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BLOW UP POINTS OF SOLUTION CURVES FOR A SEMILINEAR PROBLEM

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Abstract. We study a semilinear elliptic equation with an asymptotic linear nonlinearity. Exact multiplicity of solutions are obtained under various conditions on the nonlinearity and the spectrum set. Our method combines a bifurcation approach and Leray–Schauder degree theory.

1. Introduction

Consider a semilinear elliptic equation:

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$, $n \geq 1$, and λ is a positive parameter. We assume that f satisfies

(f1) $f \in C^1(\mathbb{R},\mathbb{R})$, $f(0) = 0$, $uf(u) > 0$ for $u \neq 0$, $\lim_{u\to\pm\infty} f(u) = \pm\infty$;
(f2)
$$\lim_{u\to\infty} \frac{f(u)}{u} = f_+ \geq 0, \qquad \lim_{u\to-\infty} \frac{f(u)}{u} = f_- \geq 0.$$

Let $0 < \lambda_1 < \ldots < \lambda_k \leq \ldots$ be the eigenvalues of the boundary value problem:

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

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We assume that all eigenvalues λ_k are simple. Define

$$\lambda_k^0 = \frac{\lambda_k}{f'(0)}. \tag{1.3}$$

Since $f'(0) > 0$, it is well-known that $(\lambda, u) = (\lambda_k^0, 0)$ is a bifurcation point where a bifurcation from the trivial solutions $u = 0$ occurs. (See Crandall, Rabinowitz [9].) Let the set of the nontrivial solutions of (1.1) be $\Sigma = \{(\lambda, u)\} \subset \mathbb{R}^+ \times X$, where $X = L^2(\Omega)$. Then $\Sigma \supset \bigcup_{k=1}^{\infty} \Sigma_k$, where Σ_k is the closure of the connected component of Σ which contains the point $(\lambda_k^0, 0)$. Since λ_k is a simple eigenvalue, near $(\lambda_k^0, 0)$, Σ_k is a curve of form $(\lambda(s), u(s))$, $(|s| \le \delta)$, with $\lambda(0) = \lambda_k^0$ and $u(0) = 0$. Away from the bifurcation point $(\lambda, u) = (\lambda_k^0, 0)$, the structure of Σ_k is very complicated in general. However, Rabinowitz [16] showed that the following alternative holds: either Σ_k is unbounded in $\mathbb{R} \times X$, or Σ_k contains another point $(\lambda_j^0, 0)$, $j \ne k$, i.e. $\Sigma_k = \Sigma_j$.

The limits f_+ and f_- has no effect on the bifurcation occurring at $(\lambda_k^0, 0)$, but they affect the asymptotic behavior of Σ_k greatly.

We consider all different cases of pair (f_+, f_-): (we assume that $f_+ \ge f_-$)

(1) (∞, ∞)-type: $f_+ = f_- = \infty$,
(2) $(0, 0)$-type: $f_+ = f_- = 0$,
(3) $(1, 1)$-type: $0 < f_- \le f_+ < \infty$,
(4) $(1, \infty)$-type: $0 < f_- < f_+ = \infty$,
(5) $(0, 1)$-type: $0 = f_- < f_+ < \infty$,
(6) $(0, \infty)$-type: $f_- = 0$, $f_+ = \infty$.

Sometimes, we call (∞, ∞)-type f *asymptotically superlinear*, $(0, 0)$-type f *asymptotically sublinear*, and $(1, 1)$-type f *asymptotically linear*[1].

We are interested in the asymptotic behavior of Σ_k. For asymptotically linear f, Shi and Wang [20] proved that if f also satisfies that

$$\frac{f(u)}{u} \text{ is increasing in } (0, \infty) \text{ and is decreasing in } (-\infty, 0), \tag{1.4}$$

and for $k = 1, \dots, m$,

$$\frac{\lambda_k}{f'(0)} \le \frac{\lambda_{k+1}}{\sup_{u \in \mathbb{R}} f'(u)}, \tag{1.5}$$

then for $k = 1, \dots, m$, $\Sigma_k = \{(\lambda(s), u(s)) : s \in \mathbb{R}\}$ is a simple curve globally, and there is no degenerate solution of (1.1) on Σ_k except $(\lambda_k^0, 0)$, i.e. $\lambda'(s) \ne 0$ for any $s \ne 0$. Moreover, $\Sigma_k \subset (\lambda_k^{\infty}, \lambda_k^0) \times X$, where $\lambda_k^{\infty} = \min(\lambda_k/f_+, \lambda_k/f_-)$. Thus from (1.5), all Σ_k's are mutually separated in λ-direction, unbounded in

[1] In other papers, $(1, 1)$-type f is often called asymptotically homogeneous nonlinearity. If $f_+ \ne f_-$, then it is also called a jumping nonlinearity (see Fučík [14]).

u-direction, and there exist $\lambda_{k,+}$ and $\lambda_{k,-}$ such that

$$\lambda_{k,+} = \lim_{s\to\infty} \lambda(s), \quad \text{and} \quad \lambda_{k,-} = \lim_{s\to-\infty} \lambda(s).$$

Obviously $||u(s)|| \to \infty$ as $s \to \pm\infty$, so we call $\lambda_{k,\pm}$ the *blow up points* of Σ_k. We define the blow up point in a more general way:

Definition 1.1. $\lambda_* \in [0,\infty]$ is a blow up point of the solution set of (1.1) if there exists a sequence $\{(\lambda^k, u^k)\}$ such that $\lim_{k\to\infty} \lambda^k = \lambda_*$ and $\lim_{k\to\infty} ||u^k|| = \infty$ as $k \to \infty$.

In the definition, we allow $\lambda_* = \infty$. The main purpose of this paper is to discuss the set of blow up points of (1.1) for f satisfying (f1) and (f2). Our motivation is two-fold. First it is related to the *a priori* estimates of the solutions. In fact, if $\lambda_* > 0$ is not a blow up point, then all solutions for this fixed λ_* are uniformly bounded, then some topological methods can be applied to obtain some or all solutions. Secondly, if λ_* is a blow up point and $\lambda_* \neq 0, \infty$, then λ_* is a point where a bifurcation from infinity occurs. Our results will be helpful for a better understanding of the bifurcation from infinity and the global structure of the solution set.

