

BLOW-UP PHENOMENA FOR THE YAMABE EQUATION

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1. INTRODUCTION

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. The Yamabe problem is concerned with finding metrics of constant scalar curvature in the conformal class of g . This problem can be reduced to a semi-linear elliptic PDE. Indeed, the metric $u^{\frac{4}{n-2}} g$ has constant scalar curvature c if and only if

$$(1) \quad \frac{4(n-1)}{n-2} \Delta_g u - R_g u + c u^{\frac{n+2}{n-2}} = 0,$$

where Δ_g is the Laplace operator with respect to g and R_g denotes the scalar curvature of g . Clearly, every solution of (1) is a critical point of the functional

$$(2) \quad E_g(u) = \frac{\int_M \left(\frac{4(n-1)}{n-2} |du|_g^2 + R_g u^2 \right) dvol_g}{\left(\int_M u^{\frac{2n}{n-2}} dvol_g \right)^{\frac{n-2}{n}}}.$$

It is well known that the PDE (1) has at least one positive solution for any choice of (M, g) . If $n \geq 6$ and (M, g) is not locally conformally flat, this follows from results of T. Aubin [3]. The remaining cases were solved by R. Schoen using the positive mass theorem [16].

Solutions to (1) are not usually unique. As an example, consider the product metric on $S^1(L) \times S^{n-1}(1)$. If L is sufficiently small, then the Yamabe PDE has a unique solution. On the other hand, there are many non-minimizing solutions if L is large. D. Pollack [14] has used gluing techniques to construct high energy solutions on more general background manifolds: given any conformal class with positive Yamabe constant and any positive integer N , there exists a new conformal class which is close to the original one in the C^0 -norm and contains at least N metrics of constant scalar curvature (see [14], Theorem 0.1).

It is an interesting question whether the set of all solutions to the Yamabe PDE is compact (in the C^2 -topology, say). A well-known conjecture states that this should be true unless (M, g) is conformally equivalent to the round sphere (see [17], [18], [19]). This conjecture has been verified in low dimensions and in the locally conformally flat case: if (M, g) is locally conformally flat, compactness follows from work of R. Schoen [17], [18]. Moreover, Schoen proposed a strategy for proving the conjecture in the non-locally conformally flat case based on the Pohozaev identity.

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In [12], Y.Y. Li and M. Zhu [12] followed this strategy to prove compactness in dimension 3. O. Druet [7] proved the conjecture in dimensions 4 and 5. Recently, F. Marques [13] showed that compactness holds up to dimension 7. The same result was obtained independently by Y.Y. Li and L. Zhang [11]. Moreover, Li and Zhang showed that compactness holds in all dimensions provided that $|W_g(p)| + |\nabla W_g(p)| > 0$ for all $p \in M$. M. Khuri, F. Marques, and R. Schoen [10] proved compactness up to dimension 24, assuming that the positive mass theorem holds.¹

In this paper, we address the opposite question: is it possible to construct Riemannian manifolds (M, g) such that the set of constant scalar curvature metrics in the conformal class of g is non-compact? So far, the only known examples where compactness fails involve non-smooth background metrics. The first result in this direction was established by A. Ambrosetti and A. Malchiodi [2]. This result was subsequently improved by M. Berti and A. Malchiodi [6]. Given positive integers n and k such that $k \geq 2$ and $n \geq 4k + 3$, Berti and Malchiodi showed that there exists a Riemannian metric g on S^n (of class C^k) for which the set of solutions to the Yamabe PDE (1) fails to be compact (see [6], Theorem 1.1). A survey of these results can be found in [1]. Recently, O. Druet and E. Hebey [8] showed that blow-up can occur for problems of the form $Lu + cu^{\frac{n+2}{n-2}} = 0$, where L is a lower order perturbation of the conformal Laplacian on S^n .

We improve the results of Berti and Malchiodi by showing that the set of solutions to the Yamabe PDE (1) can fail to be compact even if the background metric g is C^∞ smooth. In the examples we construct, the blowing-up sequence develops a singularity consisting of exactly one bubble.

Theorem. *Assume that $n \geq 52$. Then there exists a Riemannian metric g on S^n (of class C^∞) and a sequence of positive functions $v_\nu \in C^\infty(S^n)$ ($\nu \in \mathbb{N}$) with the following properties:*

- (i) g is not conformally flat,
- (ii) v_ν is a solution of the Yamabe PDE (1) for all $\nu \in \mathbb{N}$,
- (iii) $E_g(v_\nu) < Y(S^n)$ for all $\nu \in \mathbb{N}$, and $E_g(v_\nu) \rightarrow Y(S^n)$ as $\nu \rightarrow \infty$,
- (iv) $\sup_{S^n} v_\nu \rightarrow \infty$ as $\nu \rightarrow \infty$.

(Here, $Y(S^n)$ denotes the Yamabe energy of the round metric on S^n .)

Let us sketch the main steps involved in the proof of the Theorem. For convenience, we will work on \mathbb{R}^n instead of S^n . Let g be a smooth metric on \mathbb{R}^n which agrees with the Euclidean metric outside a ball of radius 1. We will assume throughout the paper that $\det g(x) = 1$ for all $x \in \mathbb{R}^n$, so that the volume form associated with g agrees with the Euclidean volume form.

Our goal is to construct solutions to the Yamabe PDE on (\mathbb{R}^n, g) . In Section 2, we show that this problem can be reduced to finding critical points of a certain function $\mathcal{F}_g(\xi, \varepsilon)$, where ξ is a vector in \mathbb{R}^n and ε is a positive real number. This idea has been used by many authors (see, e.g., [2] or [6]). In Section 3, we show that the function $\mathcal{F}_g(\xi, \varepsilon)$ can be approximated by an auxiliary function $F(\xi, \varepsilon)$. In Section 4, we prove that the function $F(\xi, \varepsilon)$ has a critical point, which is a strict local minimum. Finally, in Section 5, we use a perturbation argument to construct critical points of the function $\mathcal{F}_g(\xi, \varepsilon)$. From this the main result follows.

¹T. Aubin has recently claimed a general compactness theorem in all dimensions [4], [5]. We have, however, been unable to verify some of the arguments in [4].

2. LYAPUNOV-SCHMIDT REDUCTION

Let

$$\mathcal{E} = \left\{ w \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \cap W_{loc}^{1,2}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |dw|^2 < \infty \right\}.$$

By Sobolev’s inequality, there exists a constant K , depending only on n , such that

$$\left(\int_{\mathbb{R}^n} |w|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq K \int_{\mathbb{R}^n} |dw|^2$$

for all $w \in \mathcal{E}$. We define a norm on \mathcal{E} by $\|w\|_{\mathcal{E}}^2 = \int_{\mathbb{R}^n} |dw|^2$. It is easy to see that \mathcal{E} , equipped with this norm, is complete.

Given any pair $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$, we define a function $u_{(\xi, \varepsilon)} : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$u_{(\xi, \varepsilon)}(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x - \xi|^2} \right)^{\frac{n-2}{2}}.$$

The function $u_{(\xi, \varepsilon)}$ satisfies the elliptic PDE

$$\Delta u_{(\xi, \varepsilon)} + n(n-2) u_{(\xi, \varepsilon)}^{\frac{n+2}{n-2}} = 0.$$

It is well known that

$$\int_{\mathbb{R}^n} u_{(\xi, \varepsilon)}^{\frac{2n}{n-2}} = \left(\frac{Y(S^n)}{4n(n-1)} \right)^{\frac{n}{2}}$$

for all $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$. We next define

$$\varphi_{(\xi, \varepsilon, 0)}(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x - \xi|^2} \right)^{\frac{n+2}{2}} \frac{\varepsilon^2 - |x - \xi|^2}{\varepsilon^2 + |x - \xi|^2}$$

and

$$\varphi_{(\xi, \varepsilon, k)}(x) = \left(\frac{\varepsilon}{\varepsilon^2 + |x - \xi|^2} \right)^{\frac{n+2}{2}} \frac{2\varepsilon(x_k - \xi_k)}{\varepsilon^2 + |x - \xi|^2}$$

for $k = 1, \dots, n$. It is easy to see that the norm $\|\varphi_{(\xi, \varepsilon, k)}\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}$ is constant in ξ and ε . Finally, we define a closed subspace $\mathcal{E}_{(\xi, \varepsilon)} \subset \mathcal{E}$ by

$$\mathcal{E}_{(\xi, \varepsilon)} = \left\{ w \in \mathcal{E} : \int_{\mathbb{R}^n} \varphi_{(\xi, \varepsilon, k)} w = 0 \text{ for } k = 0, 1, \dots, n \right\}.$$

Clearly, $u_{(\xi, \varepsilon)} \in \mathcal{E}_{(\xi, \varepsilon)}$.

Proposition 1. *Consider a Riemannian metric on \mathbb{R}^n of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on \mathbb{R}^n satisfying $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha \leq 1$ for all $x \in \mathbb{R}^n$ and $h(x) = 0$ for $|x| \geq 1$. There exists a constant C , depending only on n , such that*

$$\left\| \Delta_g u_{(\xi, \varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi, \varepsilon)} + n(n-2) u_{(\xi, \varepsilon)}^{\frac{n+2}{n-2}} \right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C \alpha$$

for all pairs $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$.

Proof. Using the pointwise estimate

$$\begin{aligned} & \left| \Delta_g u_{(\xi, \varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi, \varepsilon)} + n(n-2) u_{(\xi, \varepsilon)}^{\frac{n+2}{n-2}} \right| \\ & \leq C |h| |\partial^2 u_{(\xi, \varepsilon)}| + C |\partial h| |\partial u_{(\xi, \varepsilon)}| + C (|\partial^2 h| + |\partial h|^2) u_{(\xi, \varepsilon)}, \end{aligned}$$

we obtain

$$\begin{aligned} & \left\| \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)} + n(n-2) u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\ & \leq C \|h\|_{L^\infty(\mathbb{R}^n)} \|\partial^2 u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} + C \|\partial h\|_{L^n(\mathbb{R}^n)} \|\partial u_{(\xi,\varepsilon)}\|_{L^2(\mathbb{R}^n)} \\ & \quad + C (\|\partial^2 h\|_{L^{\frac{n}{2}}(\mathbb{R}^n)} + \|\partial h\|_{L^n(\mathbb{R}^n)}) \|u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\ & \leq C \alpha. \end{aligned}$$

This proves the assertion. □

Proposition 2. *There exists a positive constant θ , depending only on n , such that*

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(|dw|^2 - n(n+2) u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} w^2 \right) \\ & \geq 2\theta \|w\|_{\mathcal{E}}^2 - \frac{16n^2}{\theta} \left(\int_{\mathbb{R}^n} u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} w \right)^2 \end{aligned}$$

for all $w \in \mathcal{E}_{(\xi,\varepsilon)}$.

Proposition 2 follows from an analysis of the eigenvalues of the Laplace operator on S^n . The details can be found in [15].

Corollary 3. *Consider a Riemannian metric on \mathbb{R}^n of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on \mathbb{R}^n satisfying $h(x) = 0$ for $|x| \geq 1$. There exists a positive constant $\alpha_0 \leq 1$, depending only on n , with the following property: if $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_0$ for all $x \in \mathbb{R}^n$, then we have*

$$\left(\int_{\mathbb{R}^n} |w|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq 2K \int_{\mathbb{R}^n} \left(|dw|_g^2 + \frac{n-2}{4(n-1)} R_g w^2 \right)$$

for all $w \in \mathcal{E}$ and

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(|dw|_g^2 + \frac{n-2}{4(n-1)} R_g w^2 - n(n+2) u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} w^2 \right) \\ & \geq \theta \|w\|_{\mathcal{E}}^2 - \frac{1}{\theta} \left(\int_{\mathbb{R}^n} \left(\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)} + n(n+2) u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \right) w \right)^2 \end{aligned}$$

for all $w \in \mathcal{E}_{(\xi,\varepsilon)}$.

