BLOWING-UP COORDINATES FOR A SIMILARITY BOUNDARY LAYER EQUATION

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Abstract

We introduce blowing-up coordinates to study the autonomous third order nonlinear differential equation : $f''' + \frac{\mathbf{m}+1}{2}ff'' - \mathbf{m}f'^2 = 0$ on $(0,\infty)$, subject to the boundary conditions $f(0) = a \in \mathbb{R}$, $f'(0) = 1$ and $f'(t) \to 0$ as $t \to \infty$. This problem arises when looking for similarity solutions to problems of boundary-layer theory in some contexts of fluids mechanics, as free convection in porous medium or flow adjacent to a stretching wall. We study the corresponding plane dynamical systems and apply the results obtained to the original boundary value problem, in order to solve questions for which direct approach fails.

1 Introduction.

We consider the autonomous third order nonlinear differential equation

$$f''' + \frac{\mathbf{m}+1}{2}ff'' - \mathbf{m}f'^2 = 0 \quad \text{on} \quad (0,\infty), \tag{1.1}$$

subject to the boundary conditions

$$f(0) = a, \tag{1.2}$$

$$f'(0) = 1, \tag{1.3}$$

$$f'(\infty) := \lim_{t\to\infty} f'(t) = 0. \tag{1.4}$$

The parameters $\mathbf{m}$ and a will be assumed to describe $\mathbb{R}$, and we are concerned by existence and uniqueness questions for the solutions of the problem (1.1)-(1.4). In the case $\mathbf{m} = 0$, equation (1.1) reduces to the so-called Blasius equation, and has been widely studied (see [6], [12], [17],

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[19], [22] and [23]). This boundary value problem arises when looking for similarity solutions in physically different contexts in fluids mechanics, as free convection about a vertical flat surface embedded in a fluid-saturated porous medium (see [10], [11], [14], [18]), or boundary-layer flow adjacent to a stretching wall (see [3], [4], [13], [16], [20]). The parameter $\mathbf{m}$ is related to some conditions given on the wall, while a correponds, for example for the stretching wall, to an impermeable wall when $a = 0$, to a permeable wall when $a \neq 0$, say suction ($a > 0$) or injection ($a < 0$) of the fluid. In these physical papers the problem (1.1)-(1.4) is essentially studied from numerical point of view, or by using formal expansions, and only some elementary results are proved. Further mathematical analysis is done in [5], [7], [15] and [9], and partial results concerning existence of one or several solutions are given. The approach consists in shooting methods and more precisely in finding values of $f''(0)$ in order to get existence of f on the whole half line $[0, \infty)$ and such that (1.4) holds. This direct approach allows to consider any value of a and solutions vanishing. Nevertheless, limitations appear and the method seems to fail in some cases (see [9]).

Noticing that for $\kappa > 0$ the function $t \longmapsto \kappa f(\kappa t)$ is a solution of (1.1) when f is, we can introduce the following blow-up coordinates: $u = \frac{f'}{f^2}$ and $v = \frac{f''}{f^3}$. More precisely, let us consider a right maximal interval $I = [\tau, \tau + T)$ on which a solution f of (1.1) does not vanish, and set

$$\forall t \in I, \qquad s = \int_\tau^t f(\xi) d\xi, \qquad u(s) = \frac{f'(t)}{f(t)^2} \qquad \text{and} \qquad v(s) = \frac{f''(t)}{f(t)^3}. \tag{1.5}$$

Then, we easily get

$$\begin{cases} \dot{u} = P(u, v) := v - 2u^2, \\ \dot{v} = Q_{\mathbf{m}}(u, v) := -\frac{\mathbf{m}+1}{2} v + \mathbf{m} u^2 - 3uv, \end{cases} \tag{1.6}$$

where the dot is for differentiating with respect to the variable s.

Our goal now is to propose proofs using the blowing-up coordinates u and v when direct approach fails.

2 The plane dynamical system (1.6).

In this section, we would like to give some results about the plane autonomous system (1.6) in order to come back to the boundary value problem (1.1)-(1.4) and pursue the study done in [7] and [9]. In this spirit we do not necessarily give complete results on the plane system, but only what we need for application to (1.1)-(1.4).

The singular points of the system (1.6) are $O = (0, 0)$ and $A = (-\frac{1}{6}, \frac{1}{18})$. The isoclinic curves $P(u, v) = 0$ and $Q_{\mathbf{m}}(u, v) = 0$ are the parabola $v = 2u^2$ and $v = \psi_{\mathbf{m}}(u)$ where $\psi_{\mathbf{m}}$ is the rational function

$$\psi_{\mathbf{m}}(u) = \frac{\mathbf{m} u^2}{3u + \frac{\mathbf{m}+1}{2}}.$$

The jacobian matrix of (1.6) to the point A is given by

$$J_A = \begin{pmatrix} \frac{2}{3} & 1 \\ -\frac{2\mathbf{m}+1}{6} & -\frac{\mathbf{m}}{2} \end{pmatrix}.$$

The eigenvalues of J_A are

$$\lambda_1 = \frac{4 - 3\mathbf{m} - \sqrt{9\mathbf{m}^2 - 24\mathbf{m} - 8}}{12}, \qquad \lambda_2 = \frac{4 - 3\mathbf{m} + \sqrt{9\mathbf{m}^2 - 24\mathbf{m} - 8}}{12},$$

if $\mathbf{m} \leq \frac{1}{3}(4 - 2\sqrt{2})$ or $\mathbf{m} \geq \frac{1}{3}(4 + 2\sqrt{2})$ and

$$\lambda_1 = \frac{4 - 3\mathbf{m} - i\sqrt{8 + 24\mathbf{m} - 9\mathbf{m}^2}}{12}, \qquad \lambda_2 = \frac{4 - 3\mathbf{m} + i\sqrt{8 + 24\mathbf{m} - 9\mathbf{m}^2}}{12},$$

if $\frac{1}{3}(4 - 2\sqrt{2}) < \mathbf{m} < \frac{1}{3}(4 + 2\sqrt{2})$. Therefore we have that

- A is an unstable node if $\mathbf{m} \leq \frac{4-2\sqrt{2}}{3}$,
- A is an unstable focus if $\frac{4-2\sqrt{2}}{3} < \mathbf{m} < \frac{4}{3}$,
- A is a stable focus if $\frac{4}{3} < \mathbf{m} < \frac{4+2\sqrt{2}}{3}$,
- A is a stable node if $\mathbf{m} \geq \frac{4+2\sqrt{2}}{3}$.

For the singular point O, the jacobian matrix is

$$J_O = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{\mathbf{m}+1}{2} \end{pmatrix},$$

of which the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -\frac{\mathbf{m}+1}{2}$. The corresponding invariant subspaces are $L_0 = \mathbb{R}(1,0)$ and $L = \mathbb{R}(1, -\frac{\mathbf{m}+1}{2})$ respectively. By looking at the vector field in the neighbourhood of O, we see that for $\mathbf{m} \neq -1$, the singular point O is a saddle-node of multiplicity 2. It has a center manifold $\mathcal{W}_0$ tangent to the subspace L_0, and a stable (resp. unstable) manifold $\mathcal{W}$ if $\mathbf{m} > -1$ (resp. $\mathbf{m} < -1$), tangent to the subspace L, (see [1] and [2]).

Concerning $\mathcal{W}$ we have the following result:

Proposition 2.1 *In the neighbourhood of O, the manifold $\mathcal{W}$ takes place below L when $\mathbf{m} < -1$ or $\mathbf{m} > -\frac{1}{3}$ and above L when $-1 < \mathbf{m} < -\frac{1}{3}$.*

Proof. Since $\mathcal{W}$ is at least of class C^2 in a neighbourhood of O and is tangent to L, we can defined it in this neighbourhood by $v = \mathrm{v}_{\mathbf{m}}(u)$, where $\mathrm{v}_{\mathbf{m}}$ is a solution of the equation

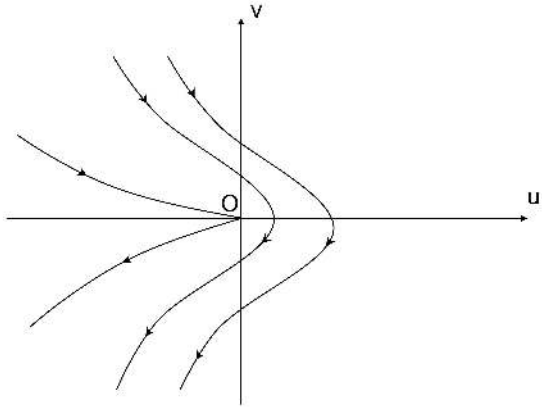
$$(v - 2u^2)v' = -\frac{\mathbf{m}+1}{2}v + \mathbf{m}u^2 - 3uv. \tag{2.1}$$

Writing $\mathrm{v}_{\mathbf{m}}(u) = -\frac{\mathbf{m}+1}{2}u + \beta u^2 + o(u^2)$ and using (2.1) we easily get $\beta = -\frac{3\mathbf{m}+1}{2(\mathbf{m}+1)}$ and the result. ■

Remark 2.1 *For $\mathbf{m} = -\frac{1}{3}$ the manifold $\mathcal{W}$ is given by*

$$\mathcal{W} = \left\{ \left(u, -\frac{u}{3}\right) \in \mathbb{R}^2 \; ; \; u > -\frac{1}{6} \right\}.$$

On the other hand, we will not consider the case $\mathbf{m} = -1$ because we know from [9] that the boundary value problem (1.1)-(1.4) has no solution. See also Part 3.1 below. Nevertheless, in this case, the center manifold is of dimension 2, and the phase portrait of the vector field in the neighbourhood of O has the form given in the figure 2.1.