The bifurcation from infinity for asymptotically linear f is well-known. Rabinowitz [17] showed that if $f_+ = f_-$, and λ_k is an eigenvalue with odd algebraic multiplicity, then $\lambda_* = \lambda_k/f_\pm$ is a blow up point, and Σ possesses an unbounded component $\mathcal{D}$ which meets (λ_*, ∞). (See Theorem 1.6 and Theorem 2.28 in [17].) A similar result holds when $f_+ \neq f_-$ and $|f_+ - f_-|$ is small, see [19] Proposition 5.1. In [19], the following result was proved:

Proposition 1.2. *Assume that f satisfies* (f1) *and* (f2)*, and λ_* is a blow up point of* (1.1)*. If f is asymptotically linear, then either $\lambda_* = \infty$, or $0 < \lambda_* < \infty$ and for $(a,b) = (\lambda_* f_+, \lambda_* f_-)$,*

$$\begin{cases} \Delta\phi + a\phi^+ - b\phi^- = 0 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

has a nontrivial solution ϕ.

This result is true without any simplicity assumption on eigenvalues. Let $\Gamma = \{(a,b) \in \mathbb{R}^2 : (1.6) \text{ has a nontrivial solution}\}$. The set Γ is usually called Fučík spectrum of $-\Delta$ on Ω. Therefore, all blow up points in this case are contained in a set associated with Fučík spectrum. In the special case of $f_+ = f_-$, it degenerates to the set of eigenvalues of $-\Delta$. Note that the fact λ_* can be ∞ is not significant since there is a sequence $\{\lambda^k\}$ such that $(\lambda^k f_+, \lambda_k f_-) \in \Gamma$ and $\lambda^k \to \infty$ as $k \to \infty$. So, in this case, ∞ is a blow up point only because it is an accumulate point of finite blow up points.

In this paper, we will first prove another general result concerning the blow up points of (1.1):

Theorem 1.3. *Assume that f satisfies* (f1) *and* (f2), *and λ_* is a blow up point of* (1.1).

(1) *If f is asymptotically sublinear, then $\lambda_* = \infty$,*
(2) *If f is $(0,1)$-type, then $\lambda_* = \infty$ or $\lambda_* = \lambda_1/f_+$. Moreover, there is a connected component $\mathcal{D}_1$ of Σ approaching $(\lambda_1/f_+, \infty)$, and the solutions on $\mathcal{D}_1$ are all positive.*

In general, it is not clear if there is only one component approaching $(\lambda_1/f_+, \infty)$ in Theorem 1.3(2), and it is also not clear if $\mathcal{D}_1$ is a curve near infinity. But if in addition, we assume that $\lim_{u\to\infty} f'(u) = f_+$, then by a result of Dancer [10], $\mathcal{D}_1$ is the only component near $(\lambda_1/f_+, \infty)$, and it is a curve. The asymptotically superlinear case is far more complicated. We have only some partial results which are consequences of some earlier works:

Theorem 1.4. *Assume that f satisfies* (f1) *and* (f2), *and f is asymptotically superlinear.*

(1) *If f also satisfies*

$$\lim_{u\to\infty} \frac{f(u)}{u^l} = 0, \quad \text{with } l = \frac{n+2}{n-2} \text{ if } n \geq 3,\ l < \infty \text{ if } n = 1, 2, \tag{1.7}$$

$$\overline{\lim_{u\to\infty}} \frac{uf(u) - \theta F(u)}{u^2 f(u)^{2/n}} \leq 0, \quad \text{for some } 0 < \theta < \frac{2n}{n-2},\ \text{if } n \geq 3, \tag{1.8}$$

and

$$\frac{uf'(u)}{f(u)} \leq \frac{n+2}{n-2}, \quad \text{if } n > 2. \tag{1.9}$$

Then $\lambda_* = 0$ is a blow up point with a branch of positive solutions approaching $(\lambda, u) = (0, \infty)$.

(2) *If f satisfies* (1.7) *and f is an odd function, then for any $\lambda_* \in [0, \infty]$, λ_* is a blow up point.*

The first result is based on *a priori* estimate of positive solutions by de Figueiredo, Lions and Nussbaum [13]. All conditions (1.7)–(1.9) are subcritical conditions. In fact, it was shown by Brezis and Nirenberg [5] that for Ω being the unit ball, $n = 3$, $f(u) = u + u^5$, the blow up point of the positive solution curve for (1.1) is $\lambda_* = \lambda_1/4 > 0$. The second result is based on the fact that, if f is odd, then (1.1) has infinitely many solutions by Lusternik–Schnirelman theory. We conjecture that the second result is true without oddness of f, but it depends on whether (1.1) has infinitely many solutions without f being odd, which has been an outstanding open question for a long time. Another related result was proved by Bahri and Lions [3]: if f also satisfies (1.4), and λ_* is a finite blow up point, then not only the norm of the blowing-up solutions are unbounded, the Morse indices are also unbounded. The blow up points for $(1,\infty)$-type and $(0,\infty)$-type f are not known in the most general case. But the blow up points of the branches of positive solutions and negative solutions may be obtained since they are only related to one of f_+ and f_-. And we will obtain the precise information of blow up points for $(1,\infty)$-type and $(0,\infty)$-type f in a special case below.

The existence and multiplicity of solutions for (1.1) has been studied in many works, see, for examples, [1], [2], [4], [6]–[8], [12], [15], [19], [20] and also the references therein.

In the second part of the paper, we consider the blow up points for a special case of (1.1): $n = 1$, and $\Omega = (0,\pi)$, which is an autonomous ordinary differential equation:

$$u'' + \lambda f(u) = 0, \quad x \in (0,\pi), \quad u(0) = u(\pi) = 0. \tag{1.10}$$

We will completely classify the blow up points for (1.10) for all f's which satisfy (f1) and (f2). And as a consequence, we determine the asymptotic behavior of all solution curves Σ_k. (In this case, we can also show that $\Sigma = \bigcup_{k=1}^{\infty} \Sigma_k$.) Our main result in that part is

Theorem 1.5. *Assume that f satisfies* (f1) *and* (f2), *and λ_* is a blow up point of* (1.10).