Proof. Using Proposition 1 and Hölder’s inequality, we obtain

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \left(\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)} + n(n+2) u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \right) w \right| \\ & \geq 4n \left| \int_{\mathbb{R}^n} u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} w \right| - C \alpha_0 \|w\|_{\mathcal{E}}. \end{aligned}$$

This implies

$$\begin{aligned} & \left(\int_{\mathbb{R}^n} \left(\Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)} + n(n+2) u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \right) w \right)^2 \\ & \geq 16n^2 \left(\int_{\mathbb{R}^n} u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} w \right)^2 - \theta^2 \|w\|_{\mathcal{E}}^2 \end{aligned}$$

if α_0 is sufficiently small. Hence, the assertion follows from Proposition 2. □

Proposition 4. Consider a Riemannian metric on \mathbb{R}^n of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on \mathbb{R}^n satisfying $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_0$ for all $x \in \mathbb{R}^n$ and $h(x) = 0$ for $|x| \geq 1$. Given any pair $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$ and any function $f \in L^{\frac{2n}{n+2}}(\mathbb{R}^n)$, there exists a unique function $w \in \mathcal{E}_{(\xi, \varepsilon)}$ such that

$$\int_{\mathbb{R}^n} \left(\langle dw, d\psi \rangle_g + \frac{n-2}{4(n-1)} R_g w \psi - n(n+2) u_{(\xi, \varepsilon)}^{\frac{4}{n-2}} w \psi \right) = \int_{\mathbb{R}^n} f \psi$$

for all test functions $\psi \in \mathcal{E}_{(\xi, \varepsilon)}$. Moreover, we have $\|w\|_{\mathcal{E}} \leq C \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}$, where C is a constant that depends only on n .

Proof. Suppose that $w \in \mathcal{E}_{(\xi, \varepsilon)}$ and

$$\int_{\mathbb{R}^n} \left(\langle dw, d\psi \rangle_g + \frac{n-2}{4(n-1)} R_g w \psi - n(n+2) u_{(\xi, \varepsilon)}^{\frac{4}{n-2}} w \psi \right) = \int_{\mathbb{R}^n} f \psi$$

for all test functions $\psi \in \mathcal{E}_{(\xi, \varepsilon)}$. This implies

$$\int_{\mathbb{R}^n} \left(|dw|_g^2 + \frac{n-2}{4(n-1)} R_g w^2 - n(n+2) u_{(\xi, \varepsilon)}^{\frac{4}{n-2}} w^2 \right) = \int_{\mathbb{R}^n} f w$$

and

$$\int_{\mathbb{R}^n} \left(\Delta_g u_{(\xi, \varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi, \varepsilon)} + n(n+2) u_{(\xi, \varepsilon)}^{\frac{n+2}{n-2}} \right) w = - \int_{\mathbb{R}^n} u_{(\xi, \varepsilon)} f.$$

Using Corollary 3, we obtain

$$\begin{aligned} \theta \|w\|_{\mathcal{E}}^2 &\leq \int_{\mathbb{R}^n} \left(|dw|_g^2 + \frac{n-2}{4(n-1)} R_g w^2 - n(n+2) u_{(\xi, \varepsilon)}^{\frac{4}{n-2}} w^2 \right) \\ &\quad + \frac{1}{\theta} \left(\int_{\mathbb{R}^n} \left(\Delta_g u_{(\xi, \varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi, \varepsilon)} + n(n+2) u_{(\xi, \varepsilon)}^{\frac{n+2}{n-2}} \right) w \right)^2 \\ &\leq \left(\int_{\mathbb{R}^n} |f|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{2n}} \left(\int_{\mathbb{R}^n} |w|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \\ &\quad + \frac{1}{\theta} \left(\int_{\mathbb{R}^n} u_{(\xi, \varepsilon)}^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \left(\int_{\mathbb{R}^n} |f|^{\frac{2n}{n+2}} \right)^{\frac{n+2}{n}} \\ &\leq K^{\frac{1}{2}} \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \|w\|_{\mathcal{E}} + \frac{1}{\theta} \left(\frac{Y(S^n)}{4n(n-1)} \right)^{\frac{n-2}{2}} \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}^2. \end{aligned}$$

Hence, it follows from Young’s inequality that

$$\frac{\theta}{2} \|w\|_{\mathcal{E}}^2 \leq \frac{K}{2\theta} \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}^2 + \frac{1}{\theta} \left(\frac{Y(S^n)}{4n(n-1)} \right)^{\frac{n-2}{2}} \|f\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}^2.$$

From this the uniqueness statement follows easily.

In order to prove the existence part, it suffices to minimize the functional

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(|dw|_g^2 + \frac{n-2}{4(n-1)} R_g w^2 - n(n+2) u_{(\xi, \varepsilon)}^{\frac{4}{n-2}} w^2 - 2fw \right) \\ &\quad + \frac{1}{\theta} \left(\int_{\mathbb{R}^n} \left(\Delta_g u_{(\xi, \varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi, \varepsilon)} + n(n+2) u_{(\xi, \varepsilon)}^{\frac{n+2}{n-2}} \right) w \right)^2 \end{aligned}$$

over all functions $w \in \mathcal{E}_{(\xi, \varepsilon)}$. □

Proposition 5. *Consider a Riemannian metric on \mathbb{R}^n of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on \mathbb{R}^n satisfying $h(x) = 0$ for $|x| \geq 1$. Moreover, let $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$. There exists a positive constant $\alpha_1 \leq \alpha_0$, depending only on n , with the following property: if $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_1$ for all $x \in \mathbb{R}^n$, then there exists a function $v_{(\xi, \varepsilon)} \in \mathcal{E}$ such that $v_{(\xi, \varepsilon)} - u_{(\xi, \varepsilon)} \in \mathcal{E}_{(\xi, \varepsilon)}$ and*

$$\int_{\mathbb{R}^n} \left(\langle dv_{(\xi, \varepsilon)}, d\psi \rangle_g + \frac{n-2}{4(n-1)} R_g v_{(\xi, \varepsilon)} \psi - n(n-2) |v_{(\xi, \varepsilon)}|^{\frac{4}{n-2}} v_{(\xi, \varepsilon)} \psi \right) = 0$$

for all test functions $\psi \in \mathcal{E}_{(\xi, \varepsilon)}$. Moreover, we have the estimate

$$\begin{aligned} & \|v_{(\xi, \varepsilon)} - u_{(\xi, \varepsilon)}\|_{\mathcal{E}} \\ & \leq C \left\| \Delta_g u_{(\xi, \varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi, \varepsilon)} + n(n-2) u_{(\xi, \varepsilon)}^{\frac{n+2}{n-2}} \right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}, \end{aligned}$$

where C is a constant that depends only on n .

Proof. Let $G_{(\xi, \varepsilon)} : L^{\frac{2n}{n+2}}(\mathbb{R}^n) \rightarrow \mathcal{E}_{(\xi, \varepsilon)}$ be the solution operator constructed in Proposition 4. We define a non-linear mapping $\Phi_{(\xi, \varepsilon)} : \mathcal{E}_{(\xi, \varepsilon)} \rightarrow \mathcal{E}_{(\xi, \varepsilon)}$ by

$$\begin{aligned} & \Phi_{(\xi, \varepsilon)}(w) \\ & = G_{(\xi, \varepsilon)} \left(\Delta_g u_{(\xi, \varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi, \varepsilon)} + n(n-2) u_{(\xi, \varepsilon)}^{\frac{n+2}{n-2}} \right) \\ & + n(n-2) G_{(\xi, \varepsilon)} \left(|u_{(\xi, \varepsilon)} + w|^{\frac{4}{n-2}} (u_{(\xi, \varepsilon)} + w) - u_{(\xi, \varepsilon)}^{\frac{n+2}{n-2}} - \frac{n+2}{n-2} u_{(\xi, \varepsilon)}^{\frac{4}{n-2}} w \right). \end{aligned}$$

It follows from Proposition 1 that $\|\Phi_{(\xi, \varepsilon)}(0)\|_{\mathcal{E}} \leq C \alpha_1$. Using the pointwise estimate

$$\begin{aligned} & \left| |u_{(\xi, \varepsilon)} + w|^{\frac{4}{n-2}} (u_{(\xi, \varepsilon)} + w) - |u_{(\xi, \varepsilon)} + \tilde{w}|^{\frac{4}{n-2}} (u_{(\xi, \varepsilon)} + \tilde{w}) \right. \\ & \quad \left. - \frac{n+2}{n-2} u_{(\xi, \varepsilon)}^{\frac{4}{n-2}} (w - \tilde{w}) \right| \\ & \leq C (|w|^{\frac{4}{n-2}} + |\tilde{w}|^{\frac{4}{n-2}}) |w - \tilde{w}|, \end{aligned}$$

we obtain

$$\begin{aligned} & \|\Phi_{(\xi, \varepsilon)}(w) - \Phi_{(\xi, \varepsilon)}(\tilde{w})\|_{\mathcal{E}} \\ & \leq C \left\| |u_{(\xi, \varepsilon)} + w|^{\frac{4}{n-2}} (u_{(\xi, \varepsilon)} + w) - |u_{(\xi, \varepsilon)} + \tilde{w}|^{\frac{4}{n-2}} (u_{(\xi, \varepsilon)} + \tilde{w}) \right. \\ & \quad \left. - \frac{n+2}{n-2} u_{(\xi, \varepsilon)}^{\frac{4}{n-2}} (w - \tilde{w}) \right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\ & \leq C \left(\|w\|_{L^{\frac{4}{n-2}}(\mathbb{R}^n)}^{\frac{4}{n-2}} + \|\tilde{w}\|_{L^{\frac{4}{n-2}}(\mathbb{R}^n)}^{\frac{4}{n-2}} \right) \|w - \tilde{w}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \end{aligned}$$

for all functions $w, \tilde{w} \in \mathcal{E}_{(\xi, \varepsilon)}$. This implies

$$\|\Phi_{(\xi, \varepsilon)}(w) - \Phi_{(\xi, \varepsilon)}(\tilde{w})\|_{\mathcal{E}} \leq C \left(\|w\|_{\mathcal{E}}^{\frac{4}{n-2}} + \|\tilde{w}\|_{\mathcal{E}}^{\frac{4}{n-2}} \right) \|w - \tilde{w}\|_{\mathcal{E}}$$

for $w, \tilde{w} \in \mathcal{E}_{(\xi, \varepsilon)}$. Hence, if α_1 is sufficiently small, then the contraction mapping principle implies that the mapping $\Phi_{(\xi, \varepsilon)}$ has a unique fixed point. From this the assertion follows easily. \square

We next define a function $\mathcal{F}_g : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$ by

$$\begin{aligned} \mathcal{F}_g(\xi, \varepsilon) &= \int_{\mathbb{R}^n} \left(|dv_{(\xi, \varepsilon)}|_g^2 + \frac{n-2}{4(n-1)} R_g v_{(\xi, \varepsilon)}^2 - (n-2)^2 |v_{(\xi, \varepsilon)}|^{\frac{2n}{n-2}} \right) \\ &\quad - 2(n-2) \left(\frac{Y(S^n)}{4n(n-1)} \right)^{\frac{n}{2}}. \end{aligned}$$

If we choose α_1 small enough, then we obtain the following result:

Proposition 6. *The function \mathcal{F}_g is continuously differentiable. Moreover, if $(\bar{\xi}, \bar{\varepsilon})$ is a critical point of the function \mathcal{F}_g , then the function $v_{(\bar{\xi}, \bar{\varepsilon})}$ is a non-negative weak solution of the equation*

$$\Delta_g v_{(\bar{\xi}, \bar{\varepsilon})} - \frac{n-2}{4(n-1)} R_g v_{(\bar{\xi}, \bar{\varepsilon})} + n(n-2) v_{(\bar{\xi}, \bar{\varepsilon})}^{\frac{n+2}{n-2}} = 0.$$

Proof. By definition of $v_{(\xi, \varepsilon)}$, we can find real numbers $a_k(\xi, \varepsilon)$, $k = 0, 1, \dots, n$, such that

