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Blowing Ups of 3-dimensional Terminal Singularities, II

By

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Abstract

We study blowing ups of 3-dimensional terminal singularities of type (cD/2) such that the exceptional loci are prime divisors and have discrepancies 1/2. We determined such blowing ups completely.

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§1. Introduction

This paper continues our study on the blowing ups of 3-dimensional terminal singularities X of indices $m \geq 2$. In our previous paper ([Hay99]), we introduced the notion of pseudo weighted valuation, which consists of an embedding $j : X \hookrightarrow \mathbb{C}^4/\mathbb{Z}_m$ and a weight σ. By using these data, we blow up

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X and get divisorial blow ups of X with small discrepancies. We also showed that, in most cases, there is a one-to-one correspondence between these divisorial blow ups and a certain set of pseudo weighted valuations, and remarked that this correspondence does not necessarily hold in the case X is of type (cD/2). Our purpose here is to study the blowing ups of terminal singularities of type (cD/2) and to determine all divisorial blow ups of X with discrepancies 1/2. Contrary to the other cases, we cannot get all these divisorial blow ups by considering the embedding $j : X \hookrightarrow \mathbb{C}^4/\mathbb{Z}_2$ and a weight σ. Instead, we have to embed X into a 5-dimensional space $\mathbb{C}^5/\mathbb{Z}_2$ as a codimension 2 subvariety and make blow ups of X in $\mathbb{C}^5/\mathbb{Z}_2$. This method works well and we can determine all prime divisors with discrepancies 1/2 by using the embeddings into 4 or 5 dimensional spaces and weights. These are the main theorems in this paper. There is a related work by [IT99] which also uses embeddings into $\mathbb{C}^5$ in order to study hypersurfaces in $\mathbb{C}^4$.

This paper, combined with the results in [Hay99], covers all 3-dimensional terminal singularities of indices $m \geq 2$. Indeed, if X is a 3-dimensional terminal singularity of index $m \geq 2$, we can determine all prime divisors with discrepancies $1/m$ and in particular we obtain the following (see (3.4)):

Theorem 1.1. *Let X be a germ of a 3-dimensional terminal singularity of index $m \geq 2$. Then there exists at least one divisorial blow up $\pi : \bar{X} \to X$ with discrepancy $1/m$. Furthermore π does not increase axial weights.*

As a consequence of this result, we obtain the following (see (3.5)):

Theorem 1.2. *Let X be a germ of a 3-dimensional terminal singularity of index $m \geq 2$. Then there is a projective birational morphism $\psi : Y \to X$ such that*

(i) *Y has only Gorenstein terminal singularities, and*

(ii) *ψ is a composition of divisorial blow ups of points of indices ≥ 2 such that their discrepancies are minimal.*

This morphism $\psi : Y \to X$ should be compared with the economic resolution by Reid ([Reid87, (6.5)]). He conjectured the existence of a projective birational morphism $\psi' : Y' \to X$ such that all exceptional prime divisors on Y' have discrepancies < 1 and Y' has only Gorenstein terminal singularities. Our morphism $\psi : Y \to X$ only satisfies the second condition, since some exceptional prime divisors of ψ may have discrepancies ≥ 1. However, the exceptional divisor of ψ contains all prime divisors with discrepancies < 1 over X, and we can determine all such divisors by using ψ. We also remark that Alexeev ([Alex94, 5.2]) obtained the same kind of birational morphisms by using Minimal Model Program. These are not necessarily the same as the above $\psi : Y \to X$.

We shall study the morphism $\psi : Y \to X$, especially the exceptional divisor of ψ, more closely in our future paper.

