t) = \zeta U(t)V(t) = V'(t)$$

and WLOG assume that  $U_0 := U(0) \geq V(0) =: V_0$ . The solution is given by

$$\begin{pmatrix} U(t) \\ V(t) \end{pmatrix} = \begin{cases} \frac{U_0 - V_0}{1 - (V_0/U_0) \exp(\zeta(U_0 - V_0)t)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ V_0 - U_0 \end{pmatrix} & \text{if } U_0 > V_0 \\ \frac{1}{1/U_0 - \zeta t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{if } U_0 = V_0 \end{cases}$$

for  $0 \leq t < \tau$  with explosion time

$$\tau = \begin{cases} \frac{\log U_0 - \log V_0}{\zeta(U_0 - V_0)} & \text{if } U_0 > V_0 \\ \frac{1}{\zeta U_0} & \text{if } U_0 = V_0. \end{cases}$$

In our scenario we have  $U_0 \geq \exp(\varepsilon_1 t_0)$ ,  $V_0 \geq t_0^{-\varepsilon_2}$  for some  $\varepsilon_1, \varepsilon_2 > 0$ , which allows to chose  $t_0$  big enough to enforce  $\tau < 1$ . Indeed if  $V_0 \geq U_0/2$  we have  $\tau \leq 2/(\zeta U_0)$ , and if  $1 \leq V_0 < U_0/2$  we can estimate  $\tau \leq (2 \log U_0)/(\zeta U_0)$ . Finally, if  $V_0 < 1$  we have  $\tau \leq (\log U_0)/(\zeta(U_0 - 1)) + \varepsilon_2 \log t_0/(\zeta(\exp(\varepsilon_1 t_0) - 1))$ .  $\square$

*Remark 7.6.* Consider instead of (7.5) the more general system

$$(7.13) \quad \begin{aligned} \frac{\partial u_t}{\partial t} &= \Delta_{\alpha_1} u_t + u_t v_t^{\beta_1} \\ \frac{\partial v_t}{\partial t} &= \Delta_{\alpha_2} v_t + u_t^{\beta_2} v_t \\ u_0 &= \varphi_1, \quad v_0 = \varphi_2, \end{aligned}$$

where  $\alpha_1, \alpha_2, \varphi_1, \varphi_2$  are as in Theorem 7.3, and  $\beta_1, \beta_2 > 0$ . Assume that  $\alpha_2 \leq \alpha_1$ . Proceeding as in the proof of Theorem 7.3 but using the simple bound (7.6) instead

of (7.9) in the Feynman-Kac representation corresponding to (7.10) one obtains quickly that (7.13) has a growing subsolution if

$$(7.14) \quad d < \max \left( \frac{\alpha_2}{\beta_1}, \left( \frac{\beta_2 - 1}{\alpha_1} + \frac{1}{\alpha_2} \right)^{-1} \right).$$

As before, from this one infers blow-up, this time by comparing with the ODE system  $U'(t) = U(t)V^{\beta_1}(t)$ ,  $V'(t) = V(t)U^{\beta_2}(t)$ .

It remains an interesting question whether the RHS of (7.14) is the critical dimension for blow-up of (7.13) and whether there is blow-up at the critical dimension. We conjecture that this is the case at least for  $\alpha_1 = \alpha_2 =: \alpha$ , in which case the RHS of (7.14) turns into  $\alpha / \min(\beta_1, \beta_2)$ . Indeed, for the special case  $\alpha = 2$ , this was proved by Escobedo and Levine [2].

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