($\mathbf{m} = -1$)
Fig 2.1

For the center manifold $\mathcal{W}_0$ we have:

Proposition 2.2 *In the neighbourhood of O, the center manifold $\mathcal{W}_0$ takes place above L_0 when $\mathbf{m} < -1$ or $\mathbf{m} > 0$, and below L_0 when $-1 < \mathbf{m} < 0$.*

Proof. Here again we use regularity of $\mathcal{W}_0$ in the neighbourhood of O, and as in Proposition 2.1, we define $\mathcal{W}_0$ by $v = \mathrm{v}_{\mathbf{m}}(u)$ for $|u|$ small enough, and we easily obtain

$$v \sim \frac{2\mathbf{m}}{\mathbf{m}+1}u^2 \qquad \text{as} \;\; u \to 0. \tag{2.2}$$

This completes the proof. ■

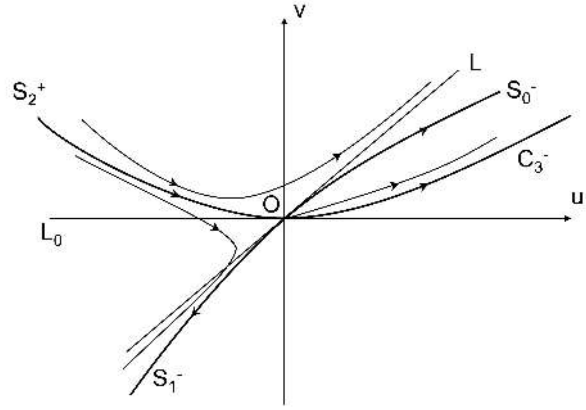
Remark 2.2 *For $\mathbf{m} = 0$ the center manifold $\mathcal{W}_0$ coincides with the u-axis.*

Let us now precise the phase portrait of the vector field in the neighbourhood of the saddle-node O. We will assume that the parabolic sector is delimited by the separatrices S_0, S_1 which are tangent to L, and the hyperbolic sectors are delimited, one by S_0 and the separatrix S_2, which is tangent to L_0, and the other by S_1 and S_2. The manifold $\mathcal{W}$ is the union of the separatrices S_0, S_1 and the singular point O, and the manifold $\mathcal{W}_0$ is the union of the separatrix S_2, the singular point O and a phase curve C_3.

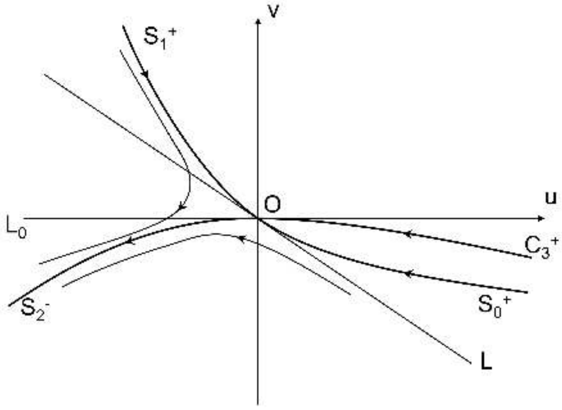
We will also write S_i^+ when the separatrix S_i is an ω-separatrix, and S_i^- when it is an α-separatrix. Taking into account the previous Propositions, we easily get the behaviors described in the figure 2.2.

In order to study the global behavior of the separatrices, let us introduce the following notations. Consider any connected piece of a phase curve C of the plane dynamical system (1.6) lying in the region $P(u,v) < 0$ (resp. $P(u,v) > 0$); then C can be characterized by $v = V_{\mathbf{m}}(u)$ (resp. $v = W_{\mathbf{m}}(u)$) with u belonging to some interval, and where $V_{\mathbf{m}}$ (resp. $W_{\mathbf{m}}$) is a solution of the differential equation

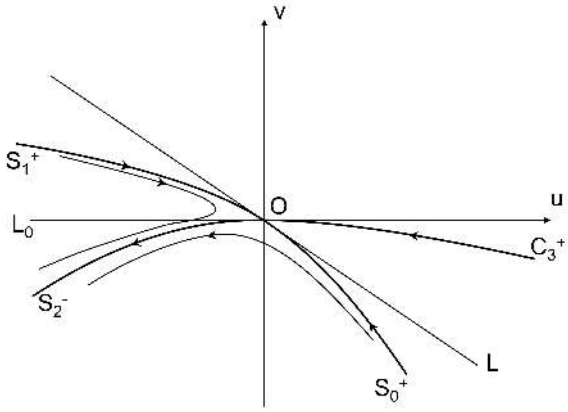
$$v' = F_{\mathbf{m}}(u,v) := \frac{Q_{\mathbf{m}}(u,v)}{P(u,v)} = \frac{-\frac{\mathbf{m}+1}{2}v + \mathbf{m}u^2 - 3uv}{v - 2u^2}. \tag{2.3}$$



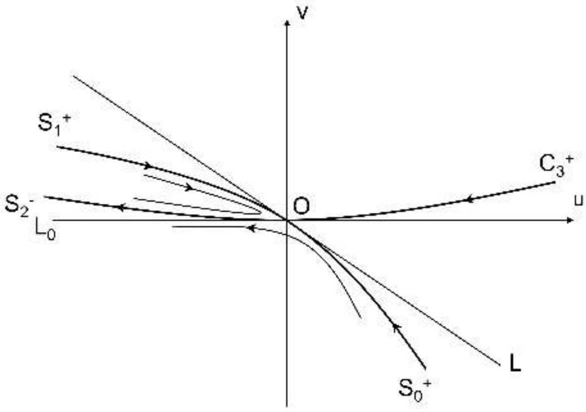
$\mathbf{m} < -1$



$-1 < \mathbf{m} < -1/3$



$-1/3 < \mathbf{m} < 0$



$\mathbf{m} > 0$

Fig 2.2

3 The boundary value problem (1.1)-(1.4).

To come back to the original problem, most of the time, we will consider the initial value problem

$$(\mathcal{P}_{\mathbf{m},a,\mu}) \qquad \begin{cases} f''' + \frac{\mathbf{m}+1}{2} ff'' - \mathbf{m}f'^2 = 0, \\ f(0) = a, \\ f'(0) = 1, \\ f''(0) = \mu, \end{cases}$$

with $a \neq 0$ and look at the trajectory $C_{a,\mu}$ of the plane dynamical system (1.6) defined by (1.5) for some τ. For the particular choice $\tau = 0$ we have

$$u(0) = \frac{1}{a^2} \qquad \text{and} \qquad v(0) = \frac{\mu}{a^3}.$$

It is clear that if $C_{a,\mu}$ is a semi-trajectory, then necessarily $T = \infty$ and f does not vanish on $[\tau, \infty)$. Conversly, if the solution f of $(\mathcal{P}_{\mathbf{m},a,\mu})$ is defined on $[0, \infty)$ and does not vanish on $[\tau, \infty)$, then $C_{a,\mu}$ is not necessarily a semi-trajectory, since the integral of f on $[\tau, \infty)$ may converge.

Let us now recall the following useful properties of solution of boundary value problem (1.1)-(1.4):

Proposition 3.1 *Let f be a solution of* (1.1)-(1.4)*; we have*
(i) *If $\mathbf{m} \le 0$, then f is strictly increasing on $[0, \infty)$, and moreover*
• *if $f''(0) \le 0$, then f is strictly concave on $[0, \infty)$* (concave solution),
• *if $f''(0) > 0$, there exists $t_0 \in (0, \infty)$ such that f is strictly convex on $[0, t_0]$ and strictly concave on $[t_0, \infty)$* (convex-concave solution).
On the other hand, if $\mathbf{m} > -1$ and $a < 0$, then f becomes positive for large t.
(ii) *If $\mathbf{m} \ge 0$, then f is bounded, $f''(0) < 0$ and moreover*
• *either f is strictly increasing and strictly concave on $[0, \infty)$* (concave solution),
• *or there exists $t_0 \in (0, \infty)$ such that f is strictly concave on $[0, t_0]$ and f is positive, strictly decreasing and strictly convex on $[t_0, \infty)$* (concave-convex solution).
(iii) *For all $\mathbf{m} \in \mathbb{R}$ one has $f''(t) \to 0$ as $t \to \infty$.*
(iv) *For all $\mathbf{m} \in \mathbb{R}$ and if f is bounded then*

$$\forall t \ge 0, \qquad f''(t) + \frac{\mathbf{m}+1}{2} f'(t) f(t) = -\frac{3\mathbf{m}+1}{2} \int_t^\infty f'(\xi)^2 \mathrm{d}\xi. \tag{3.1}$$

Proof. See [9]. ■

3.1 The case $\mathbf{m} \le -1$.

It is indicated in the appendix of [21] that one find in [24] a simple proof that problem (1.1)-(1.4) with $a = 0$ has no solutions for $\mathbf{m} \le -1$; but it is not so clear to find this result in [24]. Partial generalization can be found in [9]. In the first lemma we come back to these results and give a complementary property in terms of the blowing-up coordinates.