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\langle dv_{(\xi, \varepsilon)}, d\psi \rangle_g + \frac{n-2}{4(n-1)} R_g v_{(\xi, \varepsilon)} v_{(\xi, \varepsilon)} \psi - n(n-2) |v_{(\xi, \varepsilon)}|^{\frac{4}{n-2}} v_{(\xi, \varepsilon)} \psi \right) \\ &= \sum_{k=0}^n a_k(\xi, \varepsilon) \int_{\mathbb{R}^n} \varphi_{(\xi, \varepsilon, k)} \psi \end{aligned}$$

for all test functions $\psi \in \mathcal{E}$. This implies

$$\frac{\partial}{\partial \varepsilon} \mathcal{F}_g(\varepsilon, \xi) = 2 \sum_{k=0}^n a_k(\xi, \varepsilon) \int_{\mathbb{R}^n} \varphi_{(\xi, \varepsilon, k)} \frac{\partial}{\partial \varepsilon} v_{(\xi, \varepsilon)}$$

and

$$\frac{\partial}{\partial \xi_j} \mathcal{F}_g(\varepsilon, \xi) = 2 \sum_{k=0}^n a_k(\xi, \varepsilon) \int_{\mathbb{R}^n} \varphi_{(\xi, \varepsilon, k)} \frac{\partial}{\partial \xi_j} v_{(\xi, \varepsilon)}$$

for $j = 1, \dots, n$. On the other hand, we have

$$\int_{\mathbb{R}^n} \varphi_{(\xi, \varepsilon, k)} (v_{(\xi, \varepsilon)} - u_{(\xi, \varepsilon)}) = 0$$

since $v_{(\xi, \varepsilon)} - u_{(\xi, \varepsilon)} \in \mathcal{E}_{(\xi, \varepsilon)}$. This implies

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \frac{\partial}{\partial \varepsilon} \varphi_{(\xi, \varepsilon, k)} (v_{(\xi, \varepsilon)} - u_{(\xi, \varepsilon)}) + \int_{\mathbb{R}^n} \varphi_{(\xi, \varepsilon, k)} \frac{\partial}{\partial \varepsilon} (v_{(\xi, \varepsilon)} - u_{(\xi, \varepsilon)}) \\ &= \int_{\mathbb{R}^n} \frac{\partial}{\partial \varepsilon} \varphi_{(\xi, \varepsilon, k)} (v_{(\xi, \varepsilon)} - u_{(\xi, \varepsilon)}) + \int_{\mathbb{R}^n} \varphi_{(\xi, \varepsilon, k)} \frac{\partial}{\partial \varepsilon} v_{(\xi, \varepsilon)} \\ &\quad + \frac{n-2}{2(n+1)} \left(\frac{Y(S^n)}{4n(n-1)} \right)^{\frac{n}{2}} \varepsilon^{-1} \delta_{0k} \end{aligned}$$

and

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \frac{\partial}{\partial \xi_j} \varphi_{(\xi, \varepsilon, k)} (v_{(\xi, \varepsilon)} - u_{(\xi, \varepsilon)}) + \int_{\mathbb{R}^n} \varphi_{(\xi, \varepsilon, k)} \frac{\partial}{\partial \xi_j} (v_{(\xi, \varepsilon)} - u_{(\xi, \varepsilon)}) \\ &= \int_{\mathbb{R}^n} \frac{\partial}{\partial \xi_j} \varphi_{(\xi, \varepsilon, k)} (v_{(\xi, \varepsilon)} - u_{(\xi, \varepsilon)}) + \int_{\mathbb{R}^n} \varphi_{(\xi, \varepsilon, k)} \frac{\partial}{\partial \xi_j} v_{(\xi, \varepsilon)} \\ &\quad - \frac{n-2}{2(n+1)} \left(\frac{Y(S^n)}{4n(n-1)} \right)^{\frac{n}{2}} \varepsilon^{-1} \delta_{jk} \end{aligned}$$

for $j = 1, \dots, n$. Putting these facts together, we obtain

$$\begin{aligned} & -\frac{n-2}{n+1} \left(\frac{Y(S^n)}{4n(n-1)} \right)^{\frac{n}{2}} a_0(\xi, \varepsilon) \\ & = \varepsilon \frac{\partial}{\partial \varepsilon} \mathcal{F}_g(\xi, \varepsilon) + 2\varepsilon \sum_{k=0}^n a_k(\xi, \varepsilon) \int_{\mathbb{R}^n} \frac{\partial}{\partial \varepsilon} \varphi_{(\xi, \varepsilon, k)} (v_{(\xi, \varepsilon)} - u_{(\xi, \varepsilon)}) \end{aligned}$$

and

$$\begin{aligned} & \frac{n-2}{n+1} \left(\frac{Y(S^n)}{4n(n-1)} \right)^{\frac{n}{2}} a_j(\xi, \varepsilon) \\ & = \varepsilon \frac{\partial}{\partial \xi_j} \mathcal{F}_g(\xi, \varepsilon) + 2\varepsilon \sum_{k=0}^n a_k(\xi, \varepsilon) \int_{\mathbb{R}^n} \frac{\partial}{\partial \xi_j} \varphi_{(\xi, \varepsilon, k)} (v_{(\xi, \varepsilon)} - u_{(\xi, \varepsilon)}) \end{aligned}$$

for $j = 1, \dots, n$. Hence, if $(\bar{\xi}, \bar{\varepsilon})$ is a critical point of \mathcal{F}_g , then we have

$$\sum_{k=0}^n |a_k(\bar{\xi}, \bar{\varepsilon})| \leq C \|v_{(\bar{\xi}, \bar{\varepsilon})} - u_{(\bar{\xi}, \bar{\varepsilon})}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \sum_{k=0}^n |a_k(\bar{\xi}, \bar{\varepsilon})|,$$

where C is a constant that depends only on n . On the other hand, we have

$$\|v_{(\bar{\xi}, \bar{\varepsilon})} - u_{(\bar{\xi}, \bar{\varepsilon})}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C \alpha_1.$$

Hence, if we choose α_1 sufficiently small, then we must have $a_k(\bar{\xi}, \bar{\varepsilon}) = 0$ for $k = 0, 1, \dots, n$. Thus, we conclude that

$$\int_{\mathbb{R}^n} \left(\langle dv_{(\bar{\xi}, \bar{\varepsilon})}, d\psi \rangle_g + \frac{n-2}{4(n-1)} R_g v_{(\bar{\xi}, \bar{\varepsilon})} \psi - n(n-2) |v_{(\bar{\xi}, \bar{\varepsilon})}|^{\frac{4}{n-2}} v_{(\bar{\xi}, \bar{\varepsilon})} \psi \right) = 0$$

for all test functions $\psi \in \mathcal{E}$. It remains to show that the function $v_{(\bar{\xi}, \bar{\varepsilon})}$ is non-negative. To that end, we put $\psi = \min\{v_{(\bar{\xi}, \bar{\varepsilon})}, 0\}$. Since $v_{(\bar{\xi}, \bar{\varepsilon})} \in \mathcal{E}$, we conclude that $\psi \in \mathcal{E}$. This implies

$$\begin{aligned} & \int_{\{v_{(\bar{\xi}, \bar{\varepsilon})} < 0\}} \left(|dv_{(\bar{\xi}, \bar{\varepsilon})}|_g^2 + \frac{n-2}{4(n-1)} R_g v_{(\bar{\xi}, \bar{\varepsilon})}^2 \right) \\ & = n(n-2) \int_{\{v_{(\bar{\xi}, \bar{\varepsilon})} < 0\}} |v_{(\bar{\xi}, \bar{\varepsilon})}|^{\frac{2n}{n-2}}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} & \left(\int_{\{v_{(\bar{\xi}, \bar{\varepsilon})} < 0\}} |v_{(\bar{\xi}, \bar{\varepsilon})}|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \\ & \leq 2K \int_{\{v_{(\bar{\xi}, \bar{\varepsilon})} < 0\}} \left(|dv_{(\bar{\xi}, \bar{\varepsilon})}|_g^2 + \frac{n-2}{4(n-1)} R_g v_{(\bar{\xi}, \bar{\varepsilon})}^2 \right) \end{aligned}$$

by Corollary 3. From this we deduce that either $v_{(\bar{\xi}, \bar{\varepsilon})} \geq 0$ almost everywhere or

$$\left(\int_{\{v_{(\bar{\xi}, \bar{\varepsilon})} < 0\}} |v_{(\bar{\xi}, \bar{\varepsilon})}|^{\frac{2n}{n-2}} \right)^{\frac{2}{n}} \geq \frac{1}{2n(n-2)K}.$$

On the other hand, we have

$$\left(\int_{\{v_{(\bar{\xi}, \bar{\varepsilon})} < 0\}} |v_{(\bar{\xi}, \bar{\varepsilon})}|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \leq \left(\int_{\mathbb{R}^n} |v_{(\bar{\xi}, \bar{\varepsilon})} - u_{(\bar{\xi}, \bar{\varepsilon})}|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \leq C \alpha_1.$$

Hence, if α_1 is sufficiently small, then we have $v_{(\bar{\xi}, \bar{\varepsilon})} \geq 0$ almost everywhere. \square

3. AN ESTIMATE FOR THE ENERGY OF A “BUBBLE”

Throughout this paper, we fix a multi-linear form $W : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$. We assume that W_{ijkl} satisfy all the algebraic properties of the Weyl tensor. Moreover, we assume that some components of W are non-zero, so that

$$\sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 > 0.$$

For abbreviation, we put

$$H_{ik}(x) = \sum_{p,q=1}^n W_{ipkq} x_p x_q$$

and

$$\bar{H}_{ik}(x) = (1 - |x|^2) H_{ik}(x).$$

It is easy to see that $H_{ik}(x)$ is trace-free, $\sum_{i=1}^n x_i H_{ik}(x) = 0$, and $\sum_{i=1}^n \partial_i H_{ik}(x) = 0$ for all $x \in \mathbb{R}^n$.

We consider a Riemannian metric of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on \mathbb{R}^n satisfying $h(x) = 0$ for $|x| \geq 1$,

$$|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha_1$$

for all $x \in \mathbb{R}^n$, and

$$h_{ik}(x) = \mu (\lambda^2 - |x|^2) H_{ik}(x)$$

for $|x| \leq \rho$. We assume that the parameters λ , μ , and ρ are chosen such that $\mu \leq 1$ and $\lambda \leq \rho \leq 1$. Note that $\sum_{i=1}^n x_i h_{ik}(x) = 0$ and $\sum_{i=1}^n \partial_i h_{ik}(x) = 0$ for $|x| \leq \rho$.

Given any pair $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$, there exists a unique function $v_{(\xi, \varepsilon)}$ such that $v_{(\xi, \varepsilon)} - u_{(\xi, \varepsilon)} \in \mathcal{E}_{(\xi, \varepsilon)}$ and

$$\int_{\mathbb{R}^n} \left(\langle dv_{(\xi, \varepsilon)}, d\psi \rangle_g + \frac{n-2}{4(n-1)} R_g v_{(\xi, \varepsilon)} \psi - n(n-2) |v_{(\xi, \varepsilon)}|^{\frac{4}{n-2}} v_{(\xi, \varepsilon)} \psi \right) = 0$$

for all test functions $\psi \in \mathcal{E}_{(\xi, \varepsilon)}$ (see Proposition 5). For abbreviation, let

$$\Omega = \left\{ (\xi, \varepsilon) \in \mathbb{R}^n \times \mathbb{R} : |\xi| < 1, \frac{n-8}{3(n+4)} < \varepsilon^2 < \frac{2(n-8)}{3(n+4)} \right\}.$$

Proposition 7. *For every pair $(\xi, \varepsilon) \in \lambda \Omega$, we have*

$$\begin{aligned} & \left\| \Delta_g u_{(\xi, \varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi, \varepsilon)} + n(n-2) u_{(\xi, \varepsilon)}^{\frac{n+2}{n-2}} \right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\ & \leq C \lambda^4 \mu + C \left(\frac{\lambda}{\rho} \right)^{\frac{n-2}{2}} \end{aligned}$$

and

$$\begin{aligned} & \left\| \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)} + n(n-2) u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \right. \\ & \quad \left. + \sum_{i,k=1}^n \mu(\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k u_{(\xi,\varepsilon)} \right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\ & \leq C \lambda^8 \mu^2 + C \left(\frac{\lambda}{\rho} \right)^{\frac{n-2}{2}}. \end{aligned}$$

Proof. For abbreviation, we define two functions A_1 and A_2 by

$$A_1 = \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)} + n(n-2) u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}}$$

and

$$A_2 = \sum_{i,k=1}^n \mu(\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k u_{(\xi,\varepsilon)}.$$

