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Blowing up and desingularizing constant scalar curvature Kähler manifolds

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1. Introduction

Let (M,ω) be either an m-dimensional compact Kähler manifold or an m-dimensional compact Kähler orbifold with isolated singularities. By definition, any point $p\in M$ has a neighborhood biholomorphic to a neighborhood of the origin in $\mathbf{C}^m/\Gamma$, where Γ is a finite subgroup of $\mathrm{U}(m)$ (this last fact is a consequence of the Kähler property) acting freely on $\mathbf{C}^m\setminus\{0\}$. Observe that, when p is a smooth point of M, the group Γ reduces to the identity. In the case where M is an orbifold, the Kähler form ω lifts, near any of the singularities of M, to a Kähler form $\widetilde{\omega}$ on a punctured neighborhood of 0 in $\mathbf{C}^m$. We will always assume that $\widetilde{\omega}$ can be smoothly extended through the origin, i.e. that ω is an orbifold metric.

If we further assume that the Kähler form ω has constant scalar curvature and if we are given n distinct (smooth) points $p_1,...,p_n\in M$, one of the questions we would like to address in this paper is whether the blow-up of M at the points $p_1,...,p_n$ can still be endowed with a constant scalar curvature Kähler form. In this direction, we have obtained the following positive answer:

Theorem 1.1. *Let (M,ω) be a constant scalar curvature compact Kähler manifold or Kähler orbifold with isolated singularities. Assume that there is no nonzero holomorphic vector field vanishing somewhere on M. Then, given finitely many smooth points $p_1,...,p_n\in M$ and positive numbers $a_1,...,a_n>0$, there exists $\varepsilon_0>0$ such that the blow-up of M at $p_1,...,p_n$ carries constant scalar curvature Kähler forms*

$$\omega_\varepsilon\in\pi^*[\omega]-\varepsilon^2(a_1\mathrm{PD}[E_1]+...+a_n\mathrm{PD}[E_n]),$$

where the $\mathrm{PD}[E_j]$ are the Poincaré duals of the $(2m-2)$-homology classes of the exceptional divisors of the blow-up at p_j and $\varepsilon\in(0,\varepsilon_0)$.

If the scalar curvature of ω *is not zero then the scalar curvatures of* ω_ε *and of* ω *have the same sign.*

Following a suggestion of LeBrun, we also show that the proof of Theorem 1.1 can be used to produce zero scalar curvature Kähler metrics provided the Kähler form ω we start with has zero scalar curvature, and the first Chern class of M is not zero.

COROLLARY 1.1. *Let* (M, ω) *be a zero scalar curvature compact Kähler manifold or orbifold with isolated singularities. Assume that there is no nonzero holomorphic vector field vanishing somewhere on* M *and that the first Chern class of* M *is not zero. Then the blow-up of* M *at finitely many* (*smooth*) *points carries zero scalar curvature Kähler forms.*

Observe that, on manifolds (or orbifolds with isolated singularities) with discrete automorphism group, there are no nontrivial holomorphic vector fields. Hence, if these carry a constant scalar curvature Kähler form, they are examples to which our results do apply.

On the other hand, the assumption is verified also by some manifolds with a continuous family of automorphisms. For example Kähler flat tori can be used as base manifolds in Theorem 1.1 (but not in Corollary 1.1, since their first Chern class vanish).

Theorem 1.1 and Corollary 1.1 are consequences of a more general construction which also allows one to desingularize isolated singularities of orbifolds. This desingularization procedure combined with Theorem 1.1 is enough to prove the following result:

THEOREM 1.2. *Any compact complex surface of general type admits constant scalar curvature Kähler forms.*

It is worth pointing out that some assumption on the initial manifold (M, ω) is indeed necessary for either the desingularization or the blow-up procedure to be successful. In the first place we know from the work of Matsushima [42] and Lichnerowicz [39] that the automorphism group of a manifold with a Kähler constant scalar curvature metric must be reductive, hence, for example, the projective plane blown up at one or two points does not admit any constant scalar curvature Kähler metric (see [8, p. 331]). In the same spirit let us recall that, given a compact complex orbifold M and a fixed Kähler class $[\omega]$, there is another obstruction for the existence of a constant scalar curvature Kähler metric in the class $[\omega]$. This obstruction was discovered by Futaki in the 1980s [20], [21], [22] for smooth metrics and was extended to singular varieties by Ding and Tian [16], and to Kähler constant scalar curvature metrics by Bando, Calabi [12] and Futaki. This obstruction will be briefly described in §4, since it will play some role in our construction. The nature of this obstruction (being a character of the Lie algebra of the automorphism group) singles out two different types of Kähler manifolds or Kähler orbifolds with isolated singularities where to look for constant scalar curvature metrics: those with no nonzero holomorphic vector fields vanishing somewhere, where the obstruction is vacuous since they do not have any nontrivial holomorphic vector field, and the others where the Futaki invariant has to vanish for all holomorphic vector fields. We will say that (M,ω), a constant scalar curvature Kähler manifold or Kähler orbifold with isolated singularities, is *nondegenerate* if it does not carry any nontrivial holomorphic vector field vanishing somewhere (note that this definition does not depend on the particular Kähler class on M), and we will say that (M,ω) is *Futaki-nondegenerate* if the differential of the Futaki invariant satisfies some nondegeneracy condition (this definition, which will be made precise in §4, does depend on the Kähler class $[\omega]$).

Note that, thanks to Matsushima–Lichnerowicz's decomposition of the Lie algebra of holomorphic vector fields on a manifold which admits a constant scalar curvature Kähler metric (see [8] and [22]), a constant scalar curvature Kähler manifold is *nondegenerate* if and only if every holomorphic vector field is parallel.

Our construction gives a quite precise description of the Kähler forms we obtain on the blown-up manifold or on the desingularized orbifold. We shall now describe more carefully the general construction and some of its consequences, but also we shall give more details about the Kähler forms we construct.

The construction is obtained by choosing finitely many points $p_1, ..., p_n \in M$ and replacing a small neighborhood of each point p_j, biholomorphic to a neighborhood of the origin in $\mathbf{C}^m/\Gamma_j$, by a (suitably scaled down by a small factor ε) piece of a Kähler manifold or a Kähler orbifold with isolated singularities (N_j, η_j), biholomorphic to $\mathbf{C}^m/\Gamma_j$ away from a compact subset. This generalized connected sum yields a Kähler manifold or a Kähler orbifold with isolated singularities that we call

$$M \sqcup_{p_1,\varepsilon} N_1 \sqcup_{p_2,\varepsilon} ... \sqcup_{p_n,\varepsilon} N_n$$

and whose complex structure does not depend on $\varepsilon > 0$. We proceed to perturb the Kähler forms ω and $\varepsilon^2 \eta_j$ on the various summands, analyzing in §5 the linear part and in §6 the nonlinear part of the constant scalar curvature equation in a given Kähler class. This leads to a study of nonlinear fourth-order elliptic partial differential equations on the Kähler potentials. Then, at the end of §6, we "glue" the Kähler potentials of the perturbed Kähler forms on the different summands to get a Kähler form whose scalar curvature is constant. The most important condition that ensures this program to be successful is the following: *Each (N_j, η_j) is an "Asymptotically Locally Euclidean" (ALE) space and η_j is a zero scalar curvature Kähler form.*

Since the term ALE has often been used with slightly different meanings, we make this definition precise. In this paper, an ALE space (N, η) is an m-dimensional Kähler manifold or Kähler orbifold with isolated singularities, where $N\backslash K$ is, for some compact set K, biholomorphic to $(\mathbf{C}^m\backslash B)/\Gamma$, where B is a closed ball and where Γ is a finite subgroup of $\mathrm{U}(m)$ acting freely on $\mathbf{C}^m\backslash\{0\}$. The manifold N is assumed to be equipped with a Kähler metric η which converges to the Euclidean metric at infinity. In the case where N is an orbifold with isolated singularities, we assume that, near any singularity modeled after a neighborhood of 0 in $\mathbf{C}^m/\widetilde{\Gamma}$, the Kähler form η lifts smoothly to a neighborhood of 0 in $\mathbf{C}^m$. In addition, we will always assume that there exist complex coordinates $(u^1, ..., u^m)$ parameterizing N away from a compact set, in which the Kähler form η can be expanded as

$$\eta = i\partial\bar{\partial}\big(\tfrac{1}{2}|u|^2 + \widetilde{\varphi}(u)\big) \tag{1}$$

at infinity, where the potential $\widetilde{\varphi}$ satisfies

$$\widetilde{\varphi}(u) = \begin{cases} a|u|^{4-2m} + \mathcal{O}(|u|^{3-2m}), & \text{when } m \geqslant 3, \\ a\log|u| + \mathcal{O}(|u|^{-1}), & \text{when } m = 2. \end{cases} \tag{2}$$

Here $a \in \mathbf{R}$ and we agree that $\mathcal{O}(|u|^q)$ is a smooth function whose kth partial derivatives are bounded by a constant times $|u|^{q-k}$, for all $k \geqslant 2$. The growth (or decay) of the Kähler potentials for these models is a subtle problem of independent interest [4]. In given examples, we will see in §7 that these potentials can arise with various orders and decays, and we will show (Lemma 7.2) that, for zero scalar curvature metrics and under reasonable growth assumptions on the potential, one can change suitably the potential in order to get a potential for which (2) holds.

Let us now summarize the assumptions under which our general construction works. We will assume that:

(i) (M, ω) is an m-dimensional compact Kähler manifold or orbifold with isolated singularities.

(ii) The scalar curvature of ω is constant.

(iii) (M, ω) is either nondegenerate or is Futaki-nondegenerate.

(iv) Given points $p_1, ..., p_n \in M$ which might be either singular or regular points of M, let Γ_j be the finite subgroup of $\mathrm{U}(m)$ acting freely on $\mathbf{C}^m\backslash\{0\}$ such that a neighborhood of p_j is biholomorphic to a neighborhood of the origin in $\mathbf{C}^m/\Gamma_j$. Each $\mathbf{C}^m/\Gamma_j$ has an ALE resolution (N_j, η_j) (which might either be a manifold or an orbifold with isolated singularities) endowed with a zero scalar curvature Kähler form η_j. Furthermore, we assume that, away from a compact set, the Kähler form η_j can be expanded as in (1) with a potential satisfying (2).

Our main result reads:

THEOREM 1.3. *Assume that* (i)–(iv) *are satisfied. Then, there exists* $\varepsilon_0>0$ *and, for all* $\varepsilon\in(0,\varepsilon_0)$, *there exists a constant scalar curvature Kähler form* $\widetilde{\omega}_\varepsilon$ *defined on* $M\sqcup_{p_1,\varepsilon}N_1\sqcup_{p_2,\varepsilon}...\sqcup_{p_n,\varepsilon}N_n$.

As $\varepsilon\to 0$, *the sequence of Kähler forms* $\widetilde{\omega}_\varepsilon$ *converges* (*in* $\mathcal{C}^\infty$ *topology*) *to the Kähler metric* ω, *away from the points* p_j *and the sequence of Kähler forms* $\varepsilon^{-2}\widetilde{\omega}_\varepsilon$ *converges* (*in* $\mathcal{C}^\infty$ *topology*) *to the Kähler form* η_j, *on compact subsets of* N_j.

If ω *has positive* (*resp. negative*) *scalar curvature then the Kähler forms* $\widetilde{\omega}_\varepsilon$ *have positive* (*resp. negative*) *scalar curvature.*

Moreover, if (M,ω) *is* nondegenerate *then*

$$[\omega_\varepsilon]=[\omega]+\varepsilon^2([\eta_1]+...+[\eta_n]).$$

Note that, when (M,ω) is Futaki-nondegenerate, we cannot control the Kähler class where we find the constant scalar curvature Kähler metric.

All the previous results are consequences of this theorem.

For example the blow-up at smooth points is obtained by our generalized connected sum construction, taking N_j to be the total space of the line bundle $\mathcal{O}(-1)$ over $\mathbf{P}^{m-1}$ (in this case $\Gamma_j=\{\mathrm{id}\}$). The key property (iv) asks for an ALE zero scalar curvature metric η_j on $\mathcal{O}(-1)$ such that $[\eta_j]=-\mathrm{PD}[E_j]$ and with appropriate decay at infinity. These Kähler forms have been obtained by Calabi [11]. When $m=2$, η is usually referred to in the literature as Burns metric, and it has been described (and generalized) in a very detailed way by LeBrun [34]. In higher dimensions, $m\geqslant 3$, these metrics have been generalized by Simanca [53]. The ALE property and the issue of the rate of decay of these metrics towards the Euclidean metric can be easily derived from these papers. In the 2-dimensional case, the Kähler form η is explicit and these properties follow at once, while, in higher dimensions, it can be shown that these metrics have a potential for which (1) and (2) are satisfied. The analysis of these asymptotic properties will be done in Lemma 7.1 (Raza [49] has given an alternative proof using toric geometry). In any case, assumption (iv) is fulfilled and, given smooth points $p_1,...,p_n\in M$ and positive constants $a_1,...,a_n$, the existence of such models can be plugged into Theorem 1.3 with all ALE spaces equal to $N=\mathcal{O}(-1)$ over $\mathbf{P}^{m-1}$ with the Burns–Calabi–Simanca form $\eta_j=a_j\eta$. This leads to the results of Theorem 1.1, which then also holds for Futaki-nondegenerate manifolds (M,ω), only losing control on which Kähler class is represented.

In §8 we will observe that our gluing procedure decreases the starting scalar curvature. Therefore, if (M,ω) has zero scalar curvature, Theorem 1.3 gives (small) negative scalar curvature metrics. Nonetheless, if the first Chern class is not zero, LeBrun–Simanca [37] have shown that there exist nearby Kähler metrics ω_+ and ω_- of (small) positive and negative constant scalar curvature, respectively. We can then apply Theorem 1.1 to (M,ω_+) and (M,ω_-) to get positive and negative Kähler metrics of constant scalar curvature on the blow-up. We will show in §8 how this implies Corollary 1.1, a result which also extends to Futaki-nondegenerate manifolds with nonzero first Chern class. A similar result had been previously proved in complex dimension 2 by Rollin–Singer [51], who have shown that one can desingularize compact orbifolds of zero scalar curvature with cyclic orbifold groups, keeping the scalar curvature zero, by solving on the desingularization the Hermitian anti-selfdual equation.

To prove Theorem 1.2 we need to apply Theorem 1.3 more than once. The idea, which comes directly from algebraic geometry, is to associate with M a (possibly) singular complex surface $\overline{M}$, such that M is obtained form $\overline{M}$ by desingularizing and blowing up smooth points a finite number of times. Algebraic geometry [7] tells us that if M is a surface of general type then $\overline{M}$ (which is called the pluricanonical model of the minimal model of M) satisfies the following properties:

(i) $\overline{M}$ is again a complex variety [30];

(ii) $\overline{M}$ has only isolated singular points whose local structure groups Γ_j are in SU(2) [9];

(iii) the first Chern class of $\overline{M}$ is negative, hence it has only a discrete group of automorphisms [29, Theorem 2.1, p. 82];

(iv) $\overline{M}$ admits a Kähler–Einstein orbifold metric [29].

We will explain below how Theorem 1.3 can be used to resolve SU(2) singularities. Granted this, M is then reobtained from this desingularized manifold after a finite number of blow-ups at smooth points, and the constant scalar curvature Kähler metric is then given by Theorem 1.1.

If p_j is a singular point (and hence Γ_j is not the identity group), there is no unique way to resolve the singularity, and in fact this is an extremely rich area of algebraic geometry. Once again, whether constant scalar curvature metrics exist or not on such resolutions depends, according to Theorem 1.3, on the existence of ALE scalar-flat Kähler resolutions of $\mathbf{C}^m/\Gamma$. For a general finite subgroup $\Gamma\subset\mathrm{U}(m)$, the existence of such a resolution is unknown and this prevents us to state general existence results for constant scalar curvature Kähler metrics. Nonetheless there are large classes of discrete nontrivial groups for which a good local model is known to exist, looking at Ricci-flat metrics, very much in the spirit of noncompact versions of the Calabi conjecture.

This is the line started by Tian–Yau [57] and Bando–Kobayashi [5], culminating in Joyce's proof of the ALE Calabi conjecture [26]. Joyce himself used this approach to have good local models for his well-known special holonomy desingularization result. Joyce's theorem, recalled in Theorem 7.1, implies that given a $\mathbf{C}^m/\Gamma$ such that a Kähler crepant resolution N exists (i.e. a Kähler resolution with $c_1(N)=0$), then N has a Ricci-flat Kähler metric η which, at infinity, can be expanded as

$$\eta = i\partial\bar{\partial}\big(\tfrac{1}{2}|u|^2+\widetilde{\varphi}(u)\big)$$

for some potential $\widetilde{\varphi}$ satisfying

$$\widetilde{\varphi} = \mathcal{O}(|u|^{2-2m}),$$

so well inside the range of application of Theorem 1.3. This approach works for example when $\Gamma=\mathbf{Z}_m$, acting diagonally on $\mathbf{C}^m$ [11], and for *any* finite subgroup of SU(2) and SU(3), since in this case we know that $\mathbf{C}^m/\Gamma$ has a smooth Kähler crepant resolution. We shall refer to [50] and [26, §6.4 and Chapter 8], for these results. The 2-dimensional case can be handled directly also relying on Kronheimer's result [33].

In light of these results, and when $m=2$, we can apply Theorem 1.3 when (M,ω) is a 2-dimensional nondegenerate or Futaki-nondegenerate Kähler orbifold with isolated singularities, and $p_1, ..., p_n \in M$ is any set of points with a neighborhood biholomorphic to a neighborhood of the origin in $\mathbf{C}^2/\Gamma_j$, where Γ_j is a finite subgroup of SU(2) acting freely on $\mathbf{C}^2\setminus\{0\}$. As explained above, this is enough to prove Theorem 1.2.

Let us mention that other local models are known, for example when $\Gamma=\mathbf{Z}_k$ is act ing on $\mathbf{C}^m$ by multiplication by a kth root of unity. In this case N is the total space of the line bundle $\mathcal{O}(-k)$ over $\mathbf{P}^m$ and the metric has zero scalar curvature but, in general, is not Ricci-flat. Rollin–Singer [52] have proved the required decay properties on the corresponding Kähler potential.

Other examples should come from the work of Calderbank–Singer [13]. They have in fact shown the existence of ALE zero scalar curvature resolutions of all U(2) cyclic isolated singularities. The only piece of information missing at the moment to use them in our construction is the behavior at infinity of a Kähler potential associated with these ALE metrics.

One of the main sources of interest in Kähler metrics with constant scalar curvature lies in its relation with algebraic geometric properties of the underlying manifold, such as Chow–Mumford, Tian, or asymptotic Hilbert–Mumford stability [17], [41], [47], [48], [55]. We know for example, by Mabuchi's extension of Donaldson's work [17], [41] that if an integral class is represented by an extremal Kähler metric, then the underlying algebraic manifold is asymptotically stable in a sense which depends on the structure of the automorphism group which preserves the class. In particular if we have a Kähler manifold with discrete automorphism group, this stability reduces to the classical Chow stability. Rescaling the Kähler class $[\omega_\varepsilon]$ by a factor k to make it integral, our results have the following corollary.