Lemma 3.1 *Let $\mathbf{m} \le -1$. If $a \ge -\frac{2}{\sqrt{-\mathbf{m}-1}}$, the problem* (1.1)-(1.4) *has no solution. Moreover, if $a < -\frac{2}{\sqrt{-\mathbf{m}-1}}$ and if f is a solution of* (1.1)-(1.4)*, then necessarily $f < 0$ and the curve $s \longmapsto (u(s), v(s))$ defined by* (1.5) *with $\tau = 0$, is a negative semi-trajectory which lies for $-s$ large enough in the bounded domain*

$$\mathcal{D}_+ := \left\{ (u, v) \in \mathbb{R}^2 \ ; \ 0 < u < -\frac{\mathbf{m}+1}{4} \quad \text{and} \quad 0 \le v < -\frac{\mathbf{m}+1}{2} u \right\}. \tag{3.2}$$

Proof. Let f be a solution of (1.1)-(1.4). Using Proposition 3.1, we see that f is increasing and there exists $t_0 \ge 0$ such that $f''(t) < 0$ for $t > t_0$. On the other hand, because of $f' > 0$, we see that if $f(t_1) \ge 0$ for some point t_1, we get $f'''(t) < 0$ for $t > \max(t_0, t_1)$, and a contradiction with (iii) of Proposition 3.1 and the negativity of $f''(t)$ for large t. Consequently, $f < 0$ and necessarily $a < 0$. Since $f' > 0$ and $f''(t) \le 0$ for $t \ge t_0$, we get

$$\forall t \ge 0, \qquad \frac{f'(t)}{f(t)^2} > 0 \quad \text{and} \quad \forall t \ge t_0, \qquad \frac{f''(t)}{f(t)^3} \ge 0. \tag{3.3}$$

On the other hand, f is bounded and from (3.1) we obtain

$$\forall t \geq 0, \qquad f''(t) + \frac{\mathbf{m}+1}{2} f'(t)f(t) > 0. \tag{3.4}$$

Denoting by λ the limit of f at infinity and integrating (3.4) we get

$$-f'(t) + \frac{\mathbf{m}+1}{4}(\lambda^2 - f(t)^2) > 0 \quad \text{and} \quad f'(t) + \frac{\mathbf{m}+1}{4} f(t)^2 < \frac{\mathbf{m}+1}{4}\lambda^2 < 0. \tag{3.5}$$

For $t = 0$ this implies $a < -\frac{2}{\sqrt{-\mathbf{m}-1}}$. Finally, dividing the second inequality of (3.5) by $f(t)^2$ and (3.4) by $f(t)^3$, we obtain

$$\forall t \geq 0, \qquad \frac{f'(t)}{f(t)^2} + \frac{\mathbf{m}+1}{4} < 0 \quad \text{and} \quad \frac{f''(t)}{f(t)^3} + \frac{\mathbf{m}+1}{2}\frac{f'(t)}{f(t)^2} < 0. \tag{3.6}$$

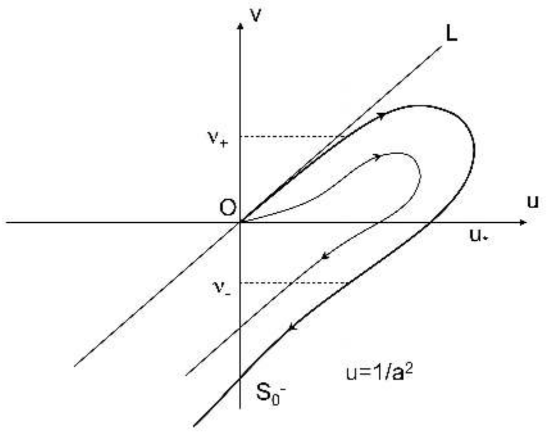
From the first inequality of (3.6) we easily deduce

$$\forall t \geq 0, \qquad f(t) \leq \frac{1}{\frac{\mathbf{m}+1}{4}t + \frac{1}{a}}$$

which implies

$$\int_0^\infty f(\xi)d\xi = -\infty.$$

Consequently, the trajectory $s \longmapsto (u(s), v(s))$ is defined on the whole interval $(-\infty, 0]$ and this together with (3.3) and (3.6) complete the proof. ■



($\mathbf{m} < -1$)
Fig 3.1.1

Lemma 3.2 *Let $\mathbf{m} < -1$. As s grows, the α-separatrix S_0^- leaves to the right the singular point O tangentially to L, and intersects successively the isoclines $Q_{\mathbf{m}}(u,v) = 0$, $P(u,v) = 0$, the u-axis and the v-axis.* (See figure 3.1.1).

Proof. From part 2, we know that close to the singular point O the separatrix S_0^- is below the straight line L and above the isoclines $Q_{\mathbf{m}}(u,v)=0$ and $P(u,v)=0$. But in the bounded area $\{2u^2 < v < -\frac{\mathbf{m}+1}{2}u\} \cap \{u>0\}$ we can define S_0^- by $v = W_{\mathbf{m}}(u)$ where $W_{\mathbf{m}}$ is a solution of (2.3). Since we have

$$F_{\mathbf{m}}(u,v) = -\frac{\mathbf{m}+1}{2} - \frac{u(3v+u)}{v-2u^2} \tag{3.7}$$

we see that $0 < W'_{\mathbf{m}}(u) < -\frac{\mathbf{m}+1}{2}$ as long as $W_{\mathbf{m}}(u) > \psi_{\mathbf{m}}(u)$, that $W'_{\mathbf{m}}$ vanishes and becomes negative. It follows that S_0^- crosses successively the isoclines $Q_{\mathbf{m}}(u,v)=0$ and $P(u,v)=0$. After that, we have $Q_{\mathbf{m}}(u,v)<0$ and $P(u,v)<0$, and if we then define S_0^- by $v=V_{\mathbf{m}}(u)$, we have

$$V'_{\mathbf{m}}(u) > -\frac{\mathbf{m}+1}{2} > 0 \quad \text{as long as} \quad V_{\mathbf{m}}(u) > -\frac{u}{3}.$$

Consequently, S_0^- intersects the u-axis and the straight line $v=-\frac{u}{3}$, and as soon as $V_{\mathbf{m}}(u) < -\frac{u}{3}$ we have

$$0 < V'_{\mathbf{m}}(u) < -\frac{\mathbf{m}+1}{2} \quad \text{as long as} \quad u>0.$$

It implies that S_0^- crosses the v-axis. See (3.7). This completes the proof. ■

Theorem 3.1 *Let* $\mathbf{m} < -1$. *There exists* $a_* < 0$ *such that the problem* (1.1)-(1.4) *has infinitely many solutions if* $a < a_*$, *one and only one solution if* $a = a_*$, *and no solution if* $a > a_*$. *Moreover, if* f *is a solution to* (1.1)-(1.4), *then* $f<0$.

Proof. First of all, if $a \geq 0$ we know by Lemma 3.1 that (1.1)-(1.4) has no solution. So, consider for $a<0$ and $\mu \in \mathbb{R}$ the initial value problem $(\mathcal{P}_{\mathbf{m},a,\mu})$, denote by f its solution and look at the corresponding trajectory $C_{a,\mu}$ of the plane system (1.6) defined by (1.5) with $\tau=0$. Let $(u_*, 2u_*^2)$ be the point where the separatrix S_0^- intersects the isocline $v=2u^2$ (see Lemma 3.2), and set $a_* = -\frac{1}{\sqrt{u_*}}$.

If $a > a_*$ then the straight line $u=\frac{1}{a^2}$ does not intersect the separatrix S_0^-, and for all $\mu \in \mathbb{R}$ the α-limit set of the trajectory $C_{a,\mu}$ cannot be O, in such way that we deduce from the Poincaré-Bendixson Theorem, that $C_{a,\mu}$ does not remain in the bounded domain $\mathcal{D}_+$. It follows from Lemma 3.1 that f cannot be a solution of (1.1)-(1.4) for any $\mu \in \mathbb{R}$.

Suppose now $a=a_*$. For $\mu \neq 2u_*^2a^3$, the previous arguments show that f is not a solution of (1.1)-(1.4), and for $\mu = 2u_*^2a^3$ the phase curve $C_{a,\mu}$ is a negative semi-trajectory which coincide with a part of the separatrix S_0^-. It follows that f exists and is negative on $[0,\infty)$, and moreover that $f'>0$ and $f''<0$. This implies that $f'(t) \to l \geq 0$ as $t \to \infty$ and if we suppose $l>0$ we get a contradiction with the fact that f is negative. Therefore f is a solution of (1.1)-(1.4).

Finally, suppose that $a < a_*$. Then the straight line $u=\frac{1}{a^2}$ intersects the separatrix S_0^- through two points $\left(\frac{1}{a^2}, \nu_-\right)$ and $\left(\frac{1}{a^2}, \nu_+\right)$. Using again the arguments above, we obtain that if $\mu \in [a^3\nu_-, a^3\nu_+]$, then f is a solution of (1.1)-(1.4) and if $\mu \notin [a^3\nu_-, a^3\nu_+]$, then f is not. (See figure 3.1.1).

This completes the proof of the theorem. ■

Remark 3.1 *From Lemma 3.1 we know that* $a_* < -\frac{2}{\sqrt{-\mathbf{m}-1}}$. *But this inequality is certainly not sharp. For example, if* $\mathbf{m}=-3$, *we have* $-\frac{2}{\sqrt{-\mathbf{m}-1}} = -\sqrt{2}$, *while numerically it seems that* $a_* < -2.5$.