Using Proposition 26 and the identity $\sum_{i=1}^n \partial_i h_{ik}(x) = 0$, we obtain

$$|R_g(x)| \leq C |h(x)|^2 |\partial^2 h(x)| + C |\partial h(x)|^2 \leq C \mu^2 (\lambda + |x|)^6$$

for $|x| \leq \rho$. This implies

$$\begin{aligned} |A_1| &= \left| \sum_{i,k=1}^n \partial_i [(g^{ik} - \delta_{ik}) \partial_k u_{(\xi,\varepsilon)}] - \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)} \right| \\ &\leq C \lambda^{\frac{n-2}{2}} \mu (\lambda + |x|)^{4-n} \end{aligned}$$

and

$$\begin{aligned} |A_1 + A_2| &= \left| \sum_{i,k=1}^n \partial_i [(g^{ik} - \delta_{ik} + h_{ik}) \partial_k u_{(\xi,\varepsilon)}] - \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)} \right| \\ &\leq C \lambda^{\frac{n-2}{2}} \mu^2 (\lambda + |x|)^{8-n} \end{aligned}$$

for $|x| \leq \rho$. Hence, we obtain

$$\|A_1\|_{L^{\frac{2n}{n+2}}(B_\rho(0))} \leq C \lambda^{\frac{n-2}{2}} \mu \left(\int_{\mathbb{R}^n} (\lambda + |x|)^{-\frac{2n(n-4)}{n+2}} \right)^{\frac{n+2}{2n}} \leq C \lambda^4 \mu$$

and

$$\|A_1 + A_2\|_{L^{\frac{2n}{n+2}}(B_\rho(0))} \leq C \lambda^{\frac{n-2}{2}} \mu^2 \left(\int_{\mathbb{R}^n} (\lambda + |x|)^{-\frac{2n(n-8)}{n+2}} \right)^{\frac{n+2}{2n}} \leq C \lambda^8 \mu^2.$$

On the other hand, we have

$$|A_1(x)| \leq C \lambda^{\frac{n-2}{2}} |x|^{-n}$$

for $\rho \leq |x| \leq 1$ and

$$|A_2(x)| \leq C \lambda^{\frac{n-2}{2}} \mu |x|^{4-n}$$

for $|x| \geq \rho$. Since the function $A_1(x)$ vanishes for $|x| \geq 1$, we conclude that

$$\|A_1\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n \setminus B_\rho(0))} \leq C \lambda^{\frac{n-2}{2}} \left(\int_{\mathbb{R}^n \setminus B_\rho(0)} |x|^{-\frac{2n^2}{n+2}} \right)^{\frac{n+2}{2n}} \leq C \left(\frac{\lambda}{\rho} \right)^{\frac{n-2}{2}}$$

and

$$\|A_2\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n \setminus B_\rho(0))} \leq C \lambda^{\frac{n-2}{2}} \mu \left(\int_{\mathbb{R}^n \setminus B_\rho(0)} |x|^{-\frac{2n(n-4)}{n+2}} \right)^{\frac{n+2}{2n}} \leq C \rho^4 \mu \left(\frac{\lambda}{\rho} \right)^{\frac{n-2}{2}}.$$

Putting these facts together, the assertion follows. □

Corollary 8. *The function $v_{(\xi, \varepsilon)} - u_{(\xi, \varepsilon)}$ satisfies the estimate*

$$\|v_{(\xi, \varepsilon)} - u_{(\xi, \varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C \lambda^4 \mu + C \left(\frac{\lambda}{\rho} \right)^{\frac{n-2}{2}}$$

for $(\xi, \varepsilon) \in \lambda\Omega$.

Proof. It follows from Proposition 5 that

$$\begin{aligned} & \|v_{(\xi, \varepsilon)} - u_{(\xi, \varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\ & \leq C \left\| \Delta_g u_{(\xi, \varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi, \varepsilon)} + n(n-2) u_{(\xi, \varepsilon)}^{\frac{n+2}{n-2}} \right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}, \end{aligned}$$

where C is a constant that depends only on n . Hence, the assertion follows from Proposition 7. □

We now prove a more refined estimate for the difference $v_{(\xi, \varepsilon)} - u_{(\xi, \varepsilon)}$. Using Proposition 4 with $h = 0$, we conclude that there exists a unique function $w_{(\xi, \varepsilon)} \in \mathcal{E}_{(\xi, \varepsilon)}$ such that

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(\langle dw_{(\xi, \varepsilon)}, d\psi \rangle - n(n+2) u_{(\xi, \varepsilon)}^{\frac{4}{n-2}} w_{(\xi, \varepsilon)} \psi \right) \\ (3) \quad & = - \int_{\mathbb{R}^n} \sum_{i, k=1}^n \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k u_{(\xi, \varepsilon)} \psi \end{aligned}$$

for all test functions $\psi \in \mathcal{E}_{(\xi, \varepsilon)}$.

Proposition 9. *The function $w_{(\xi, \varepsilon)}$ is smooth. Moreover, if $(\xi, \varepsilon) \in \lambda\Omega$, then we have*

$$\begin{aligned} |w_{(\xi, \varepsilon)}(x)| & \leq C \lambda^{\frac{n-2}{2}} \mu (\lambda + |x|)^{6-n}, \\ |\partial w_{(\xi, \varepsilon)}(x)| & \leq C \lambda^{\frac{n-2}{2}} \mu (\lambda + |x|)^{5-n}, \\ |\partial^2 w_{(\xi, \varepsilon)}(x)| & \leq C \lambda^{\frac{n-2}{2}} \mu (\lambda + |x|)^{4-n} \end{aligned}$$

for all $x \in \mathbb{R}^n$.

Proof. Let $\varphi_{(\xi, \varepsilon, k)}$ be the functions defined in Section 2. We can find real numbers $b_k(\xi, \varepsilon)$, $k = 0, 1, \dots, n$, such that

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(\langle dw_{(\xi, \varepsilon)}, d\psi \rangle - n(n+2) u_{(\xi, \varepsilon)}^{\frac{4}{n-2}} w_{(\xi, \varepsilon)} \psi \right) \\ & = - \int_{\mathbb{R}^n} \sum_{i, k=1}^n \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k u_{(\xi, \varepsilon)} \psi + \sum_{k=0}^n b_k(\xi, \varepsilon) \int_{\mathbb{R}^n} \varphi_{(\xi, \varepsilon, k)} \psi \end{aligned}$$

for all test functions $\psi \in \mathcal{E}$. It follows from standard elliptic regularity theory that $w_{(\xi, \varepsilon)}$ is smooth.

In the next step, we establish quantitative estimates for $w_{(\xi,\varepsilon)}$. To that end, we consider a pair $(\xi, \varepsilon) \in \lambda\Omega$. A straightforward calculation yields

$$(4) \quad \left\| \sum_{i,k=1}^n \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k u_{(\xi,\varepsilon)} \right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C \lambda^4 \mu.$$

From this we deduce that $\|w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C \lambda^4 \mu$ and $\sum_{k=0}^n |b_k(\xi, \varepsilon)| \leq C \lambda^4 \mu$. This implies

$$\begin{aligned} & \left| \Delta w_{(\xi,\varepsilon)} + n(n+2) u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} w_{(\xi,\varepsilon)} \right| \\ &= \left| \sum_{i,k=1}^n \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k u_{(\xi,\varepsilon)} - \sum_{k=0}^n b_k(\xi, \varepsilon) \varphi_{(\xi,\varepsilon,k)} \right| \\ &\leq C \lambda^{\frac{n-2}{2}} \mu (\lambda + |x|)^{4-n} \end{aligned}$$

for all $x \in \mathbb{R}^n$. We claim that

$$\sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{\frac{n-2}{2}} |w_{(\xi,\varepsilon)}(x)| \leq C \lambda^4 \mu.$$

To show this, we fix a point $x_0 \in \mathbb{R}^n$ and put $r = \frac{1}{2}(\lambda + |x_0|)$. Clearly, $\lambda + |x| \geq r$ for all $x \in B_r(x_0)$. This implies

$$u_{(\xi,\varepsilon)}(x)^{\frac{4}{n-2}} \leq C r^{-2}$$

and

$$\left| \Delta w_{(\xi,\varepsilon)} + n(n+2) u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} w_{(\xi,\varepsilon)} \right| \leq C \lambda^{\frac{n-2}{2}} \mu r^{4-n}$$

for all $x \in B_r(x_0)$. Using standard interior estimates, we obtain

$$\begin{aligned} r^{\frac{n-2}{2}} |w_{(\xi,\varepsilon)}(x_0)| &\leq C \|w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(B_r(x_0))} \\ &+ C r^{\frac{n+2}{2}} \left\| \Delta w_{(\xi,\varepsilon)} + n(n+2) u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} w_{(\xi,\varepsilon)} \right\|_{L^\infty(B_r(x_0))} \\ &\leq C \lambda^4 \mu + C \lambda^{\frac{n-2}{2}} \mu r^{-\frac{n-10}{2}} \\ &\leq C \lambda^4 \mu. \end{aligned}$$

Thus, we conclude that

$$\sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{\frac{n-2}{2}} |w_{(\xi,\varepsilon)}(x)| \leq C \lambda^4 \mu,$$

as claimed. Since $\sup_{x \in \mathbb{R}^n} |x|^{\frac{n-2}{2}} |w_{(\xi,\varepsilon)}(x)| < \infty$, we can express the function $w_{(\xi,\varepsilon)}$ in the form

$$(5) \quad w_{(\xi,\varepsilon)}(x) = -\frac{1}{(n-2)|S^{n-1}|} \int_{\mathbb{R}^n} |x-y|^{2-n} \Delta w_{(\xi,\varepsilon)}(y) dy$$

for all $x \in \mathbb{R}^n$.

We can now use a bootstrap argument to prove the desired estimate for $w_{(\xi,\varepsilon)}$. It follows from (5) that

$$\sup_{x \in \mathbb{R}^n} (\lambda + |x|)^\beta |w_{(\xi,\varepsilon)}(x)| \leq C \sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{\beta+2} |\Delta w_{(\xi,\varepsilon)}(x)|$$

for all $0 < \beta < n - 2$. Since

$$|\Delta w_{(\xi,\varepsilon)}(x)| \leq n(n+2) u_{(\xi,\varepsilon)}(x)^{\frac{4}{n-2}} |w_{(\xi,\varepsilon)}(x)| + C \lambda^{\frac{n-2}{2}} \mu (\lambda + |x|)^{4-n}$$

for all $x \in \mathbb{R}^n$, we conclude that

$$\sup_{x \in \mathbb{R}^n} (\lambda + |x|)^\beta |w_{(\xi,\varepsilon)}(x)| \leq C \lambda^2 \sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{\beta-2} |w_{(\xi,\varepsilon)}(x)| + C \lambda^{\beta - \frac{n-10}{2}} \mu$$

for all $0 < \beta \leq n - 6$. Iterating this inequality, we obtain

$$\sup_{x \in \mathbb{R}^n} (\lambda + |x|)^{n-6} |w_{(\xi,\varepsilon)}(x)| \leq C \lambda^{\frac{n-2}{2}} \mu.$$

The estimates for the first and second derivatives of $w_{(\xi,\varepsilon)}$ follow now from standard interior estimates. □

Corollary 10. *The function $v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}$ satisfies the estimate*

$$\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C \lambda^{\frac{4(n+2)}{n-2}} \mu^{\frac{n+2}{n-2}} + C \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}$$

for $(\xi, \varepsilon) \in \lambda \Omega$.