Corollary 1.2. *Let (M,L) be a polarized compact algebraic manifold of complex dimension $m\geqslant 2$, with discrete automorphism group, and ω a Kähler form with constant scalar curvature in an integral class. Then all the manifolds obtained by blowing up any finite set of points are asymptotically Chow-stable relative to the polarizing class $k\pi^*[\omega]-(\mathrm{PD}[E_1]+...+\mathrm{PD}[E_n])$, where the E_j are the exceptional divisors of the blow-up and k is sufficiently big.*

Note that playing with the weights a_j in Theorem 1.1, one gets an abundance of different polarizing classes for which the above corollary holds (in the above statement we have used $a_1=...=a_n$). Moreover similar results for the different versions of stability, which are known to be implied by the existence of constant scalar curvature Kähler metrics, follow from our theorems. In this setting it is interesting to observe that Mabuchi's and Donaldson's results do not apply to Kähler orbifolds, due to the failure of the Tian–Catling–Zelditch expansion [54]. Nonetheless if a *full* desingularizing process were to go through, then we would get the stability of the smooth polarized manifold obtained.

Another important phenomenon concerning constant scalar curvature Kähler metrics is that they are unique in their class, up to automorphisms. This result was proved for Kähler–Einstein metrics thanks to the work of Calabi [10] when $c_1\leqslant 0$, and Bando–Mabuchi [6] when $c_1>0$. Uniqueness of constant scalar curvature Kähler metrics was then proved by Donaldson and Mabuchi [40] (for extremal metrics) in integral classes (with either discrete or continuous automorphism group), and by Chen [14] for any Kähler class on manifolds with $c_1\leqslant 0$. Recently Chen and Tian [15] have proved it for any Kähler class, even for extremal metrics. This implies that all the constant scalar curvature Kähler metrics produced in this paper are the unique such representative of their Kähler class, up to automorphisms.

Despite these important works, our knowledge of concrete examples is still limited and mainly confined in complex dimension 2. For example, Hong [24], [25] has proved the existence of such metrics in some Kähler classes of ruled manifolds, and Fine [19] has studied this problem for complex surfaces projecting over Riemann surfaces with fibres of genus at least 2. The only case completely understood, and giving a rich source of examples, is the one of zero scalar curvature Kähler surfaces thanks to the work of Kim, LeBrun, Pontecorvo and Singer [27], [28], [35], [38], and the recent results of Rollin–Singer [51]. Our construction allows one to produce many new constant scalar curvature Kähler manifolds.

In our construction, it is possible to keep track of the geometric meaning of the parameter ε. Indeed, for the blow-up construction $a_j^{m-1}\varepsilon^{2m-2}$ gives the volume of the exceptional divisor E_j (up to a universal constant depending only on the dimension). The role of ε in our results has a direct analogue in Fine–Hong's papers [19], [24], [25], replacing the exceptional divisor with the fiber of the projection onto the Riemann surface or the projectivized fiber of the vector bundle.

The last sources of interest in our results we would like to mention is that they give a reverse picture to Tian–Viaclovsky's [56] and Anderson's [1] study of degenerations of critical metrics, in the special case of constant scalar curvature Kähler metrics in (real) dimension 4. If we look at the sequence of Kähler forms ω_ε on $M\sqcup_{p_1,\varepsilon}N_1\sqcup_{p_2,\varepsilon}...\sqcup_{p_n,\varepsilon}N_n$ seen as a fixed smooth manifold, we get from our analysis that they degenerate in the Gromov–Hausdorff sense to the orbifold (M,ω), so they provide many examples of the phenomenon studied in these works.

A natural generalization of these results is to look for gluing theorems for Kähler metrics with nontrivial automorphisms. The technical difficulties of this extension give rise to some new interesting phenomena and will be the subject of a forthcoming paper [3].

After the first version of these results were posted in electronic form, Claude LeBrun has indicated some implications of our main theorem that we first missed (notably Corollary 1.1 and Theorem 1.2). We wish to thank him for his suggestions.

2. Gluing the orbifold and the ALE spaces together

We start by describing the Kähler orbifold near each of its singularities and we proceed with a description of the ALE spaces near infinity.

Let (M,ω) be an m-dimensional Kähler manifold or Kähler orbifold with isolated singularities. We choose points $p_1,...,p_n\in M$. By assumption, near p_j, the orbifold M is biholomorphic to a neighborhood of 0 in $\mathbf{C}^m/\Gamma_j$, where Γ_j is a finite subgroup of $\mathrm{U}(m)$ acting freely on $\mathbf{C}^m\setminus\{0\}$. The group Γ_j depends on the point p_j as the subscript is meant to remind the reader. In the particular case where p_j is a regular point of M, the group Γ_j reduces to the identity.

We can choose complex coordinates $z:=(z^1,...,z^m)$ in a neighborhood of 0 in $\mathbf{C}^m$ to parameterize a neighborhood of p_j in M and, in these coordinates, the Kähler form ω can be expanded as

$$\omega=\frac{i}{2}\sum_a dz^a\wedge d\bar{z}^a+\sum_{a,b}\mathcal{O}_{j,a,b}(|z|^2)\,dz^a\wedge d\bar{z}^b \tag{3}$$

near $0\in\mathbf{C}^m$ [23]. The complex-valued functions $\mathcal{O}_{j,a,b}(|z|^2)$ are smooth functions which depend on j, a and b, vanish at the origin and whose first order partial derivatives also vanish at the origin. Even though the coordinates z do depend on p_j, we shall not make this dependence apparent in the notation and we hope that the meaning will be clear from the context.

It will be convenient to denote by

$$\begin{aligned} B_{j,r} &:= \{z \in \mathbf{C}^m/\Gamma_j : |z| < r\}, \\ B^*_{j,r} &:= \{z \in \mathbf{C}^m/\Gamma_j : 0 < |z| < r\}, \\ \overline{B}^*_{j,r} &:= \{z \in \mathbf{C}^m/\Gamma_j : 0 < |z| \leqslant r\}, \end{aligned} \tag{4}$$

the open ball, the punctured open ball and the punctured closed ball of radius r centered at p_j (in the above defined coordinates which parameterize a neighborhood of p_j in M). We define, for all $r>0$ small enough (say $r\in(0, r_0)$),

$$M_r := M \setminus \bigcup_{j=1}^n B_{j,r}. \tag{5}$$

In other words, M_r is obtained from M by excising small balls centered at the points p_j. The boundaries of M_r will be denoted by $\partial B_{1,r}, ..., \partial B_{n,r}$.

As promised, we now turn to the description of the ALE spaces near infinity. We assume that, for each $j=1, ..., n$, we are given an m-dimensional Kähler manifold or Kähler orbifold with isolated singularities (N_j, η_j), with one end biholomorphic to a neighborhood of infinity in $\mathbf{C}^m/\Gamma_j$. We further assume that the Kähler metric g_j, which is associated with the Kähler form η_j, converges at order $2-2m$ towards the Euclidean metric. These assumptions imply that one can choose complex coordinates $u:=(u^1, ..., u^m)$ defined outside a neighborhood of 0 in $\mathbf{C}^m$ to parameterize a neighborhood of infinity in N_j and, in these coordinates, the Kähler form η_j can be expanded as

$$\eta_j = \frac{i}{2} \sum_a du^a \wedge d\bar{u}^a + \sum_{a,b} \mathcal{O}_{j,a,b}(|u|^{2-2m})\, du^a \wedge d\bar{u}^b, \tag{6}$$

outside a fixed neighborhood of the origin in $\mathbf{C}^m$. Here, the complex-valued function $\mathcal{O}_{j,a,b}(|u|^{2-2m})$ is a smooth function which depends on j, a and b, is bounded by a constant times $|u|^{2-2m}$ and whose kth partial derivatives are bounded by a constant (depending on k) times $|u|^{2-2m-k}$. As will be explained in §7, this decay assumption is a natural one and, under some mild assumption, one can prove that this rate of decay is indeed achieved.

It will be convenient to denote by

$$\begin{aligned} C_{j,R} &:= \{u \in \mathbf{C}^m/\Gamma_j : |u| > R\}, \\ \overline{C}_{j,R} &:= \{u \in \mathbf{C}^m/\Gamma_j : |u| \geqslant R\}, \end{aligned} \tag{7}$$

the complement of a closed large ball and the complement of an open large ball in N_j (in the coordinates which parameterize a neighborhood of infinity in N_j). We define, for all $R>0$ large enough (say $R>R_0$),

$$N_{j,R} := N_j \setminus C_{j,R}, \tag{8}$$

which corresponds to the manifold N_j whose end has been truncated. The boundary of $N_{j,R}$ is denoted by $\partial C_{j,R}$.

We are now in a position to describe the generalized connected sum construction. For all $\varepsilon>0$ small enough, we define a complex manifold by removing from M small balls centered at the points p_j, for $j=1,...,n$, and by replacing them by properly rescaled versions of the ALE spaces N_j. More precisely, for all $\varepsilon\in(0,r_0/R_0)$, we choose $r_\varepsilon\in(\varepsilon R_0,r_0)$ and define

$$R_\varepsilon := \frac{r_\varepsilon}{\varepsilon}. \tag{9}$$

By construction

$$M_\varepsilon := M\sqcup_{p_1,\varepsilon}N_1\sqcup_{p_2,\varepsilon}...\sqcup_{p_n,\varepsilon}N_n$$

is obtained by connecting M_{r_ε} with the truncated ALE spaces $N_{1,R_\varepsilon},...,N_{n,R_\varepsilon}$. The identification of the boundary $\partial B_{j,r_\varepsilon}$ in M_{r_ε} with the boundary $\partial C_{j,R_\varepsilon}$ of N_{j,R_ε} is performed using the change of variables

$$(z^1,...,z^m)=\varepsilon(u^1,...,u^m),$$

where $(z^1,...,z^m)$ are the coordinates in B_{j,r_0} and $(u^1,...,u^m)$ are the coordinates in C_{j,R_0}. Observe that, when all singularities of M are in the set $\{p_1,...,p_n\}$ and the N_j are all smooth manifolds, then M_ε is a manifold, otherwise M_ε is still an orbifold.

3. Weighted spaces

In this section, we describe weighted spaces on (M^*,ω), where

$$M^* := M\setminus\{p_j : j=1,...,n\}, \tag{10}$$

as well as weighted spaces on each (N_j,η_j).

To begin with, we define the weighted space on (M^*,ω). These weighted spaces are by now well known and have been extensively used in many connected sum constructions. Roughly speaking, we are interested in functions whose rate of decay or blow-up near any of the points p_j is controlled by a power of the distance to p_j. To make this definition precise, we first need the following one.

Definition 3.1. Given $\bar r>0$, $k\in\mathbf{N}$, $\alpha\in(0,1)$ and $\delta\in\mathbf{R}$, the space $\mathcal{C}^{k,\alpha}_\delta(\overline{B}^*_{j,\bar r})$ is defined to be the space of functions $\varphi\in\mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\overline{B}^*_{j,\bar r})$ for which the norm

$$\|\varphi\|_{\mathcal{C}^{k,\alpha}_\delta(\overline{B}^*_{j,\bar r})} := \sup_{0<r\leqslant\bar r} r^{-\delta}\|\varphi(r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_{j,1}\setminus B_{j,1/2})}$$

is finite.

Observe that the function

$$z \longmapsto |z|^{\delta'},$$

belongs to $\mathcal{C}^{k,\alpha}_{\delta}(\overline{B}^*_{j,\bar{r}})$ if and only if $\delta \leqslant \delta'$. This being understood, we have the following definition.

Definition 3.2. Given $k \in \mathbf{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbf{R}$, the weighted space $\mathcal{C}^{k,\alpha}_{\delta}(M^*)$ is defined to be the space of functions $\varphi \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(M^*)$ for which the norm

$$\|\varphi\|_{\mathcal{C}^{k,\alpha}_{\delta}(M^*)} := \|w\|_{\mathcal{C}^{k,\alpha}(M_{r_0/2})} + \sum_{j=1}^{n} \|\varphi|_{\overline{B}^*_{j,r_0}}\|_{\mathcal{C}^{k,\alpha}_{\delta}(\overline{B}^*_{j,r_0})}$$

is finite.

With this definition in mind, we can now give a quantitative statement about the rate of convergence of a potential associated with ω toward the potential associated with the standard Kähler form on $\mathbf{C}^m$, at any of the points p_j. More precisely, near p_j, we can write

$$\omega = i\partial\bar{\partial}\big(\tfrac{1}{2}|z|^2 + \varphi_j\big), \tag{11}$$

where φ_j is a function which lifts smoothly to a neighborhood of 0 in $\mathbf{C}^m$.

We claim that, without loss of generality, it is possible to choose the potential φ_j in such a way that $\varphi_j \in \mathcal{C}^{4,\alpha}_{4}(\overline{B}^*_{j,r_0})$ (more precisely, $\varphi_j \in \mathcal{C}^{4,\alpha}(\overline{B}_{j,r_0})$ and has all its partial derivatives up to order 3 vanishing at 0). Indeed, the potential φ_j lifts to a smooth potential defined on a neighborhood of 0 in $\mathbf{C}^m$. We can then perform the Taylor expansion of this potential at 0, namely

$$\varphi_j = \sum_{k=0}^{3} \varphi_j^{(k)} + \varphi_j',$$

where the polynomial $\varphi_j^{(k)}$ is homogeneous of degree k and φ_j', together with its partial derivatives up to order 3, vanish at 0. Obviously $\varphi_j^{(0)}$ and $\varphi_j^{(1)}$ are not relevant for the computation of the Kähler form ω, since $\partial\bar{\partial}(\varphi_j^{(0)} + \varphi_j^{(1)}) = 0$, hence we might as well assume that $\varphi_j^{(0)} \equiv 0$ and $\varphi_j^{(1)} \equiv 0$. Next, making use of the fact that the coordinates $(z^1, ..., z^m)$ are chosen so that (3) holds, we see that

$$\partial\bar{\partial}(\varphi_j^{(2)} + \varphi_j^{(3)}) = \mathcal{O}(|z|^2)$$

but, as $\partial\bar{\partial}\varphi_j^{(2)}$ and $\partial\bar{\partial}\varphi_j^{(3)}$ are homogeneous polynomials of degree 0 and 1 respectively, we conclude that $\partial\bar{\partial}(\varphi_j^{(2)} + \varphi_j^{(3)}) \equiv 0$. Considering φ_j' instead of φ_j, we have found a potential which satisfies the desired property.

Similarly, we define weighted spaces on the ALE spaces (N_j, η_j). This time we are interested in functions which decay or blow up near the end of N_j at a rate which is controlled by a power of the distance from a fixed point in N_j. To be more specific, we first make the following definition.

Definition 3.3. Given $\bar{R}>0$, $k\in\mathbf{N}$, $\alpha\in(0,1)$ and $\delta\in\mathbf{R}$, the space $\mathcal{C}^{k,\alpha}_{\delta}(\overline{C}_{j,\bar{R}})$ is defined to be the space of functions $\varphi\in\mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\overline{C}_{j,\bar{R}})$ such that the norm

$$\|\varphi\|_{\mathcal{C}^{k,\alpha}_{\delta}(\overline{C}_{j,\bar{R}})} := \sup_{R\geqslant\bar{R}} R^{-\delta}\|\varphi(R\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_{j,2}\setminus B_{j,1})}$$

is finite.

Again, observe that the function

$$u\longmapsto |u|^{\delta'}$$

belongs to $\mathcal{C}^{k,\alpha}_{\delta}(\overline{C}_{j,\bar{R}})$ if and only if $\delta'\leqslant\delta$. We can now make the following definition.

Definition 3.4. Given $k\in\mathbf{N}$, $\alpha\in(0,1)$ and $\delta\in\mathbf{R}$, the weighted space $\mathcal{C}^{k,\alpha}_{\delta}(N_j)$ is defined to be the space of functions $\varphi\in\mathcal{C}^{k,\alpha}_{\mathrm{loc}}(N_j)$ for which the norm

$$\|\varphi\|_{\mathcal{C}^{k,\alpha}_{\delta}(N_j)} := \|\varphi\|_{\mathcal{C}^{k,\alpha}(N_{j,2R_0})} + \|\varphi|_{\overline{C}_{j,R_0}}\|_{\mathcal{C}^{k,\alpha}_{\delta}(\overline{C}_{j,R_0})}$$

is finite.

We can now explain the assumption on the rate of convergence at infinity of the Kähler form η_j toward the standard Kähler form on $\mathbf{C}^m$. We will assume that, away from a compact set in N_j,

$$\eta_j = i\partial\bar{\partial}\big(\tfrac{1}{2}|u|^2+\widetilde{\varphi}_j\big), \tag{12}$$

for some potential $\widetilde{\varphi}_j$ which satisfies

$$\widetilde{\varphi}_j + a_j|\cdot|^{4-2m} \in \mathcal{C}^{4,\alpha}_{3-2m}(\overline{C}_{j,R_0}), \tag{13}$$

when $m\geqslant 3$, and

$$\widetilde{\varphi}_j - a_j\log|\cdot| \in \mathcal{C}^{4,\alpha}_{-1}(\overline{C}_{j,R_0}), \tag{14}$$

when $m=2$, for some $a_j\in\mathbf{R}$. As already mentioned in the introduction, this is a rather natural assumption which is fulfilled in many important examples.

Remark 3.1. We will show in Section 7 that, if one simply assumes that the potential $\widetilde{\varphi}_j$ associated with η_j satisfies

$$\widetilde{\varphi}_j \in \mathcal{C}^{4,\alpha}_{2-\gamma}(\overline{C}_{j,R_0})$$

for some $\gamma>0$, then one can always replace $\widetilde{\varphi}_j$ by some potential $\widetilde{\varphi}'_j$ satisfying (13)–(14).

4. The geometry of the equation

The material contained in this section is well known (see for example [22]); we include it for completeness and to introduce the reader to the objects entering into the proofs of our results. Recall that (M,ω) is an m-dimensional compact Kähler manifold or a Kähler orbifold with isolated singularities. We will indicate by g the Riemannian metric associated with ω, Ric_g its Ricci tensor, ϱ_g the Ricci form, and $\mathbf{s}(\omega)$ its scalar curvature.

Following [36] and [8], we want to understand the behavior of the scalar curvature under deformations of the Kähler form ω, of the form

$$\widetilde{\omega} := \omega + i\partial\bar{\partial}\varphi + \beta,$$

where β is a closed $(1,1)$-form and φ a function defined on M. In local coordinates $(v^1, \dots, v^m)$, if we write

$$\widetilde{\omega} = \frac{i}{2}\sum_{a,b} \tilde{g}_{a\bar{b}}\, dv^a \wedge d\bar{v}^b,$$

then the scalar curvature of $\widetilde{\omega}$ is given by

$$\mathbf{s}(\widetilde{\omega}) = -\sum_{a,b} \tilde{g}^{a\bar{b}}\, \partial_{v^a}\partial_{\bar{v}^b} \log(\det(\tilde{g})), \tag{15}$$

where $\tilde{g}^{a\bar{b}}$ are the coefficients of the inverse of the matrix $(\tilde{g}_{a\bar{b}})$. The following result is proven in [36] and [8, Lemma 2.158].