(1) *If f is asymptotically superlinear, then the set of λ_* is $[0,\infty]$.*
(2) *If f is $(1,\infty)$-type, then $\lambda_* = 0$ or $\lambda_* = \infty$ or $\lambda_* = \lambda_k/f_-$ for $k \in \mathbb{N}$. Moreover, there is only one connected component of Σ approaching $(0,\infty)$, and the solutions on that component are all positive.*
(3) *If f is $(0,\infty)$-type, then $\lambda_* = 0$ or $\lambda_* = \infty$. Moreover, there is only one connected component of Σ approaching $(0,\infty)$, and the solutions on that component are all positive.*

A more complete classification will be found in Theorem 3.6. It is interesting that we obtain finite blow up points for $(1,\infty)$-type f. A consequence is that there is a bifurcation from infinity occurring at such a point, and moreover we can show that at each $\lambda_* = \lambda_k/f_-$, there are four branches of solutions bifurcating from there. (See Figure 2.) Recall that if $f_+ = f_- \in (0,\infty)$, then there are only two branches of solutions bifurcating from $\lambda_* = \lambda_k/f_\pm$. (See Figure 1.) The bifurcation from infinity for $(1,\infty)$-type f does not seem to be found in the literature. The asymptotic profile of solution on such branch is that the solution has large negative humps, and very small positive humps as $||u|| \to \infty$, $u(x) \approx -M|\sin(kx)|$ for $k \in \mathbb{N}$, where $M = ||u||$. In the asymptotically linear case, the sizes of positive humps and negative humps are in the same order as $||u|| \to \infty$.

We prove Proposition 1.2 and Theorems 1.3 and 1.4 in Section 2, and study (1.10) in Section 3. And in Section 4, we obtain an exact multiplicity result for (1.10).

Figure 1 Figure 2

2. Blow up points for PDE

In this section, we prove Proposition 1.2 and Theorems 1.3 and 1.4.

Proof of Proposition 1.2. Let λ_* be a blow up point and $\lambda_* < \infty$. Then there exists a sequence $\{(\lambda^k, u^k)\}$ such that $\lim_{k\to\infty} \lambda^k = \lambda_*$ and $\lim_{k\to\infty} ||u^k|| = \infty$ as $k \to \infty$. We define $\phi_k(x) = ||u^k||^{-1} u^k(x)$, then ϕ_k satisfies

$$\Delta\phi_k + \lambda^k \frac{f(u^k)}{u^k}\phi^k = 0. \tag{2.1}$$

We multiply (2.1) by ϕ_k and integrate over Ω, then we obtain

$$\int_\Omega |\nabla\phi_k|^2 dx - \lambda^k \int_\Omega \frac{f(u^k)}{u^k}\phi_k^2 dx = 0. \tag{2.2}$$

Since $f(u)/u$ is bounded, then $||\phi_k||_{H_0^1(\Omega)}$ is uniformly bounded. Thus there exists $\phi \in H_0^1(\Omega)$ such that $\{\phi_k\}$ has a subsequence (which we still denote by $\{\phi_k\}$) converging to ϕ strongly in $L^2(\Omega)$, and weakly in $H_0^1(\Omega)$. Let $\Omega^+ = \{x \in \Omega : \phi(x) > 0\}$ and $\Omega^- = \{x \in \Omega : \phi(x) < 0\}$. Then $u^k(x) = ||u^k||\phi_k(x) \to \pm\infty$ as $k \to \infty$ for $x \in \Omega^+ \cup \Omega^-$, thus

$$\frac{f(u^k(x))}{u^k(x)} \to f_+, \quad x \in \Omega^+, \quad \text{and} \quad \frac{f(u^k(x))}{u^k(x)} \to f_-, \quad x \in \Omega^-, \tag{2.3}$$

by Lebesgue Control Convergence Theorem.

Let $\psi \in C_0^1(\Omega)$. We multiply (2.1) by ψ and integrate over Ω, then we obtain (here $\Omega^0 = \Omega \backslash (\Omega^+ \bigcup \Omega^-)$)

$$\begin{aligned} \int_\Omega \nabla\phi_k \cdot \nabla\psi\, dx - \lambda^k \int_{\Omega^+} \frac{f(u^k)}{u^k}\phi_k\psi\, dx \\ - \lambda^k \int_{\Omega^-} \frac{f(u^k)}{u^k}\phi_k\psi\, dx - \lambda^k \int_{\Omega^0} \frac{f(u^k)}{u^k}\phi_k\psi\, dx = 0. \end{aligned} \tag{2.4}$$

By the weak convergence of ϕ_k and (2.3), we obtain

$$\int_\Omega \nabla\phi\cdot\nabla\psi dx - \int_\Omega (\lambda_* f_+\phi^+ - \lambda_* f_-\phi^-)\psi dx = 0, \tag{2.5}$$

and we conclude that ϕ is a weak solution of (1.6) with $(a,b) = (\lambda_* f_+, \lambda_* f_-)$.□

Proof of Theorem 1.3. We follow the similar way as in Proposition 1.2. Let λ^k, u^k, ϕ_k be the same as in the proof of Proposition 1.2. Suppose that the blow up point $\lambda_* \in [0,\infty)$, then $\{\lambda^k\}$ is bounded, and we can still conclude $\{\phi_k\}$ has a subsequence (which we still denote by $\{\phi_k\}$) converging to some ϕ strongly in $L^2(\Omega)$, and weakly in $H_0^1(\Omega)$. Let $\Omega_\pm$ be the same as in the proof of Proposition 1.2. Since f is sublinear, then (2.3) holds with $f_+ = f_- = 0$. Then (2.4) and (2.5) are also true with $f_+ = f_- = 0$. That implies $\Delta\phi = 0$, which has only zero solution in $H_0^1(\Omega)$, contradicting $||\phi|| = 1$. So for any finite $\lambda_* \geq 0$, λ_* is not a blow up point.