Proof. Consider the functions

$$B_1 = \sum_{i,k=1}^n \partial_i [(g^{ik} - \delta_{ik}) \partial_k w_{(\xi,\varepsilon)}] - \frac{n-2}{4(n-1)} R_g w_{(\xi,\varepsilon)}$$

and

$$B_2 = \sum_{i,k=1}^n \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k u_{(\xi,\varepsilon)}.$$

Using (3), we obtain

$$\int_{\mathbb{R}^n} \left(\langle dw_{(\xi,\varepsilon)}, d\psi \rangle_g + \frac{n-2}{4(n-1)} R_g w_{(\xi,\varepsilon)} \psi - n(n+2) u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} w_{(\xi,\varepsilon)} \psi \right) = - \int_{\mathbb{R}^n} (B_1 + B_2) \psi$$

for all functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$. Since $w_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$, it follows that

$$w_{(\xi,\varepsilon)} = -G_{(\xi,\varepsilon)}(B_1 + B_2).$$

Moreover, we have

$$v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} = G_{(\xi,\varepsilon)}(B_3 + n(n-2) B_4),$$

where

$$B_3 = \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)} + n(n-2) u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}}$$

and

$$B_4 = |v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}} v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} - \frac{n+2}{n-2} u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}).$$

Thus, we conclude that

$$v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)} = G_{(\xi,\varepsilon)}(B_1 + B_2 + B_3 + n(n-2) B_4),$$

where $G_{(\xi,\varepsilon)} : L^{\frac{2n}{n+2}}(\mathbb{R}^n) \rightarrow \mathcal{E}_{(\xi,\varepsilon)}$ denotes the solution operator constructed in Proposition 4. In particular, we have

$$\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C \|B_1 + B_2 + B_3 + n(n-2) B_4\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}$$

by Proposition 4. Using Proposition 9, we obtain

$$|B_1(x)| \leq C \lambda^{\frac{n-2}{2}} \mu^2 (\lambda + |x|)^{8-n}$$

for $|x| \leq \rho$ and

$$|B_1(x)| \leq C \lambda^{\frac{n-2}{2}} \mu |x|^{4-n}$$

for $\rho \leq |x| \leq 1$. Since the function $B_1(x)$ vanishes for $|x| \geq 1$, we conclude that

$$\|B_1\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C \lambda^8 \mu^2 + C \rho^4 \mu \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}.$$

Moreover, we have

$$\|B_2 + B_3\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \leq C \lambda^8 \mu^2 + C \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}}$$

by Proposition 7. Finally, the function B_4 satisfies a pointwise estimate of the form

$$|B_4| \leq C |v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}|^{\frac{n+2}{n-2}},$$

where C is a constant that depends only on n . Hence, it follows from Corollary 8 that

$$\begin{aligned} \|B_4\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} &\leq C \|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}^{\frac{n+2}{n-2}} \\ &\leq C \lambda^{\frac{4(n+2)}{n-2}} \mu^{\frac{n+2}{n-2}} + C \left(\frac{\lambda}{\rho}\right)^{\frac{n+2}{2}}. \end{aligned}$$

Putting these facts together, we obtain

$$\|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C \lambda^{\frac{4(n+2)}{n-2}} \mu^{\frac{n+2}{n-2}} + C \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}},$$

as claimed. □

Proposition 11. *We have*

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} \left(|dv_{(\xi,\varepsilon)}|_g^2 - |du_{(\xi,\varepsilon)}|_g^2 + \frac{n-2}{4(n-1)} R_g (v_{(\xi,\varepsilon)}^2 - u_{(\xi,\varepsilon)}^2) \right) \right. \\ &\quad + \int_{\mathbb{R}^n} n(n-2) (|v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{4}{n-2}}) u_{(\xi,\varepsilon)} v_{(\xi,\varepsilon)} \\ &\quad - \int_{\mathbb{R}^n} n(n-2) (|v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}}) \\ &\quad \left. - \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k u_{(\xi,\varepsilon)} w_{(\xi,\varepsilon)} \right| \\ &\leq C \lambda^{\frac{8n}{n-2}} \mu^{\frac{2n}{n-2}} + C \lambda^4 \mu \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} + C \left(\frac{\lambda}{\rho}\right)^{n-2} \end{aligned}$$

for $(\xi, \varepsilon) \in \lambda \Omega$.

Proof. Using Proposition 5 with $\psi = v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}$, we obtain

$$\int_{\mathbb{R}^n} \left(|dv_{(\xi,\varepsilon)}|_g^2 - \langle du_{(\xi,\varepsilon)}, dv_{(\xi,\varepsilon)} \rangle_g + \frac{n-2}{4(n-1)} R_g v_{(\xi,\varepsilon)} (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) \right) - \int_{\mathbb{R}^n} n(n-2) |v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}} v_{(\xi,\varepsilon)} (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) = 0.$$

Moreover, it follows from Proposition 7 and Corollary 8 that

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \left(\langle du_{(\xi,\varepsilon)}, dv_{(\xi,\varepsilon)} \rangle_g - |du_{(\xi,\varepsilon)}|_g^2 + \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)} (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) \right) \right. \\ & \quad - \int_{\mathbb{R}^n} n(n-2) u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) \\ & \quad \left. - \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k u_{(\xi,\varepsilon)} (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}) \right| \\ & \leq \left\| \Delta_g u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)} + n(n-2) u_{(\xi,\varepsilon)}^{\frac{n+2}{n-2}} \right. \\ & \quad \left. + \sum_{i,k=1}^n \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k u_{(\xi,\varepsilon)} \right\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \\ & \quad \cdot \|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\ & \leq C \lambda^{12} \mu^3 + C \lambda^4 \mu \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} + C \left(\frac{\lambda}{\rho}\right)^{n-2}. \end{aligned}$$

Finally, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k u_{(\xi,\varepsilon)} (v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}) \right| \\ & \leq C \lambda^4 \mu \|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)} - w_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \\ & \leq C \lambda^{\frac{8n}{n-2}} \mu^{\frac{2n}{n-2}} + C \lambda^4 \mu \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} \end{aligned}$$

by (4) and Corollary 10. Putting these facts together, the assertion follows. □

Proposition 12. *We have*

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \left(|v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} \right) u_{(\xi,\varepsilon)} v_{(\xi,\varepsilon)} - \frac{2}{n} \int_{\mathbb{R}^n} \left(|v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}} \right) \right| \\ & \leq C \lambda^{\frac{8n}{n-2}} \mu^{\frac{2n}{n-2}} + C \left(\frac{\lambda}{\rho}\right)^n \end{aligned}$$

for $(\xi, \varepsilon) \in \lambda \Omega$.

Proof. We have the pointwise estimate

$$\begin{aligned} & \left| \left(|v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{4}{n-2}} \right) u_{(\xi,\varepsilon)} v_{(\xi,\varepsilon)} - \frac{2}{n} \left(|v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}} \right) \right| \\ & \leq C |v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}}, \end{aligned}$$

where C is a constant that depends only on n . This implies

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} (|v_{(\xi,\varepsilon)}|^{\frac{4}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{4}{n-2}}) u_{(\xi,\varepsilon)} v_{(\xi,\varepsilon)} - \frac{2}{n} \int_{\mathbb{R}^n} (|v_{(\xi,\varepsilon)}|^{\frac{2n}{n-2}} - u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}}) \right| \\ & \leq C \|v_{(\xi,\varepsilon)} - u_{(\xi,\varepsilon)}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)}^{\frac{2n}{n-2}} \\ & \leq C \lambda^{\frac{8n}{n-2}} \mu^{\frac{2n}{n-2}} + C \left(\frac{\lambda}{\rho}\right)^n \end{aligned}$$

by Corollary 8. □

Proposition 13. *We have*

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} \left(|du_{(\xi,\varepsilon)}|_g^2 + \frac{n-2}{4(n-1)} R_g u_{(\xi,\varepsilon)}^2 - n(n-2) u_{(\xi,\varepsilon)}^{\frac{2n}{n-2}} \right) \right. \\ & \quad - \int_{B_\rho(0)} \frac{1}{2} \sum_{i,k,l=1}^n h_{il} h_{kl} \partial_i u_{(\xi,\varepsilon)} \partial_k u_{(\xi,\varepsilon)} \\ & \quad \left. + \int_{B_\rho(0)} \frac{n-2}{16(n-1)} \sum_{i,k,l=1}^n (\partial_l h_{ik})^2 u_{(\xi,\varepsilon)}^2 \right| \\ & \leq C \lambda^{12} \mu^3 + C \left(\frac{\lambda}{\rho}\right)^{n-2} \end{aligned}$$

for all $(\xi, \varepsilon) \in \lambda\Omega$.

Proof. Note that

$$\begin{aligned} & \left| g^{ik}(x) - \delta_{ik} + h_{ik}(x) - \frac{1}{2} \sum_{l=1}^n h_{il}(x) h_{kl}(x) \right| \\ & \leq C |h(x)|^3 \leq C \mu^3 (\lambda + |x|)^{12} \end{aligned}$$

for $|x| \leq \rho$. This implies

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} (|du_{(\xi,\varepsilon)}|_g^2 - |du_{(\xi,\varepsilon)}|^2) + \int_{\mathbb{R}^n} \sum_{i,k=1}^n h_{ik} \partial_i u_{(\xi,\varepsilon)} \partial_k u_{(\xi,\varepsilon)} \right. \\ & \quad \left. - \int_{B_\rho(0)} \frac{1}{2} \sum_{i,k,l=1}^n h_{il} h_{kl} \partial_i u_{(\xi,\varepsilon)} \partial_k u_{(\xi,\varepsilon)} \right| \\ & \leq C \lambda^{n-2} \mu^3 \int_{B_\rho(0)} (\lambda + |x|)^{14-2n} + C \lambda^{n-2} \int_{\mathbb{R}^n \setminus B_\rho(0)} (\lambda + |x|)^{2-2n} \\ & \leq C \lambda^{12} \mu^3 + C \left(\frac{\lambda}{\rho}\right)^{n-2}. \end{aligned}$$

By Proposition 26, the scalar curvature of g satisfies the estimate

$$\begin{aligned} & \left| R_g(x) + \frac{1}{4} \sum_{i,k,l=1}^n (\partial_l h_{ik}(x))^2 \right| \\ & \leq C |h(x)|^2 |\partial^2 h(x)| + C |h(x)| |\partial h(x)|^2 \\ & \leq C \mu^3 (\lambda + |x|)^{10} \end{aligned}$$

for $|x| \leq \rho$. This implies

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} R_g u_{(\xi,\varepsilon)}^2 + \int_{B_\rho(0)} \frac{1}{4} \sum_{i,k,l=1}^n (\partial_l h_{ik})^2 u_{(\xi,\varepsilon)}^2 \right| \\ & \leq C \lambda^{12} \mu^3 \int_{B_\rho(0)} (\lambda + |x|)^{14-2n} + C \lambda^{n-2} \int_{\mathbb{R}^n \setminus B_\rho(0)} (\lambda + |x|)^{4-2n} \\ & \leq C \lambda^{12} \mu^3 + C \rho^2 \left(\frac{\lambda}{\rho}\right)^{n-2}. \end{aligned}$$

At this point, we use the formula

$$\begin{aligned} & \partial_i u_{(\xi,\varepsilon)} \partial_k u_{(\xi,\varepsilon)} - \frac{n-2}{4(n-1)} \partial_i \partial_k (u_{(\xi,\varepsilon)}^2) \\ & = \frac{1}{n} \left(|du_{(\xi,\varepsilon)}|^2 - \frac{n-2}{4(n-1)} \Delta(u_{(\xi,\varepsilon)}^2) \right) \delta_{ik}. \end{aligned}$$

Since h_{ik} is trace-free, we obtain

$$\sum_{i,k=1}^n h_{ik} \partial_i u_{(\xi,\varepsilon)} \partial_k u_{(\xi,\varepsilon)} = \frac{n-2}{4(n-1)} \sum_{i,k=1}^n h_{ik} \partial_i \partial_k (u_{(\xi,\varepsilon)}^2);$$

hence

$$\int_{\mathbb{R}^n} \sum_{i,k=1}^n h_{ik} \partial_i u_{(\xi,\varepsilon)} \partial_k u_{(\xi,\varepsilon)} = \int_{\mathbb{R}^n} \frac{n-2}{4(n-1)} \sum_{i,k=1}^n \partial_i \partial_k h_{ik} u_{(\xi,\varepsilon)}^2.$$

Since $\sum_{i=1}^n \partial_i h_{ik}(x) = 0$ for $|x| \leq \rho$, it follows that

$$\left| \int_{\mathbb{R}^n} \sum_{i,k=1}^n h_{ik} \partial_i u_{(\xi,\varepsilon)} \partial_k u_{(\xi,\varepsilon)} \right| \leq C \int_{\mathbb{R}^n \setminus B_\rho(0)} u_{(\xi,\varepsilon)}^2 \leq C \rho^2 \left(\frac{\lambda}{\rho}\right)^{n-2}.$$

Putting these facts together, the assertion follows. □

Corollary 14. *The function $\mathcal{F}_g(\xi, \varepsilon)$ satisfies the estimate*

$$\begin{aligned} & \left| \mathcal{F}_g(\xi, \varepsilon) - \int_{B_\rho(0)} \frac{1}{2} \sum_{i,k,l=1}^n h_{il} h_{kl} \partial_i u_{(\xi,\varepsilon)} \partial_k u_{(\xi,\varepsilon)} \right. \\ & \quad + \int_{B_\rho(0)} \frac{n-2}{16(n-1)} \sum_{i,k,l=1}^n (\partial_l h_{ik})^2 u_{(\xi,\varepsilon)}^2 \\ & \quad \left. - \int_{\mathbb{R}^n} \sum_{i,k=1}^n \mu (\lambda^2 - |x|^2) H_{ik}(x) \partial_i \partial_k u_{(\xi,\varepsilon)} w_{(\xi,\varepsilon)} \right| \\ & \leq C \lambda^{\frac{8n}{n-2}} \mu^{\frac{2n}{n-2}} + C \lambda^4 \mu \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} + C \left(\frac{\lambda}{\rho}\right)^{n-2} \end{aligned}$$

for $(\xi, \varepsilon) \in \lambda \Omega$.