Proposition 4.1. *The scalar curvature of $\widetilde{\omega}$ can be expanded in terms of β and φ as*

$$\mathbf{s}(\widetilde{\omega}) = \mathbf{s}(\omega) - \left(\tfrac{1}{2}\Delta_g^2\varphi + \mathrm{Ric}_g \cdot \nabla_g^2\varphi + \Delta_g(\omega,\beta) + 2(\varrho_g,\beta)\right) + Q_g(\nabla^2\varphi,\beta),$$

where Q_g collects all the nonlinear terms and where all the operators on the right-hand side of this identity are computed with respect to the Kähler metric g.

Being a local calculation, this formula holds for orbifolds with isolated singularities too. Of crucial importance will be the two linear operators which appear in this formula. First, we set

$$\mathcal{L}_g := \Delta_g(\omega,\cdot\,) + 2(\varrho_g,\cdot\,), \tag{16}$$

which is a linear operator acting on closed $(1,1)$-forms, and we also define the operator

$$\mathbf{L}_g := \tfrac{1}{2}\Delta_g^2 + \mathrm{Ric}_g \cdot \nabla_g^2, \tag{17}$$

which acts on functions.

For a general Kähler metric it can be very difficult to analyze these operators. Nevertheless geometry comes to the rescue for a constant scalar curvature metric. Indeed, in this case we have

$$\mathbf{L}_g = 2(\bar{\partial}\partial_g^{\#})^*(\bar{\partial}\partial_g^{\#}), \tag{18}$$

where $\partial_g^{\#}\varphi$ denotes the $(1,0)$-part of the g-gradient of φ. In other words,

$$\partial_g^{\#} := (\bar{\partial}\,\cdot\,)_g^{\#},$$

where $\#$ is the inverse of

$$\begin{aligned} \flat: TM\otimes\mathbf{C} &\longrightarrow T^*M\otimes\mathbf{C}, \\ \Xi &\longmapsto g(\Xi,\,\cdot\,). \end{aligned}$$

Using (18) one observes that with any element $\varphi\in\operatorname{Ker}\mathbf{L}_g$ one can associate a holomorphic vector field, namely $\partial_g^{\#}\varphi$, which vanishes somewhere on M. Indeed, just multiply $\mathbf{L}_g\varphi=0$ by φ and integrate the result over M using (18) to conclude that

$$\int_M |\bar{\partial}\partial_g^{\#}\varphi|^2\,dv_g = 0,$$

where dv_g denotes the volume form associated with g. Therefore, $\bar{\partial}(\partial_g^{\#}\varphi)=0$. Moreover, if g is of constant scalar curvature and $\varphi\in\operatorname{Ker}\mathbf{L}_g$, [36, Proposition 1] proves that $\operatorname{Im}(\partial_g^{\#}\varphi)$ is a Killing vector field, and that any Killing vector field vanishing somewhere on M arises in this way. This means that the image of $\operatorname{Ker}\mathbf{L}_g$ by $\partial_g^{\#}$ is equal to the real span of the holomorphic vector fields vanishing somewhere whose imaginary part are Killing. By the Matsushima–Lichnerowicz theorem [39], the complexification of this space gives the space $\mathfrak{h}_0(M)$ of holomorphic vector fields vanishing somewhere on M, so that $\dim_{\mathbf{R}}(\operatorname{Ker}\mathbf{L}_g)-1=\dim_{\mathbf{C}}(\mathfrak{h}_0(M))$.

We define the nonlinear mapping

$$\begin{aligned} S_\omega: \mathcal{C}^{4,\alpha}(M) &\longrightarrow \mathcal{C}^{0,\alpha}(M)/\,\mathbf{R}, \\ \varphi &\longmapsto \mathbf{s}(\omega+i\partial\bar{\partial}\varphi) \text{ modulo constant.} \end{aligned}$$

LeBrun–Simanca applied the implicit function theorem to the map S_ω and proved, in the case of manifolds, the following result which extends immediately to orbifolds with isolated singularities.

PROPOSITION 4.2. ([36]) *Assume that (M,ω) is nondegenerate and further assume that its scalar curvature $\mathbf{s}(\omega)$ is constant. Then, the operator*

$$\varphi \longmapsto DS_\omega|_0\,\varphi = -\mathbf{L}_g\varphi$$

is surjective and has a kernel which is spanned by a constant function.

Let ψ_g be the function (up to constants) which gives

$$\varrho_g = \varrho_g^h + i\partial\bar{\partial}\psi_g$$

(where ϱ_g^h is the harmonic representative for $[\varrho_g]$) and let $\Xi \in \mathfrak{h}(M)$, where $\mathfrak{h}(M)$ is the space of holomorphic vector fields on M. Then, we can define the Futaki invariant

$$\mathcal{F}(\Xi, [\omega]) := \int_M \Xi\psi_g \, dv_g,$$

where dv_g denotes the volume form associated with g. Recall that $\mathfrak{h}_0(M)$ denotes the space of holomorphic vector fields which vanish somewhere on M. By definition, (M, ω) is Futaki-nondegenerate if the "linearization" of the Futaki invariant

$$D\mathcal{F}_{[\omega]} : \mathfrak{h}_0(M) \longrightarrow (H^{(1,1)}(M, \mathbf{C}))^*$$

is injective. It is a standard fact, though not obvious, that $\mathcal{F}(\Xi, [\omega])$ only depends on the Kähler class and does not depend on its representative. On the other hand, if $[\omega]$ has a representative with constant scalar curvature, then $\mathcal{F}(\Xi, [\omega])$ vanishes for any $\Xi \in \mathfrak{h}(M)$.

Now define the nonlinear mapping

$$\begin{aligned} \widehat{S}_\omega : \mathcal{C}^{4,\alpha}(M) \times \mathcal{H}^{1,1}(M, \mathbf{C}) &\longrightarrow \mathcal{C}^{0,\alpha}(M)/\mathbf{R}, \\ (\varphi, \beta) &\longmapsto \mathbf{s}(\omega + i\partial\bar{\partial}\varphi + \beta) \text{ modulo constant,} \end{aligned}$$

where $\mathcal{H}^{1,1}(M, \mathbf{C})$ is the space of ω-harmonic $(1,1)$-forms. The result of [36] again extends to orbifolds with isolated singularities, and we have the following result.

PROPOSITION 4.3. ([36]) *Assume that (M, ω) is Futaki-nondegenerate and further assume that its scalar curvature $\mathbf{s}(\omega)$ is constant. Then, the operator*

$$(\varphi, \beta) \longmapsto D\widehat{S}_\omega|_{(0,[0])}(\varphi, \beta) = -(\mathbf{L}_g\varphi + \mathcal{L}_g\beta)$$

is surjective.

5. Mapping properties

We construct right inverses for the operator $\mathbf{L}_g$ defined in the previous section.

5.1. Analysis of the operators defined on (M^*, ω)

Assume that (M, ω) is a compact Kähler manifold or Kähler orbifold with isolated singularities and further assume that ω has constant scalar curvature. We first construct a right inverse for the operator $\mathbf{L}_g$ when $m\geqslant 3$ and when (M,ω) is *nondegenerate*, i.e. when there are no nontrivial holomorphic vector field vanishing somewhere on M. Next, we proceed with the proof of the corresponding result when $m=2$ and with the modifications which are needed to handle the case where the kernel of $\mathbf{L}_g$ is nontrivial but (M,ω) is *Futaki-nondegenerate*, i.e. when the linearized Futaki invariant is nondegenerate.

The mapping properties of $\mathbf{L}_g$, when defined between weighted spaces, heavily depend on the choice of the weight parameter. Recall that, by definition, $\zeta\in\mathbf{R}$ is an indicial root of $\mathbf{L}_g$ at p_j if there exists some nontrivial function $v\in\mathcal{C}^\infty(\partial B_{j,1})$ such that

$$\mathbf{L}_g(|z|^\zeta v)=\mathcal{O}(|z|^{\zeta-3}) \tag{19}$$

near 0 (here we have implicitly used the coordinates defined in §2 to parameterize M close to the point p_j).

Let Δ_0 denote the Laplacian in $\mathbf{C}^m$ with its standard Kähler form. Using (3), it is easy to check that, near each p_j, (19) holds for some function v if and only if

$$\Delta_0^2(|z|^\zeta v)=\mathcal{O}(|z|^{\zeta-3}).$$

Therefore, the set of indicial roots of $\mathbf{L}_g$ at p_j is equal to the set of indicial roots at the origin for the operator Δ_0^2 defined on $\mathbf{C}^m/\Gamma_j$. This later turns out to be included in $\mathbf{Z}\setminus\{5-2m,...,-1\}$ when $m\geqslant 3$ and included in $\mathbf{Z}$ when $m=2$ (observe that the set of indicial roots depends on the group Γ_j). Indeed, let e be an eigenfunction of $\Delta_{S^{2m-1}}$ which is invariant under the action of Γ_j and is associated with the eigenvalue $\gamma(2m-2+\gamma)$, where $\gamma\in\mathbf{N}$, hence

$$\Delta_{S^{2m-1}}e=-\gamma(2m-2+\gamma)e.$$

If we identify S^{2m-1} with the unit sphere in $\mathbf{C}^m$, then

$$\Delta_0^2(|z|^\zeta e)=(\zeta-\gamma)(\zeta-\gamma-2)(\zeta-2+2m+\gamma)(\zeta-4+2m+\gamma)|z|^{\zeta-4}e.$$

Therefore, we find that γ, $\gamma+2$, $2-2m-\gamma$ and $4-2m-\gamma$ are indicial roots of Δ_0^2 at 0. Since the eigenfunctions of the Laplacian on the sphere constitute a Hilbert basis of $L^2(S^{2m-1})$, we have obtained all the indicial roots of Δ_0^2 at the origin.

It is clear that the operator

$$\begin{aligned} L'_\delta\colon \mathcal{C}^{4,\alpha}_\delta(M^*) &\longrightarrow \mathcal{C}^{0,\alpha}_{\delta-4}(M^*),\\ \varphi &\longmapsto \mathbf{L}_g\varphi, \end{aligned}$$

is well defined. It follows from the general theory in [44], where weighted Sobolev spaces are considered instead of weighted Hölder spaces, and in [43], where the corresponding analysis in weighted Hölder spaces is performed (see also [46]) that the operator L'_δ has closed range and is Fredholm, provided δ is not an indicial root of $\mathbf{L}_g$ at the points $p_1, ..., p_n$. Under this condition, a duality argument (in weighted Sobolev spaces) shows that the operator L'_δ is surjective if and only if the operator $L'_{4-2m-\delta}$ is injective. And, still under this assumption,

$$\dim \operatorname{Ker} L'_\delta = \dim \operatorname{Coker} L'_{4-2m-\delta}. \tag{20}$$

Using these, we obtain the following result.

PROPOSITION 5.1. *Assume that* $m\geqslant 3$, $\delta\in(4-2m, 0)$ *and assume that* (M,ω) *is nondegenerate so that the kernel of* $\mathbf{L}_g$ *is spanned by a constant function. Then, the operator*

$$\begin{aligned} L_\delta\colon \mathcal{C}^{4,\alpha}_\delta(M^*)\times\mathbf{R} &\longrightarrow \mathcal{C}^{0,\alpha}_{\delta-4}(M^*), \\ (\varphi,\nu) &\longmapsto \mathbf{L}_g\varphi+\nu, \end{aligned}$$

is surjective and has a 1*-dimensional kernel spanned by a constant function.*

Proof. We claim that, when $\delta\in(4-2m, 0)$, the operator L'_δ has a 1-dimensional kernel spanned by a constant function. Indeed, when $\delta\in(4-2m, 0)$, standard regularity theory implies that the isolated singularities of any element of the kernel of L'_δ are removable, and hence the elements of the kernel of L'_δ are in fact smooth functions in M. Therefore, it follows from our assumption that the kernel of L'_δ reduces to the constant functions. It follows from (20) that the operator L'_δ also has a 1-dimensional cokernel, which is easily seen to be spanned by a constant function since (by (18))

$$\int_M \mathbf{L}_g\varphi\, dv_g = 0,$$

for any $\varphi\in\mathcal{C}^{4,\alpha}_\delta(M^*)$. This completes the proof of the result. □

When $m=2$, the above result has to be modified (since $4-2m=0$ in this case). We set

$$\mathcal{D} := \operatorname{Span}\{\chi_1, ..., \chi_n\},$$

where χ_j is a cutoff function which is identically equal to 1 in $B_{j,r_0/2}$ and identically equal to 0 in $M\setminus B_{j,r_0}$. This time, we have the following result.

PROPOSITION 5.2. *Assume that* $m=2$, $\delta\in(0,1)$ *and assume that* (M,ω) *is nondegenerate so that the kernel of* $\mathbf{L}_g$ *is spanned by a constant function. Then*

$$\begin{aligned} L_\delta\colon (\mathcal{C}^{4,\alpha}_\delta(M^*)\oplus\mathcal{D})\times\mathbf{R} &\longrightarrow \mathcal{C}^{0,\alpha}_{\delta-4}(M^*), \\ (\varphi,\nu) &\longmapsto \mathbf{L}_g\varphi+\nu, \end{aligned}$$

is surjective and has a 1*-dimensional kernel spanned by a constant function.*

Proof. We keep the notation of the previous proof. Assume that $\delta>0$. Then the operator L'_δ is injective (since we have assumed that the kernel of $\mathbf{L}_g$ is spanned by a constant function and a nonzero function does not belong to $\mathcal{C}^{4,\alpha}_\delta(M^*)$ when $\delta>0$). Therefore, when $\delta>0$, $\delta\notin\mathbf{N}$, the operator $L'_{-\delta}$ is surjective and admits a right inverse, which, unfortunately, is not unique.

Moreover, when $\delta\in(0,1)$, a relative index argument [44] shows that the dimension of the kernel of $L'_{-\delta}$ and the dimension of the cokernel of L'_δ are both equal to n. The kernel of $L'_{-\delta}$ is rather explicit since it is spanned by a constant function and, for $j=1,...,n-1$, the unique function γ_j which is a solution (in the sense of distributions) of

$$\mathbf{L}_g\gamma_j=\delta_{p_{j+1}}-\delta_{p_j}$$

and whose mean value over M is 0.

Let us now assume that $\delta\in(0,1)$. Given $\psi\in\mathcal{C}^{0,\alpha}_\delta(M^*)$, we choose $\nu\in\mathbf{R}$ to be equal to the mean value of the function ψ. Since $L'_{-\delta}$ is surjective, we have the existence of a solution of

$$\mathbf{L}_g\varphi=\psi-\nu,$$

which belongs to $\mathcal{C}^{4,\alpha}_{-\delta}(M^*)$ (this solution is for example obtained by applying to $\psi-\nu$ a given right inverse for $L'_{-\delta}$). It follows from elliptic regularity theory that, near any p_j, the function φ can be expanded as

$$\varphi(z)=d_j+b_j\log|z|+\widetilde{\varphi}_j(z),$$

where $d_j,b_j\in\mathbf{R}$ and $\widetilde{\varphi}_j\in\mathcal{C}^{4,\alpha}_\delta(B^*_{j,r_0})$. This implies that the function φ is a solution (in the sense of distributions) of

$$\mathbf{L}_g\varphi+\nu=\psi-c_2\sum_{j=1}^{n}b_j\,\delta_{p_j},\tag{21}$$

where $c_2=2|S^3|\neq0$. Using the fact that the functions γ_j are in the kernel of $L'_{-\delta}$, we may assume without loss of generality that the b_j at the different points p_j are all equal, by adding to φ a suitable linear combination of the functions γ_j (this amounts to choosing a particular right inverse of $L'_{-\delta}$). Integration of (21) over M implies that $0=-c_2\sum_{j=1}^n b_j$. Hence, all b_j are equal to 0 and, near p_j, the function φ can be expanded as

$$\varphi(z)=d_j+\widetilde{\varphi}_j(z).$$

This shows that there exists a choice of the right inverse $G'_{-\delta}$ of $L'_{-\delta}$ such that, if $\psi\in\mathcal{C}^{0,\alpha}_{\delta-4}(M^*)$ and if ν is the mean value of ψ, then

$$G'_{-\delta}(\psi-\nu)\in\mathcal{C}^{4,\alpha}_\delta(M^*)\oplus\mathcal{D}.$$

This completes the proof of the result. □

Remark 5.1. Observe that, given $\psi\in\mathcal{C}^{0,\alpha}_{\delta-4}(M^*)$, the constant $\nu\in\mathbf{R}$ in the equation $\mathbf{L}_g\,\varphi+\nu=\psi$ is equal to the mean value of ψ so that $\psi-\nu$ is L^2-orthogonal to the kernel of $\mathbf{L}_g$ which is spanned by a constant function.

We turn to the case where the kernel of $\mathbf{L}_g$ is not only spanned by a constant function and we now assume that (M,ω) is Futaki-nondegenerate. The proof relies on the following result which replaces Proposition 5.1 and whose proof is identical.

PROPOSITION 5.3. *Assume that $m\geqslant 3$ and that (M,ω) is Futaki-nondegenerate. Then, for all $\delta\in(4-2m,0)$ the operator*

$$\begin{aligned} L_\delta\colon \mathcal{C}^{4,\alpha}_{\delta}(M^*)\times\mathcal{H}^{1,1}(M,\mathbf{C})\times\mathbf{R} &\longrightarrow \mathcal{C}^{0,\alpha}_{\delta-4}(M^*),\\ (\varphi,\beta,\nu) &\longmapsto \mathbf{L}_g\varphi+\mathcal{L}_g\beta+\nu, \end{aligned}$$

is surjective and has a kernel which is equal to the kernel of $\mathbf{L}_g$.

Given $\psi\in\mathcal{C}^{0,\alpha}_{\delta-4}(M^*)$, the $(1,1)$-form $\beta\in\mathcal{H}^{1,1}(M,\mathbf{C})$ and the constant $\nu\in\mathbf{R}$ in the equation $\mathbf{L}_g\varphi+\mathcal{L}_g\beta+\nu=\psi$ are chosen in such a way that $\psi-\mathcal{L}_M\beta-\nu$ is L^2-orthogonal to the elements of the kernel of $\mathbf{L}_g$.