The detailed contents of this paper are as follows: In Section 2, we recall the results on the classification of 3-dimensional terminal singularities, and also review the notion of weighted blow ups, discrepancies, divisorial blow ups, etc. These are used in the following sections. Almost all notions are the same as in [Hay99]. The only new notion is about the embedding of X into a 5-dimensional space $\mathbb{C}^5/\mathbb{Z}_m$, which we shall call the generalized liftable embedding. In Section 3, we state our main results and give an outline of their proofs. Sections 4 and 5 are devoted to studying the blowing ups of terminal singularities X of type (cD/2), and we determine all prime divisors with discrepancies 1/2 over X. We shall give a proof of (1.2) in Section 6.

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Notation.

(1) For a real number α, we denote the round down of α by $\lfloor\alpha\rfloor$, and the fractional part of α by $\langle\alpha\rangle$, i.e., $\lfloor\alpha\rfloor$ is the integer satisfying $\lfloor\alpha\rfloor \leq \alpha < \lfloor\alpha\rfloor + 1$, and $\langle\alpha\rangle = \alpha - \lfloor\alpha\rfloor$.

(2) $\delta_{i,j}$ denotes the Kronecker's symbol, i.e., $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$.

§2. Classification of Terminal Singularities and Weighted Blow Ups

The purpose of this section is to state the results on classification of 3-dimensional terminal singularities and to introduce some notation in order to state our main results. Almost all of them are contained in [Hay99] and the only new notion is the generalized liftable embedding which we shall define in (2.8).

2.1. Let $\mathbb{Z}_m$ be the cyclic group of order m and let $(x_1, \ldots, x_n)$ be the complex space $\mathbb{C}^n$ with coordinates $x_1, \ldots, x_n$. We define the action of $\mathbb{Z}_m$ on $(x_1, \ldots, x_n)$ by

$$(x_1, x_2, \ldots, x_n) \mapsto (\zeta^{\alpha_1} x_1, \ldots, \zeta^{\alpha_n} x_n)$$

where ζ is a primitive m-th root of unity and $\alpha_1, \ldots, \alpha_n$ are integers. The quotient space is denoted by $(x_1, \ldots, x_n)/\mathbb{Z}_m(\alpha_1, \ldots, \alpha_n)$ or simply $(x_1, \ldots, x_n)/\mathbb{Z}_m$.

If $\varphi_1, \ldots, \varphi_r \in \mathbb{C}\{x_1, \ldots, x_n\}$ are semi-invariant elements with respect to the action of $\mathbb{Z}_m$ (we shall abbreviate this as $\mathbb{Z}_m$-*semi-invariants*), then $\mathbb{Z}_m$ also acts on the germ of $\{\varphi_1 = \cdots = \varphi_r = 0\}$ at the origin (0). We denote the quotient space by $\{\varphi_1 = \cdots = \varphi_r = 0\}/\mathbb{Z}_m(\alpha_1, \ldots, \alpha_n)$.

Now we recall the results on classification of 3-dimensional terminal singularities, which are due to [Reid83], [Dan83], [MS84] and [Mori85].

Theorem 2.2. *A 3-dimensional singularity is terminal of index* 1 *if and only if it is an isolated cDV point.*

Theorem 2.3. *Let X be a germ of a 3-dimensional terminal singularity of index ≥ 2. Then there is an embedding $j : X \hookrightarrow (x, y, z, u)/\mathbb{Z}_m$ such that one of the following holds*:

(cA/m) $X \simeq \{xy + f(z, u) = 0\}/\mathbb{Z}_m(\alpha, -\alpha, 1, 0)$ *where α is an integer prime to m and $f(z, u) \in \mathbb{C}\{z, u\}$ is a $\mathbb{Z}_m$-invariant.*

(cAx/4) $X \simeq \{x^2 + y^2 + f(z, u) = 0\}/\mathbb{Z}_4(1, 3, 1, 2)$ *where $f(z, u) \in \mathbb{C}\{z, u\}$ is a $\mathbb{Z}_4$-semi-invariant and $u \notin f(z, u)$.*