Proof. This follows by combining Proposition 11, Proposition 12, and Proposition 13. □

4. FINDING A CRITICAL POINT OF AN AUXILIARY FUNCTION

We define a function $F : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$ as follows: given any pair $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$, we define

$$\begin{aligned} F(\xi, \varepsilon) &= \int_{\mathbb{R}^n} \frac{1}{2} \sum_{i,k,l=1}^n \overline{H}_{il}(x) \overline{H}_{kl}(x) \partial_i u_{(\xi,\varepsilon)}(x) \partial_k u_{(\xi,\varepsilon)}(x) \\ &\quad - \int_{\mathbb{R}^n} \frac{n-2}{16(n-1)} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2 u_{(\xi,\varepsilon)}(x)^2 \\ &\quad + \int_{\mathbb{R}^n} \sum_{i,k=1}^n \overline{H}_{ik}(x) \partial_i \partial_k u_{(\xi,\varepsilon)}(x) z_{(\xi,\varepsilon)}(x), \end{aligned}$$

where $z_{(\xi,\varepsilon)} \in \mathcal{E}_{(\xi,\varepsilon)}$ satisfies the relation

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\langle dz_{(\xi,\varepsilon)}, d\psi \rangle - n(n+2) u_{(\xi,\varepsilon)}(x)^{\frac{4}{n-2}} z_{(\xi,\varepsilon)} \psi \right) \\ &= - \int_{\mathbb{R}^n} \sum_{i,k=1}^n \overline{H}_{ik} \partial_i \partial_k u_{(\xi,\varepsilon)} \psi \end{aligned}$$

for all test functions $\psi \in \mathcal{E}_{(\xi,\varepsilon)}$. Our goal in this section is to show that the function $F(\xi, \varepsilon)$ has a critical point.

Proposition 15. *The function $F(\xi, \varepsilon)$ satisfies $F(\xi, \varepsilon) = F(-\xi, \varepsilon)$ for all $(\xi, \varepsilon) \in \mathbb{R}^n \times (0, \infty)$. Consequently, we have $\frac{\partial}{\partial \xi_p} F(0, \varepsilon) = 0$ and $\frac{\partial^2}{\partial \varepsilon \partial \xi_p} F(0, \varepsilon) = 0$ for all $\varepsilon > 0$ and $p = 1, \dots, n$.*

Proof. This follows immediately from the relation $\overline{H}_{ik}(-x) = \overline{H}_{ik}(x)$. □

Proposition 16. *We have*

$$\begin{aligned} &\int_{\partial B_r(0)} \sum_{i,k,l=1}^n (\partial_l H_{ik}(x))^2 x_p x_q \\ &= \frac{2}{n(n+2)} |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp}) (W_{iqkl} + W_{ilkq}) r^{n+3} \\ &\quad + \frac{1}{n(n+2)} |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \delta_{pq} r^{n+3} \end{aligned}$$

and

$$\begin{aligned} &\int_{\partial B_r(0)} \sum_{i,k=1}^n H_{ik}(x)^2 x_p x_q \\ &= \frac{2}{n(n+2)(n+4)} |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp}) (W_{iqkl} + W_{ilkq}) r^{n+5} \\ &\quad + \frac{1}{2n(n+2)(n+4)} |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \delta_{pq} r^{n+5}. \end{aligned}$$

Proof. By definition of $H_{ik}(x)$, we have

$$\begin{aligned} & \int_{\partial B_r(0)} \sum_{i,k,l=1}^n (\partial_l H_{ik}(x))^2 x_p x_q \\ &= \int_{\partial B_r(0)} \sum_{i,j,k,l,m=1}^n (W_{ijkl} + W_{ilkj})(W_{imkl} + W_{ilkm}) x_j x_m x_p x_q \\ &= \frac{2}{n(n+2)} |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp})(W_{iqkl} + W_{ilkq}) r^{n+3} \\ &+ \frac{1}{n(n+2)} |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \delta_{pq} r^{n+3}. \end{aligned}$$

Moreover, it follows from Corollary 29 that

$$\begin{aligned} & \int_{\partial B_r(0)} \sum_{i,k=1}^n H_{ik}(x)^2 x_p x_q \\ &= \int_{\partial B_r(0)} \sum_{i,j,k,l,m,s=1}^n W_{ijkl} W_{imks} x_j x_l x_m x_s x_p x_q \\ &= \frac{2}{n(n+2)(n+4)} |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp})(W_{iqkl} + W_{ilkq}) r^{n+5} \\ &+ \frac{1}{2n(n+2)(n+4)} |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \delta_{pq} r^{n+5}. \end{aligned}$$

This completes the proof. □

Proposition 17. *We have*

$$\begin{aligned} & \int_{\partial B_r(0)} \sum_{i,k,l=1}^n (\partial_l \bar{H}_{ik}(x))^2 x_p x_q \\ &= \frac{2}{n(n+2)} |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp})(W_{iqkl} + W_{ilkq}) \\ &\quad \cdot \left[r^{n+3} - \frac{2(n+8)}{n+4} r^{n+5} + \frac{n+16}{n+4} r^{n+7} \right] \\ &+ \frac{1}{n(n+2)} |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \delta_{pq} \\ &\quad \cdot \left[r^{n+3} - \frac{2(n+6)}{n+4} r^{n+5} + \frac{n+10}{n+4} r^{n+7} \right]. \end{aligned}$$

Proof. Using the identity

$$\partial_l \bar{H}_{ik}(x) = (1 - |x|^2) \partial_l H_{ik}(x) - 2 H_{ik}(x) x_l$$

and Euler’s theorem, we obtain

$$\begin{aligned} & \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2 \\ &= (1 - |x|^2)^2 \sum_{i,k,l=1}^n (\partial_l H_{ik}(x))^2 \\ & - 4(1 - |x|^2) \sum_{i,k,l=1}^n H_{ik}(x) x_l \partial_l H_{ik}(x) + 4|x|^2 \sum_{i,k=1}^n H_{ik}(x)^2 \\ &= (1 - |x|^2)^2 \sum_{i,k,l=1}^n (\partial_l H_{ik}(x))^2 - 4(2 - 3|x|^2) \sum_{i,k=1}^n H_{ik}(x)^2. \end{aligned}$$

Hence, the assertion follows from the previous proposition. □

Corollary 18. *We have*

$$\begin{aligned} \int_{\partial B_r(0)} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2 &= \frac{1}{n} |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \\ & \cdot \left[r^{n+1} - \frac{2(n+4)}{n+2} r^{n+3} + \frac{n+8}{n+2} r^{n+5} \right]. \end{aligned}$$

Proposition 19. *We have*

$$\begin{aligned} F(0, \varepsilon) &= -\frac{(n-2)(n+4)}{16n(n-1)(n+2)} |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \\ & \cdot \left[\frac{n-8}{n+4} \varepsilon^4 - 2\varepsilon^6 + \frac{n+8}{n-10} \varepsilon^8 \right] \int_0^\infty (1+r^2)^{2-n} r^{n+3} dr. \end{aligned}$$

Proof. Note that $z_{(0,\varepsilon)}(x) = 0$ for all $x \in \mathbb{R}^n$. This implies

$$F(0, \varepsilon) = - \int_{\mathbb{R}^n} \frac{n-2}{16(n-1)} \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{2-n} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2.$$

Using Corollary 18, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^n} \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{2-n} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2 \\ &= \frac{1}{n} |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \\ & \cdot \int_0^\infty (1+r^2)^{2-n} \left[\varepsilon^4 r^{n+1} - \frac{2(n+4)}{n+2} \varepsilon^6 r^{n+3} + \frac{n+8}{n+2} \varepsilon^8 r^{n+5} \right] dr. \end{aligned}$$

Moreover, we have

$$\int_0^\infty (1+r^2)^{2-n} r^{n+1} dr = \frac{n-8}{n+2} \int_0^\infty (1+r^2)^{2-n} r^{n+3} dr$$

and

$$\int_0^\infty (1+r^2)^{2-n} r^{n+5} dr = \frac{n+4}{n-10} \int_0^\infty (1+r^2)^{2-n} r^{n+3} dr$$

by Proposition 27. From this the assertion follows. □

Corollary 20. *Assume that $n \geq 52$. Moreover, suppose that $\varepsilon_* > 0$ is defined by*

$$(6) \quad \left(3 + \sqrt{9 - \frac{8(n+8)(n-8)}{(n+4)(n-10)}} \right) \varepsilon_*^2 = \frac{2(n-8)}{n+4}.$$

Then $(0, \varepsilon_)$ is a critical point of the function $F(\xi, \varepsilon)$. Moreover, we have $\frac{\partial^2}{\partial \varepsilon^2} F(0, \varepsilon_*) > 0$.*

In the next step, we show that $(0, \varepsilon_*)$ is a strict local minimum of the function F . To that end, we compute the Hessian of F at a point $(0, \varepsilon)$.

Proposition 21. *The second order partial derivatives of the function $F(\xi, \varepsilon)$ are given by*

$$\begin{aligned} \frac{\partial^2}{\partial \xi_p \partial \xi_q} F(0, \varepsilon) &= \int_{\mathbb{R}^n} (n-2)^2 \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{-n} \sum_{l=1}^n \bar{H}_{pl}(x) \bar{H}_{ql}(x) \\ &\quad - \int_{\mathbb{R}^n} \frac{(n-2)^2}{4} \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{-n} \sum_{i,k,l=1}^n (\partial_l \bar{H}_{ik}(x))^2 x_p x_q \\ &\quad + \int_{\mathbb{R}^n} \frac{(n-2)^2}{8(n-1)} \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{1-n} \sum_{i,k,l=1}^n (\partial_l \bar{H}_{ik}(x))^2 \delta_{pq}. \end{aligned}$$

Proof. Using the identity

$$\begin{aligned} &\sum_{i,k,l=1}^n \bar{H}_{il}(x) \bar{H}_{kl}(x) \partial_i u_{(\xi,\varepsilon)}(x) \partial_k u_{(\xi,\varepsilon)}(x) \\ &= (n-2)^2 \varepsilon^{n-2} (\varepsilon^2 + |x - \xi|^2)^{-n} \sum_{i,k,l=1}^n \bar{H}_{il}(x) \bar{H}_{kl}(x) (x_i - \xi_i) (x_k - \xi_k) \\ &= (n-2)^2 \varepsilon^{n-2} (\varepsilon^2 + |x - \xi|^2)^{-n} \sum_{i,k,l=1}^n \bar{H}_{il}(x) \bar{H}_{kl}(x) \xi_i \xi_k, \end{aligned}$$

we obtain

$$\begin{aligned} &\frac{\partial^2}{\partial \xi_p \partial \xi_q} \left(\frac{1}{2} \sum_{i,k,l=1}^n \bar{H}_{il}(x) \bar{H}_{kl}(x) \partial_i u_{(\xi,\varepsilon)}(x) \partial_k u_{(\xi,\varepsilon)}(x) \right) \Big|_{\xi=0} \\ &= (n-2)^2 \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{-n} \sum_{l=1}^n \bar{H}_{pl}(x) \bar{H}_{ql}(x). \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\frac{\partial^2}{\partial \xi_p \partial \xi_q} \left(\frac{n-2}{16(n-1)} \sum_{i,k,l=1}^n (\partial_l \bar{H}_{ik}(x))^2 u_{(\xi,\varepsilon)}(x)^2 \right) \Big|_{\xi=0} \\ &= \frac{(n-2)^2}{4} \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{-n} \sum_{i,k,l=1}^n (\partial_l \bar{H}_{ik}(x))^2 x_p x_q \\ &\quad - \frac{(n-2)^2}{8(n-1)} \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{1-n} \sum_{i,k,l=1}^n (\partial_l \bar{H}_{ik}(x))^2 \delta_{pq}. \end{aligned}$$