Remark 3.2 *Let* $\mathbf{m} < -1$*,* $a \le a_*$ *and* f *be a solution of* (1.1)-(1.4) *corresponding, in the phase plane* (u, v)*, to the separatrix* S_0^-*. Then*

$$\lambda := \lim_{t\to\infty} f(t) < 0.$$

Indeed, let us assume that $\lambda = 0$*. We have*

$$(u(s), v(s)) \to (0,0) \qquad \text{and} \qquad \frac{v(s)}{u(s)} \to -\frac{\mathbf{m}+1}{2} \qquad \text{as} \qquad s \to -\infty.$$

This implies that

$$\frac{f''(t)}{f(t)f'(t)} \to -\frac{\mathbf{m}+1}{2} \qquad \text{as} \qquad t \to \infty,$$

and thus there exists t_0 *such that for* $t \ge t_0$ *we have*

$$-f''(t) \ge \frac{\mathbf{m}+1}{4} f(t)f'(t).$$

Integrating between $t \ge t_0$ *and* ∞ *we get*

$$\frac{f'(t)}{f(t)^2} \ge -\frac{\mathbf{m}+1}{8},$$

and a contradiction with the fact that $u(s) \to 0$ *as* $s \to -\infty$*.*

Consider now, when $a < a_*$*, a solution* f *of* (1.1)-(1.4)*, corresponding to a phase curve which is not the separatrix. Then* $f(t) \to 0$ *as* $t \to \infty$*. On the contrary suppose that* $f(t) \to \lambda < 0$ *as* $t \to \infty$*. Since the phase curve we have considered tends to the singular point* O *tangentially to the* u*-axis, we have*

$$\frac{f''(t)}{f(t)f'(t)} \to 0 \qquad \text{as} \qquad t \to \infty. \tag{3.8}$$

But, from (3.1) *and for* t *large enough we have*

$$\begin{aligned} \frac{f''(t)}{f(t)f'(t)} + \frac{\mathbf{m}+1}{2} &= -\frac{3\mathbf{m}+1}{2f(t)} \int_t^\infty \frac{f'(\xi)^2}{f'(t)} \mathrm{d}\xi \\ &\ge -\frac{3\mathbf{m}+1}{2f(t)} \int_t^\infty f'(\xi)\mathrm{d}\xi = -\frac{3\mathbf{m}+1}{2} \left(\frac{\lambda - f(t)}{f(t)} \right). \end{aligned}$$

Letting $t \to \infty$ *and using* (3.8) *we get* $\frac{\mathbf{m}+1}{2} \ge 0$ *and a contradiction.*

3.2 The case $-1 < \mathbf{m} < -\frac{1}{3}$.

In this case, the value $\mathbf{m} = -\frac{1}{2}$ plays a central role. In [10] and [20], numerical investigations allow the authors to conjecture existence results. In [9], one find mathematical nonexistence proof for $-1 < \mathbf{m} \le -\frac{1}{2}$ and $a \le 0$, and partial existence result for $-\frac{1}{2} \le \mathbf{m} < -\frac{1}{3}$ and $a > 0$. Here we complete this study in the case $a > 0$ for all $\mathbf{m} \in (-1, -\frac{1}{3})$.

First we precise the behavior of the ω-separatrix S_0^+.

Lemma 3.3 *If* $\mathbf{m} = -\frac{1}{2}$*, the separatrix* S_0^+ *is defined by*

$$S_0^+ = \left\{ (u,v) \in \mathbb{R}^2 \ ; \ v = V_{-\frac{1}{2}}(u) := \frac{u^2}{2} - \frac{u}{4}, \ u > 0 \right\}.$$

Proof. We immediately verify that the curve $v = V_{-\frac{1}{2}}(u)$ $(u > 0)$ is a phase curve, and coincide with the separatrix S_0^+ since it is tangent to L at O. ■

Lemma 3.4 *If* $-\frac{1}{2} < \mathbf{m} < -\frac{1}{3}$*, the separatrix* S_0^+ *is defined by*

$$S_0^+ = \left\{ (u,v) \in \mathbb{R}^2 \ ; \ v = V_{\mathbf{m}}(u), \ u > 0 \right\},$$

where $V_{\mathbf{m}} \le V_{-\frac{1}{2}}$ *and* $V_{\mathbf{m}}(u) \to \infty$ *as* $u \to \infty$*.*

Proof. We know from Proposition 2.1 that in the neighbourhood of O, the separatrix S_0^+ lies in the region $\{-\frac{\mathbf{m}+1}{2}u < v < 2u^2\} \cap \{u > 0\}$ and as long as S_0^+ stays below the isocline $v = 2u^2$ we can define it by $v = V_{\mathbf{m}}(u)$ where $V_{\mathbf{m}}$ is a solution of (2.3). On the other hand, we have

$$V_{\mathbf{m}}(u) - V_{-\frac{1}{2}}(u) = -\frac{2\mathbf{m}+1}{4} + o(u) \quad \text{as} \quad u \to 0^+, \tag{3.9}$$

from which we get $V_{\mathbf{m}}(u_0) \le V_{-\frac{1}{2}}(u_0)$ for u_0 close to 0^+ and since

$$F_{\mathbf{m}}(u,v) - F_{-\frac{1}{2}}(u,v) = -\frac{2\mathbf{m}+1}{4} \le 0, \tag{3.10}$$

we deduce from classical differential inequalities (see [17] or [23]) that $V_{\mathbf{m}}$ is defined on the whole interval $(0,\infty)$ and that $V_{\mathbf{m}} \le V_{-\frac{1}{2}}$.

To see that $V_{\mathbf{m}}(u) \to \infty$ as $u \to \infty$, it is sufficient to look at the values of the vector field in the region $\{-\frac{\mathbf{m}+1}{2} < v < 2u^2\} \cap \{u > 0\}$, and remark that when u is growing, then the phase curve $v = V_{\mathbf{m}}(u)$ intersects the isocline $Q_{\mathbf{m}}(u,v) = 0$, the u-axis, and next $V_{\mathbf{m}}$ increases to ∞, since in the region $\{0 < v < 2u^2\} \cap \{u > 0\}$ we have

$$F_{\mathbf{m}}(u,v) - \left(-\frac{\mathbf{m}}{2}\right) = \frac{1}{v - 2u^2}\left(-\frac{v}{2} - 3uv\right) > 0 \tag{3.11}$$

which implies that $V_{\mathbf{m}}'(u) > -\frac{\mathbf{m}}{2}$ for u large enough. ■

Lemma 3.5 *If* $-1 < \mathbf{m} < -\frac{1}{2}$*, the separatrix* S_0^+ *crosses the isocline* $v = 2u^2$ *through a point* $(u_*, 2u_*^2)$ *and next intersects the* v*-axis. Moreover, as long as it stays below the isocline,* S_0^+ *is defined by* $v = V_{\mathbf{m}}(u)$ *for* $0 < u < u_*$ *where* $V_{\mathbf{m}} \ge V_{-\frac{1}{2}}$*, and as soon as it has intersected the isocline, is defined by* $v = W_{\mathbf{m}}(u)$*, with* $W_{\mathbf{m}}' < 0$ *for* $0 < u < u_*$*.* (See figure 3.2.1).

Proof. First we remark that, using (3.9) and (3.10), we get $V_{\mathbf{m}}(u) \ge V_{-\frac{1}{2}}(u)$ for $u \in (0, u_*)$ with either $u_* = \infty$ if S_0^+ stays below the isocline, or $u_* < \infty$ if S_0^+ crosses the isocline through the point $(u_*, 2u_*^2)$. We have to prove that u_* is finite. Suppose on the contrary that $u_* = \infty$. Therefore we have

$$\forall u > 0, \qquad \frac{u^2}{2} - \frac{u}{4} \le V_{\mathbf{m}}(u) < 2u^2. \tag{3.12}$$

Taking into account (3.12), easy calculations give, for $\tilde{V}_{\mathbf{m}}(u) = u^{-2}V_{\mathbf{m}}(u)$ and $u > \frac{1}{2}$

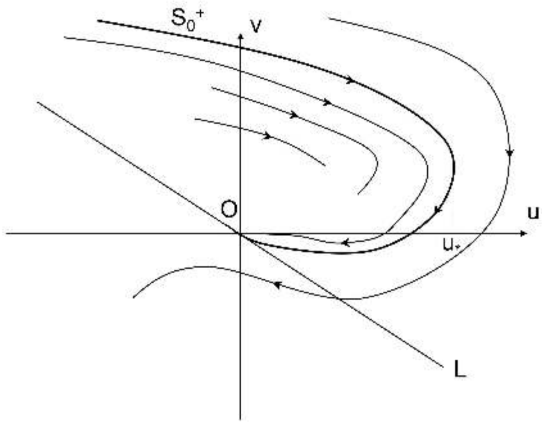
$$\begin{aligned}\tilde{V}'_{\mathbf{m}}(u) &= \frac{1}{u^3(V_{\mathbf{m}}(u) - 2u^2)}\left(-\frac{\mathbf{m}+1}{2}uV_{\mathbf{m}}(u) + \mathbf{m}u^3 - 2V_{\mathbf{m}}(u)^2 + u^2V_{\mathbf{m}}(u)\right)\\ &= \frac{1}{u^3(V_{\mathbf{m}}(u) - 2u^2)}\left(-\frac{\mathbf{m}}{2}uV_{\mathbf{m}}(u) + \mathbf{m}u^3 - 2V_{\mathbf{m}}(u)\left(V_{\mathbf{m}}(u) - \frac{u^2}{2} + \frac{u}{4}\right)\right)\\ &\geq \frac{1}{u^3(V_{\mathbf{m}}(u) - 2u^2)}\left(-\frac{\mathbf{m}}{2}uV_{\mathbf{m}}(u) + \mathbf{m}u^3\right) = -\frac{\mathbf{m}}{2u^2} > 0.\end{aligned}$$