(cAx/2) $X \simeq \{x^2+y^2+f(z, u) = 0\}/\mathbb{Z}_2(0, 1, 1, 1)$ *where $f(z, u) \in (z, u)^4\mathbb{C}\{z, u\}$ is a $\mathbb{Z}_2$-invariant.*

(cD/3) $X \simeq \{\varphi(x, y, z, u) = 0\}/\mathbb{Z}_3(1, 2, 2, 0)$ *where φ has one of the following forms*:

(cD/3-1) $\varphi = u^2 + x^3 + yz(y + z)$,

(cD/3-2) $\varphi = u^2+x^3+yz^2+xy^4\lambda(y^3)+y^6\mu(y^3)$ *where* $\lambda(y^3)$, $\mu(y^3) \in \mathbb{C}\{y^3\}$ *and* $4\lambda^3 + 27\mu^2 \neq 0$,

(cD/3-3) $\varphi = u^2 + x^3 + y^3 + xyz^3\alpha(z^3) + xz^4\beta(z^3) + yz^5\gamma(z^3) + z^6\delta(z^3)$ *where* $\alpha(z^3)$, $\beta(z^3)$, $\gamma(z^3)$, $\delta(z^3) \in \mathbb{C}\{z^3\}$.

(cD/2) $X \simeq \{\varphi(x, y, z, u) = 0\}/\mathbb{Z}_2(1, 1, 0, 1)$ *where φ has one of the following forms*:

(cD/2-1) $\varphi = u^2 + xyz + x^{2a} + y^{2b} + z^c$ *where* a, $b \geq 2$, $c \geq 3$,

(cD/2-2) $\varphi = u^2 + y^2z + \lambda yx^{2a+1} + g(x, z)$ *where* $\lambda \in \mathbb{C}$, $a \geq 1$, $g(x, z) \in (x^4, x^2z^2, z^3)\mathbb{C}\{x, z\}$.

(cE/2) $X \simeq \{u^2 + x^3 + g(y, z)x + h(y, z) = 0\}/\mathbb{Z}_2(0, 1, 1, 1)$ *where* $g(y, z) \in (y, z)^4\mathbb{C}\{y, z\}$, $h(y, z) \in (y, z)^4\mathbb{C}\{y, z\} \setminus (y, z)^5\mathbb{C}\{y, z\}$.

The index of X is equal to the order of the cyclic group $\mathbb{Z}_m$.

In this paper, we shall study 3-dimensional terminal singularities of type (cD/2).

The following theorem by [KSB88] (see also [Ste88]) completes the classification of 3-dimensional terminal singularities.

Theorem 2.4. *Let X be one of the hyperquotient singularity $\{\varphi(x, y, z, u) = 0\}/\mathbb{Z}_m$ listed in* (2.3). *Assume that $\varphi(x, y, z, u) = 0$ defines an isolated singularity at* (0) *and that the action of $\mathbb{Z}_m$ is free outside* (0). *Then X is terminal.*

2.5. Let X be a germ of a 3-dimensional terminal singularity of index $m \geq 2$ at $P \in X$. Then there is an embedding $j : X \hookrightarrow (x, y, z, u)/\mathbb{Z}_m$ as in (2.3). We fix one of such embedding and call it a *standard embedding* of X.

We also see from (2.3) that X can be deformed to a collection of cyclic quotient terminal singularities (see [Hay99, 2.6] for details). We call the number of singularities in such a deformation the *axial weight* of X at P and we shall denote this by $\mathrm{aw}(X, P)$ or simply $\mathrm{aw}(X)$ if there is no danger of confusion. If X is smooth or has an isolated cDV point at P, then we shall define $\mathrm{aw}(X, P) = 1$. The explicit value of $\mathrm{aw}(X, P)$ can be found in [Hay99, 2.6].