Next we prove ∞ is a blow up point. It suffices to prove for λ large, (1.1) has a positive solution $u(\lambda)$ such that $||u(\lambda)|| \to \infty$ as $\lambda \to \infty$. We use the sub-supersolution method. Let $\phi(x)$ be the principle eigenfunction of $\Delta\phi + \lambda_1\phi = 0$, $\phi = 0$ on $\partial\Omega$. Then for any $K > 0$, if $\lambda > 0$ is large enough, $\Delta(K\phi) + \lambda f(K\phi) = -\lambda_1 K\phi + \lambda f(K\phi) > 0$. So $K\phi$ is a subsolution. On the other hand, we choose $v(x) = -M(x\cdot x) + CM$, where $x\cdot x = \sum_{i=1}^n x_i^2$ for $x = (x_1,\dots,x_n)$, M and C are positive constants specified later. Then $\Delta v(x) = -2nM$, and $\Delta v + \lambda f(v) = -2nM + \lambda(f(v)/v)v = M[-2n + \lambda(f(v)/v)(C - x\cdot x)]$. First we choose $C > 0$ such that for any $x \in \overline{\Omega}$, $C_2 > C - x\cdot x \geq C_1 > 0$. This can be achieved since Ω is bounded. Next we choose $M > 0$ for fixed $K, \lambda > 0$, such that

(a) $MC_1 > K\phi(x)$ for $x \in \overline{\Omega}$,
(b) $\lambda C_2(f(MC_1)/MC_1) < 2n$.

All these can be met just letting M large enough, since f is asymptotically sublinear. Therefore v is a supersolution for such $M, C > 0$, and $v > K\phi$. Therefore, there exists a solution u of (1.1) such that $v \geq u \geq K\phi$ for any $K > 0$, and $||u|| \geq K||\phi||$. Since K can be arbitrarily large, ∞ is a blow up point.

The proof for $(0,1)$-type f is similar. The same proof will yield that ϕ satisfies $\Delta\phi + \lambda_* f_+\phi^+ = 0$ for some $\phi \in H_0^1(\Omega)$. But the only possibility is that $\lambda_* f_+ = \lambda_1$ and $\phi = \phi^+$ is the principle eigenfunction. So $\lambda_* = \lambda_1/f_+$ is the only possible finite blow up point. On the oereth hand, from Theorem 2.28 in [17], λ_1/f_+ is a point where bifurcation form infinity occurs, and the solutions on the component approaching $(\lambda_1/f_+, \infty)$ are positive. The same proof as in the last paragraph shows that ∞ is a blow up point for $(0,1)$-type f. □

Proof of Theorem 1.4. For the first part, we apply Theorem 2.1 of [13]. When f satisfies the conditions in Theorem 1.4(1), (1.1) has at least one positive solution u_λ for $\lambda \in (0, \lambda_1^0)$. Moreover, for λ in any compact subset of $(0, \lambda_1^0)$, u_λ is uniformly bounded because of the *a priori* estimate in Theorem 1.2 of [13]. However as $\lambda \to 0^+$, the solution u_λ cannot be bounded anymore since (1.1) has no solution when $\lambda = 0$. So $\lambda = 0$ is a blow up point with a branch of positive solutions approaching $(\lambda, u) = (0, \infty)$.

For the second part, we apply Theorem 9.38 of [18]. Then for any fixed $\lambda \in (0, \infty)$, (1.1) has a sequence of unbounded solutions, so λ is a blow up point. And 0 and ∞ are also blow up points since they are in the closure of $(0, \infty)$. □

3. Blow up points of ODE

In this section, we consider the blow up points of (1.10). In fact, we will determine most of the profile of the solution set Σ of (1.10). We starts with some preliminaries on (1.10). First, since $f(u)$ is at least C^1, then all solutions of (1.10) are smooth, so in this section, we will work in space $X = C^1[0, \pi]$ instead of L^2 space. The eigenvalue problem

$$\phi'' + \lambda\phi = 0, \quad x \in (0, \pi), \quad \phi(0) = \phi(\pi) = 0 \tag{3.1}$$

possesses a sequence of eigenvalues $\{\lambda_j = j^2\}$ such that $\lambda_1 < \cdots < \lambda_j \to \infty$ as $j \to \infty$, λ_j is a simple eigenvalue, the eigenfunction ϕ_j corresponding to λ_j has exactly $(j-1)$ zeros in $(0, \pi)$ and all zeros of ϕ_j in $[0, \pi]$ are simple. (A simple zero of ϕ_j is a point $x \in [0, \pi]$ such that $\phi_j(x) = 0$ and $\phi_j'(x) \neq 0$). We define

$$\begin{aligned} S_j^+ = &\{v \in X : v(0) = v(\pi) = 0,\ v_x(0) > 0,\ v \text{ has exactly } j-1 \text{ zeros in } (0,\pi), \\ &\text{and all zeros of } v \in [0,\pi] \text{ are simple }\}, \\ S_j^- = &-S_j^+, \text{ and } S_j = S_j^+ \cup S_j^-. \end{aligned}$$

Then Theorem 2.3 of [16] can be applied to (1.10), and we have the following lemma:

Lemma 3.1. *For any $k \in \mathbb{N}$, (1.10) possesses a component of solutions Σ_k in $\mathbb{R} \times X$ with $\Sigma_k \subset (\mathbb{R} \times S_k) \cup \{(\lambda_k^0, 0)\}$ and Σ_k is unbounded.*

Let $\Sigma_k^+ = \Sigma_k \bigcap S_k^+$ and $\Sigma_k^- = \Sigma_k \bigcap S_k^-$. We show that we can use the $\max u(\lambda, \cdot)$ to parameterize Σ_k^+, and $\min u(\lambda, \cdot)$ to parameterize Σ_k^-. If $u(\lambda, \cdot) \in S_k$ is a nontrivial solution of (1.10) with $k \geq 2$, then $u(\lambda, \cdot)$ is a rescaling and periodic extension of a positive solution and a negative solution. In fact, (u, u_x) is a solution of a first order system:

$$u' = v, \quad v' = -\lambda f(u), \quad u(0) = 0, \quad v(0) \neq 0. \tag{3.2}$$

For the function f which we consider here, each solution orbit of (3.2) in (u, v) plane is a periodic orbit centered at origin from the phase portrait analysis of (3.2).