Clearly, in the above statement, one can replace $\mathcal{H}^{1,1}(M,\mathbf{C})$ by a finite-dimensional subspace $D\subset\mathcal{H}^{1,1}(M,\mathbf{C})$ whose dimension is equal to $\dim_{\mathbf{R}}(\operatorname{Ker}\mathbf{L}_g)-1=\dim_{\mathbf{C}}(\mathfrak{h}_0)$. We claim that one can further replace the space D by a subspace $D_{\bar r_0}$ of the space of closed $(1,1)$-forms which are supported in $M_{\bar r_0}$, provided $\bar r_0$ is fixed small enough. Indeed, near each p_j, any element $\beta\in\mathcal{H}^{1,1}(M,\mathbf{C})$ can be decomposed as

$$\beta=d\pi_j.$$

We truncate the potential π_j between $2\bar r_0$ and $\bar r_0$ and define

$$\beta_{\bar r_0}:=d((1-\chi_{\bar r_0})\pi_j),$$

where $\chi_{\bar r_0}$ is a cutoff function identically equal to 0 in $M_{2\bar r_0}$ and identically equal to 1 in each $B_{j,\bar r_0}$. If $\beta^{(1)},...,\beta^{(b)}$ is a basis of D, we set

$$D_{\bar r_0}:=\operatorname{Span}\{\beta^{(1)}_{\bar r_0},...,\beta^{(b)}_{\bar r_0}\}.$$

When $m\geqslant 3$, it is easy to check that, given $\delta\in(4-2m,0)$, the operator

$$\begin{aligned} L'_\delta\colon \mathcal{C}^{4,\alpha}_{\delta}(M^*)\times D_{\bar r_0}\times\mathbf{R} &\longrightarrow \mathcal{C}^{0,\alpha}_{\delta-4}(M^*),\\ (\varphi,\beta,\nu) &\longmapsto \mathbf{L}_M\varphi+\mathcal{L}_M\beta+\nu, \end{aligned}$$

is surjective provided $\bar r_0$ is chosen small enough.

In dimension $m=2$, this result has to be modified. As above we find that, given $\delta\in(0,1)$, the operator

$$\begin{aligned} L'_\delta\colon (\mathcal{C}^{4,\alpha}_{\delta}(M^*)\oplus\mathcal{D})\times D_{\bar r_0}\times\mathbf{R} &\longrightarrow \mathcal{C}^{0,\alpha}_{\delta-4}(M^*),\\ (\varphi,\beta,\nu) &\longmapsto \mathbf{L}_g\varphi+\mathcal{L}_g\beta+\nu, \end{aligned}$$

is surjective and has a kernel which is equal to the kernel of $\mathbf{L}_g$.

5.2. Operators defined on (N_j,η_j)

Assume that (N_j,η_j) is an ALE space with zero scalar curvature Kähler metric η_j. Further assume that, at infinity, the Kähler form η_j can be expanded as

$$\eta_j = i\partial\bar\partial\big(\tfrac12|u|^2+\widetilde\varphi_j(u)\big),$$

where $\widetilde\varphi_j$ satisfies

$$\nabla^2\widetilde\varphi_j\in\mathcal{C}^{2,\alpha}_{2-2m}(C_{j,R_0}).$$

We denote by g_j the metric associated with the Kähler form η_j. Again the analysis of $\mathbf{L}_{g_j}$, when defined between weighted spaces, follows from the general theory developed in [44] and [43] (see also [46]) and the mapping properties of $\mathbf{L}_{g_j}$, when defined between weighted spaces, heavily depend on the choice of the weight parameter.

Recall that $\zeta\in\mathbf{R}$ is an indicial root of $\mathbf{L}_{g_j}$ at infinity if there exists some nontrivial function $v\in\mathcal{C}^\infty(\partial B_{j,1})$ such that

$$\mathbf{L}_{g_j}(|u|^\zeta v)=\mathcal{O}(|u|^{\zeta-5}) \tag{22}$$

near ∞ (we have implicitly used the coordinates defined in §2 to parameterize N_j near its end).

Again, it is easy to check that (22) holds for some function v if and only if

$$\Delta_0^2(|u|^\zeta v)=\mathcal{O}(|u|^{\zeta-5})$$

(here one uses the fact that $g_j=g_{\mathrm{Eucl}}+\mathcal{O}(|z|^{2-2m})$ at infinity and hence the coefficients of the Ricci tensor at infinity are bounded by a constant times $|u|^{-2m}$). Therefore, the set of indicial roots of $\mathbf{L}_{g_j}$ at infinity is equal to the set of indicial roots at infinity for the operator Δ_0^2 defined on $\mathbf{C}^m/\Gamma_j$. Again, this set is included in $\mathbf{Z}\setminus\{5-2m,...,-1\}$ when $m\geqslant3$ and is included in $\mathbf{Z}$ when $m=2$ (the set of indicial roots depends on the group Γ_j). The proof of this fact follows the analysis done in §5.1.

The operator

$$\begin{aligned}\tilde L_\delta\colon\mathcal{C}^{4,\alpha}_\delta(N_j)&\longrightarrow\mathcal{C}^{0,\alpha}_{\delta-4}(N_j),\\ \varphi&\longmapsto\mathbf{L}_{g_j}\varphi,\end{aligned}$$

is well defined (again one uses the fact that $g_j=g_{\mathrm{Eucl}}+\mathcal{O}(|z|^{2-2m})$ at infinity). Moreover, according to [44] and [43] (see also [46]), this operator has closed range and is Fredholm, provided δ is not an indicial root of $\mathbf{L}_{g_j}$ at infinity. Under this condition, a duality argument (in weighted Sobolev spaces) shows that the operator $\tilde L_\delta$ is surjective if and only if the operator $\tilde L_{4-2m-\delta}$ is injective. And, still under this assumption,

$$\dim\operatorname{Ker}\tilde L_\delta=\dim\operatorname{Coker}\tilde L_{4-2m-\delta}. \tag{23}$$

The construction of a right inverse for the operator $\mathbf{L}_{g_j}$ relies on the following result whose proof is essentially borrowed from [32].

Proposition 5.4. *Assume that (N_j,η_j) is a constant scalar curvature ALE Kähler manifold or Kähler orbifold with isolated singularities. Then, there is no nontrivial solution of $\mathbf{L}_{g_j}\varphi=0$, which belongs to $\mathcal{C}^{4,\alpha}_\delta(N_j)$ for some $\delta<0$.*

Proof. Assume that $\mathbf{L}_{g_j}\varphi=0$ and that $\varphi\in\mathcal{C}^{4,\alpha}_\delta(N_j)$ for some $\delta<0$. Then, as explained in §4, the vector field $\partial^\#_{g_j}\varphi$ is a holomorphic vector field which tends to 0 at infinity. Indeed, we have

$$\mathbf{L}_{g_j}=2(\bar\partial\partial^\#_{g_j})^*(\bar\partial\partial^\#_{g_j})$$

and, multiplying $\mathbf{L}_{g_j}\varphi=0$ by φ and integrating by parts, we get

$$\int_{N_j}|\bar\partial\partial^\#_{g_j}\varphi|^2\,dv_{g_j}=0$$

All integrations are justified because of the decaying behavior of φ at infinity which implies that $\varphi\in\mathcal{C}^{4,\alpha}_{4-2m}(N_j)$ when $m\geqslant 3$. Therefore $\partial^\#_{g_j}\varphi=0$. Using Hartogs' theorem, the restriction of $\partial^\#_{g_j}\varphi$ to C_{j,R_0} can be extended to a holomorphic vector field on $\mathbf{C}^m$. Since this vector field decays at infinity, it has to be identically equal to 0. This implies that $\partial^\#_{g_j}\varphi$ is identically equal to 0 on C_{j,R_0}. However, φ being a real-valued function, this implies that $\partial\varphi=\bar\partial\varphi=0$ in C_{j,R_0}. Hence the function φ is constant in C_{j,R_0} and decays at infinity. This implies that φ is identically equal to 0 in C_{j,R_0} and satisfies $\mathbf{L}_{g_j}\varphi=0$ in N_j. Now, we use the unique continuation theorem for solutions of linear elliptic equations to conclude that φ is identically equal to 0 in N_j. $\square$

This being understood, we have the following result.

Proposition 5.5. *Assume that $\delta\in(0,1)$. Then*

$$\begin{aligned}\tilde L_\delta\colon \mathcal{C}^{4,\alpha}_\delta(N_j) &\longrightarrow \mathcal{C}^{0,\alpha}_{\delta-4}(N_j),\\ \varphi &\longmapsto \mathbf{L}_{g_j}\varphi,\end{aligned}$$

is surjective and has a 1*-dimensional kernel spanned by the constant function.*

Proof. It follows from Proposition 5.4 that, when $\delta'<0$, the operator $\tilde L_{\delta'}$ is injective, and this implies that $\tilde L_\delta$ is surjective whenever $\delta>4-2m$ is not an indicial root of $\mathbf{L}_{g_j}$ at infinity. $\square$

5.3. Biharmonic extensions

We end this section with the following simple result whose proof follows at once from the application of the maximum principle. Here, as usual, Γ is a finite subgroup of $\mathrm{U}(m)$

acting freely on $\mathbf{C}^m \setminus \{0\}$. We define

$$\overline{B}_\Gamma := \{z \in \mathbf{C}^m/\Gamma : |z| \leqslant 1\},$$
$$\overline{B}^*_\Gamma := \{z \in \mathbf{C}^m/\Gamma : |z| \leqslant 1\},$$
$$\overline{C}_\Gamma := \{z \in \mathbf{C}^m/\Gamma : |z| \geqslant 1\}.$$

Therefore, when $\Gamma = \Gamma_j$, we have $\overline{B}_{\Gamma_j} = \overline{B}_{j,1}$, $\overline{B}^*_{\Gamma_j} = \overline{B}^*_{j,1}$ and $\overline{C}_{\Gamma_j} = \overline{C}_{j,1}$. Recall that Δ_0 denotes the Laplacian in $\mathbf{C}^m$ with the standard Kähler form. With this notation in mind, we have the following result.

PROPOSITION 5.6. *Assume that $m \geqslant 3$. Given $h \in \mathcal{C}^{4,\alpha}(\partial B_\Gamma)$ and $k \in \mathcal{C}^{2,\alpha}(\partial B_\Gamma)$, there exist biharmonic functions $H^i_{h,k} \in \mathcal{C}^{4,\alpha}(\overline{B}_\Gamma)$ and $H^o_{h,k} \in \mathcal{C}^{4,\alpha}_{4-2m}(\overline{C}_\Gamma)$ such that*

$$\Delta_0^2 H^i_{h,k} = 0 \quad \textit{in } B_\Gamma,$$
$$\Delta_0^2 H^o_{h,k} = 0 \quad \textit{in } C_\Gamma,$$

with

$$H^i_{h,k} = H^o_{h,k} = h \quad \textit{and} \quad \Delta_0 H^i_{h,k} = \Delta_0 H^o_{h,k} = k \quad \textit{on } \partial B_\Gamma.$$

Moreover,

$$\|H^i_{h,k}\|_{\mathcal{C}^{4,\alpha}(\overline{B}_\Gamma)} + \|H^o_{h,k}\|_{\mathcal{C}^{4,\alpha}_{4-2m}(\overline{C}_\Gamma)} \leqslant c(\|h\|_{\mathcal{C}^{4,\alpha}(\partial B_\Gamma)} + \|k\|_{\mathcal{C}^{2,\alpha}(\partial B_\Gamma)}).$$

For later use, it will be convenient to get explicit formulas for $H^i_{h,k}$ and $H^o_{h,k}$. We decompose both functions h and k over eigenfunctions of the Laplacian on the sphere. Namely

$$h = \sum_{\gamma=0}^{\infty} h^{(\gamma)} e_\gamma \quad \text{and} \quad k = \sum_{\gamma=0}^{\infty} k^{(\gamma)} e_\gamma,$$

where the function e_γ satisfies

$$\Delta_{S^{2m-1}} e_\gamma = -\gamma(2m-2+\gamma) e_\gamma$$

and is normalized so that $\|e_\gamma\|_{L^2} = 1$. Observe that we only have to consider the eigenvalues corresponding to eigenfunctions which are invariant under the action of Γ. Then, the functions $H^i_{h,k}$ and $H^o_{h,k}$ are explicitly given by

$$H^i_{h,k}(z) = \sum_{\gamma=0}^{\infty} \left(\left(h^{(\gamma)} - \frac{k^{(\gamma)}}{4(m+\gamma)} \right) |z|^\gamma + \frac{k^{(\gamma)}}{4(m+\gamma)} |z|^{\gamma+2} \right) e_\gamma \tag{24}$$

and

$$H^o_{h,k}(z) = \sum_{\gamma=0}^{\infty} \left(\left(h^{(\gamma)} + \frac{k^{(\gamma)}}{4(\gamma+m-2)} \right) |z|^{2-2m-\gamma} - \frac{k^{(\gamma)}}{4(\gamma+m-2)} |z|^{4-2m-\gamma} \right) e_\gamma. \tag{25}$$

Proof of Proposition 5.6. The existence of $H^i_{h,k}$ is clear and the estimate follows at once. The explicit expression of $H^o_{h,k}$ provides a direct proof of the estimate of this function. First observe that elliptic regularity implies that there exists $c=c(m)>0$ and $N=N(m)\in\mathbf{N}$ such that

$$\|e_\gamma\|_{L^\infty}\leqslant c(1+\gamma)^N\|e_\gamma\|_{L^2}=c(1+\gamma)^N,$$

since we have normalized the functions e_γ to have L^2-norm equal to 1. In addition, Cauchy–Schwartz inequality yields

$$|h^{(\gamma)}|+|k^{(\gamma)}|\leqslant c(\|h\|_{\mathcal{C}^{4,\alpha}}+\|k\|_{\mathcal{C}^{2,\alpha}}),$$

for some constant which does not depend on γ. Using this information together with (25), we conclude that

$$\sup_{|z|\geqslant 2}(|z|^{2m-4}|H^o_{h,k}|+|z|^{2m-2}|\Delta_0 H^o_{h,k}|)\leqslant c(\|h\|_{\mathcal{C}^{4,\alpha}}+\|k\|_{\mathcal{C}^{2,\alpha}}),$$

since the series are absolutely convergent for $|z|$ larger than 1. The maximum principle applied in $\{z\in C_\Gamma:|z|\in[1,2]\}$ then allows us to fill in the gap in the estimate, and we conclude that

$$\sup_{|z|\geqslant 1}(|z|^{2m-4}|H^o_{h,k}|+|z|^{2m-2}|\Delta_0 H^o_{h,k}|)\leqslant c(\|h\|_{\mathcal{C}^{4,\alpha}}+\|k\|_{\mathcal{C}^{2,\alpha}}).$$

The estimates for the derivatives of $H^o_{h,k}$ follow from Schauder's estimates. $\square$

When $m=2$, the result has to be slightly modified since in this case we can choose

$$H^o_{h,k}(z)=h^{(0)}|z|^{-2}+\frac{k^{(0)}}{2}\log|z|+\sum_{\gamma=1}^{\infty}\left(\left(h^{(\gamma)}+\frac{k^{(\gamma)}}{4\gamma}\right)|z|^{-2-\gamma}-\frac{k^{(\gamma)}}{4\gamma}|z|^{-\gamma}\right)e_\gamma. \tag{26}$$

This time, one can check that

$$H^o_{h,k}\in\mathcal{C}^{4,\alpha}_{-1}(\overline{C}_\Gamma)\oplus\operatorname{Span}\{\log|z|\} \tag{27}$$

and that

$$\|H^o_{h,k}\|_{\mathcal{C}^{4,\alpha}_{-1}(\overline{C}_\Gamma)\oplus\operatorname{Span}\{\log|z|\}}\leqslant c(\|h\|_{\mathcal{C}^{4,\alpha}(\partial B_\Gamma)}+\|k\|_{\mathcal{C}^{2,\alpha}(\partial B_\Gamma)}). \tag{28}$$

6. Constant scalar curvature Kähler metrics

We set

$$r_\varepsilon:=\varepsilon^{(m-1)/m}\quad\text{and}\quad R_\varepsilon:=\frac{r_\varepsilon}{\varepsilon}=\varepsilon^{-1/m}.$$

6.1. Perturbation of ω

We will now use the result of the previous sections to perturb ω, the Kähler form on M_{r_ε}, into infinite families of constant scalar curvature Kähler forms which are defined on M_{r_ε} and which are parameterized by the boundary data of their potentials. We carry out this analysis when (M,ω) is Futaki-nondegenerate, since this case clearly includes the nondegenerate case. We consider the perturbed Kähler form

$$\widetilde{\omega}=\omega+i\partial\bar{\partial}\varphi+\beta, \tag{29}$$

where β is a closed $(1,1)$-form and φ is a function defined on M_{r_ε}. The scalar curvature of $\widetilde{\omega}$ is given by

$$\mathbf{s}(\widetilde{\omega})=\mathbf{s}(\omega)-(\mathbf{L}_g\varphi+\mathcal{L}_g\beta)+Q_g(\nabla^2\varphi,\beta), \tag{30}$$

where the operators $\mathbf{L}_g$ and $\mathcal{L}_g$ have been defined in (16) and (17), and where Q_g collects all the nonlinear terms. The structure of Q_g is quite complicated; however, away from the support of the elements of $D_{\bar{r}_0}$ (i.e. in each $\overline{B}_{j,\bar{r}_0}$), we have $Q_g(\nabla^2\varphi,\beta)=Q_g(\nabla^2\varphi,0)$ and this operator, only acting on the function φ, enjoys the following decomposition:

$$\begin{aligned} Q_g(\nabla^2\varphi,0)=\sum_q & B_{q,4,2}(\nabla^4\varphi,\nabla^2\varphi)C_{q,4,2}(\nabla^2\varphi)\\ &+\sum_q B_{q,3,3}(\nabla^3\varphi,\nabla^3\varphi)C_{q,3,3}(\nabla^2\varphi)\\ &+|z|\sum_q B_{q,3,2}(\nabla^3\varphi,\nabla^2\varphi)C_{q,3,2}(\nabla^2\varphi)\\ &+\sum_q B_{q,2,2}(\nabla^2\varphi,\nabla^2\varphi)C_{q,2,2}(\nabla^2\varphi), \end{aligned} \tag{31}$$

where the sum over q is finite, the operators $(U,V)\mapsto B_{q,a,b}(U,V)$ are bilinear in the entries and have coefficients which are smooth functions on $\overline{B}_{j,\bar{r}_0}$. The nonlinear operators $W\mapsto C_{q,a,b}(W)$ have Taylor expansions (with respect to W) whose coefficients are smooth functions on $\overline{B}_{j,\bar{r}_0}$. These facts follow at once from the expression of the scalar curvature of $\widetilde{\omega}$ in local coordinates as given in (15).

We would like to solve the equation

$$\mathbf{s}(\widetilde{\omega})=\mathbf{s}(\omega)+\nu \tag{32}$$

in M_{r_ε}, where $\nu\in\mathbf{R}$.