Since (3.12) can be rewritten as

$$\forall u > 0, \qquad \frac{1}{2} - \frac{1}{4u} \leq \tilde{V}_{\mathbf{m}}(u) < 2$$

we get $\tilde{V}_{\mathbf{m}}(u) \to \mu$ as $u \to \infty$ with some $\mu \in \left[\frac{1}{2}, 2\right]$ and moreover we have $\tilde{V}_{\mathbf{m}}(u) \leq \mu$ for $u > \frac{1}{2}$. In other words, we have

$$V_{\mathbf{m}}(u) \sim \mu u^2 \quad \text{as} \quad u \to \infty \qquad \text{and} \qquad V_{\mathbf{m}}(u) \leq \mu u^2 \quad \text{for} \quad u > \frac{1}{2}. \tag{3.13}$$



$(-1 < \mathbf{m} < -1/2)$
Fig 3.2.1

To calculate the value of μ, we remark that (3.13) gives

$$V'_{\mathbf{m}}(u) = \frac{-\frac{\mathbf{m}+1}{2}V_{\mathbf{m}}(u) + \mathbf{m}u^2 - 3uV_{\mathbf{m}}(u)}{V_{\mathbf{m}}(u) - 2u^2} \sim \frac{3\mu}{2-\mu}u \quad \text{as} \quad u \to \infty$$

and thus by integrating and coming back to (3.13) we easily get $\mu = \frac{1}{2}$, and thanks to (3.12) we obtain

$$\forall u > \frac{1}{2}, \qquad \frac{u^2}{2} - \frac{u}{4} \leq V_{\mathbf{m}}(u) \leq \frac{u^2}{2}. \tag{3.14}$$

To conclude we have to look more precisely at the asymptotic behavior of $V_{\mathbf{m}}(u)$ as $u \to \infty$. Let us set, for $u > \frac{1}{2}$

$$\tilde{W}_{\mathbf{m}}(u) = \frac{V_{\mathbf{m}}(u)}{u} - \frac{u}{2}.$$

We have

$$\tilde{W}'_{\mathbf{m}}(u) = \frac{V'_{\mathbf{m}}(u)}{u} - \frac{V_{\mathbf{m}}(u)}{u^2} - \frac{1}{2}$$

and suppose that $\tilde{W}'_{\mathbf{m}}(u_0) = 0$ for some $u_0 > \frac{1}{2}$. Therefore,

$$\frac{V_{\mathbf{m}}(u_0)}{u_0} + \frac{u_0}{2} = V'_{\mathbf{m}}(u_0) = \frac{-\frac{\mathbf{m}+1}{2}V_{\mathbf{m}}(u_0) + \mathbf{m}u_0^2 - 3u_0V_{\mathbf{m}}(u_0)}{V_{\mathbf{m}}(u_0) - 2u_0^2}$$

which gives

$$2V_{\mathbf{m}}(u_0)^2 + 3u_0^2V_{\mathbf{m}}(u_0) - 2u_0^4 + (\mathbf{m}+1)u_0V_{\mathbf{m}}(u_0) - 2\mathbf{m}u_0^3 = 0.$$

Using (3.14) we get

$$2\left(\frac{u_0^2}{2} - \frac{u_0}{4}\right)^2 + 3u_0^2\left(\frac{u_0^2}{2} - \frac{u_0}{4}\right) - 2u_0^4 + (\mathbf{m}+1)u_0\left(\frac{u_0^2}{2} - \frac{u_0}{4}\right) - 2\mathbf{m}u_0^3 \leq 0,$$

which implies

$$-\frac{2\mathbf{m}+1}{8}u_0^2(6u_0+1) \leq 0,$$

and gives a contradiction. Consequently, $\tilde{W}'_{\mathbf{m}}$ does not vanish on $(\frac{1}{2}, \infty)$ and since (3.14) is equivalent to

$$-\frac{1}{4} \leq \tilde{W}_{\mathbf{m}}(u) \leq 0,$$

we get that $\tilde{W}_{\mathbf{m}}(u) \to \nu$ as $u \to \infty$, for some $\nu \in [-\frac{1}{4}, 0]$. To compute ν, let us write

$$V_{\mathbf{m}}(u) = \frac{u^2}{2} + \nu u + u\eta(u) \tag{3.15}$$

where $\eta(u) \to 0$ as $u \to \infty$. Therefore, we have

$$\begin{aligned}
V'_{\mathbf{m}}(u) - u =& \frac{-\frac{\mathbf{m}+1}{2}V_{\mathbf{m}}(u) + \mathbf{m}u^2 - 4uV_{\mathbf{m}}(u) + 2u^3}{V_{\mathbf{m}}(u) - 2u^2} \\
=& \frac{-\frac{\mathbf{m}+1}{2}\left(\frac{u^2}{2} + \nu u + u\eta(u)\right) + \mathbf{m}u^2 - 4u\left(\frac{u^2}{2} + \nu u + u\eta(u)\right) + 2u^3}{\left(\frac{u^2}{2} + \nu u + u\eta(u)\right) - 2u^2} \\
=& \frac{\left(\frac{3\mathbf{m}-1}{4} - 4\nu\right)u^2 - \frac{\mathbf{m}+1}{2}\nu u - 4u^2\eta(u) - \frac{\mathbf{m}+1}{2}u\eta(u)}{-\frac{3u^2}{2} + \nu u + u\eta(u)} \\
\longrightarrow& -\frac{2}{3}\left(\frac{3\mathbf{m}-1}{4} - 4\nu\right) \qquad \text{as} \quad u \to \infty.
\end{aligned}$$

By integrating and comparing with (3.15) we arrive to

$$\nu = -\frac{2}{3}\left(\frac{3\mathbf{m}-1}{4} - 4\nu\right)$$

and $\nu = \frac{3\mathbf{m}-1}{10}$. But then $\nu \geq -\frac{1}{4}$ gives $\mathbf{m} \geq -\frac{1}{2}$ and a contradiction. Thus u_* is finite.

To complete the proof, it is sufficient to remark that in the region $\{v > 2u^2\} \cap \{u > 0\}$, we have $Q_{\mathbf{m}}(u,v) < 0$, in such a way that in this region S_0^+ is characterized by $v = W_{\mathbf{m}}(u)$ with $W'_{\mathbf{m}} < 0$ and S_0^+ has to cross the v-axis, because on the contrary we should have $W_{\mathbf{m}}(u) \to \infty$ as $u \to u_1$ for some $u_1 \in (0, u_*)$; but in this case we get

$$W'_{\mathbf{m}}(u) \sim -\frac{\mathbf{m}+1}{2} - 3u_1 \quad \text{as} \quad u \to u_1^+,$$

and a contradiction. ■

Theorem 3.2 *Let* $\mathbf{m} \in (-1, -\frac{1}{3})$.

• *If* $-1 < \mathbf{m} < -\frac{1}{2}$, *then there exists* $a_* > 0$ *such that problem* (1.1)-(1.4) *has no solution for* $0 < a < a_*$, *one and only one solution which is bounded for* $a = a_*$, *and two bounded solutions and infinitely many unbounded solutions for* $a > a_*$.

• *If* $-\frac{1}{2} \leq \mathbf{m} < -\frac{1}{3}$, *then for every* $a > 0$, *the problem* (1.1)-(1.4) *has one bounded solution and infinitely many unbounded solutions.*

Proof. • Let us start with the second case: $-\frac{1}{2} \leq \mathbf{m} < -\frac{1}{3}$. Consider for $a > 0$ the initial value problem $(\mathcal{P}_{\mathbf{m},a,\mu})$ and look at the corresponding trajectory $C_{a,\mu}$ of the plane system (1.6) defined by (1.5) with $\tau = 0$. From Lemmas 3.3 and 3.4, we know that the straight line $u = \frac{1}{a^2}$ intersects the separatrix S_0^+ through a point $\left(\frac{1}{a^2}, \nu\right)$.

Claim 1. If $\mu = a^3\nu$ then f is a bounded solution of (1.1)-(1.4). Indeed, since in this case $C_{a,\mu}$ tends to the point O as $s \to \infty$, tangentially to the line L, we can assert that for t large enough we have $f'(t) > 0$, $f''(t) < 0$ and moreover

$$\frac{f'(t)}{f(t)^2} \to 0 \quad \text{and} \quad \frac{f''(t)}{f(t)f'(t)} \to -\frac{\mathbf{m}+1}{2} \quad \text{as} \quad t \to \infty. \tag{3.16}$$

Therefore, we get $f'(t) \to l \geq 0$ as $t \to \infty$ and if we suppose $l > 0$ it follows from (3.16) that

$$f''(t) \sim -\frac{\mathbf{m}+1}{2} l^2 t \quad \text{as} \quad t \to \infty,$$

which contradicts the fact that $f'(t) \to l > 0$ as $t \to \infty$. So $l = 0$ and f is a solution to (1.1)-(1.4). Suppose now f were unbounded, i.e. $f(t) \to \infty$ as $t \to \infty$. According to (3.16), there exists $t_0 > 0$ such that

$$\forall t \geq t_0, \qquad f''(t) \leq -\frac{\mathbf{m}+1}{4} f(t)f'(t).$$

Integrating and dividing by $f(t)^2$ we get

$$\forall t \geq t_0, \qquad \frac{f'(t)}{f(t)^2} - \frac{f'(t_0)}{f(t)^2} \leq -\frac{\mathbf{m}+1}{8}\left(1 - \frac{f(t_0)^2}{f(t)^2}\right).$$

This and (3.16) give a contradiction when $t \to \infty$.