Finally, we have

$$\begin{aligned} & \sum_{i,k=1}^n \overline{H}_{ik}(x) \partial_i \partial_k u_{(\xi,\varepsilon)}(x) \\ &= n(n-2) \varepsilon^{\frac{n-2}{2}} (\varepsilon^2 + |x - \xi|^2)^{-\frac{n+2}{2}} \sum_{i,k=1}^n \overline{H}_{ik}(x) (x_i - \xi_i) (x_k - \xi_k) \\ &= n(n-2) \varepsilon^{\frac{n-2}{2}} (\varepsilon^2 + |x - \xi|^2)^{-\frac{n+2}{2}} \sum_{i,k=1}^n \overline{H}_{ik}(x) \xi_i \xi_k \end{aligned}$$

since $\overline{H}_{ik}(x)$ is trace-free. Thus, we conclude that

$$\begin{aligned} & \left. \frac{\partial^2}{\partial \xi_p \partial \xi_q} \left(\sum_{i,k=1}^n \overline{H}_{ik}(x) \partial_i \partial_k u_{(\xi,\varepsilon)}(x) z_{(\xi,\varepsilon)}(x) \right) \right|_{\xi=0} \\ &= 2n(n-2) \varepsilon^{\frac{n-2}{2}} (\varepsilon^2 + |x|^2)^{-\frac{n+2}{2}} \sum_{i,k=1}^n \overline{H}_{pq}(x) z_{(0,\varepsilon)}(x) = 0. \end{aligned}$$

From this the assertion follows. □

Proposition 22. *The second order partial derivatives of the function $F(\xi, \varepsilon)$ are given by*

$$\begin{aligned} & \frac{\partial^2}{\partial \xi_p \partial \xi_q} F(0, \varepsilon) \\ &= \frac{4(n-2)^2}{n(n+2)(n+4)} |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp}) (W_{iqkl} + W_{ilkq}) \\ & \quad \cdot \left[\varepsilon^4 - \frac{3(n+6)}{2(n-8)} \varepsilon^6 \right] \int_0^\infty (1+r^2)^{-n} r^{n+5} dr \\ &+ \frac{(n-2)^2}{n(n+2)(n+4)} |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \delta_{pq} \\ & \quad \cdot \left[\varepsilon^4 - \frac{n+7}{n-8} \varepsilon^6 \right] \int_0^\infty (1+r^2)^{-n} r^{n+5} dr. \end{aligned}$$

Proof. Using the identity

$$\begin{aligned} & \int_{\partial B_r(0)} \sum_{l=1}^n \overline{H}_{pl}(x) \overline{H}_{ql}(x) \\ &= \int_{\partial B_r(0)} \sum_{i,j,k,l,m=1}^n W_{ipkl} W_{jqml} x_i x_j x_k x_m (1 - |x|^2)^2 \\ &= \frac{1}{n(n+2)} |S^{n-1}| \\ & \quad \cdot \sum_{i,j,k,l,m=1}^n W_{ipkl} W_{jqml} (\delta_{ij} \delta_{km} + \delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk}) r^{n+3} (1 - r^2)^2 \\ &= \frac{1}{2n(n+2)} |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp}) (W_{iqkl} + W_{ilkq}) r^{n+3} (1 - r^2)^2, \end{aligned}$$

we obtain

$$\begin{aligned} & \int_{\mathbb{R}^n} \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{-n} \sum_{i,k,l=1}^n \overline{H}_{pl}(x) \overline{H}_{ql}(x) \\ &= \frac{1}{2n(n+2)} |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp}) (W_{iqkl} + W_{ilkq}) \\ & \quad \cdot \int_0^\infty (1+r^2)^{-n} \left[\varepsilon^2 r^{n+3} - 2\varepsilon^4 r^{n+5} + \varepsilon^6 r^{n+7} \right] dr. \end{aligned}$$

Similarly, it follows from Proposition 17 that

$$\begin{aligned} & \int_{\mathbb{R}^n} \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{-n} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2 x_p x_q \\ &= \frac{2}{n(n+2)} |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp}) (W_{iqkl} + W_{ilkq}) \\ & \quad \cdot \int_0^\infty (1+r^2)^{-n} \left[\varepsilon^2 r^{n+3} - \frac{2(n+8)}{n+4} \varepsilon^4 r^{n+5} + \frac{n+16}{n+4} \varepsilon^6 r^{n+7} \right] dr \\ &+ \frac{1}{n(n+2)} |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \delta_{pq} \\ & \quad \cdot \int_0^\infty (1+r^2)^{-n} \left[\varepsilon^2 r^{n+3} - \frac{2(n+6)}{n+4} \varepsilon^4 r^{n+5} + \frac{n+10}{n+4} \varepsilon^6 r^{n+7} \right] dr. \end{aligned}$$

Moreover, we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \varepsilon^{n-2} (\varepsilon^2 + |x|^2)^{1-n} \sum_{i,k,l=1}^n (\partial_l \overline{H}_{ik}(x))^2 \delta_{pq} \\ &= \frac{1}{n} |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \delta_{pq} \\ & \quad \cdot \int_0^\infty (1+r^2)^{1-n} \left[\varepsilon^2 r^{n+1} - \frac{2(n+4)}{n+2} \varepsilon^4 r^{n+3} + \frac{n+8}{n+2} \varepsilon^6 r^{n+5} \right] dr. \end{aligned}$$

by Corollary 18. Using Proposition 21 and the identity

$$\int_0^\infty (1+r^2)^{1-n} r^{n+1} dr = \frac{2(n-1)}{n+2} \int_0^\infty (1+r^2)^{-n} r^{n+3} dr,$$

we obtain

$$\begin{aligned} & \frac{\partial^2}{\partial \xi_p \partial \xi_q} F(0, \varepsilon) \\ &= \frac{4(n-2)^2}{n(n+2)(n+4)} |S^{n-1}| \sum_{i,k,l=1}^n (W_{ipkl} + W_{ilkp}) (W_{iqkl} + W_{ilkq}) \\ & \quad \cdot \int_0^\infty (1+r^2)^{-n} \left[\varepsilon^4 r^{n+5} - \frac{3}{2} \varepsilon^6 r^{n+7} \right] dr \\ &+ \frac{(n-2)^2}{4n(n+2)} |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \delta_{pq} \\ & \quad \cdot \int_0^\infty (1+r^2)^{-n} \left[\frac{2(n+6)}{n+4} \varepsilon^4 r^{n+5} - \frac{n+10}{n+4} \varepsilon^6 r^{n+7} \right] dr \\ &- \frac{(n-2)^2}{8n(n-1)} |S^{n-1}| \sum_{i,j,k,l=1}^n (W_{ijkl} + W_{ilkj})^2 \delta_{pq} \\ & \quad \cdot \int_0^\infty (1+r^2)^{1-n} \left[\frac{2(n+4)}{n+2} \varepsilon^4 r^{n+3} - \frac{n+8}{n+2} \varepsilon^6 r^{n+5} \right] dr. \end{aligned}$$

Hence, the assertion follows from the identities

$$\begin{aligned} \int_0^\infty (1+r^2)^{-n} r^{n+7} dr &= \frac{n+6}{n-8} \int_0^\infty (1+r^2)^{-n} r^{n+5} dr, \\ \int_0^\infty (1+r^2)^{1-n} r^{n+3} dr &= \frac{2(n-1)}{n+4} \int_0^\infty (1+r^2)^{-n} r^{n+5} dr, \\ \int_0^\infty (1+r^2)^{1-n} r^{n+5} dr &= \frac{2(n-1)}{n-8} \int_0^\infty (1+r^2)^{-n} r^{n+5} dr. \end{aligned}$$

□

Corollary 23. *Assume that $n \geq 52$ and $\varepsilon_* > 0$ is defined by (6). Then the function $F(\xi, \varepsilon)$ has a strict local minimum at the point $(0, \varepsilon_*)$.*

Proof. It follows from Corollary 20 that $(0, \varepsilon_*)$ is a critical point of the function $F(\xi, \varepsilon)$. Moreover, we have $\frac{\partial^2}{\partial \varepsilon^2} F(0, \varepsilon_*) > 0$. Since $n \geq 52$, we have

$$\frac{6}{n+4} < \sqrt{9 - \frac{8(n+8)(n-8)}{(n+4)(n-10)}}.$$

This implies

$$\frac{3(n+6)}{n+4} \varepsilon_*^2 < \left(3 + \sqrt{9 - \frac{8(n+8)(n-8)}{(n+4)(n-10)}} \right) \varepsilon_*^2 = \frac{2(n-8)}{n+4}.$$

Thus, we conclude that

$$\frac{n+7}{n-8} \varepsilon_*^2 < \frac{3(n+6)}{2(n-8)} \varepsilon_*^2 < 1.$$

Hence, it follows from Proposition 22 that the matrix $\frac{\partial^2}{\partial \xi_p \partial \xi_q} F(0, \varepsilon_*)$ is positive definite. This proves the assertion. □

5. PROOF OF THE MAIN THEOREM

Proposition 24. *Assume that $n \geq 52$. Moreover, let g be a smooth metric on \mathbb{R}^n of the form $g(x) = \exp(h(x))$, where $h(x)$ is a trace-free symmetric two-tensor on \mathbb{R}^n such that $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha \leq \alpha_1$ for all $x \in \mathbb{R}^n$, $h(x) = 0$ for $|x| \geq 1$, and $h_{ik}(x) = \mu(\lambda^2 - |x|^2)H_{ik}(x)$ for $|x| \leq \rho$. As above, we assume that $\lambda \leq \rho \leq 1$ and $\mu \leq 1$. If α and $\rho^{2-n}\mu^{-2}\lambda^{n-10}$ are sufficiently small, then there exists a positive function v such that*

$$\Delta_g v - \frac{n-2}{4(n-1)} R_g v + n(n-2)v^{\frac{n+2}{n-2}} = 0,$$

$$\int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} < \left(\frac{Y(S^n)}{4n(n-1)}\right)^{\frac{n}{2}},$$

and $\sup_{|x| \leq \lambda} v(x) \geq c\lambda^{\frac{2-n}{2}}$. Here, c is a positive constant that depends only on n .