We fix a (large) constant $\varkappa>0$. Assume that we are given boundary data $h_j\in\mathcal{C}^{4,\alpha}(\partial B_{\Gamma_j})$ and $k_j\in\mathcal{C}^{2,\alpha}(\partial B_{\Gamma_j})$, for $j=1,...,n$, satisfying

$$\|h_j\|_{\mathcal{C}^{4,\alpha}(\partial B_{\Gamma_j})}\leqslant\varkappa r_\varepsilon^4 \quad\text{and}\quad \|k_j\|_{\mathcal{C}^{2,\alpha}(\partial B_{\Gamma_j})}\leqslant\varkappa r_\varepsilon^4. \tag{33}$$

When $m \geqslant 3$, we define

$$H_{\mathbf{h},\mathbf{k}} := \sum_{j=1}^{n} \chi_j H^{o}_{h_j,k_j}(\,\cdot\,/r_\varepsilon), \tag{34}$$

where we have set

$$\mathbf{h} := (h_1, ..., h_n) \quad \text{and} \quad \mathbf{k} := (k_1, ..., k_n),$$

and where we recall that the cutoff functions χ_j are identically equal to 1 in $B_{j,r_0/2}$ and identically equal to 0 in $M \setminus B_{j,r_0}$. When $m=2$, some modifications are necessary. We decompose each k_j as

$$k_j = k_j^{(0)} + k_j^{\perp},$$

where $k_j^{(0)}$ is a constant function and $k_j^{\perp}$ has mean 0 on ∂B_{Γ_j}. With this decomposition in mind, we define

$$H_{\mathbf{h},\mathbf{k}} := \sum_{j=1}^{n} \chi_j \left(H^{o}_{h_j,k_j^{\perp}}(\,\cdot\,/r_\varepsilon) + \frac{k_j^{(0)}}{2} \log|\cdot| \right). \tag{35}$$

We replace in (29) the function φ by $H_{\mathbf{h},\mathbf{k}}+\varphi$. Then, (32) leads to the equation

$$\mathbf{L}_g(H_{\mathbf{h},\mathbf{k}}+\varphi) + \mathcal{L}_g\beta + \nu = Q_g(H_{\mathbf{h},\mathbf{k}}+\varphi, \beta), \tag{36}$$

which we would like to solve in M_{r_ε}.

Definition 6.1. Given $\bar r \in (0, r_0/2)$, $k \in \mathbf{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbf{R}$, the weighted space $\mathcal{C}^{k,\alpha}_{\delta}(M_{\bar r})$ is defined to be the space of functions $\varphi \in \mathcal{C}^{k,\alpha}(M_{\bar r})$ endowed with the norm

$$\|\varphi\|_{\mathcal{C}^{k,\alpha}_{\delta}(M_{\bar r})} := \|\varphi\|_{\mathcal{C}^{k,\alpha}(M_{r_0/2})} + \sum_{j=1}^{n} \sup_{2\bar r \leqslant r \leqslant r_0} r^{-\delta} \|\varphi|_{(B_{j,r_0}-B_{j,\bar r})}(r\,\cdot\,)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_{j,1} \setminus B_{j,1/2})}.$$

For each $\bar r \in (0, r_0/2)$, it will be convenient to define an "extension" (linear) operator

$$\mathcal{E}_{\bar r} : \mathcal{C}^{0,\alpha}_{\delta'}(M_{\bar r}) \longrightarrow \mathcal{C}^{0,\alpha}_{\delta'}(M^*)$$

as follows:

(i) in $M_{\bar r}$, we set $\mathcal{E}_{\bar r}(\psi) = \psi$;

(ii) in each $B_{j,\bar r} \setminus B_{j,\bar r/2}$, we set

$$\mathcal{E}_{\bar r}(\psi)(z) = \frac{2|z|-\bar r}{\bar r} \psi\left(\bar r \frac{z}{|z|} \right);$$

(iii) in each $B_{j,\bar r/2}$, we set $\mathcal{E}_{\bar r}(\psi) = 0$.

It is easy to check that there exists a constant $c=c(\delta')>0$, independent of $\bar r\in(0,r_0/2)$, such that

$$\|\mathcal{E}_{\bar r}(\psi)\|_{\mathcal{C}^{0,\alpha}_{\delta'}(M^*)}\leqslant c\|\psi\|_{\mathcal{C}^{0,\alpha}_{\delta'}(M_{\bar r})}. \tag{37}$$

We fix

$$\delta\in(4-2m,5-2m).$$

With the above notation and definitions, we rephrase the equation we would like to solve as

$$L_\delta(\varphi,\beta,\nu)=\mathcal{E}_{r_\varepsilon}(Q_g(\nabla^2(H_{\mathbf{h},\mathbf{k}}+\varphi),\beta)-\mathbf{L}_g H_{\mathbf{h},\mathbf{k}}), \tag{38}$$

where $\varphi\in\mathcal{C}^{4,\alpha}_\delta(M^*)$ when $m\geqslant3$, and $\varphi\in\mathcal{C}^{4,\alpha}_\delta(M^*)\oplus\mathcal{D}$ when $m=2$, $\beta\in D_{\bar r_0}$ and $\nu\in\mathbf{R}$ have to be determined. Recall that $\mathcal{D}$ and $D_{\bar r_0}$ have been defined in §5.1. Observe that any solution of (38) is a solution of (36). The advantage of the latter versus the former is that we can now make use of the analysis of §6.1 which allows us to find G_δ, a right inverse for the operator L_δ, and rephrase the solvability of (38) as a fixed point problem:

$$(\varphi,\beta,\nu)=\mathcal{N}(\varepsilon,\mathbf{h},\mathbf{k};\varphi,\beta),$$

where the nonlinear operator $\mathcal{N}$ is defined by

$$\mathcal{N}(\varepsilon,\mathbf{h},\mathbf{k};\varphi,\beta):=G_\delta(\mathcal{E}_{r_\varepsilon}(Q_g(\nabla^2(H_{\mathbf{h},\mathbf{k}}+\varphi),\beta)-\mathbf{L}_gH_{\mathbf{h},\mathbf{k}})).$$

It will be convenient to set

$$\mathcal{F}:=\begin{cases}\mathcal{C}^{4,\alpha}_\delta(M^*)\times D_{\bar r_0}\times\mathbf{R}, & \text{when } m\geqslant3,\\ (\mathcal{C}^{4,\alpha}_\delta(M^*)\oplus\mathcal{D})\times D_{\bar r_0}\times\mathbf{R}, & \text{when } m=2.\end{cases}$$

This space is naturally endowed with the product norm.

We first estimate the terms on the right-hand side of (38) when $\varphi=0$ and $\beta=0$, and next show that $\mathcal{N}$ is a contraction on a suitable small ball in $\mathcal{F}$. This is the content of the following lemma.

LEMMA 6.1. *There exist $c_\varkappa=c(\varkappa)>0$, $\tilde c_\varkappa=\tilde c(\varkappa)>0$ and $\varepsilon_\varkappa=\varepsilon(\varkappa)>0$ such that, for all $\varepsilon\in(0,\varepsilon_\varkappa)$,*

$$\|\mathcal{N}(\varepsilon,\mathbf{h},\mathbf{k};0,0)\|_{\mathcal{F}}\leqslant c_\varkappa r_\varepsilon^{2m}. \tag{39}$$

In addition,

$$\|\mathcal{N}(\varepsilon,\mathbf{h},\mathbf{k};\varphi,\beta)-\mathcal{N}(\varepsilon,\mathbf{h},\mathbf{k};\varphi',\beta')\|_{\mathcal{F}}\leqslant\tilde c_\varkappa r_\varepsilon^2\|(\varphi-\varphi',\beta-\beta')\|_{\mathcal{F}} \tag{40}$$

and

$$\|\mathcal{N}(\varepsilon,\mathbf{h},\mathbf{k};\varphi,\beta)-\mathcal{N}(\varepsilon,\mathbf{h}',\mathbf{k}';\varphi,\beta)\|_{\mathcal{F}}\leqslant\tilde c_\varkappa r_\varepsilon^{2m-4}\|(\mathbf{h}-\mathbf{h}',\mathbf{k}-\mathbf{k}')\|_{(\mathcal{C}^{4,\alpha})^n\times(\mathcal{C}^{2,\alpha})^n} \tag{41}$$

provided $(\varphi,\beta,0),(\varphi',\beta',0)\in\mathcal{F}$ *satisfy*

$$\|(\varphi,\beta,0)\|_{\mathcal{F}}\leqslant 2c_\varkappa r_\varepsilon^{2m} \quad \textit{and} \quad \|(\varphi',\beta',0)\|_{\mathcal{F}}\leqslant 2c_\varkappa r_\varepsilon^{2m},$$

and $\mathbf{h}:=(h_1,...,h_n)$, $\mathbf{h}':=(h'_1,...,h'_n)$, $\mathbf{k}:=(k_1,...,k_n)$ *and* $\mathbf{k}':=(k'_1,...,k'_n)$ *satisfy* (33).

Proof. We give a precise proof of the first estimate. The other estimates follow from similar considerations. In the proof, the constants $c_\varkappa^{(l)}>0$ only depend on $\varkappa$.

Step 1. We first estimate $\mathbf{L}_g H_{\mathbf{h},\mathbf{k}}$. Using the result of Proposition 5.6, together with (33), we obtain

$$\|\nabla^2 H_{\mathbf{h},\mathbf{k}}\|_{\mathcal{C}^{2,\alpha}_{2-2m}(M_{r_\varepsilon})}\leqslant c_\varkappa^{(1)} r_\varepsilon^{2m}. \tag{42}$$

Now observe that, by construction, $\nabla^2 H_{\mathbf{h},\mathbf{k}}=0$ in M_{r_0} and hence $\mathbf{L}_g H_{\mathbf{h},\mathbf{k}}=0$ in this set. Next,

$$\Delta_0^2 H_{\mathbf{h},\mathbf{k}}=0$$

in each $B_{j,r_0/2}\setminus B_{j,r_\varepsilon}$, hence

$$\mathbf{L}_g H_{\mathbf{h},\mathbf{k}}=\big(\mathbf{L}_g-\tfrac{1}{2}\Delta_0^2\big)H_{\mathbf{h},\mathbf{k}}$$

in each such set. Using (3), we conclude that

$$\|\mathbf{L}_g H_{\mathbf{h},\mathbf{k}}\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(M_{r_\varepsilon})}\leqslant c_\varkappa^{(2)} r_\varepsilon^{2m}$$

and

$$\int_M |\mathcal{E}_{r_\varepsilon}(\mathbf{L}_g H_{\mathbf{h},\mathbf{k}})|\,dv_g\leqslant c_\varkappa^{(2)} r_\varepsilon^{2m}.$$

These two estimates together with the properties of G_δ immediately imply that

$$\|G_\delta(\mathcal{E}_{r_\varepsilon}(\mathbf{L}_g H_{\mathbf{h},\mathbf{k}}))\|_{\mathcal{F}}\leqslant c_\varkappa^{(3)} r_\varepsilon^{2m}.$$

Step 2. We turn to the estimate of $Q_g(\nabla^2 H_{\mathbf{h},\mathbf{k}},0)$. To this end, we use the structure of Q_g as described in (31) together with (42) to get

$$\|Q_g(\nabla^2 H_{\mathbf{h},\mathbf{k}},0)\|_{\mathcal{C}^{0,\alpha}(M_{\bar r_0/2})}\leqslant c_\varkappa^{(4)} r_\varepsilon^{4m},$$

and

$$\|\mathcal{E}_{r_\varepsilon}(B_{q,a,b}(\nabla^{2+a} H_{\mathbf{h},\mathbf{k}},\nabla^{2+b} H_{\mathbf{h},\mathbf{k}})C_{q,a,b}(\nabla^2 H_{\mathbf{h},\mathbf{k}}))\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(\bar B_{j,\bar r_0})}\leqslant c_\varkappa^{(4)} r_\varepsilon^{8-a-b-\delta}.$$

Therefore, we conclude that

$$\|\mathcal{E}_{r_\varepsilon}(Q_g(\nabla^2 H_{\mathbf{h},\mathbf{k}},0))\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(M^*)}\leqslant c_\varkappa^{(5)} r_\varepsilon^{6-\delta},$$

as well as

$$\int_M |\mathcal{E}_{r_\varepsilon}(Q_g(\nabla^2 H_{\mathbf{h},\mathbf{k}},0))|\,dv_g \leqslant c_{\varkappa}^{(5)} r_\varepsilon^{2m+2}.$$

The properties of G_δ yield

$$\|G_\delta(\mathcal{E}_{r_\varepsilon}(Q_g(\nabla^2 H_{\mathbf{h},\mathbf{k}},0)))\|_{\mathcal{F}} \leqslant c_{\varkappa}^{(6)} r_\varepsilon^{6-\delta}.$$

This completes the proof of the first estimate.

Step 3. We now turn to the derivation of the second estimate. Again, we use the structure of Q_g as described in (31) to get

$$\|Q_g(\nabla^2(H_{\mathbf{h},\mathbf{k}}+\varphi),\beta)-Q_g(\nabla^2(H_{\mathbf{h},\mathbf{k}}+\varphi'),\beta')\|_{\mathcal{C}^{0,\alpha}(M_{\bar r_0})} \leqslant c_{\varkappa}^{(7)} r_\varepsilon^2 \|(\varphi-\varphi',\beta-\beta',0)\|_{\mathcal{F}},$$

and, arguing as above, we find that

$$\|\mathcal{E}_{r_\varepsilon}(Q_g(\nabla^2(H_{\mathbf{h},\mathbf{k}}+\varphi),\beta)-Q_g(\nabla^2(H_{\mathbf{h},\mathbf{k}}+\varphi'),\beta'))\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(\overline{B}_{j,\bar r_0})} \leqslant c_{\varkappa}^{(7)} r_\varepsilon^2 \|(\varphi-\varphi',0,0)\|_{\mathcal{F}}.$$

Therefore, we conclude that

$$\|\mathcal{E}_{r_\varepsilon}(Q_g(\nabla^2(H_{\mathbf{h},\mathbf{k}}+\varphi),\beta)-Q_g(\nabla^2(H_{\mathbf{h},\mathbf{k}}+\varphi'),\beta'))\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(M^*)} \leqslant c_{\varkappa}^{(8)} r_\varepsilon^2 \|(\varphi-\varphi',\beta-\beta',0)\|_{\mathcal{F}},$$

as well as

$$\int_M |\mathcal{E}_{r_\varepsilon}(Q_g(\nabla^2(H_{\mathbf{h},\mathbf{k}}+\varphi),\beta)-Q_g(\nabla^2(H_{\mathbf{h},\mathbf{k}}+\varphi'),\beta'))|\,dv_g \\ \leqslant c_{\varkappa}^{(8)} r_\varepsilon^{2m-2+\delta} \|(\varphi-\varphi',\beta-\beta',0)\|_{\mathcal{F}}.$$

Observe that, in order to derive the second estimate, we have implicitly used the fact that the computation of the scalar curvature only involves second and higher partial differential of the functions φ and φ', and hence, in dimension $m=2$, the effect of the elements of $\mathcal{D}$ have no influence in $\overline{B}_{j,r_0}\setminus B_{j,r_\varepsilon}$. The estimate then follows from the boundedness of G_δ.

Step 4. In order to prove the third estimate, we first observe that

$$\|\mathbf{L}_g(H_{\mathbf{h},\mathbf{k}}-H_{\mathbf{h}',\mathbf{k}'})\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(M_{r_\varepsilon})} \leqslant c_{\varkappa}^{(9)} r_\varepsilon^{2m-4} \|(\mathbf{h}-\mathbf{h}',\mathbf{k}-\mathbf{k}')\|_{(\mathcal{C}^{4,\alpha})^n\times(\mathcal{C}^{2,\alpha})^n}$$

and

$$\int_M |\mathcal{E}_{r_\varepsilon}(\mathbf{L}_g(H_{\mathbf{h},\mathbf{k}})-H_{\mathbf{h}',\mathbf{k}'}))|\,dv_g \leqslant c_{\varkappa}^{(9)} r_\varepsilon^{2m-4} \|(\mathbf{h}-\mathbf{h}',\mathbf{k}-\mathbf{k}')\|_{(\mathcal{C}^{4,\alpha})^n\times(\mathcal{C}^{2,\alpha})^n}.$$

Next, we have

$$\|\mathcal{E}_{r_\varepsilon}(Q_g(\nabla^2(H_{\mathbf{h},\mathbf{k}}+\varphi),\beta)-Q_g(\nabla^2(H_{\mathbf{h}',\mathbf{k}'}+\varphi),\beta))\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(M^*)} \\ \leqslant c_{\varkappa}^{(10)} r_\varepsilon^{2-\delta} \|(\mathbf{h}-\mathbf{h}',\mathbf{k}-\mathbf{k}')\|_{(\mathcal{C}^{4,\alpha})^n\times(\mathcal{C}^{2,\alpha})^n},$$

as well as

$$\int_M |\mathcal{E}_{r_\varepsilon}(Q_g(\nabla^2(H_{\mathbf{h},\mathbf{k}}+\varphi),\beta)-Q_g(\nabla^2(H_{\mathbf{h}',\mathbf{k}'}+\varphi),\beta))|\,dv_g \\ \leqslant c_{\varkappa}^{(10)} r_\varepsilon^{2m-2} \|(\mathbf{h}-\mathbf{h}',\mathbf{k}-\mathbf{k}')\|_{(\mathcal{C}^{4,\alpha})^n\times(\mathcal{C}^{2,\alpha})^n}.$$

The third estimate now follows from the boundedness of G_δ.

This completes the proof of the result. □

Reducing $\varepsilon_\varkappa>0$ if necessary, we can assume that

$$\tilde{c}_\varkappa r_\varepsilon^2 \leqslant \tfrac{1}{2} \tag{43}$$

for all $\varepsilon\in(0,\varepsilon_\varkappa)$. Then, the estimates (39) and (40) in the above lemma are enough to show that

$$(\varphi,\beta,\nu)\longmapsto \mathcal{N}(\varepsilon,\mathbf{h},\mathbf{k};\varphi,\beta)$$

is a contraction from

$$\{(\varphi,\beta,\nu)\in\mathcal{F}: \|(\varphi,\beta,\nu)\|_{\mathcal{F}} \leqslant 2c_\varkappa r_\varepsilon^{2m}\}$$

into itself, and hence has a unique fixed point $(\varphi_{\varepsilon,\mathbf{h},\mathbf{k}},\beta_{\varepsilon,\mathbf{h},\mathbf{k}},\nu_{\varepsilon,\mathbf{h},\mathbf{k}})$ in this set. This fixed point is a solution of (36) in M_{r_ε} and hence provides a constant scalar curvature Kähler form on M_{r_ε}.

Remark 6.1. When $m=2$, $\varphi_{\varepsilon,\mathbf{h},\mathbf{k}}$ can be decomposed as

$$\varphi_{\varepsilon,\mathbf{h},\mathbf{k}} = \widehat{\varphi}_{\varepsilon,\mathbf{h},\mathbf{k}} + c_{\varepsilon,\mathbf{h},\mathbf{k}} + \sum_{j=1}^n \chi_j \frac{k_j^{(0)}}{2}\log r_\varepsilon,$$

where $\widehat{\varphi}_{\varepsilon,\mathbf{h},\mathbf{k}}\in\mathcal{C}^{4,\alpha}_\delta(M^*)$ and $c_{\varepsilon,\mathbf{h},\mathbf{k}}\in\mathcal{D}$. When $m\geqslant 3$, we agree that $\widehat{\varphi}_{\varepsilon,\mathbf{h},\mathbf{k}}=\varphi_{\varepsilon,\mathbf{h},\mathbf{k}}$.

To summarize, we have obtained the following result.