2.6. Let X be a germ of a normal variety such that K_X is $\mathbb{Q}$-Cartier.

For a projective birational morphism $\psi : Z \to X$ from a normal variety Z such that K_Z is $\mathbb{Q}$-Cartier, we write

$$K_Z = \psi^*(K_X) + \sum a(F, X)F$$

where F runs over prime divisors on Z and $a(F, X) \in \mathbb{Q}$. The coefficient $a(F, X)$ is called the *discrepancy* of F over X, this depends only on the discrete valuation of the function field of X, and not on the particular choice of ψ. We sometimes identify prime divisors with the corresponding valuations when we speak about divisors over X.

A projective birational morphism $\pi : \bar{X} \to X$ is called a *divisorial* (resp. *pre-divisorial*) *blow up with discrepancy* k if

(i) $\bar{X}$ has only terminal (resp. canonical) singularities,

(ii) the exceptional set of π is an irreducible divisor E, and

(iii) $K_{\bar{X}} = \pi^*(K_X) + kE$.

We also say that π is divisorial (resp. pre-divisorial) with discrepancy k.

2.7. Let $N = \mathbb{Z}^n + \frac{1}{m}(\alpha_1, \dots, \alpha_n)\mathbb{Z}$ be the lattice and let $e_i = (0, \dots, \overset{i\text{-th}}{1}, \dots, 0) \in N$ $(i = 1, \dots, n)$. An element $\sigma = \frac{1}{m}(a_1, \dots, a_n) \in N$ is called a *weight* if $a_1, \dots, a_n > 0$ and if $e_1, \dots, e_n$ and σ generate N.

For each weight σ, we can associate the function $\sigma\text{-}wt : \mathbb{C}\{x_1, \dots, x_n\} \to \mathbb{Q}$. This function is determined by the values $\sigma\text{-}wt(x_1), \dots, \sigma\text{-}wt(x_n)$ (see [Hay99, 3.4]). For $f = \sum_I \alpha_I M_I \in \mathbb{C}\{x_1, \dots, x_n\}$, $\alpha_I \in \mathbb{C}$, M_I : monomials, and for $l \in \mathbb{Q}$, we define

$$f_{\sigma\text{-}wt=l} = \sum_{\sigma\text{-}wt(M_I)=l} \alpha_I M_I \quad \text{and} \quad f_{\sigma\text{-}wt\geq l} = \sum_{\sigma\text{-}wt(M_I)\geq l} \alpha_I M_I.$$

For $f \in \mathbb{C}\{x_1, \dots, x_n\}$ and a monomial M, we shall write $M \in f$ when the coefficient of M in the power series expansion of f is non-zero.

If a weight σ is given, we have a projective birational morphism

$$\bar{\pi} : \bar{Y} \to Y = (x_1, \dots, x_n)/\mathbb{Z}_m(\alpha_1, \dots, \alpha_n)$$

which is called the *weighted blow up associated to* σ or simply the *σ-blow up* of Y (see [Hay99, 3.2]). The variety $\bar{Y}$ is covered by n affine open sets $\bar{U}_1, \dots, \bar{U}_n$.

These affine open sets and $\bar{\pi}$ are described as follows:

$$\bar{U}_i = (\bar{x}_1, \dots, \bar{x}_n)/\mathbb{Z}_{a_i}(-a_1, \dots, \overset{i\text{-th}}{m}, \dots, -a_n)$$
$$\bar{\pi}\mid_{\bar{U}_i}: \bar{U}_i \ni (\bar{x}_1, \dots, \bar{x}_n) \mapsto (\bar{x}_1\bar{x}_i^{a_1/m}, \dots, \overset{i\text{-th}}{\bar{x}_i^{a_i/m}}, \dots, \bar{x}_n\bar{x}_i^{a_n/m}) \in Y.$$

The exceptional divisor $\bar{E}$ of $\bar{\pi}$ is isomorphic to the weighted projective space $\mathbb{P}(a_1, \dots, a_n)$.