Lemma 3.2. *Suppose that f satisfies* (f1), (f2). *Given $k \in \mathbb{N}$, for any $d > 0$ there exists exactly one $\lambda > 0$ such that* (1.1) *has a solution $u(\lambda, \cdot\,) \in S_k^+$ and $\max_{x\in[0,\pi]} u(\lambda, x) = d$. Similar result hold for $d < 0$ and S_k^-.*

Proof. We prove the lemma for $k = 2m + 1$ where $m \in \mathbb{N}$. The case when k is an even number is similar. Consider the initial value problem:

$$u' = v, \quad v' = -f(u), \quad u(0) = d, \quad v(0) = 0. \tag{3.3}$$

Let $(u(x), v(x))$ be the unique solution of (3.3). For any $d > 0$, there exists a unique $T_1 = T_1(d) > 0$ such that $u(T_1) = 0$, $u(x) > 0$ and $v(x) < 0$ for $x \in [0, T_1)$, and there exists a unique $T_2 = T_2(d)$ such that $v(T_1 + T_2) = 0$, $u(x) < 0$ and $v(x) < 0$ for $x \in (T_1, T_1 + T_2)$. Let $v(T_1) = -v_0 < 0$ for some $v_0 > 0$. Then

$$u' = v, \quad v' = -f(u), \quad u(0) = 0, \quad v(0) = v_0 \tag{3.4}$$

has a unique solution $(u_1(x), v_1(x))$ for $x > 0$. In particular, $(u_1(x), v_1(x)) = (u(x + T_1), v(x + T_1))$ from the property of (3.4). Therefore, $u_1(x)$ with $x \in [0, 2(m + 1)T_1 + 2mT_2]$ is a solution of $u'' + f(u) = 0$, $u(0) = 0$, $u(2(m + 1)T_1 + 2mT_2) = 0$, and u_1 has exactly $2m + 1$ zeros in $(0, 2(m + 1)T_1 + 2mT_2)$. Let $T = 2(m + 1)T_1 + 2mT_2$, $u_2(x) = u_1(Tx)$, then u_2 is a solution of (1.10) with $\lambda = \pi^{-2}T^2$. Since T_1 and T_2 are uniquely determined by $d > 0$, thus T and λ are also uniquely determined by d. □

Lemma 3.2 implies that Σ_k^+ can be represented as a graph $\{(\lambda(d), d) : d > 0\}$ in $\mathbb{R}^+ \times \mathbb{R}^+$. We define $\Sigma_k^+ = \{(\lambda_{k,+}(d), d) : d > 0\}$, $\Sigma_k^- = \{(\lambda_{k,-}(d), d) : d < 0\}$. To determine the asymptotic behavior of $\lambda_{k,\pm}(d)$, we derive the explicit formula of $\lambda_{k,\pm}(d)$. For any $d \neq 0$, let $(u(x), v(x))$ be the unique solution of (3.3). From the proof of Lemma 3.2, there is a unique $T_1 = T_1(d) > 0$ such that $u(T_1) = 0$, $u(x) \neq 0$ and $v(x) \neq 0$ for $x \in [0, T_1)$. T_1 can be calculated from the equation (1.10):

$$T_1(d) = \frac{1}{\sqrt{2}} \int_0^d \frac{du}{\sqrt{F(d) - F(u)}}. \tag{3.5}$$

where $F(u) = \int_0^u f(t)dt$. From the symmetry of the orbit of (3.3), we also have $T_1(R(d)) = T_2(d)$, where $T_2(d)$ is defined in the proof of Lemma 3.2, and $R(d)$ is the unique point satisfying $F(d) = F(R(d))$. Then $\lambda_{k,\pm}(d)$ can be determined as follows: $(m \geq 1)$

$$\begin{aligned} \lambda_{2m-1,\pm}(d) &= \pi^{-2}[2mT_1(d) + 2(m-1)T_1(R(d))]^2, \\ \lambda_{2m,\pm}(d) &= \pi^{-2}[2mT_1(d) + 2mT_1(R(d))]^2. \end{aligned} \tag{3.6}$$

To compute $T_1(d)$, we have the following basic estimates:

LEMMA 3.3. *Suppose that f satisfies*

$$(a-\eta)u \le f(u) \le (a+\eta)u, \quad \text{for } u \ge M, \tag{3.7}$$

where $a, M > 0$, $0 < \eta < a/2$. Then

$$\frac{a-\eta}{2}(u^2-v^2) \le F(u)-F(v) \le \frac{a+\eta}{2}(u^2-v^2), \tag{3.8}$$

for any $u > v \ge M$.

PROOF. Define $G(u) = F(u)-F(v)-(1/2)(a+\eta)(u^2-v^2)$. Then $G(v)=0$, $G'(u) = f(u)-(a+\eta)u \le 0$ for $u \in [v,\infty)$ by (3.7). Hence (3.8) holds. □

LEMMA 3.4. *Suppose that f satisfies* (f1) *and* (f2).