PROPOSITION 6.1. *Given $\varkappa>0$, there exist $\hat{c}_\varkappa>0$ and $\varepsilon_\varkappa>0$ such that, for all $\varepsilon\in(0,\varepsilon_\varkappa)$, for all $h_j\in\mathcal{C}^{4,\alpha}(\partial B_{\Gamma_j})$ and all $k_j\in\mathcal{C}^{2,\alpha}(\partial B_{\Gamma_j})$ satisfying* (33)*, the Kähler form*

$$\omega_{\varepsilon,\mathbf{h},\mathbf{k}} := \omega + i\partial\bar\partial\varphi_{\varepsilon,\mathbf{h},\mathbf{k}} + \beta_{\varepsilon,\mathbf{h},\mathbf{k}},$$

defined on M_{r_ε}, has constant scalar curvature equal to

$$\mathbf{s}(\omega_{\varepsilon,\mathbf{h},\mathbf{k}}) = \mathbf{s}(\omega) + \nu_{\varepsilon,\mathbf{h},\mathbf{k}}.$$

Moreover, $\beta_{\varepsilon,\mathbf{h},\mathbf{k}}\in D_{\bar r_0}$,

$$\|\widehat{\varphi}_{\varepsilon,\mathbf{h},\mathbf{k}}|_{\overline{B}_{j,2r_\varepsilon}\setminus B_{j,r_\varepsilon}}(r_\varepsilon\,\cdot\,) - H^o_{h_j,k_j}\|_{\mathcal{C}^{4,\alpha}(\overline{B}_{j,2}\setminus B_{j,1})} \leqslant \hat{c}_\varkappa r_\varepsilon^{2m+\delta}$$

and

$$|\nu_{\varepsilon,\mathbf{h},\mathbf{k}}| \leqslant \hat{c}_\varkappa r_\varepsilon^{2m}.$$

Using (40) and (41), and increasing $\hat{c}_{\varkappa}$ if necessary, one can check that

$$\begin{aligned}\|(\widehat{\varphi}_{\varepsilon,\mathbf{h},\mathbf{k}}-\widehat{\varphi}_{\varepsilon,\mathbf{h}',\mathbf{k}'})|_{\overline{B}_{j,2r_\varepsilon}\setminus B_{j,r_\varepsilon}}(r_\varepsilon\,\cdot)-H^o_{h_j-h'_j,k_j-k'_j}\|_{\mathcal{C}^{4,\alpha}(\overline{B}_{j,2}\setminus B_{j,1})}&\\ \leqslant \hat{c}_{\varkappa} r_\varepsilon^{2m-4+\delta}\|(\mathbf{h}-\mathbf{h}',\mathbf{k}-\mathbf{k}')\|_{(\mathcal{C}^{4,\alpha})^n\times(\mathcal{C}^{2,\alpha})^n}&\end{aligned} \tag{44}$$

and

$$|\nu_{\varepsilon,\mathbf{h},\mathbf{k}}-\nu_{\varepsilon,\mathbf{h}',\mathbf{k}'}|\leqslant \hat{c}_{\varkappa} r_\varepsilon^{2m-4}\|(\mathbf{h}-\mathbf{h}',\mathbf{k}-\mathbf{k}')\|_{(\mathcal{C}^{4,\alpha})^n\times(\mathcal{C}^{2,\alpha})^n}. \tag{45}$$

Indeed, if

$$(\varphi,\beta,\nu)=\mathcal{N}(\varepsilon,\mathbf{h},\mathbf{k};\varphi,\beta)\quad\text{and}\quad(\varphi',\beta',\nu')=\mathcal{N}(\varepsilon,\mathbf{h}',\mathbf{k}';\varphi',\beta'),$$

then we can write

$$\begin{aligned}(\varphi'-\varphi,\beta'-\beta,\nu'-\nu)=(\mathcal{N}(\varepsilon,\mathbf{h}',\mathbf{k}';\varphi',\beta')-\mathcal{N}(\varepsilon,\mathbf{h}',\mathbf{k}';\varphi,\beta))&\\ +(\mathcal{N}(\varepsilon,\mathbf{h}',\mathbf{k}';\varphi,\beta)-\mathcal{N}(\varepsilon,\mathbf{h},\mathbf{k};\varphi,\beta)).&\end{aligned}$$

Using (40), we get

$$\|(\varphi'-\varphi,\beta'-\beta,\nu'-\nu)\|_{\mathcal{F}}\leqslant 2\|\mathcal{N}(\varepsilon,\mathbf{h}',\mathbf{k}';\varphi,\beta)-\mathcal{N}(\varepsilon,\mathbf{h},\mathbf{k};\varphi,\beta)\|_{\mathcal{F}},$$

and the result follows from (41).

6.2. Perturbation of η_j

Now, we would like to perturb the Kähler form on N_{j,R_ε} into some infinite-dimensional family of constant scalar curvature Kähler forms which are parameterized by their scalar curvature and the boundary data of their potentials.

We consider the perturbed Kähler form

$$\tilde{\eta}_j=\eta_j+i\partial\bar{\partial}\varphi. \tag{46}$$

The scalar curvature of $\tilde{\eta}_j$ is given by

$$\mathbf{s}(\tilde{\eta}_j)=-\mathbf{L}_{g_j}\varphi+Q_{g_j}(\nabla^2\varphi), \tag{47}$$

since the scalar curvature of η_j is identically equal to 0. Again, the structure of the non-linear operator Q_{g_j} is quite complicated but, in C_{j,R_0}, it enjoys a decomposition similar to the one described in (31). Indeed, using (12)–(15), we see that we can decompose

$$\begin{aligned}Q_{g_j}(\nabla^2\varphi) = &\sum_q B_{q,4,2}(\nabla^4\varphi,\nabla^2\varphi)C_{q,4,2}(\nabla^2\varphi)\\ &+\sum_q B_{q,3,3}(\nabla^3\varphi,\nabla^3\varphi)C_{q,3,3}(\nabla^2\varphi)\\ &+\sum_q |u|^{1-2m}B_{q,3,2}(\nabla^3\varphi,\nabla^2\varphi)C_{q,3,2}(\nabla^2\varphi)\\ &+\sum_q |u|^{-2m}B_{q,2,2}(\nabla^2\varphi,\nabla^2\varphi)C_{q,2,2}(\nabla^2\varphi),\end{aligned}$$

where the sum over q is finite, the operators $(U,V)\mapsto B_{q,a,b}(U,V)$ are bilinear in the entries and have coefficients which are bounded functions in $\mathcal{C}^{0,\alpha}(\overline{C}_{j,R_0})$. The nonlinear operators $W\mapsto C_{q,a,b}(W)$ have Taylor expansion (with respect to W) whose coefficients are bounded functions on $\mathcal{C}^{0,\alpha}(\overline{C}_{j,R_0})$. Even though these operators do depend on j we have not made this dependence apparent in the notation.

We would like to solve the equation

$$\mathbf{s}(\tilde\eta_j) = \varepsilon^2\nu \tag{48}$$

in N_{j,R_ε}, where $\nu\in\mathbf{R}$ and where we recall that $R_\varepsilon := r_\varepsilon/\varepsilon$.

We fix a constant $\varkappa>0$ large enough and assume that we are given $\nu\in\mathbf{R}$ and boundary data $h\in\mathcal{C}^{4,\alpha}(\partial B_{\Gamma_j})$ and $k\in\mathcal{C}^{2,\alpha}(\partial B_{\Gamma_j})$ satisfying

$$|\nu|\leqslant|\mathbf{s}(\omega)|+1,\quad \|h\|_{\mathcal{C}^{4,\alpha}(\partial B_{\Gamma_j})}\leqslant\varkappa R_\varepsilon^{4-2m}\quad\text{and}\quad \|k\|_{\mathcal{C}^{2,\alpha}(\partial B_{\Gamma_j})}\leqslant\varkappa R_\varepsilon^{4-2m}. \tag{49}$$

We decompose

$$h = h^{(0)} + h^{\perp},$$

where $h^{(0)}$ is a constant function and $h^\perp$ has mean 0 on ∂B_{Γ_j}, and we define

$$\widetilde{H}_{h,k} := \tilde\chi_j H^i_{h^\perp,k}(\,\cdot\,/R_\varepsilon)+h^{(0)} = \tilde\chi_j(H^i_{h,k}(\,\cdot\,/R_\varepsilon)-H^i_{h,k}(0))+H^i_{h,k}(0), \tag{50}$$

where $\tilde\chi_j$ is a cutoff function which is identically equal to 1 in $C_{j,2R_0}$ and identically equal to 0 in N_{j,R_0}.

Replacing in (46) the function φ by $\widetilde{H}_{h,k}+\varphi$, we see that (45) can be written as

$$\mathbf{L}_{g_j}(\widetilde{H}_{h,k}+\varphi) = Q_{g_j}(\nabla^2(\widetilde{H}_{h,k}+\varphi))-\varepsilon^2\nu, \tag{51}$$

which we would like to solve in N_{j,R_ε}. Here $\varphi\in\mathcal{C}^{4,\alpha}_\delta(N_j)$ for some $\delta\in\mathbf{R}$ which has to be determined.

Definition 6.2. Given $\bar{R}>2R_0$, $k\in\mathbf{N}$, $\alpha\in(0,1)$ and $\delta\in\mathbf{R}$, the weighted space $\mathcal{C}^{k,\alpha}_\delta(N_{j,\bar{R}})$ is defined to be the space of functions $\varphi\in\mathcal{C}^{k,\alpha}(N_{j,\bar{R}})$ endowed with the norm

$$\|\varphi\|_{\mathcal{C}^{k,\alpha}_\delta(N_{j,\bar{R}})} := \|\varphi\|_{\mathcal{C}^{k,\alpha}(N_{j,2R_0})} + \sup_{2R_0\leqslant R\leqslant\bar{R}} R^{-\delta}\|\varphi|_{\bar{C}_{j,R_0}\setminus C_{j,\bar{R}}}(R\,\cdot\,)\|_{\mathcal{C}^{k,\alpha}(\bar{B}_{j,1}\setminus B_{j,1/2})}.$$

For each $\bar{R}\geqslant 2R_0$, it will be convenient to define an “extension” (linear) operator

$$\widetilde{\mathcal{E}}_{\bar{R}}\colon \mathcal{C}^{0,\alpha}_{\delta'}(N_{j,\bar{R}}) \longrightarrow \mathcal{C}^{0,\alpha}_{\delta'}(N_j)$$

as follows:

(i) in N_{j,R_0}, we set $\widetilde{\mathcal{E}}_{\bar{R}}(\psi)=\psi$;

(ii) in $C_{j,\bar{R}}\setminus C_{j,2\bar{R}}$, we set

$$\widetilde{\mathcal{E}}_{\bar{R}}(\psi)(u) = \frac{2\bar{R}-|u|}{\bar{R}}\psi\left(\bar{R}\frac{u}{|u|}\right);$$

(iii) in $C_{j,2\bar{R}}$, we set $\widetilde{\mathcal{E}}_{\bar{R}}(\psi)=0$.

It is easy to check that there exists a constant $c=c(\delta')>0$, independent of $\bar{R}\geqslant 2R_0$, such that

$$\|\widetilde{\mathcal{E}}_{\bar{R}}(\psi)\|_{\mathcal{C}^{0,\alpha}_{\delta'}(N_j)} \leqslant c\|\psi\|_{\mathcal{C}^{0,\alpha}_{\delta'}(N_{j,\bar{R}})}. \tag{52}$$

We fix $\delta\in(0,1)$. The equation we would like to solve can be rewritten as

$$\tilde{L}_\delta\varphi = \widetilde{\mathcal{E}}_{R_\varepsilon}(Q_{g_j}(\nabla^2(\widetilde{H}_{h,k}+\varphi)) - \mathbf{L}_{g_j}\widetilde{H}_{h,k} - \varepsilon^2\nu), \tag{53}$$

where $\varphi\in\mathcal{C}^{4,\alpha}_\delta(N_j)$ has to be determined. Observe that any solution of (53) is a solution of (50). Again, we make use of the analysis of §6.2 in order to find a right inverse $\widetilde{G}_\delta$ for the operator $\tilde{L}_\delta$ and rephrase the solvability of (53) as a fixed point problem:

$$\varphi = \widetilde{\mathcal{N}}_j(\varepsilon,h,k,\nu;\varphi), \tag{54}$$

where the nonlinear operator $\widetilde{\mathcal{N}}$ is defined by

$$\widetilde{\mathcal{N}}(\varepsilon,h,k,\nu;\varphi) := \widetilde{G}_\delta(\widetilde{\mathcal{E}}_{R_\varepsilon}(Q_{g_j}(\nabla^2(\widetilde{H}_{h,k}+\varphi)) - \mathbf{L}_{g_j}\widetilde{H}_{h,k} - \varepsilon^2\nu)).$$

To keep the notation short, it will be convenient to define

$$\widetilde{\mathcal{F}} := \mathcal{C}^{4,\alpha}_\delta(N_j).$$

We first estimate the terms on the right-hand side of (54) when $\varphi=0$, and next show that $\widetilde{\mathcal{N}}$ is a contraction from a suitable small ball in $\widetilde{\mathcal{F}}$. This is the content of the following lemma.

Lemma 6.2. *There exist* $c>0$ (*independent of* $\varkappa$), $\tilde{c}_\varkappa=\tilde{c}(\varkappa)>0$ *and* $\varepsilon_\varkappa=\varepsilon(\varkappa)>0$ *such that, for all* $\varepsilon\in(0,\varepsilon_\varkappa)$,

$$\|\widetilde{\mathcal{N}}(\varepsilon,h,k,\nu;0)\|_{\widetilde{\mathcal{F}}}\leqslant cR_\varepsilon^{4-2m-\delta}. \tag{55}$$

Moreover, for all $\varphi,\varphi'\in\widetilde{\mathcal{F}}$, *satisfying*

$$\|\varphi\|_{\widetilde{\mathcal{F}}}\leqslant 2cR^{4-2m-\delta} \quad \text{and} \quad \|\varphi'\|_{\widetilde{\mathcal{F}}}\leqslant 2cR_\varepsilon^{4-2m-\delta},$$

we have

$$\|\widetilde{\mathcal{N}}(\varepsilon,h,k,\nu;\varphi)-\widetilde{\mathcal{N}}(\varepsilon,h,k,\nu;\varphi')\|_{\widetilde{\mathcal{F}}}\leqslant \tilde{c}_\varkappa R_\varepsilon^{4-2m-\delta}\|\varphi-\varphi'\|_{\widetilde{\mathcal{F}}} \tag{56}$$

and

$$\begin{aligned}\|\widetilde{\mathcal{N}}(\varepsilon,h,k,\nu;\varphi)-\widetilde{\mathcal{N}}(\varepsilon,h',k',\nu';\varphi)\|_{\widetilde{\mathcal{F}}}&\\ \leqslant \tilde{c}_\varkappa(R_\varepsilon^{-1}\|(h-h',k-k')\|_{\mathcal{C}^{4,\alpha}\times\mathcal{C}^{2,\alpha}}+R_\varepsilon^{4-2m-\delta}|\nu'-\nu|)&\end{aligned} \tag{57}$$

provided h, h', k *and* k' *satisfy* (49).

Proof. The proof is identical to the proof of Lemma 6.1. We give details about the derivation of the first estimate and leave the two other estimates to the reader.

It follows from the analysis of §5.3, together with (49), that

$$\|\nabla^2\widetilde{H}_{h,k}\|_{\mathcal{C}_0^{2,\alpha}(N_{j,R_\varepsilon})}\leqslant c_\varkappa^{(1)}R_\varepsilon^{2-2m} \tag{58}$$

and also that

$$\|\nabla^2\widetilde{H}_{h,k}\|_{\mathcal{C}_0^{2,\alpha}(\overline{C}_{j,2R_0}\setminus C_{j,R_0})}\leqslant c_\varkappa^{(1)}R_\varepsilon^{3-2m}. \tag{59}$$

We use the fact that, in $C_{j,2R_0}\setminus C_{j,R_\varepsilon}$, we can write

$$\mathbf{L}_{g_j}H_{h,k}=\big(\mathbf{L}_{g_j}-\tfrac{1}{2}\Delta_0^2\big)\widetilde{H}_{h,k}.$$

Then, (13)–(14) together with (58) yields

$$\|\mathbf{L}_{g_j}\widetilde{H}_{h,k}\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(N_{j,R_\varepsilon})}\leqslant cR_\varepsilon^{3-2m}.$$

Next, we use the structure of Q_{g_j}, together with (58), to estimate

$$\|\widetilde{\mathcal{E}}_{R_\varepsilon}(Q_{g_j}(\nabla^2(\widetilde{H}_{h,k})+\varphi))\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(N_j)}\leqslant c_\varkappa^{(2)}R_\varepsilon^{6-4m}.$$

Finally, we estimate

$$\|\widetilde{\mathcal{E}}_{R_\varepsilon}(\varepsilon^2\nu)\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(N_j)}\leqslant \tilde{c}R_\varepsilon^{4-2m-\delta},$$

for some constant $\tilde{c}>0$ which does not depend on ε, since $|\nu|\leqslant 1+|\mathbf{s}(\omega)|$. This completes the proof of the estimate. □

Reducing $\varepsilon_\varkappa>0$ if necessary, we may assume that

$$\tilde{c}_\varkappa R_\varepsilon^{4-2m-\delta} \leqslant \tfrac{1}{2} \tag{60}$$

for all $\varepsilon\in(0,\varepsilon_\varkappa)$. Then, the estimates (55) and (56) in the above lemma are enough to show that

$$\varphi \longmapsto \widetilde{N}(\varepsilon, h, k, \nu; \varphi)$$

is a contraction from

$$\{\varphi\in\widetilde{\mathcal{F}} : \|\varphi\|_{\widetilde{\mathcal{F}}} \leqslant 2cR_\varepsilon^{4-2m-\delta}\}$$

into itself and hence has a unique fixed point $\widetilde{\varphi}_{\varepsilon,h,k,\nu}$ in this set. This fixed point is a solution of (51) in N_{j,R_ε}, and hence provides a constant scalar curvature Kähler form on N_{j,R_ε}.

We have obtained the following result.

PROPOSITION 6.2. *There exist $c>0$ (independent of $\varkappa$) and $\varepsilon_\varkappa=\varepsilon(\varkappa)>0$ such that, for all $\varepsilon\in(0,\varepsilon_\varkappa)$, for all $h\in\mathcal{C}^{4,\alpha}(\partial B_{\Gamma_j})$, $k\in\mathcal{C}^{2,\alpha}(\partial B_{\Gamma_j})$ and $\nu\in\mathbf{R}$ satisfying* (49), *the Kähler form*

$$\eta_{h,k,\nu} := \eta_j + i\partial\bar{\partial}\widetilde{\varphi}_{\varepsilon,h,k,\nu},$$

defined on N_{j,R_ε}, has constant scalar curvature equal to $\varepsilon^2\nu$. Moreover,

$$\|\widetilde{\varphi}_{\varepsilon,h,k,\nu}|_{\overline{C}_{j,R_\varepsilon/2}\setminus C_{j,R_\varepsilon}}(R_\varepsilon\,\cdot) - H^i_{h,k}\|_{\mathcal{C}^{4,\alpha}(\overline{B}_{j,1}\setminus B_{j,1/2})} \leqslant cR_\varepsilon^{4-2m},$$

for some constant $c>0$ independent of $\varkappa$ and ν.