Claim 2. If $\mu > a^3\nu$ then f is a unbounded solution of (1.1)-(1.4). Because of the behavior of the vector field in the area $\{u > 0\} \cap \{v > 0\}$ (cf. (3.11)), we see that the phase curve $C_{a,\mu}$ has to cross the u-axis and go to the singular point O as $s \to \infty$ tangentially and below this axis. It means that for t large enough we have $f'(t) > 0$, $f''(t) < 0$ and moreover

$$\frac{f'(t)}{f(t)^2} \to 0 \quad \text{and} \quad \frac{f''(t)}{f(t)f'(t)} \to 0 \quad \text{as} \quad t \to \infty. \tag{3.17}$$

Consequently, we have $f'(t) \to l \geq 0$ as $t \to \infty$ and if we suppose $l > 0$, we deduce from the following identity

$$f''(t) + \frac{\mathbf{m}+1}{2} f(t)f'(t) = \mu + \frac{\mathbf{m}+1}{2} a + \frac{3\mathbf{m}+1}{2} \int_0^t f'(\xi)^2 d\xi$$

that

$$f''(t) \sim -\frac{\mathbf{m}+1}{2} l^2 t + \frac{3\mathbf{m}+1}{2} l^2 t = \mathbf{m} l^2 t \quad \text{as} \quad t \to \infty$$

which is a contradiction with the fact that $f'(t) \to l$ as $t \to \infty$. It follows that f is a solution of (1.1)-(1.4). We next show that f is unbounded. On the contrary suppose that f is bounded, and denote by λ the limit of f at infinity. Multiplying the equation (1.1) by f' and integrating between t and ∞ we obtain

$$-f(t)f''(t) + \frac{1}{2} f'(t)^2 - \frac{\mathbf{m}+1}{2} f'(t)f(t)^2 = (2\mathbf{m}+1) \int_t^\infty f(\xi)f'(\xi)^2 d\xi \geq 0. \tag{3.18}$$

Dividing by $f'(t)f(t)^2$ and using (3.17) we immediately get a contradiction. Therefore, f is an unbounded solution of (1.1)-(1.4).

Claim 3. If $\mu < a^3\nu$ then f is not a solution of (1.1)-(1.4). In this case the trajectory $C_{a,\mu}$, which lies below the separatrix S_0^+, has to cross the v-axis, in such a way that f' vanishes at some point t_1 and f is not a solution of (1.1)-(1.4).

• Let us consider now the case: $-1 < \mathbf{m} < -\frac{1}{2}$. First, if we denote by $(u_*, 2u_*^2)$ the point where the separatrix S_0^+ crosses the isocline $v = 2u^2$, and set $a_* = \frac{1}{\sqrt{u_*}}$, we see that the line $u = \frac{1}{a^2}$ does not intersect the separatrix S_0^+ if $a < a*$, is tangent to it if $a = a*$, and intersects it through two points $\left(\frac{1}{a^2}, \nu_-\right)$ and $\left(\frac{1}{a^2}, \nu_+\right)$ if $a > a*$ (see Lemma 3.5). Using the arguments invoked in the first part, we easily get that problem (1.1)-(1.4) has no solution for $0 < a < a_*$, one and only one solution which is bounded for $a = a_*$ (for $\mu = 2u_*^2 a^3$), and infinitely many solutions for $a > a_*$ (for $\mu \in [a^3\nu_-, a^3\nu_+]$). Since the inequality (3.18) does not hold for $\mathbf{m} < -\frac{1}{2}$, we still have to prove that solutions corresponding to the positive semi-trajectory $C_{a,\mu}$ with $a^3\nu_- < \mu < a^3\nu_+$ are unbounded. For that we come back to the equality in (3.18), we divide again by $f'(t)f(t)^2$ and using (3.17) we deduce that

$$\int_t^\infty f(\xi)f'(\xi)^2 d\xi \sim -\frac{\mathbf{m}+1}{2(2\mathbf{m}+1)} f'(t)f(t)^2 \quad \text{as} \quad t \to \infty,$$

and since $f(t) \to \lambda$ as $t \to \infty$ we get

$$\int_t^\infty f'(\xi)^2 d\xi \sim -\frac{\mathbf{m}+1}{2(2\mathbf{m}+1)} \lambda f'(t) \quad \text{as} \quad t \to \infty. \tag{3.19}$$

On the other hand we have from (3.1)

$$\int_t^\infty f'(\xi)^2 d\xi = -\frac{2}{3\mathbf{m}+1}\left(f''(t) + \frac{\mathbf{m}+1}{2}f'(t)f(t)\right).$$

Combining this equality with (3.19) yields to

$$\frac{f''(t)}{f(t)f'(t)} \to -\frac{(\mathbf{m}+1)^2}{4(2\mathbf{m}+1)} \neq 0,$$

which contradicts (3.17). Therefore, f is an unbounded solution of (1.1)-(1.4). ∎

Remark 3.3 *For $-1 < \mathbf{m} < -\frac{1}{2}$, the critical value a_* is depending on $\mathbf{m}$, and a_* decreases from ∞ to 0 when $\mathbf{m}$ goes from -1 to $-\frac{1}{2}$.*

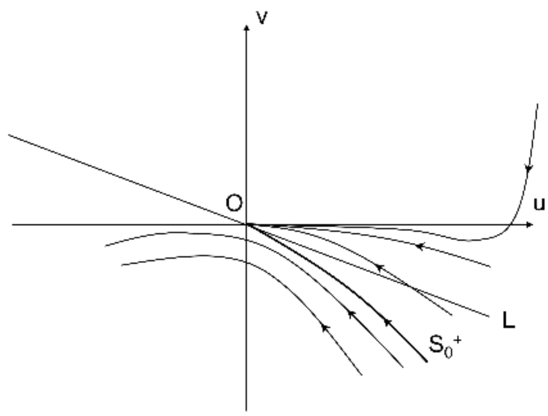
3.3 The case $-\frac{1}{3} \leq \mathbf{m} < 0$.

This case is almost completely solved. To our knowledge, the only open question is uniqueness of bounded solution when $a < 0$. We summarize in the following theorem the results of [9].

Theorem 3.3 *Let $-\frac{1}{3} \leq \mathbf{m} < 0$, then for every $a \in \mathbb{R}$, the problem (1.1)-(1.4) has an infinite number of solutions. Moreover, if $a \geq 0$ one and only one solution is bounded, and if $a < 0$ at least one is bounded, many infinitely are unbounded.*

Proof. See [9]. ∎

Remark 3.4 *It is easy to recover the previous results for $a > 0$ from the system (1.6), by looking at the phase curves in the region $\{u > 0\}$; see figure 3.3.1.*



$(-1/3 < \mathbf{m} < 0)$
Fig 3.3.1

3.4 The case $\mathbf{m} \geq 0$.

In this case we know from [9] that problem (1.1)-(1.4) has one and only one concave solution, for any $a \in \mathbb{R}$. Our main goal in this section is to give existence or nonexistence results of concave-convex solutions. The value $\mathbf{m} = 1$ plays a particular role in this study. First of all we give some preparatory lemmas in order to prove, in the case $\mathbf{m} \in [0, 1]$, the uniqueness result suggested in [7], [8] and [9], and in the case $\mathbf{m} > 1$, that concave-convex solutions exist for $a > 0$.

Lemma 3.6 *Let $\mathbf{m} \geq 0$ and f be a concave-convex solution of* (1.1)-(1.4)*. If we denote by t_0 the point satisfying $f''(t_0) = 0$, then the curve $s \longmapsto (u(s), v(s))$ defined by* (1.5) *with $\tau = t_0$ is a positive semi-trajectory which lies in the bounded domain*

$$\mathcal{D}_- := \left\{ (u, v) \in \mathbb{R}^2 \ ; \ -\frac{\mathbf{m}+1}{4} < u < 0 \quad \text{and} \quad 0 \leq v < -\frac{\mathbf{m}+1}{2} u \right\}. \tag{3.20}$$

Proof. From Proposition 3.1, we know that f is positive, decreasing and convex on $[t_0, \infty)$, from which it follows that

$$\forall t \geq t_0, \qquad \frac{f'(t)}{f(t)^2} < 0 \quad \text{and} \quad \frac{f''(t)}{f(t)^3} \geq 0. \tag{3.21}$$

On the other hand, since f is bounded, we deduce from (3.1) that

$$\forall t \geq 0, \qquad f''(t) + \frac{\mathbf{m}+1}{2} f'(t) f(t) < 0, \tag{3.22}$$

and if λ denotes the limit of f at infinity, we get by integrating

$$-f'(t) + \frac{\mathbf{m}+1}{4}(\lambda^2 - f(t)^2) < 0 \quad \text{and} \quad f'(t) + \frac{\mathbf{m}+1}{4} f(t)^2 > \frac{\mathbf{m}+1}{4}\lambda^2 \geq 0. \tag{3.23}$$

Relations (3.22) and (3.23) give

$$\forall t \geq t_0, \qquad \frac{f'(t)}{f(t)^2} + \frac{\mathbf{m}+1}{4} > 0 \quad \text{and} \quad \frac{f''(t)}{f(t)^3} + \frac{\mathbf{m}+1}{2} \frac{f'(t)}{f(t)^2} < 0, \tag{3.24}$$

and from the first inequality of (3.24) we easily get

$$\forall t \geq t_0, \qquad f(t) \geq \frac{1}{\frac{\mathbf{m}+1}{4}(t - t_0) + \frac{1}{f(t_0)}}$$

which implies

$$\int_{t_0}^{\infty} f(\xi) d\xi = \infty.$$

Consequently, the trajectory $s \longmapsto (u(s), v(s))$ is defined on the whole interval $[0, \infty)$ and this together with (3.21) and (3.24) complete the proof. ■

The following lemmas describe the global behavior of the separatrices S_0^+, S_1^+ and S_2^-.