(1) *Suppose $0 < f_+ < \infty$, then for $d > 0$ large enough,*

$$T_1(d) = \frac{\pi}{2} f_+^{-1/2} + o(1). \tag{3.9}$$

(2) *Suppose $f_+ = \infty$, then for any fixed $A > 0$, for $d > 0$ large enough,*

$$T_1(d) \le A^{-1/2}\left(\frac{\pi}{2} + o(1)\right). \tag{3.10}$$

(3) *Suppose $f_+ = 0$, then for any fixed $A > 0$, for $d > 0$ large enough,*

$$T_1(d) \ge A^{-1/2}\left(\frac{\pi}{2} + o(1)\right). \tag{3.11}$$

PROOF. First we consider the case of $0 < f_+ < \infty$. For any $\eta > 0$, there is $M_1 > 0$ such that (3.7) holds for $a = f_+$, $M = M_1$. Then for $d > M_1$,

$$\sqrt{2}T_1(d) = \int_0^d \frac{du}{\sqrt{F(d)-F(u)}} = \left[\int_0^{M_1} + \int_{M_1}^d\right] \equiv [I_1 + I_2], \tag{3.12}$$

$$\begin{aligned} I_1 &= \int_0^{M_1} \frac{du}{\sqrt{F(d)-F(u)}} \\ &\le \frac{M_1}{\sqrt{F(d)-F(M_1)}} \le M_1\sqrt{\frac{2}{(f_+-\eta)(d^2-M_1^2)}}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} I_2 &= \int_{M_1}^d \frac{du}{\sqrt{F(d)-F(u)}} \le \sqrt{\frac{2}{f_+-\eta}} \int_{M_1}^d \frac{du}{\sqrt{d^2-u^2}} \\ &= \sqrt{\frac{2}{f_+-\eta}} \int_{M_1/d}^1 \frac{dw}{\sqrt{1-w^2}} \quad (\text{let } w = u/d) \\ &= \sqrt{\frac{2}{f_+-\eta}}\left[\frac{\pi}{2} - \arcsin\left(\frac{M_1}{d}\right)\right]. \end{aligned} \tag{3.14}$$

Similarly, we obtain

$$I_2 \geq \sqrt{\frac{2}{f_+ + \eta}}\left[\frac{\pi}{2} - \arcsin\left(\frac{M_1}{d}\right)\right]. \tag{3.15}$$

Therefore, from (3.13), (3.14) and (3.15)

$$\begin{aligned} T_1(d) &= (f_+^{-1/2} + C(\eta))\left[\frac{\pi}{2} - \arcsin\left(\frac{M_1}{d}\right)\right] \\ &\quad + O\left((f_+^{-1/2} + C(\eta))\sqrt{\left(\frac{M_1}{d}\right)^2 - 1}\right) \\ &= \frac{\pi}{2} f_+^{-1/2} + C(\eta) + O\left(\frac{M_1}{d}\right), \end{aligned} \tag{3.16}$$

where $M_1/d \ll 1$, $C(\eta)$ is a constant and $C(\eta) \to 0$ as $\eta \to 0$.

Next we consider the case of $f_+ = \infty$. For any $A > 0$, there exists $M_1 > 0$ such that $f(u) \geq Au$ for any $u > M_1$. Then (3.10) can be established from (3.13), (3.14). Similarly we can prove (3.11). □

We are ready to prove our main result on $\lambda_{k,\pm}(d)$:

Theorem 3.5. *Suppose that f satisfies* (f1) *and* (f2).

(1) *If f is of (∞,∞)-type, then as $d \to \pm\infty$, $\lambda_{k,\pm}(d) \to 0$ for $k \in \mathbb{N}$.*
(2) *If f is of $(0,0)$-type, then as $d \to \pm\infty$, $\lambda_{k,\pm}(d) \to \infty$ for $k \in \mathbb{N}$.*
(3) *If f is of $(1,1)$-type, then for $m \geq 1$,*

$$\begin{aligned} \lambda_{2m-1,+}(d) &= [m f_+^{-1/2} + (m-1) f_-^{-1/2}]^2 + o(1), \quad && d \to \infty, \\ \lambda_{2m-1,-}(d) &= [(m-1) f_+^{-1/2} + m f_-^{-1/2}]^2 + o(1), && d \to -\infty, \\ \lambda_{2m,\pm}(d) &= [m f_+^{-1/2} + m f_-^{-1/2}]^2 + o(1), && d \to \pm\infty. \end{aligned} \tag{3.17}$$

(4) *If f is of $(1,\infty)$-type, then for $m \geq 1$,*

$$\begin{aligned} \lambda_{2m-1,+}(d) &= (m-1)^2 f_-^{-1} + o(1), \quad && d \to \infty, \\ \lambda_{2m-1,-}(d) &= m^2 f_-^{-1} + o(1), && d \to -\infty, \\ \lambda_{2m,\pm}(d) &= m^2 f_-^{-1} + o(1), && d \to \pm\infty. \end{aligned} \tag{3.18}$$

(5) *If f is of $(0,1)$-type, then for $\lambda_{1,+}(d) = f_+^{-1} + o(1)$ as $d \to \infty$, $\lambda_{1,-}(d) \to \infty$ as $d \to -\infty$ and $\lambda_{k,\pm}(d) \to \infty$ as $d \to \pm\infty$ for $k \geq 2$.*
(6) *If f is of $(0,\infty)$-type, then for $\lambda_{1,+}(d) \to 0$ as $d \to \infty$, $\lambda_{1,-}(d) \to \infty$ as $d \to -\infty$ and $\lambda_{k,\pm}(d) \to \infty$ as $d \to \pm\infty$ for $k \geq 2$.*

Proof. The proofs are all similar and right from the estimates in Lemma 3.4. Note that the estimates in Lemma 3.4 is also true if we consider $u \in (-\infty, 0)$, and as $d \to \pm\infty$, then $R(d) \to \mp\infty$. We prove the case of f being $(1,\infty)$. From (3.6),

$$\lambda_{2m-1,+}(d) = \pi^{-2}[2m T_1(d) + 2(m-1) T_1(R(d))]^2,$$

By (3.10), $T_1(d) \leq [(\pi/2) + O(d^{-1})]A^{-1/2}$ for any $A > 0$ as $d \to \infty$, and by (3.9), $T_1(R(d)) = (\pi/2)f_-^{-1/2} + o(1)$. So we choose any large $A > 0$, we obtain $\lambda_{2m-1,+}(d) = (m-1)^2 f_-^{-1} + o(1)$ as $d \to \infty$. Other proofs are similar, so we omit them. □

We now determine the blow up points of (1.10) for all cases.

Theorem 3.6. *Suppose that f satisfies* (f1) *and* (f2), *and $B(f) \subset [0,\infty]$ is the set of blow up points* (1.10).