The important fact is that the last estimate involves a constant times R_ε^{4-2m}, where the constant does not depend on $\varkappa$ provided $\varepsilon\in(0,\varepsilon_\varkappa)$.

Using (56) and (57), and increasing $\tilde{c}_\varkappa$ if necessary, one checks that

$$\begin{aligned} &\|(\widetilde{\varphi}_{\varepsilon,h,k,\nu} - \widetilde{\varphi}_{\varepsilon,h',k',\nu'})|_{\overline{C}_{j,R_\varepsilon/2}\setminus C_{j,R_\varepsilon}}(R_\varepsilon\,\cdot) - H^i_{h-h',k-k'}\|_{\mathcal{C}^{4,\alpha}(\overline{B}_{j,1}\setminus B_{j,1/2})} \\ &\qquad\qquad \leqslant \tilde{c}_\varkappa(R_\varepsilon^{\delta-1}\|(h-h',k-k')\|_{\mathcal{C}^{4,\alpha}\times\mathcal{C}^{2,\alpha}} + R_\varepsilon^{4-2m}|\nu-\nu'|). \end{aligned} \tag{61}$$

6.3. Cauchy data matching: the proof of Theorem 1.3

Building on the analysis of the previous subsections we complete the proof of Theorem 1.3.

Granted the results of Propositions 6.1 and 6.2, it remains to explain how to choose

$$\mathbf{h} := (h_1, ..., h_n) \quad\text{and}\quad \mathbf{k} := (k_0, ..., k_n)$$

satisfying (33), and

$$\tilde{\mathbf{h}} := (\tilde{h}_1, ..., \tilde{h}_n) \quad\text{and}\quad \tilde{\mathbf{k}} := (\tilde{k}_1, ..., \tilde{k}_n)$$

satisfying (49) in such a way that, for each $j=1, ..., n$, the function

$$\psi_j^o := (\varphi_j + \widehat{\varphi}_{\varepsilon,\mathbf{h},\mathbf{k}})(r_\varepsilon\, \cdot),$$

defined in $\overline{B}_{j,2} \setminus B_{j,1}$ (see Proposition 6.1 and Remark 6.1 for the definition of $\widehat{\varphi}_{\varepsilon,\mathbf{h},\mathbf{k}}$) on the one hand, and for

$$\nu := \mathbf{s}(\omega_{\varepsilon,\mathbf{h},\mathbf{k}})$$

the function

$$\psi_j^i := \varepsilon^2 (\widetilde{\varphi}_j + \widetilde{\varphi}_{\varepsilon,\tilde{h}_j,\tilde{k}_j,\nu})(R_\varepsilon\, \cdot),$$

defined in $\overline{B}_{j,1} \setminus B_{j,1/2}$ (see Proposition 6.1) on the other hand, have their partial derivatives up to order 3 which coincide on $\partial B_{j,1}$.

Remark 6.2. In dimension 2, a slight modification is necessary since the functions involve some log terms. In view of (14), we consider the function ψ_j^i defined by

$$\psi_j^i := \varepsilon^2 (\widetilde{\varphi}_j + \widetilde{\varphi}_{\varepsilon,\tilde{h}_j,\tilde{k}_j,\nu})(R_\varepsilon\, \cdot) - \varepsilon^2 a_j \log R_\varepsilon.$$

There is no loss of generality in doing so, since changing locally the potential by some constant function does not alter the corresponding Kähler forms.

In fact, we shall solve the following system of equations

$$\psi_j^o = \psi_j^i, \quad \partial_r \psi_j^o = \partial_r \psi_j^i, \quad \Delta_0 \psi_j^o = \Delta_0 \psi_j^i \quad \text{and} \quad \partial_r \Delta_0 \psi_j^o = \partial_r \Delta_0 \psi_j^i \tag{62}$$

on $\partial B_{j,1}$, where $r=|v|$ and $v=(v^1, ..., v^m)$ are coordinates in $B_{j,2}$.

Let us assume that we have already solved this problem. The first identity in (62) implies that ψ_j^o and ψ_j^i as well as all their kth order partial derivatives with respect any vector field tangent to $\partial B_{j,1}$, with $k \leqslant 4$, agree on $\partial B_{j,1}$. The second identity in (62) then shows that $\partial_r \psi_j^o$ and $\partial_r \psi_j^i$ as well as all their kth order partial derivatives with respect to any vector field tangent to $\partial B_{j,1}$, with $k \leqslant 3$, agree on $\partial B_{j,1}$. Using the decomposition of the Laplacian in polar coordinates, it is easy to check that the third identity implies that $\partial_r^2 \psi_j^o$ and $\partial_r^2 \psi_j^i$ as well as all their kth order partial derivatives with respect to any vector field tangent to $\partial B_{j,1}$, with $k \leqslant 2$, agree on $\partial B_{j,1}$. And finally, the last identity in (62) implies that $\partial_r^3 \psi_j^o$ and $\partial_r^3 \psi_j^i$ as well as all their first order partial derivatives with respect to any vector field tangent to $\partial B_{j,1}$, agree on $\partial B_{j,1}$.

Moreover, the Kähler form

$$i\partial\bar{\partial}\bigl(\tfrac{1}{2}|v|^2 + \psi_j^o\bigr),$$

defined in $B_{j,2} \setminus B_{j,1}$, and the Kähler form

$$i\partial\bar{\partial}\bigl(\tfrac{1}{2}|v|^2 + \psi_j^i\bigr),$$

defined in $B_{j,1}\backslash B_{j,1/2}$, both have the same constant scalar curvature equal to $\mathbf{s}(\omega_{\varepsilon,\mathbf{h},\mathbf{k}})$. This then implies that any kth order partial derivatives of the functions ψ_j^o and ψ_j^i, with $k\leqslant 4$, coincide on $\partial B_{j,1}$.

Therefore, we conclude that the function ψ defined by $\psi:=\psi_j^o$ in $B_{j,2}\backslash B_{j,1}$ and $\psi:=\psi_j^i$ in $B_{j,1}\backslash B_{j,1/2}$ is $\mathcal{C}^4$ in $B_{j,2}\backslash B_{j,1/2}$, and is a solution of the nonlinear elliptic partial differential equation

$$\mathbf{s}\big(i\partial\bar\partial\big(\tfrac12|v|^2+\psi\big)\big)=\mathbf{s}(\omega_{\varepsilon,\mathbf{h},\mathbf{k}})=\text{constant}.$$

It then follows from elliptic regularity theory, together with a bootstrap argument, that the function ψ is in fact smooth. Hence, by gluing the Kähler metrics $\omega_{\mathbf{h},\mathbf{k}}$ and $\omega_{\tilde h_j,\tilde k_j}$ on the different pieces constituting M_{r_ε}, we have produced a Kähler metric on M_{r_ε} which has constant scalar curvature. This will end the proof of Theorem 1.3.

It remains to explain how to find the boundary data

$$\mathbf{h}=(h_1,...,h_n),\quad \mathbf{k}=(k_1,...,k_n),\quad \tilde{\mathbf{h}}=(\tilde h_1,...,\tilde h_n)\quad\text{and}\quad \tilde{\mathbf{k}}=(\tilde k_1,...,\tilde k_n)$$

which satisfy (62). We will make use of the following result.

LEMMA 6.3. *Assume that* Γ *is a discrete subgroup of* $\mathrm{U}(m)$ *acting freely on* $\mathbf{C}^m\backslash\{0\}$. *Then, the mapping*

$$\begin{aligned}\mathcal{P}\colon \mathcal{C}^{4,\alpha}(\partial B_\Gamma)\times\mathcal{C}^{2,\alpha}(\partial B_\Gamma)&\longrightarrow \mathcal{C}^{3,\alpha}(\partial B_\Gamma)\times\mathcal{C}^{1,\alpha}(\partial B_\Gamma),\\ (h,k)&\longmapsto (\partial_r(H^i_{h,k}-H^o_{h,k}),\partial_r\Delta_0(H^i_{h,k}-H^o_{h,k})),\end{aligned}$$

is an isomorphism.

Proof. There are many different ways to prove this result (see [18] for example). Let us concentrate on the case where $m\geqslant 3$, since the case $m=2$ is essentially the same. We use the formulas (24) and (25) to compute

$$\partial_r(H^i_{h,k}-H^o_{h,k})=\sum_{\gamma=0}^{\infty}2(\gamma+m-1)\bigg(h^{(\gamma)}+\frac{k^{(\gamma)}}{2(\gamma+m)(\gamma+m-2)}\bigg)e_\gamma$$

and

$$\partial_r\Delta_0(H^i_{h,k}-H^o_{h,k})=\sum_{\gamma=0}^{\infty}2(\gamma+m-1)k^{(\gamma)}e_\gamma.$$

It is then easy to see that

$$\begin{aligned}\mathcal{P}\colon W^{4,2}(\partial B_\Gamma)\times W^{2,2}(\partial B_\Gamma)&\longrightarrow W^{3,2}(\partial B_\Gamma)\times W^{1,2}(\partial B_\Gamma),\\ (h,k)&\longmapsto (\partial_r(H^i_{h,k}-H^o_{h,k}),\partial_r\Delta_0(H^i_{h,k}-H^o_{h,k})),\end{aligned}$$

is well defined and invertible. Recall that the norm in $W^{l,2}(\partial B_\Gamma)$ can be taken to be

$$\|f\|_{W^{l,2}} = \left(\sum_{\gamma=0}^{\infty} (1+\gamma)^{2l} |f^{(\gamma)}|^2\right)^{1/2}$$

if the function f is decomposed as

$$f = \sum_{\gamma=0}^{\infty} f^{(\gamma)} e_\gamma.$$

Elliptic regularity theory then implies that the same result is true when the operator is defined between Hölder spaces. □

It will be convenient to observe that ψ_j^o satisfies

$$\|\psi_j^o - H_{h_j,k_j}\|_{\mathcal{C}^{4,\alpha}(\overline{B}_{j,2}\setminus B_{j,1})} \leqslant c r_\varepsilon^4, \tag{63}$$

and also that

$$\|\psi_j^i - \varepsilon^2 \widetilde{H}_{\tilde{h}_j,\tilde{k}_j}\|_{\mathcal{C}^{4,\alpha}(\overline{B}_{j,1}\setminus B_{j,1/2})} \leqslant c\varepsilon^2 R_\varepsilon^{4-2m} = c r_\varepsilon^4, \tag{64}$$

for some constant $c>0$ which does not depend on $\varkappa$, provided ε is chosen small enough, say $\varepsilon\in(0,\varepsilon_\varkappa)$. These two estimates follow at once from the estimates in Propositions 6.1 and 6.2, and also from the choice of r_ε.

We use the following notation for the rescaled boundary data

$$(\mathbf{h}', \tilde{\mathbf{h}}', \mathbf{k}', \tilde{\mathbf{k}}') := (\mathbf{h}, \varepsilon^2 \tilde{\mathbf{h}}, \mathbf{k}, \varepsilon^2 \tilde{\mathbf{k}}).$$

Using Lemma 6.3, the solvability of (62) reduces to a fixed point problem which can be written as

$$(\mathbf{h}', \tilde{\mathbf{h}}', \mathbf{k}', \tilde{\mathbf{k}}') = S_\varepsilon(\mathbf{h}', \tilde{\mathbf{h}}', \mathbf{k}', \tilde{\mathbf{k}}),$$

and we know from (63) and (64) that the nonlinear operator S_ε satisfies

$$\|S_\varepsilon(\mathbf{h}', \tilde{\mathbf{h}}', \mathbf{k}', \tilde{\mathbf{k}}')\|_{(\mathcal{C}^{4,\alpha})^{2n}\times(\mathcal{C}^{2,\alpha})^{2n}} \leqslant c_0 r_\varepsilon^4,$$

for some constant $c_0>0$ which does not depend on $\varkappa$, provided $\varepsilon\in(0,\varepsilon_\varkappa)$. We finally choose $\varkappa=2c_0$ and $\varepsilon\in(0,\varepsilon_\varkappa)$. We have therefore proved that S_ε is a map from

$$A_\varepsilon := \{(\mathbf{h}', \tilde{\mathbf{h}}', \mathbf{k}', \tilde{\mathbf{k}}') \in (\mathcal{C}^{4,\alpha})^{2n}\times(\mathcal{C}^{2,\alpha})^{2n} : \|(\mathbf{h}', \tilde{\mathbf{h}}', \mathbf{k}', \tilde{\mathbf{k}}')\|_{(\mathcal{C}^{4,\alpha})^{2n}\times(\mathcal{C}^{2,\alpha})^{2n}} \leqslant \varkappa r_\varepsilon^4\}$$

into itself. It follows from (44), (45) and (61) that, reducing $\varepsilon_\varkappa$ if necessary, S_ε is a contraction mapping from A_ε into itself for all $\varepsilon\in(0,\varepsilon_\varkappa)$. Therefore, S_ε has a fixed point in this set. This completes the proof of the existence of a solution of (62).

The proof of the existence on M_{r_ε} of a Kähler form ω_ε which has constant scalar curvature is therefore complete. Observe that the scalar curvature of ω and ω_ε are close, since the estimate

$$|\mathbf{s}(\omega_\varepsilon) - \mathbf{s}(\omega)| \leqslant c r_\varepsilon^{2m}$$

follows directly from the construction.

7. Refined asymptotics for ALE spaces

Let us now describe in detail (N,η), the blow-up at the origin of $\mathbf{C}^m$ endowed with the Burns–Calabi–Simanca metric. Away from the exceptional divisor, the Kähler form η is given by

$$\eta = i\partial\bar{\partial}A_m(|v|^2),$$

where $v=(v_1,...,v_n)$ are complex coordinates in $\mathbf{C}^m\setminus\{0\}$ and the function $s\mapsto A_m(s)$ is a solution of the ordinary differential equation

$$s^2(s\partial_s A_m)^{m-1}\partial_s^2 A_m+(m-1)s\partial_s A_m-(m-2)=0,$$

which satisfies $A_m\sim\log s$ near 0. We refer to [53] for a derivation of this equation. It turns out that, when $m=2$, the function A_2 is explicitly given by

$$A_2(s)=\log s+\lambda s,$$

where $\lambda>0$, while in dimension $m\geqslant 3$, even though there is no explicit formula for A_m, we have the following simple result.

LEMMA 7.1. *Assume that $m\geqslant 3$. Then the function A_m can be expanded as*

$$A_m(s)=\lambda s-\lambda^{2-m}\frac{s^{2-m}}{m-2}+\mathcal{O}(s^{1-m})$$

for $s>1$, where $\lambda>0$.

Proof. Define the function ζ by $s\zeta:=s\partial_s A_m-1$. A direct computation shows that ζ solves

$$(1+s\zeta)^{m-1}s^2\partial_s\zeta=(1+s\zeta)^{m-1}-1-(m-1)s\zeta.$$

If in addition we take $\zeta(0)=1$, then $\partial_s\zeta$ remains positive and one can check that ζ is well defined for all time and converges to some positive constant λ, as s tends to ∞. This immediately implies that $s\partial_s A_m=\lambda s+\mathcal{O}(1)$ at infinity. The expansion then follows easily. □

Changing variables $u:=v\sqrt{2\lambda}$, we see from the previous lemma that the Kähler form η can be expanded near infinity as

$$\eta=i\partial\bar{\partial}\big(\tfrac{1}{2}|u|^2+\log|u|^2\big) \tag{65}$$

in dimension $m=2$, and as

$$\eta=i\partial\bar{\partial}\bigg(\frac{1}{2}|u|^2-2^{m-2}\frac{|u|^{4-2m}}{m-2}+\mathcal{O}(|u|^{2-2m})\bigg) \tag{66}$$

in dimension $m\geqslant 3$.

We now recall the following results of Joyce [26] (which is a corollary of his Theorem 8.2.3 in our notation).

Theorem 7.1. *Let Γ be a finite subgroup of* $\mathrm{SU}(m)$ *acting freely on* $\mathbf{C}^m\setminus\{0\}$ *and* $\pi\colon X\to\mathbf{C}^m/\Gamma$ *a Kähler crepant resolution of* $\mathbf{C}^m/\Gamma$. *Then, there exists a Ricci-flat Kähler metric* η *such that*

$$\eta = i\partial\bar\partial\big(\tfrac{1}{2}|u|^2+\widetilde\varphi(u)\big)$$

outside a compact neighborhood of $\pi^{-1}(0)$.

Moreover

$$\widetilde\varphi(u) = |u|^{2-2m}+\mathcal{O}(|u|^{\gamma})$$

for some $\gamma\in(1-2m,2-2m)$.

We end this section with a proof of Remark 3.1.

Lemma 7.2. *Let φ be a potential defined on C_Γ such that $\varphi\in\mathcal{C}^{4,\alpha}_{2-\gamma}(\overline{C}_\Gamma)$ for some $\gamma>0$. Further assume that*

$$\eta := i\partial\bar\partial\big(\tfrac{1}{2}|u|^2+\varphi\big) \tag{67}$$

is a zero scalar curvature Kähler form. Then, the function φ can be expanded as

$$\varphi = a\cdot u+b+c|u|^{4-2m}+\mathcal{O}(|u|^{3-2m}) \tag{68}$$

when $m\geqslant 3$, and as

$$\varphi = a\cdot u+b+c\log|u|+\mathcal{O}(|u|^{-1}) \tag{69}$$

when $m=2$. Here $a\in\mathbf{C}$ and $b\in\mathbf{R}$. In particular, the potential $\widetilde\varphi:=\varphi-a\cdot u-b$ satisfies

$$\eta := i\partial\bar\partial\big(\tfrac{1}{2}|u|^2+\widetilde\varphi\big).$$

Proof. The key point is that, since η has zero scalar curvature, the potential φ is a solution of some nonlinear fourth order elliptic differential equation and satisfies some *a priori* bound. It is then possible to get "refined asymptotics" for the potential φ in the spirit of what has been done in [31] for constant scalar curvature metrics. These refined asymptotics are obtained by using a bootstrap argument in Hölder weighted spaces.

Using (15), we see that the scalar curvature of η can be expanded in powers of φ as

$$\mathbf{s}(\eta) = \tfrac{1}{2}\Delta_0^2\varphi+Q_{g_{\mathrm{Eucl}}}(\nabla^2\varphi),$$

where the nonlinear operator $Q_{g_{\mathrm{Eucl}}}$ collects all the nonlinear terms. We shall now be more specific about the structure of $Q_{g_{\mathrm{Eucl}}}$. Indeed, it follows from the explicit computation of the Ricci curvature that the nonlinear operator $Q_{g_{\mathrm{Eucl}}}$ can be decomposed as

$$Q_{g_{\mathrm{Eucl}}}(\nabla^2\varphi) = \sum_q B_{q,4,2}(\nabla^4\varphi,\nabla^2\varphi)C_{q,4,2}(\nabla^2\varphi)+\sum_q B_{q,3,3}(\nabla^3\varphi,\nabla^3\varphi)C_{q,3,3}(\nabla^2\varphi), \tag{70}$$

where the sum over q is finite, the operators $(U,V)\mapsto B_{q,a,b}(U,V)$ are bilinear in the entries and have coefficients which are bounded functions in $\mathcal{C}^{0,\alpha}(\overline{C}_\Gamma)$. The nonlinear operators $W\mapsto C_{q,a,b}(W)$ have Taylor expansion (with respect to W) whose coefficients are bounded functions on $\mathcal{C}^{0,\alpha}(\overline{C}_\Gamma)$.