Lemma 3.7 *If $\mathbf{m} \geq 0$, the separatrix S_0^+ is defined by $v = V_{\mathbf{m}}(u)$ for $u > 0$, where the function $V_{\mathbf{m}}$ is such that*

$$\forall u > 0, \qquad -3u - \frac{\mathbf{m}+1}{2} < V'_{\mathbf{m}}(u) < -\frac{\mathbf{m}+1}{2}.$$

(See figure 3.4.1)

Proof. Since S_0^+ leaves the singular point O tangentially to L and below it, we deduce from the positivity of $\mathbf{m}$ and (3.7) that

$$\forall u > 0, \qquad V'_{\mathbf{m}}(u) < -\frac{\mathbf{m}+1}{2}.$$

On the other hand, we have

$$\forall u > 0, \qquad V'_{\mathbf{m}}(u) - \left(-3u - \frac{\mathbf{m}+1}{2}\right) = \frac{-6u^3 - u^2}{V_{\mathbf{m}}(u) - 2u^2} > 0,$$

which completes the proof. ■

Lemma 3.8 *If $\mathbf{m} = 1$, the separatrices S_1^+ and S_2^- coincide, and the functions V_1, W_1 allowing to characterized them, are defined for $-\frac{1}{4} < u < 0$ by*

$$V_1(u) = \frac{-u + u\sqrt{1+4u}}{2} \qquad \textit{and} \qquad W_1(u) = \frac{-u - u\sqrt{1+4u}}{2}.$$

Proof. Let (u, v) be a solution of (1.6). If we set $w = v^2 + uv - u^3$ we get $\dot{w} = -(1 + 6u)w$. Consequently, the set $\{(u, v) \in \mathbb{R}^2 \ ; \ v^2 + uv - u^3 = 0\}$ is an union of phase curves and it is easy to see that these curves are the separatrices and the singular point O. ■

Lemma 3.9 *Let $\mathbf{m} \in [0, 1]$.*

• *As s increases, the α-separatrix S_2^- leaves to the left the singular point O tangentially to L_0, and either does not cross the isocline $P(u, v) = 0$, or crosses it through a point $(u_*, 2u_*^2)$ such that $u_* \leq -\frac{1}{4}$ and next crosses the straight line L.*

• *As s decreases, the ω-separatrix S_1^+ leaves to the left the singular point O tangentially to L, and crosses the isocline $P(u, v) = 0$ through a point $(u_*, 2u_*^2)$ such that $-\frac{1}{4} \leq u_* < 0$ and next stays in the bounded region $\mathcal{D}_-$.*

(See figure 3.4.1)

Proof. Let $\mathbf{m} \in [0, 1]$. We know from section 2 that in the neighbourhood of O, the separatrix S_2^- lies in the region $\{0 \leq v < 2u^2\} \cap \{u < 0\}$ and as long as S_2^- stays below the isocline $v = 2u^2$ we can define it by $v = V_{\mathbf{m}}(u)$ where $V_{\mathbf{m}}$ is a solution of the equation (2.3). Thanks to (2.2) we have $V_{\mathbf{m}}(u_0) \leq V_1(u_0)$ for u_0 close to 0^- and since

$$F_{\mathbf{m}}(u, v) - F_1(u, v) = \frac{1 - \mathbf{m}}{2} \geq 0, \tag{3.25}$$

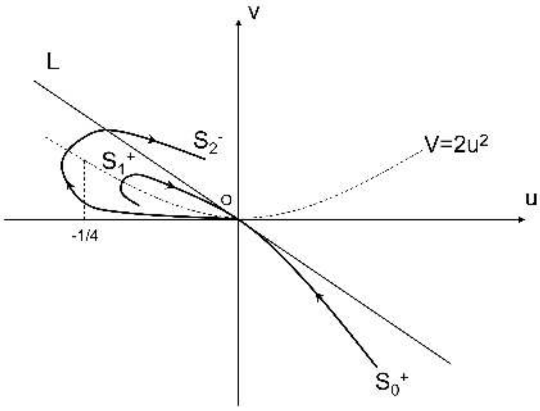
we deduce from classical differential inequalities (see [17] or [23]) that $V_{\mathbf{m}} \leq V_1$ on the left maximal interval $(-\frac{1}{4}, u_0]$ on which V_1 is defined (see Lemma 3.8). It follows that if the separatrix S_2^- crosses the isocline $v = 2u^2$ through a point $(u_*, 2u_*^2)$, then we have $u_* \leq -\frac{1}{4}$.

For S_1^+ we use similar arguments. Writing $v = W_{\mathbf{m}}(u)$, we have

$$W_{\mathbf{m}}(u) - W_1(u) = -\frac{\mathbf{m}+1}{2}u + u + o(u) = \frac{1-\mathbf{m}}{2}u + o(u)$$

in such a way that $W_{\mathbf{m}}(u_0) \leq W_1(u_0)$ for u_0 close to 0^-. Therefore, it follows from (3.25) that $W_{\mathbf{m}} \leq W_1$ as long as $W_{\mathbf{m}}$ and W_1 are defined, and thanks to Lemma 3.8, we see that S_1^+ has to cross the isocline $P(u,v) = 0$ through a point $(u_*, 2u_*^2)$ such that $u_* \geq -\frac{1}{4}$.

To complete the proof, we remark by looking at the vector field that S_1^+ must stay in $\mathcal{D}_-$ and that if S_2^- does not stay below the parabola $v = 2u^2$, then it has to intersect L. ■



$(0 < \mathbf{m} < 1)$
Fig 3.4.1

Lemma 3.10 *Let* $\mathbf{m} > 1$.

• *As s increases, the α-separatrix S_2^- leaves to the left the singular point O tangentially to L_0, and crosses the isocline $P(u,v) = 0$ through a point $(u_*, 2u_*^2)$ such that $-\frac{1}{4} \leq u_* < 0$ and next stays in the bounded region $\mathcal{D}_-$.*

• *As s decreases, the ω-separatrix S_1^+ leaves to the left the singular point O tangentially to L, crosses the isocline $P(u,v) = 0$ through a point $(u_*, 2u_*)$ such that $u_* \leq -\frac{1}{4}$, intersects successively the u-axis and the v-axis, and next stays in the quadrant $\{u > 0\} \cap \{v < 0\}$, going to infinity with a slope less than $-\frac{\mathbf{m}+1}{2}$ and greater than $-3u - \frac{\mathbf{m}+1}{2}$.*
(See figure 3.4.2).

Proof. The separatrix S_2^- starts to the left from O, tangentially to L_0 and above it. Similar arguments to the ones used in the proof of Lemma 3.9 show that S_2^- can be characterized by $v = V_{\mathbf{m}}(u)$ with $V_{\mathbf{m}} \geq V_1$, and thus crosses the isocline $P(u,v) = 0$ through a point $(u_*, 2u_*^2)$ such that $-\frac{1}{4} \leq u_* < 0$.

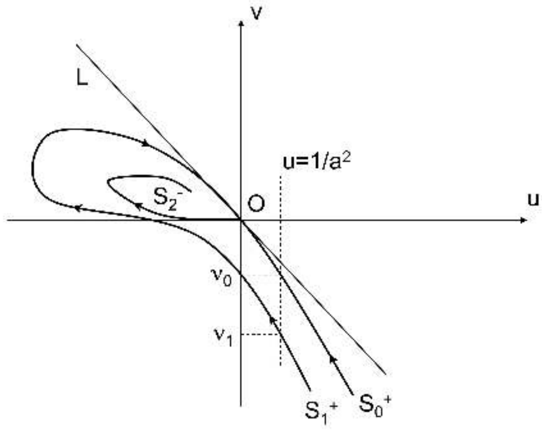
The separatrix S_1^+ starts to the left from the singular point O tangentially to L. Moreover S_1^+ is below L and above the isoclines $Q_{\mathbf{m}}(u,v) = 0$ and $P(u,v) = 0$, and in the bounded area $\{2u^2 < v < -\frac{\mathbf{m}+1}{2}u\} \cap \{u < 0\}$ we have $v = V_{\mathbf{m}}(u)$ where $V_{\mathbf{m}}$ is a solution of the equation (2.3). Since $\mathbf{m} > 1$ we deduce from (3.7) that $-\frac{\mathbf{m}+1}{2} < V'_{\mathbf{m}}(u) < 0$ as long as $V_{\mathbf{m}}(u) > \psi_{\mathbf{m}}(u)$, in such a way that S_1^+ intersects the curve $Q_{\mathbf{m}}(u,v) = 0$ through a point $(\bar{u}, \psi_{\mathbf{m}}(\bar{u}))$ with $\bar{u} < -\frac{1}{6}$.