Proof. By Corollary 23, the function $F(\xi, \varepsilon)$ has a strict local minimum at $(0, \varepsilon_*)$. Hence, we can find an open set $\Omega' \subset \Omega$ such that $(0, \varepsilon_*) \in \Omega'$ and

$$F(0, \varepsilon_*) < \inf_{(\xi, \varepsilon) \in \partial\Omega'} F(\xi, \varepsilon) < 0.$$

Using Corollary 14, we obtain

$$|\mathcal{F}_g(\lambda\xi, \lambda\varepsilon) - \lambda^8 \mu^2 F(\xi, \varepsilon)|$$

$$\leq C \lambda^{\frac{8n}{n-2}} \mu^{\frac{2n}{n-2}} + C \lambda^4 \mu \left(\frac{\lambda}{\rho}\right)^{\frac{n-2}{2}} + C \left(\frac{\lambda}{\rho}\right)^{n-2}$$

for all $(\xi, \varepsilon) \in \Omega$. This implies

$$|\lambda^{-8} \mu^{-2} \mathcal{F}_g(\lambda\xi, \lambda\varepsilon) - F(\xi, \varepsilon)|$$

$$\leq C \lambda^{\frac{16}{n-2}} \mu^{\frac{4}{n-2}} + C \rho^{\frac{2-n}{2}} \mu^{-1} \lambda^{\frac{n-10}{2}} + C \rho^{2-n} \mu^{-2} \lambda^{n-10}$$

for all $(\xi, \varepsilon) \in \Omega$. Hence, if $\rho^{2-n}\mu^{-2}\lambda^{n-10}$ is sufficiently small, then we have

$$\mathcal{F}_g(0, \lambda\varepsilon_*) < \inf_{(\xi, \varepsilon) \in \partial\Omega'} \mathcal{F}_g(\lambda\xi, \lambda\varepsilon) < 0.$$

Consequently, there exists a point $(\bar{\xi}, \bar{\varepsilon}) \in \Omega'$ such that

$$\mathcal{F}_g(\lambda\bar{\xi}, \lambda\bar{\varepsilon}) = \inf_{(\xi, \varepsilon) \in \Omega'} \mathcal{F}_g(\lambda\xi, \lambda\varepsilon) < 0.$$

By Proposition 6, the function $v = v_{(\lambda\bar{\xi}, \lambda\bar{\varepsilon})}$ is a non-negative weak solution of the partial differential equation

$$\Delta_g v - \frac{n-2}{4(n-1)} R_g v + n(n-2)v^{\frac{n+2}{n-2}} = 0.$$

Using a result of N. Trudinger, we conclude that v is smooth (see [20], Theorem 3 on p. 271). Moreover, we have

$$2(n-2) \int_{\mathbb{R}^n} v^{\frac{2n}{n-2}} = 2(n-2) \left(\frac{Y(S^n)}{4n(n-1)}\right)^{\frac{n}{2}} + \mathcal{F}_g(\lambda\bar{\xi}, \lambda\bar{\varepsilon})$$

$$< 2(n-2) \left(\frac{Y(S^n)}{4n(n-1)}\right)^{\frac{n}{2}}.$$

Finally, it follows from Proposition 5 that $\|v - u_{(\lambda\bar{\xi}, \lambda\bar{\varepsilon})}\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \leq C\alpha$. This implies

$$|B_\lambda(0)|^{\frac{n-2}{2n}} \sup_{|x| \leq \lambda} v(x) \geq \|v\|_{L^{\frac{2n}{n-2}}(B_\lambda(0))} \geq \|u_{(\lambda\bar{\xi}, \lambda\bar{\varepsilon})}\|_{L^{\frac{2n}{n-2}}(B_\lambda(0))} - C\alpha.$$

Hence, if α is sufficiently small, then we obtain $\lambda^{\frac{n-2}{2}} \sup_{|x| \leq \lambda} v(x) \geq c$. □

Proposition 25. *Let $n \geq 52$. Then there exists a smooth metric g on \mathbb{R}^n with the following properties:*

- (i) $g_{ik}(x) = \delta_{ik}$ for $|x| \geq \frac{1}{2}$,
- (ii) g is not conformally flat,
- (iii) there exists a sequence of non-negative smooth functions v_ν ($\nu \in \mathbb{N}$) such that

$$\Delta_g v_\nu - \frac{n-2}{4(n-1)} R_g v_\nu + n(n-2) v_\nu^{\frac{n+2}{n-2}} = 0$$

for all $\nu \in \mathbb{N}$,

$$\int_{\mathbb{R}^n} v_\nu^{\frac{2n}{n-2}} < \left(\frac{Y(S^n)}{4n(n-1)} \right)^{\frac{n}{2}}$$

for all $\nu \in \mathbb{N}$, and $\sup_{|x| \leq 1} v_\nu(x) \rightarrow \infty$ as $\nu \rightarrow \infty$.

Proof. Choose a smooth cutoff function $\eta : \mathbb{R} \rightarrow \mathbb{R}$ such that $\eta(t) = 1$ for $t \leq 1$ and $\eta(t) = 0$ for $t \geq 2$. We define a trace-free symmetric two-tensor on \mathbb{R}^n by

$$h_{ik}(x) = \sum_{N=N_0}^{\infty} \eta(4N^2|x - y_N|) 2^{-N} (2^{-N} - |x - y_N|^2) H_{ik}(x - y_N),$$

where $y_N = (\frac{1}{N}, 0, \dots, 0) \in \mathbb{R}^n$. It is straightforward to verify that $h(x)$ is C^∞ smooth.

Let α be the constant appearing in Proposition 24. If N_0 is sufficiently large, then we have $|h(x)| + |\partial h(x)| + |\partial^2 h(x)| \leq \alpha$ for all $x \in \mathbb{R}^n$ and $h(x) = 0$ for $|x| \geq \frac{1}{2}$. Moreover, we have $h_{ik}(x) = 2^{-N} (2^{-N} - |x - y_N|^2) H_{ik}(x - y_N)$ provided that $N \geq N_0$ and $|x - y_N| \leq \frac{1}{4N^2}$. Hence, we can apply Proposition 24 with $\lambda = 2^{-N/2}$, $\mu = 2^{-N}$, and $\rho = \frac{1}{4N^2}$. From this the assertion follows. □

APPENDIX A. AN ASYMPTOTIC EXPANSION FOR THE SCALAR CURVATURE

Suppose that $h(x)$ is a trace-free symmetric two-tensor defined on \mathbb{R}^n satisfying $|h(x)| \leq 1$ for all $x \in \mathbb{R}^n$. We define a Riemannian metric g on \mathbb{R}^n by $g(x) = \exp(h(x))$. In this section, we derive an approximate formula for the scalar curvature of this metric. A similar formula is derived in [2].

Proposition 26. *Let R_g be the scalar curvature of g . There exists a constant C , depending only on n , such that*

$$\begin{aligned} & \left| R_g - \partial_i \partial_k h_{ik} + \partial_i (h_{il} \partial_k h_{kl}) - \frac{1}{2} \partial_i h_{il} \partial_k h_{kl} + \frac{1}{4} \partial_l h_{ik} \partial_l h_{ik} \right| \\ & \leq C |h|^2 |\partial^2 h| + C |h| |\partial h|^2. \end{aligned}$$

Proof. The Riemann curvature tensor is defined as

$$\partial_i \Gamma_{jk}^m - \partial_j \Gamma_{ik}^m + \Gamma_{jk}^l \Gamma_{il}^m - \Gamma_{ik}^l \Gamma_{jl}^m.$$

Hence, the scalar curvature of g is given by

$$R_g = g^{jk} (\partial_i \Gamma_{jk}^i - \partial_j \Gamma_{ik}^i + \Gamma_{jk}^l \Gamma_{il}^i - \Gamma_{ik}^l \Gamma_{jl}^i).$$

Since h is trace-free, we have $\det g(x) = 1$ for all $x \in \mathbb{R}^n$. This implies $\Gamma_{ik}^i = \frac{1}{2} g^{il} \partial_k g_{il} = \frac{1}{2} \partial_k \log \det g = 0$. Therefore, we obtain

$$\begin{aligned} R_g &= g^{jk} \partial_i \Gamma_{jk}^i - g^{jk} \Gamma_{ik}^l \Gamma_{jl}^i \\ &= \partial_i (g^{jk} \Gamma_{jk}^i) + g^{jk} \Gamma_{ik}^l \Gamma_{jl}^i. \end{aligned}$$

Note that

$$g^{jk} \Gamma_{jk}^i = g^{il} g^{jk} \partial_k g_{jl}.$$

From this it follows that

$$\begin{aligned} &\left| \partial_i (g^{jk} \Gamma_{jk}^i) - \partial_i \partial_k h_{ik} + \frac{1}{2} \partial_i (h_{il} \partial_k h_{kl}) + \frac{1}{2} \partial_i (h_{kl} \partial_k h_{il}) \right| \\ &\leq C |h|^2 |\partial^2 h| + C |h| |\partial h|^2; \end{aligned}$$

hence

$$\begin{aligned} &\left| \partial_i (g^{jk} \Gamma_{jk}^i) - \partial_i \partial_k h_{ik} + \partial_i (h_{il} \partial_k h_{kl}) - \frac{1}{2} \partial_i h_{il} \partial_k h_{kl} + \frac{1}{2} \partial_i h_{kl} \partial_k h_{il} \right| \\ &\leq C |h|^2 |\partial^2 h| + C |h| |\partial h|^2. \end{aligned}$$

Moreover, we have

$$\left| g^{jk} \Gamma_{ik}^l \Gamma_{jl}^i + \frac{1}{4} \partial_l h_{ik} \partial_l h_{ik} - \frac{1}{2} \partial_i h_{kl} \partial_k h_{il} \right| \leq C |h| |\partial h|^2.$$

Putting these facts together, we obtain

$$\begin{aligned} &\left| R_g - \partial_i \partial_k h_{ik} + \partial_i (h_{il} \partial_k h_{kl}) - \frac{1}{2} \partial_i h_{il} \partial_k h_{kl} + \frac{1}{4} \partial_l h_{ik} \partial_l h_{ik} \right| \\ &\leq C |h|^2 |\partial^2 h| + C |h| |\partial h|^2. \end{aligned}$$

This completes the proof. □

APPENDIX B. SOME USEFUL IDENTITIES

Proposition 27. *Suppose that α and β are real numbers satisfying $2\alpha - 2 > \beta + 1 > 0$. Then*

$$\int_0^\infty (1 + r^2)^{1-\alpha} r^\beta dr = \frac{2\alpha - 2}{2\alpha - \beta - 3} \int_0^\infty (1 + r^2)^{-\alpha} r^\beta dr$$

and

$$\int_0^\infty (1 + r^2)^{-\alpha} r^{\beta+2} dr = \frac{\beta + 1}{2\alpha - \beta - 3} \int_0^\infty (1 + r^2)^{-\alpha} r^\beta dr.$$

Proof. Using the fundamental theorem of calculus, we obtain

$$\begin{aligned} 0 &= \int_0^\infty \frac{d}{dr} [(1 + r^2)^{1-\alpha} r^{\beta+1}] dr \\ &= (\beta + 1) \int_0^\infty (1 + r^2)^{1-\alpha} r^\beta dr - (2\alpha - 2) \int_0^\infty (1 + r^2)^{-\alpha} r^{\beta+2} dr. \end{aligned}$$

From this the assertion follows. □

Proposition 28. *Suppose that $p(x)$ is a homogenous polynomial of degree d . Then*

$$\int_{\partial B_1(0)} p(x) = \frac{1}{d(n + d - 2)} \int_{\partial B_1(0)} \Delta p(x).$$

Proof. Using the divergence theorem, we obtain

$$\begin{aligned} \int_{\partial B_1(0)} \Delta p(x) &= (n+d-2) \int_{B_1(0)} \Delta p(x) \\ &= (n+d-2) \int_{\partial B_1(0)} \sum_{k=1}^n x_k \partial_k p(x) \\ &= d(n+d-2) \int_{\partial B_1(0)} p(x). \end{aligned}$$

□

Corollary 29. *We have*

$$\begin{aligned} \int_{\partial B_1(0)} x_i x_j &= \frac{1}{n} |S^{n-1}| \delta_{ij}, \\ \int_{\partial B_1(0)} x_i x_j x_k x_l &= \frac{1}{n(n+2)} |S^{n-1}| (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \end{aligned}$$

and

$$\begin{aligned} \int_{\partial B_1(0)} x_i x_j x_k x_l x_p x_q &= \frac{1}{n(n+2)(n+4)} |S^{n-1}| (\delta_{ij} \delta_{kl} \delta_{pq} + \delta_{ij} \delta_{kp} \delta_{lq} + \delta_{ij} \delta_{kq} \delta_{lp} \\ &\quad + \delta_{ik} \delta_{jl} \delta_{pq} + \delta_{ik} \delta_{jp} \delta_{lq} + \delta_{ik} \delta_{jq} \delta_{lp} \\ &\quad + \delta_{il} \delta_{jk} \delta_{pq} + \delta_{il} \delta_{jp} \delta_{kq} + \delta_{il} \delta_{jq} \delta_{kp} \\ &\quad + \delta_{ip} \delta_{jk} \delta_{lq} + \delta_{ip} \delta_{jl} \delta_{kq} + \delta_{ip} \delta_{jq} \delta_{kl} \\ &\quad + \delta_{iq} \delta_{jk} \delta_{lp} + \delta_{iq} \delta_{jl} \delta_{kp} + \delta_{iq} \delta_{jp} \delta_{kl}). \end{aligned}$$

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