If we assume that $\varphi\in\mathcal{C}^{4,\alpha}_{2-\gamma}(\overline{C}_\Gamma)$, then we see that

$$Q_{g_{\mathrm{Eucl}}}(\nabla^2\varphi)\in\mathcal{C}^{0,\alpha}_{-2-2\gamma}(\overline{C}_\Gamma).$$

Therefore, $\Delta_0^2\varphi\in\mathcal{C}^{0,\alpha}_{-2-2\gamma}(\overline{C}_\Gamma)$.

Now, if $\Delta_0^2\varphi\in\mathcal{C}^{0,\alpha}_{\gamma'-4}(\overline{C}_\Gamma)$ and $\varphi\in\mathcal{C}^{4,\alpha}_{2-\gamma}(\overline{C}_\Gamma)$ for some $\gamma>0$ then, depending on the value of γ', the following alternatives hold [43]:

(i) If $\gamma'\in(1,2)$, then $\varphi\in\mathcal{C}^{4,\alpha}_{\gamma'}(\overline{C}_\Gamma)$.

(ii) If $\gamma'\in(0,1)$, then

$$\varphi\in\mathcal{C}^{4,\alpha}_{\gamma'}(\overline{C}_\Gamma)\oplus\{u\mapsto a\cdot u:a\in\mathbf{C}\}.$$

(iii) If $m\geqslant3$ and $\gamma'\in(4-2m,0)$, then

$$\varphi\in\mathcal{C}^{4,\alpha}_{\gamma'}(\overline{C}_\Gamma)\oplus\{u\mapsto a\cdot u:a\in\mathbf{C}\}\oplus\mathbf{R}.$$

(iv) If $m\geqslant3$ and $\gamma'\in(3-2m,4-2m)$, then

$$\varphi\in\mathcal{C}^{4,\alpha}_{\gamma'}(\overline{C}_\Gamma)\oplus\{u\mapsto a\cdot u:a\in\mathbf{C}\}\oplus\mathbf{R}\oplus\operatorname{Span}\{u\mapsto|u|^{4-2m}\}.$$

(v) If $m=2$ and $\gamma'\in(-1,0)$, then

$$\varphi\in\mathcal{C}^{4,\alpha}_{\gamma'}(\overline{C}_\Gamma)\oplus\{u\mapsto a\cdot u:a\in\mathbf{C}\}\oplus\mathbf{R}\oplus\operatorname{Span}\{u\mapsto\log|u|\}.$$

Using these alternatives together with a bootstrap argument, we conclude that (68) and (69) hold. The result then follows by taking $\widetilde{\varphi}:=\varphi-a\cdot u-b$. □

8. Applications, examples and comments

8.1. Blow-up of smooth manifolds

Theorem 1.1 follows at once from Theorem 1.3 and the analysis of Lemma 7.1 by taking $(N_j,\eta_j)=(N,a_j\eta)$, where (N,η) is the blow-up at the origin of $\mathbf{C}^m$ endowed with the Burns–Calabi–Simanca metric, and $a_j>0$. Observe that the points of blow-up $p_1,\dots,p_n$ and the coefficients $a_1,\dots,a_n$ are parameters of our construction.

A first natural question is to which base smooth manifolds Theorem 1.1 can be applied! Here, we do not make a comprehensive list but we highlight some large classes of manifolds:

(i) All the Kähler–Einstein manifolds with discrete automorphism group. This means any manifold with negative first Chern class and many families of examples of positive first Chern class [2], [45], [55]. We should note that there are no Futaki-nondegenerate Kähler–Einstein manifolds except the ones with discrete automorphisms, as observed by LeBrun–Simanca [36].

(ii) Most of the zero scalar curvature Kähler surfaces which have been proved by Kim, LeBrun, Pontecorvo, Rollin and Singer [27], [28], [35], [38] to admit such constant scalar curvature metric. In particular any blow-up of a non Ricci-flat Kähler surface whose integral of the scalar curvature is non-negative has blow-ups which admit zero scalar curvature Kähler metrics. Of course, if the number of blow-ups is sufficiently large, then no continuous families of automorphisms survive and we can apply Theorem 1.3.

(iii) Note that also flat tori of any dimension can be used as base manifolds, since, despite the presence of continuous automorphisms, there are no nonzero holomorphic vector fields vanishing somewhere. The Cheeger–Gromoll splitting theorem and the above remark imply that, on any Kähler Ricci-flat manifold, no nonzero holomorphic vector field vanishing somewhere exists (see [8, Corollary 6.67]). Therefore to any Kähler Ricci-flat manifold one can apply Theorem 1.1 (but not Corollary 1.2).

(iv) Some important classes of manifolds on which there are constant scalar curvature Kähler metrics have been provided by Fine [19]. Indeed, he has proved the existence of Kähler constant scalar curvature metrics on complex surfaces with a holomorphic submersion onto a Riemann surface Σ with smooth fibres of genus at least 2. If the genus of Σ is larger than or equal to 2, the automorphism group is indeed discrete.

(v) Another family of examples of constant scalar curvature Kähler manifolds with discrete automorphism group has been given by Hong [24], [25]. These are ruled manifolds given by the projectivization of some vector bundles over constant scalar curvature Kähler manifolds.

(vi) In [36], LeBrun and Simanca gave examples of (and strategies to construct new) Futaki-nondegenerate manifolds with constant scalar curvature Kähler metrics.

(vii) Recall that the space of holomorphic vector fields on a blown manifold is isomorphic to the space of those holomorphic vector fields on the base manifold vanishing at the blown-up points. Hence our procedure applied to any of the nondegenerate manifolds above gives new nondegenerate manifolds (with constant scalar curvature), so our procedure can be iterated.

(viii) Riemannian products of nondegenerate Kähler manifolds of constant scalar curvature are again nondegenerate Kähler manifolds of constant scalar curvature. By taking factors with scalar curvature of different signs and scaling, one can then produce Kähler metrics of any nonzero scalar curvature also on the blow-ups.

In addition to the above examples, one can apply LeBrun–Simanca's implicit function argument [36] to get an open subset of the Kähler cone of a fixed complex manifold and also an open subset of the moduli of complex structures for which constant scalar curvature Kähler metrics exist, providing a wealth of new examples.

8.2. Zero scalar curvature examples. Proof of Corollary 1.1

Let us now focus on the effect of our construction on the size of the scalar curvature when we blow up smooth points. Let us denote by π the standard projection from the blow-up manifold $\widetilde{M}$ to the base manifold M. To this end, let us recall that the average of the scalar curvature of a Kähler metric is a cohomological number given by

$$\mathbf{s}(\omega)=\frac{mc_1(M)\cup[\omega]^{m-1}([M])}{[\omega]^m([M])}.$$

Our gluing procedure constructs on $\widetilde{M}$ metrics in the Kähler classes

$$[\omega_\varepsilon]=\pi^*[\omega]-\varepsilon^2(a_1\mathrm{PD}[E_1]+\ldots+a_n\mathrm{PD}[E_n]),$$

while the first Chern class behaves like

$$c_1(\widetilde{M})=\pi^*(c_1(M))-(m-1)(\mathrm{PD}[E_1]+\ldots+\mathrm{PD}[E_n]).$$

Recalling (see [23, p. 475]) that for any $j=1,\ldots,n$,

$$(\mathrm{PD}[E_j])^m[\widetilde{M}]=(-1)^{m-1},$$

we get

$$(c_1(\widetilde{M})\cup[\omega_\varepsilon]^{m-1})([\widetilde{M}])=(c_1(M)\cup[\omega]^{m-1})([M])-\varepsilon^{2m-2}(m-1)\left(\sum_{j=1}^n a_j\right)$$

and

$$[\omega_\varepsilon]^m([\widetilde{M}])=[\omega]^m([M])+(-1)^{m-1}\varepsilon^{2m}\left(\sum_{j=1}^n a_j\right).$$

The scalar curvature of this metric is hence given by

$$\mathbf{s}(\omega_\varepsilon)=m\frac{(c_1(M)\cup[\omega]^{m-1})([M])-\varepsilon^{2m-2}(m-1)\left(\sum_{j=1}^n a_j\right)}{[\omega]^m([M])+(-1)^{m-1}\varepsilon^{2m}\left(\sum_{j=1}^n a_j\right)}.$$

It is easily seen that, since $a_j>0$, this gives a decreasing function of ε, for ε close to 0 (and of course it gives the old scalar curvature for $\varepsilon=0$).

The direct application of Theorem 1.1 would then give small negative scalar curvature if (M,ω) had zero scalar curvature. Nonetheless, changing the Kähler class we can bypass this problem provided the first Chern class of the base orbifold is nonzero, forcing the scalar curvature to vanish in the gluing procedure.

COROLLARY 8.1. *Any blow-up* (*at a finite set of smooth points*) *of a compact smooth Kähler manifold* (*or orbifold*) *of zero scalar curvature of discrete type with nonzero first Chern class, has a Kähler metric of zero constant scalar curvature.*

Proof. Let us denote by $\omega(0)$ the zero scalar curvature Kähler metric on the base manifold M, and by ϱ the harmonic representative of the first Chern class $c_1(M)$ (and hence nonzero by our assumption). LeBrun–Simanca have proved ([37, Corollary 1]) that, if the first Chern class is nonzero, then the automorphism group is discrete and for $|t|$ sufficiently small (say $t\in[-t_0,t_0]$) each Kähler class $[\omega(0)-t\varrho]$ contains a metric $\omega(t)$ of constant scalar curvature; this constant is positive for $t>0$ and negative for $t<0$. Moreover, $\omega(t)$ depends continuously on t.

We can apply Theorem 1.3 to the continuous family of Kähler forms $\omega(t)$. Given $t\in[-t_0,t_0]$, this yields the existence of $\varepsilon_0(t)>0$ and a family of Kähler metrics $\omega(t,\varepsilon)$ of constant scalar curvature for all $\varepsilon\in(0,\varepsilon_0(t))$. It turns out that the constant $\varepsilon_0(t)$ is uniformly bounded from below by some positive constant $\varepsilon_0>0$, since $\varepsilon_0(t)$ only depends on the $\mathcal{C}^{2,\alpha}$ norm of the coefficients of the Kähler form $\omega(t)$, and these are uniformly bounded as $t\in[-t_0,t_0]$. We claim that, reducing ε_0 if necessary, $\omega(t,\varepsilon)$ depends continuously on t. This easily follows from the fact that the Kähler forms on the blown-up manifold are obtained by solving nonlinear problems using a fixed point theorem for a contraction mapping. Therefore, they depend continuously on any of the parameters of our construction such as the Kähler class, the parameter ε, the points which are blown up, the coefficients $a_j>0$, etc.

Let us then look at the family of constant scalar curvature metrics $\omega(t,\varepsilon)$. We know that, reducing ε_0 if necessary, $\omega(-t_0,\varepsilon)$ has constant negative scalar curvature while $\omega(t_0,\varepsilon)$ has positive scalar curvature, for all $\varepsilon\in(0,\varepsilon_0)$. Moreover $\mathbf{s}(\omega(t,\varepsilon))$ depends continuously on t. Therefore, for each $\varepsilon\in(0,\varepsilon_0)$, there exists $t_\varepsilon\in[-t_0,t_0]$ such that $\mathbf{s}(\omega(t_\varepsilon,\varepsilon))=0$ as claimed. □

Note that the above corollary can be applied to most of the examples described in (ii) and (viii) in §8.1.

8.3. Desingularization of orbifolds

More delicate is the situation for singular orbifolds, since few examples even of Kähler–Einstein orbifolds are known. As mentioned in the introduction, the clearest picture is in complex dimension 2 and 3, where, thanks to the work of Kronheimer [33] and Joyce [26], we know how to handle $\mathrm{SU}(m)$ singular points. We summarize this in the following result.

COROLLARY 8.2. *Let* (M,ω) *be a nondegenerate compact* m*-dimensional constant scalar curvature Kähler orbifold with* $m=2$ *or* $m=3$ *and isolated singularities. Let* $p_1,...,p_n\in M$ *be any set of points with a neighborhood biholomorphic to a neighborhood of the origin in* $\mathbf{C}^m/\Gamma_j$, *where* Γ_j *is a finite subgroup of* $\mathrm{SU}(m)$. *Let further* N_j *be a Kähler crepant resolution of* $\mathbf{C}^m/\Gamma_j$ (*which always exists*; *see* [7] *for* $m=2$ *and* [50] *for* $m=3$), *and* η_j *given by Theorem* 7.1.

Then, there exists $\varepsilon_0>0$ *such that, for all* $\varepsilon\in(0,\varepsilon_0)$, *there exists a constant scalar curvature Kähler form* ω_ε *on* $M\sqcup_{p_1,\varepsilon}N_1\sqcup_{p_2,\varepsilon}...\sqcup_{p_n,\varepsilon}N_n$.

Moreover,

(i) *if* ω *has positive* (*resp. negative*) *scalar curvature, then* ω_ε *has positive* (*resp. negative*) *scalar curvature;*

(ii) *if* $c_1(M)\neq 0$ *and* ω_M *has zero scalar curvature, then* ω_ε *can be chosen to have zero scalar curvature too.*

The range of applicability of Corollary 8.2 is very large, even if we look just at Kähler–Einstein orbifolds of nonpositive scalar curvature, thanks to Aubin–Yau's solution of the Calabi conjecture (which holds in the orbifold category). In fact, we can use it to prove the following general result mentioned in the introduction.

COROLLARY 8.3. *Any compact complex surface of general type admits constant scalar curvature Kähler metrics.*

The proof of the above result requires some notions from algebraic geometry which can be found for example in [7] and which we quickly recall for the reader's convenience.

First of all, a complex surface M is called *minimal* if it does not contain a smooth rational curve of self-intersection -1. A fundamental result in complex surface theory (the Enriques–Castelnuovo criterion; see [23, p. 476]) says that any such curve is in fact the exceptional divisor of a blow-up at a smooth point of a smooth surface. Moreover, one can apply the above procedure ("blowing down") a finite number of times to be left with a minimal surface called a *minimal model* of M.

From a different perspective, one can study an algebraic surface by looking at its images into projective spaces, via maps given by evaluating holomorphic sections of line bundles as in the celebrated Kodaira's embedding theorem. In particular, if K_M is the canonical line bundle of M, one has (rational) maps $\phi_{K_M^{\otimes k}}$ from M into $\mathbf{P}(H^0(M, K_M^{\otimes k}))$. These in general may not be defined at points which annihilate all holomorphic sections of $K_M^{\otimes k}$, but for minimal surfaces of general type they are indeed globally defined holomorphic maps for $k \geqslant 5$ (see [7, p. 220]).

A complex surface M is said to be *of general type* if $\dim(\phi_{K_M^{\otimes k}}(M))=2$ for k large enough. It is not hard to see that all minimal models of a fixed surface of general type are isomorphic (see [7, Proposition 4.6]). If M is a minimal surface of general type, Kodaira [30] has proved that $\phi_{K_M^{\otimes k}}$ is an embedding away from smooth rational curves of self-intersection -2, and Brieskorn [9] has proved that the image of these curves are isolated singular points of the image surface with local structure groups Γ_j, with $\Gamma_j \subset \mathrm{SU}(2)$. We are now in position to give the proof of the above corollary.

Proof of Corollary 8.3. Let us first assume that M is a minimal complex surface of general type, and suppose k is chosen big enough to guarantee that the image of the pluricanonical rational map $\phi_{K_M^{\otimes k}}$ is an embedding away from the set of (-2)-curves of M, which get collapsed to points, giving the singularities of $\phi_{K_M^{\otimes k}}$.

Kobayashi [29] has proved that $\phi_{K_M^{\otimes k}}(M)$ has a Kähler–Einstein orbifold metric of negative scalar curvature, extending Aubin's proof of the Calabi conjecture. Moreover $c_1(M)<0$ implies, as in the smooth case, the existence of only a discrete group of automorphisms.

As already observed, being the structure groups of the singularities in SU(2), we have an ALE local model with the required decay at infinity. We can then apply Theorem 1.3. The complex manifold produced by our gluing construction is easily seen to be minimal, hence getting a constant negative scalar curvature Kähler metric on the minimal resolution Y of M. But M is already a minimal model of M, therefore the minimal model of M, and so M is in fact Y proving our result.

If M is not minimal, we apply the previous discussion to its minimal model Y, which is a complex surface with discrete automorphism group, to get a Kähler constant negative scalar curvature metric. Recalling that M is obtained from Y applying a finite number of blow-ups, Theorem 1.1 (possibly applied more than once in case one needs to blow up at a point on the exceptional divisor of the previous blow-up, and of course blowing up preserves the property of having only discrete automorphism groups) gives the conclusion. □

Going back to the problem of resolving singularities in the Kähler constant scalar curvature setting, in dimension greater than 3 only a few examples can be dealt with at the moment.

Other types of singularities which can be dealt with are, for example, those locally modeled on $\mathbf{C}^m/\mathbf{Z}_m$, where $\mathbf{Z}_m$ acts diagonally on $\mathbf{C}^m$, by multiplication by a fixed mth root of unity $\zeta = e^{2\pi i/m}$. Putting $r = (|z^1|^2 + ... + |z^m|^2)^{1/2}$, Calabi [11] defined a Kähler potential on $X \setminus \{\text{exceptional divisor}\}$ by

$$\varphi = (r^{2m}+1)^{1/m} + \frac{1}{m} \sum_{j=0}^{m-1} \zeta^j \log[(r^{2m}+1)^{1/m} - \zeta^j].$$

We can then observe that $\eta = \frac{1}{2} i\partial\bar{\partial}\varphi$ is indeed a Kähler form which extends through the exceptional divisor, and is ALE, Ricci-flat and asymptotic to $\mathbf{C}^m/\mathbf{Z}_m$. We can then glue (X, η) to any smooth Kähler orbifold (M, ω) of constant scalar curvature, provided the Futaki obstructions vanish as described in §4.

The above example has been recently generalized by Rollin–Singer [52]. They have shown that if $G = \{1, \lambda, ..., \lambda^{k-1}\}$, $\lambda = e^{2\pi i/k}$, then $\mathbf{C}^m/G$ has an ALE scalar-flat (in general not Ricci-flat) Kähler resolution whose metric decays at infinity of order $2-2m$.

These last examples can be used to produce compact orbifolds by taking global quotients of some of the smooth manifolds described in the first section (for example tori or Kähler–Einstein manifolds with negative first Chern class or with positive first Chern class and discrete automorphism group containing a group as above).

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