(1) *If f is of (∞,∞)-type, then $B(f) = [0,\infty]$.*
(2) *If f is of $(0,0)$-type, then $B(f) = \infty$.*
(3) *If f is of $(1,1)$-type, then $B(f) = \{\infty\} \cup \{[mf_+^{-1/2} + (m-1)f_-^{-1/2}]^2, [(m-1)f_+^{-1/2} + mf_-^{-1/2}]^2, [mf_+^{-1/2} + mf_-^{-1/2}]^2 : m \in \mathbb{N}\}$.*
(4) *If f is of $(1,\infty)$-type, then $B(f) = \{0,\infty\} \cup \{m^2 f_-^{-1} : m \in \mathbb{N}\}$.*
(5) *If f is of $(0,1)$-type, then $B(f) = \{f_+^{-1}, \infty\}$.*
(6) *If f is of $(0,\infty)$-type, then $B(f) = \{0,\infty\}$.*

Proof. First we prove for $(0,0)$-type f. From Theorem 3.5, $\lambda_{1,\pm}(d) \to \infty$ as $d \to \pm\infty$. For any fixed $\lambda_* > 0$, there is a maximal $d_* > 0$ such that $\lambda_{1,+}(d_*) = \lambda_*$, then for any $k \geq 1$, if $\lambda_{k,+}(d) = \lambda_*$, then $d \in (0, d_*)$. (Notice that $\lambda_{k,+}(d) > \lambda_{j,+}(d)$ for $k > j$, so $\lambda_{1,+}(\,\cdot\,)$ serves as an envelope of other $\lambda_{k,+}$.) And such d_* can be found for any point in a compact neighborhood of λ_*. Same arguments can be applied to $\lambda_{k,-}(\,\cdot\,)$. Thus λ_* is not a blow up point. $\lambda = 0$ is not a blow up point either, since $\lambda_{1,\pm}(d)$ has global minimum which is positive. On the other hand, it is easy to see that ∞ is a blow up point. The proofs for $(0,1)$-type and $(0,\infty)$-type are similar, since in each case we have an envelope which goes to infinity, and there is only one curve is outside of that envelope, which makes f_+^{-1} (or 0) also being a blow up point in respective cases.

For $(1,1)$-type f, the result can be obtained by combining Theorem 3.5 and Proposition 1.2. For $(1,\infty)$-type f, we know that all of 0, ∞ and $m^2 f_-^{-1}$ are blow up points. Suppose that there is another one $\lambda_* \in (0,\infty)$, then by the definition, there is a solution sequence $\{(\lambda^k, u^k)\}$ such that $\lambda^k \to \lambda_*$ and $||u|| \to \infty$ as $k \to \infty$. Then eventually all (λ^k, u^k) will be on the same curve $\lambda_{i,\pm}$, since otherwise $|\lambda^k - \lambda^j| \geq f_-^{-1} + o(1)$ from (3.18). Then $\lambda_* = m^2 f_-^{-1}$ for some $m \in \mathbb{N}$.

Finally we deal with (∞,∞)-type f. From Theorem 3.5, we see that for any $\lambda > 0$, (1.10) has infinitely many nontrivial solutions, and these solutions must be unbounded. Indeed, for any fixed $\lambda > 0$ and $d_0 > 0$, $T_1(d_0)$ and $T_1(R(d))$ are well defined and finite, then there exists $m \in \mathbb{N}$ such that $\lambda_{m,+}(d_0) > \lambda$ from the formula of $\lambda_{m,\pm}(d)$. On the other hand, $\lim_{d\to\infty} \lambda_{m,+}(d) = 0$, thus there exists $d_1 > d_0$ such that $\lambda_{m,+}(d_1) = \lambda$, which implies any $d_0 > 0$ is not an upper bound of L^∞ norm of solutions of (1.10). Thus any $\lambda \in [0,\infty]$ is a blow up point. □

We finish this section by summarizing our results in this section and describe the set Σ of the nontrivial solutions of (1.10). From Lemmas 3.1 and 3.2, $\Sigma = \bigcup_{k=1}^{\infty} \Sigma_{k,\pm}$, where $\Sigma_{k,+} \subset S_k^+$ and $\Sigma_{k,-} \subset S_k^-$. Each $\Sigma_{k,\pm}$ is a curve in $\mathbb{R}^+ \times X$, with a global parameter $d = \max_{x\in[0,\pi]} u(\lambda, x)$ for $\Sigma_{k,+}$ or $d = \min_{x\in[0,\pi]} u(\lambda, x)$ for $\Sigma_{k,-}$. So we can write

$$\Sigma_{k,+} = \{(\lambda_{k,+}(d), d) : d > 0\} \quad \text{and} \quad \Sigma_{k,-} = \{(\lambda_{k,-}(d), d) : d < 0\}.$$

$\Sigma_{k,+}$ bifurcates from $(\lambda_k^0, 0)$, and terminates at a blow up point which is determined by Theorem 3.6. Under merely (f1) and (f2), there may be turning points on $\Sigma_{k,+}$, which is where $\lambda_{k,+}'(d) = 0$. However, if we also assume (1.4) (superlinear) or the condition with opposite sign (sublinear), then there is no turning points on $\Sigma_{k,+}$ except at $(\lambda_k^0, 0)$. (See [20] Theorem 1.4 and [19] Theorem 7.1, also Section 4 of the present paper.)

From Theorem 3.6, there are several cases where we have a blow up point $\lambda_* \in (0, \infty)$. For $(0, 1)$-type, there is a unique finite blow up point, where the branch of positive solutions blows up, which is not surprising from the result of [17]. For $(1, 1)$-type, there are a sequence of blow up points related to the Fučík spectrum. In fact, the Fučík spectrum in one dimensional space is well-known, which consists of two curves in $\mathbb{R}^+ \times \mathbb{R}^+$ emanating from (λ_k, λ_k). (See [19].) The blow up points in Theorem 3.6 can be regarded as all the intersection points of a ray $y = (f_+/f_-)x$, $x > 0$ and the curves of the Fučík spectrum. For the blow up points of form $[mf_+^{-1/2} + (m-1)f_-^{-1/2}]^2$ or $[(m-1)f_+^{-1/2} + mf_-^{-1/2}]^2$, there is exactly one solution curve blowing up at that point, and for the points of form $[mf_+^{-1/2} + mf_-^{-1/2}]^2$, there are two curves, one with $d > 0$ and one with $d < 0$. See Theorem 3.5 for corresponding curves.