Let $\varphi_1, \dots, \varphi_r \in \mathbb{C}\{x_1, \dots, x_n\}$ be $\mathbb{Z}_m$-semi-invariants and let $X = \{\varphi_1 = \dots = \varphi_r = 0\}/\mathbb{Z}_m(\alpha_1, \dots, \alpha_n)$. Since $X \subseteq (x_1, \dots, x_n)/\mathbb{Z}_m(\alpha_1, \dots, \alpha_n)$, we can define the projective birational morphism $\pi : \bar{X} \to X$ by taking the proper transform. This morphism π is also called the *weighted blow up associated to* σ or σ*-blow up* of X. Thus $\bar{X}$ is covered by $U_1, \dots, U_n$ where $U_i = \bar{U}_i \mid_{\bar{X}}$ $(i = 1, \dots, n)$, and U_i is called the x_i*-chart* of $\bar{X}$.

In this paper, we only use the cases $n = 4$ and $n = 5$, and we usually denote the coordinates of $\mathbb{C}^4$ (resp. $\mathbb{C}^5$) by (x, y, z, u) (resp. (x, y, z, u, t)) (in this order) instead of $(x_1, \dots, x_n)$. We denote the x-chart (resp. y-chart, z-chart, u-chart, t-chart) by U_1 (resp. U_2, U_3, U_4, U_5), and the origin of U_i is denoted by Q_i $(i = 1, \dots, 5)$.

2.8. Let X be a germ of a 3-dimensional terminal singularity of index $m \geq 2$. Then there is a standard embedding $j : X \hookrightarrow (x, y, z, u)/\mathbb{Z}_m(\alpha, \beta, \gamma, \delta)$.

We say that $j_1 : X \hookrightarrow (x_1, y_1, z_1, u_1)/\mathbb{Z}_m(\alpha, \beta, \gamma, \delta)$ is a *liftable embedding* if there is a $\mathbb{Z}_m$-equivariant automorphism $\bar{\chi} : (x, y, z, u) \to (x_1, y_1, z_1, u_1)$ such that $j_1 = \chi \circ j$ where $\chi : (x, y, z, u)/\mathbb{Z}_m \to (x_1, y_1, z_1, u_1)/\mathbb{Z}_m$ is the automorphism induced by $\bar{\chi}$. We can write

$$\begin{aligned} j_1 : X &\simeq \{\varphi'(x_1, y_1, z_1, u_1) = 0\}/\mathbb{Z}_m(\alpha, \beta, \gamma, \delta) \\ &\subseteq (x_1, y_1, z_1, u_1)/\mathbb{Z}_m(\alpha, \beta, \gamma, \delta) \end{aligned}$$

for some $\mathbb{Z}_m$-semi-invariant $\varphi' \in \mathbb{C}\{x_1, y_1, z_1, u_1\}$. Let $v_1 = (j_1, \sigma_1)$ be a pair consisting of a liftable embedding $j_1 : X \hookrightarrow (x_1, y_1, z_1, u_1)/\mathbb{Z}_m$ and a weight $\sigma_1 \in \mathbb{Z}^4 + \frac{1}{m}(\alpha, \beta, \gamma, \delta)\mathbb{Z}$. We define

$$d(v_1) = \sigma_1\text{-}wt(x_1y_1z_1u_1) - \sigma_1\text{-}wt(\varphi') - 1.$$

For a positive rational number k, we define

$$\mathcal{W}_k = \left\{ v_1 = (j_1, \sigma_1) \;\middle|\; \begin{array}{l} j_1 : X \hookrightarrow (x_1, y_1, z_1, u_1)/\mathbb{Z}_m \text{ liftable embedding} \\ \sigma_1 \in \mathbb{Z}^4 + \frac{1}{m}(\alpha, \beta, \gamma, \delta)\mathbb{Z} \text{ weight, } d(v_1) = k \end{array} \right\}.$$