For $u < \bar{u}$ we have $V'_{\mathbf{m}}(u) > 0$ and S_1^+ has to cross the isocline $P(u,v) = 0$ through a point $(u_*, 2u_*^2)$. Similar arguments to the ones used in the proof of Lemma 3.9 show that $u_* \leq -\frac{1}{4}$. It is then easy to see that after having intersected the parabola, S_2^- stays in the bounded region $\mathcal{D}_-$.

After having crossed the parabola, we define S_1^+ by $v = W_{\mathbf{m}}(u)$ and we deduce from the behavior of S_2^- that S_1^+ has to intersect the u-axis. Next, thanks to (3.7) we see that, as soon as $W_{\mathbf{m}}(u) < 0$, we have

$$-\frac{\mathbf{m}+1}{2} < W'_{\mathbf{m}}(u) < 0 \quad \text{as long as} \quad u < 0.$$

Consequently, S_1^+ intersects the v-axis and we conclude as in the proof of Lemma 3.7. ■



$(\mathbf{m} > 1)$
Fig 3.4.2

Theorem 3.4 *If* $\mathbf{m} \in [0,1]$*, then for any* $a \in \mathbb{R}$ *the problem* (1.1)-(1.4) *has one and only one solution, which is concave.*

Proof. Taking into account the fact that for $\mathbf{m} \geq 0$ problem (1.1)-(1.4) has one and only one concave solution (see [9]), we just have to prove that concave-convex solutions cannot exist when $\mathbf{m} \in [0,1]$. To this end, suppose that f is a concave-convex solution of (1.1)-(1.4) and denote by t_0 the point such that $f''(t_0) = 0$. Consider the positive semi-trajectory $s \longmapsto (u(s), v(s))$ defined in Lemma 3.6. We have

$$u(0) = \frac{f'(t_0)}{f(t_0)^2} < 0 \qquad \text{and} \qquad v(0) = 0.$$

In view of Lemma 3.9 we see that this semi-trajectory cannot remain in the bounded domain $\mathcal{D}_-$ defined by (3.20). This is a contradiction. ■

Remark 3.5 *Recall that for* $\mathbf{m} = 1$ *the unique solution of* (1.1)-(1.4) *is given by* $f(t) = a + (c-a)(1-e^{-t})$ *with* $c = \frac{1}{2}(a + \sqrt{a^2+4})$. *See* [16], [20] *and also* [7], [9].

Theorem 3.5 *If* $\mathbf{m} > 1$*, then for any* $a > 0$ *the problem* (1.1)-(1.4) *has one and only one concave solution and an infinite number of concave-convex solutions.*

Proof. Let $a > 0$. Consider the initial value problem $(\mathcal{P}_{\mathbf{m},a,\mu})$ and the corresponding trajectory $C_{a,\mu}$ of the plane system (1.6) defined by (1.5) with $\tau = 0$. From Lemma 3.10 the straight line $u = \frac{1}{a^2}$ crosses the separatrices S_0^+ and S_1^+ through points $\left(\frac{1}{a^2}, \nu_0\right)$ and $\left(\frac{1}{a^2}, \nu_1\right)$ respectively, with $\nu_1 < \nu_0 < 0$ (see figure 3.4.2).

It is easy to see that for $\mu = a^3\nu_0$ the function f is the concave bounded solution of (1.1)-(1.4), exhibited in [9]. Indeed, since $C_{a,\mu}$ is a positive semi-trajectory corresponding to a part of S_0^+, it follows that f is positive, defined on $[0, \infty)$ and moreover $f' > 0$, $f'' < 0$ and

$$\frac{f'(t)}{f(t)^2} \to 0 \quad \text{and} \quad \frac{f''(t)}{f(t)f'(t)} \to -\frac{\mathbf{m}+1}{2} \quad \text{as} \quad t \to \infty,$$

and we conclude as in the proof of Theorem 3.2.

Now, for $\mu \in [a^3\nu_1, a^3\nu_0)$, we see that the trajectory $C_{a,\mu}$ intersects the u-axis for some s_0 and remains in the domain defined by the separatrix S_1^+ for $s > s_0$. It follows from the Poincaré-Bendixson Theorem that $C_{a,\mu}$ is a positive semi-trajectory whose ω-limit set is the point O if $\mu = a^3\nu_1$, and either the singular point A or a limit cycle surrounding A if $a^3\nu_1 < \mu < a^3\nu_0$. Since $F_{\mathbf{m}}(u, 0) = -\frac{\mathbf{m}}{2}$ such a limit cycle cannot cross the u-axis and therefore, f is defined on $[0, \infty)$, is positive and there exists $t_0 > 0$ such that $f'(t) < 0$ and $f''(t) > 0$ for $t > t_0$. Thus $f'(t) \to l \leq 0$ as $t \to \infty$ and if we suppose $l < 0$ we get a contradiction with the positivity of f. Consequently, if $\mu \in [a^3\nu_1, a^3\nu_0)$ then f is a concave-convex solution of (1.1)-(1.4). To complete the proof, let us remark that for $\mu \notin [a^3\nu_1, a^3\nu_0]$, the function f cannot be a solution of (1.1)-(1.4) in accordance with Lemma 3.6. ■

Remark 3.6 *Let* $\mathbf{m} > 1$ *and let* f *be a concave-convex solution of* (1.1)-(1.4). *Since* f *is positive and decreasing at infinity, then* $f(t) \to \lambda \geq 0$ *as* $t \to \infty$. *If* f *corresponds to the separatrix* S_1^+ *(i.e.* $f''(0) = a^3\nu_1$*) then we prove as in Remark 3.2 that* $\lambda > 0$*, and if* $a^3\nu_1 < f''(0) < a^3\nu_0$*, there exists* $c > 0$ *such that* $|f'(t)| > c|f(t)^2|$ *for* t *large enough, in such a way that* $\lambda = 0$.

Remark 3.7 *For* $1 < \mathbf{m} < \frac{4}{3}$ *the singular point* A *is an unstable focus, which implies that at least one cycle surrounding* A *has to exist. If* $\mathbf{m} > \frac{4}{3}$ *then* A *is attractif and it seems that cycles do not exist. If it is the case, we have*

$$\frac{f'(t)}{f(t)^2} \sim -\frac{1}{6} \quad \textit{and} \quad \frac{f''(t)}{f(t)^3} \sim \frac{1}{18} \quad \textit{as} \quad t \to \infty,$$

which easily give

$$f(t) \sim \frac{6}{t} \quad \textit{as} \quad t \to \infty.$$

4 Conclusion

Based on [9] and on the previous investigations, the following conclusions can be drawn:

• For $\mathbf{m} < -1$, there exists $a_* < 0$ such that the problem (1.1)-(1.4) has infinitely many solutions if $a < a_*$, one and only one solution if $a = a_*$, and no solution if $a > a_*$. Moreover, if f is a solution to (1.1)-(1.4), then $f < 0$.

• For $\mathbf{m} = -1$ and for every $a \in \mathbb{R}$, the problem (1.1)-(1.4) has no solution.

• For $-1 < \mathbf{m} \leq -\frac{1}{2}$ and for every $a \leq 0$, the problem (1.1)-(1.4) has no solution.

• For $-1 < \mathbf{m} < -\frac{1}{2}$, there exists $a_* > 0$ such that problem (1.1)-(1.4) has no solution for $0 < a < a_*$, one and only one solution which is bounded for $a = a_*$, and two bounded solutions and infinitely many unbounded solutions for $a > a_*$.

• For $-\frac{1}{2} \leq \mathbf{m} < -\frac{1}{3}$ and for every $a > 0$, the problem (1.1)-(1.4) has one bounded solution and infinitely many unbounded solutions.

• For $-\frac{1}{3} \leq \mathbf{m} < 0$ and for every $a \in \mathbb{R}$, the problem (1.1)-(1.4) has an infinite number of solutions. Moreover, if $a \geq 0$ one and only one solution is bounded, and if $a < 0$ at least one is bounded, many infinitely are unbounded.

• For $\mathbf{m} \in [0, 1]$ and for every $a \in \mathbb{R}$, the problem (1.1)-(1.4) has one and only one solution.

• For $\mathbf{m} > 1$ and for every $a > 0$, the problem (1.1)-(1.4) has one and only one concave solution and an infinite number of concave-convex solutions.

We see from these results, that the questions we have not solved concern the case $a \leq 0$. More precisely, it should be interesting to try to answer to the following points:

(a) For $-\frac{1}{2} < \mathbf{m} < -\frac{1}{3}$, what happens for $a \leq 0$?

(b) For $-\frac{1}{3} \leq \mathbf{m} < 0$ and $a < 0$, is there one or more bounded solutions ?

(c) For $\mathbf{m} > 1$ and $a \leq 0$, do concave-convex solutions exist ?

Another purpose is to compute the critical values a_* appearing in the results above. Considerations about analycity of the manifold $\mathcal{W}$ could allow to estimate a_*.

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