For $(1, \infty)$-type, there are four curves blowing up at $m^2 f_-^{-1}$ for $m \in \mathbb{N}$, and they are $\Sigma_{2m,\pm}$, $\Sigma_{2m-1,-}$ and $\Sigma_{2m+1,+}$. From the point of view of bifurcation from infinity, a multiple bifurcation occurs at $m^2 f_-^{-1}$. Note that $m^2 f_-^{-1} = \lambda_m/f_-$, it would be an interesting question whether $\lambda = \lambda_m/f_-$ is a point where a bifurcation from infinity occurs for general domain Ω, and if so, what is the multiplicity of bifurcation at such a point.

4. An exact multiplicity result

In this section, we are concerned with the multiplicity of the nontrivial solutions of (1.1) for a fixed $\lambda > 0$. To obtain the exact multiplicity results, we assume f also satisfies (1.4), or

$$\frac{f(u)}{u} \text{ is decreasing in } (0, \infty) \text{ and is increasing in } (-\infty, 0). \tag{4.1}$$

We will only consider (1.10). The exact multiplicity for (1.1) for general domain was established in [20] and [19], but with much restricted conditions on the relation between the nonlinear function $f(u)$ and the spectral set $\{\lambda_i\}$.

The bifurcation version of the exact multiplicity for (1.10) is also proved in [20] and [19], which we quote here:

Theorem 4.1. *Suppose that f satisfies* (f1), (f2) *and* (1.4). *Then there exists a solution curve $\Sigma_k = \{(\lambda_k(d), u(d)) : d \in \mathbb{R}\}$ bifurcating from $(\lambda, u) = (\lambda_k^0, 0)$, $\lambda_k'(d) < 0$ for $d > 0$ and $\lambda_k'(d) > 0$ for $d < 0$. If $(\lambda, u) \in \Sigma_k \backslash \{(\lambda_k^0, 0)\}$, then $u \in S_k$. Let $\lambda_{k,\pm} = \lim_{d\to\pm\infty} \lambda_k(d)$, then $\lambda_{k,+}$ and $\lambda_{k,-}$ belongs to the blow up point set in Theorem* 3.5 *according to the different behavior of (f_+, f_-). The same results hold if we replace* (1.4) *by* (4.1), *except $\lambda_k'(d) > 0$ for $d > 0$ and $\lambda_k'(d) < 0$ for $d < 0$.*

Theorem 4.1 excludes the possibility of turning points on Σ_k, and Σ_k is a curve which bends to the left if (1.4) holds and bends to the right if (4.1) holds. So for fixed λ, we can examine the location of all bifurcation points and determine the exact count of the nontrivial solutions. Similar to [19], we consider

$$u'' + f(u) = 0, \quad x \in (0, \pi), \quad u(0) = u(\pi) = 0. \tag{4.2}$$

We have the following result:

Theorem 4.2. *Suppose that f satisfies* (f1), (f2) *and* (1.4).

(1) *If f is of (∞, ∞)-type, then* (4.2) *has infinitely many solutions.*
(2) *If f is of $(0, 0)$-type, then* (4.2) *has exactly $2k$ nontrivial solutions if $\lambda_k < f'(0) \le \lambda_{k+1}$.*
(3) *If f is of $(1, \infty)$-type, then* (4.2) *has exactly* 1 *nontrivial solution if $f_- \le \lambda_1$, and has $4k-2m+1$ nontrivial solutions if $\lambda_m < f'(0) \le \lambda_{m+1}$ and $\lambda_k < f_- \le \lambda_{k+1}$.*
(4) *If f is of $(0, 1)$-type, then* (4.2) *has no nontrivial solution if $f'(0) < \lambda_1$ and $f_+ \le \lambda_1$, has exactly* 1 *nontrivial solution if $f'(0) < \lambda_1$ and $\lambda_1 < f_+$, has no nontrivial solution if $f'(0) = \lambda_1$, and has exactly $2k - 1$ nontrivial solutions if $\lambda_1 < f'(0) \le \lambda_{k+1}$ for $k \ge 1$.*
(5) *If f is of $(0, \infty)$-type, then* (4.2) *has exactly* 1 *nontrivial solution if $f'(0) < \lambda_1$, has no nontrivial solution if $f'(0) = \lambda_1$, and has exactly $2k - 1$ nontrivial solutions if $\lambda_1 < f'(0) \le \lambda_{k+1}$ for $k \ge 1$.*

For $(1, 1)$-type f, a similar result holds, see Theorem 7.7 in [19]. The proof of the theorem is quite simple by just counting the number of bifurcations. For example, we sketch a proof for $(0, 1)$-type f: we can embed (4.2) into (1.10), then $\lambda_m < f'(0) \le \lambda_{m+1}$ and $\lambda_k < f_- \le \lambda_{k+1}$ are equivalent to $\lambda_m/f'(0) < \lambda \le \lambda_{m+1}/f'(0)$ and $\lambda_k/f_- < \lambda \le \lambda_{k+1}/f_-$. For $\lambda \le \lambda_k/f_-$, there is one solution bifurcated from $\lambda = 0$ and four solutions bifurcated from $\lambda = \lambda_i/f_-$ for $i = 1, \dots, k$, so there are $4k + 1$ solutions bifurcated from infinity before λ reaches $\lambda = \lambda_k/f_-$. On the other hand, each time λ crosses $\lambda = \lambda_i/f'(0)$, two solutions bifurcate into $u = 0$, so among these $4k + 1$ solutions, $2m$ solutions bifurcate into 0 before $\lambda = \lambda_m/f'(0)$, so the final count is $4k - 2m + 1$.

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