We need another notion to study terminal singularities of type (cD/2). An embedding $j_2 : X \hookrightarrow (x_2, y_2, z_2, u_2, t_2)/\mathbb{Z}_m(\alpha, \beta, \gamma, \delta, \varepsilon)$ is called a *generalized* *liftable embedding* if there is a liftable embedding $j_1 : X \hookrightarrow (x_1, y_1, z_1, u_1)/\mathbb{Z}_m(\alpha, \beta, \gamma, \delta)$ and a $\mathbb{Z}_m$-equivariant morphism $\bar{\chi} : (x_1, y_1, z_1, u_1) \to (x_2, y_2, z_2, u_2, t_2)$ of the form

$$\bar{\chi}^*(x_2) = x_1,\ \bar{\chi}^*(y_2) = y_1,\ \bar{\chi}^*(z_2) = z_1,\ \bar{\chi}^*(u_2) = u_1$$
$$\text{and } \bar{\chi}^*(t_2) = \psi(x_1, y_1, z_1, u_1)$$

such that $j_2 = \chi \circ j_1$ where $\chi : (x_1, y_1, z_1, u_1)/\mathbb{Z}_m \to (x_2, y_2, z_2, u_2, t_2)/\mathbb{Z}_m$ is the morphism induced by $\bar{\chi}$.

If $j_2 : X \hookrightarrow (x_2, y_2, z_2, u_2, t_2)/\mathbb{Z}_m(\alpha, \beta, \gamma, \delta, \varepsilon)$ is a generalized liftable embedding, then we can write

$$j_2 : X \simeq \{\varphi_1(x_2, y_2, z_2, u_2, t_2) = \varphi_2(x_2, y_2, z_2, u_2, t_2) = 0\}/\mathbb{Z}_m(\alpha, \beta, \gamma, \delta, \varepsilon)$$
$$\subseteq (x_2, y_2, z_2, u_2, t_2)/\mathbb{Z}_m(\alpha, \beta, \gamma, \delta, \varepsilon)$$

where $\varphi_1, \varphi_2 \in \mathbb{C}\{x_2, y_2, z_2, u_2, t_2\}$ are $\mathbb{Z}_m$-semi-invariants and we assume that $\varphi_2 = t_2 - \psi(x_2, y_2, z_2, u_2)$. We note that φ_1 is not uniquely determined by j_2. Let $v_2 = (j_2, \sigma_2)$ be a pair consisting of a generalized liftable embedding $j_2 : X \hookrightarrow (x_2, y_2, z_2, u_2, t_2)/\mathbb{Z}_m(\alpha, \beta, \gamma, \delta, \varepsilon)$ and a weight $\sigma_2 \in \mathbb{Z}^5 + \frac{1}{m}(\alpha, \beta, \gamma, \delta, \varepsilon)\mathbb{Z}$. We define

$$d(v_2) = \sigma_2\text{-}wt(x_2 y_2 z_2 u_2 t_2) - \sigma_2\text{-}wt(\varphi_1 \varphi_2) - 1.$$

For a positive rational number k, we define

$$\mathcal{W}'_k = \left\{ v_2 = (j_2, \sigma_2) \;\middle|\; \begin{array}{l} j_2 : X \simeq \{\varphi_1 = \varphi_2 = 0\}/\mathbb{Z}_m \subseteq (x_2, y_2, z_2, u_2, t_2)/\mathbb{Z}_m \\ \quad \text{generalized liftable embedding} \\ \sigma_2 \in \mathbb{Z}^5 + \frac{1}{m}(\alpha, \beta, \gamma, \delta, \varepsilon)\mathbb{Z} \text{ weight, } d(v_2) = k \end{array} \right\}.$$

Each element $v_1 \in \mathcal{W}_k$ (resp. $v_2 \in \mathcal{W}'_k$), determines the projective birational morphism as in (2.7). This is called the v_1-*blow up* (resp. v_2-*blow up*) of X.

Since all embeddings in this paper are liftable or generalized liftable, we often omit the word "liftable" and "generalized liftable".

The following proposition is due to Danilov and Barlow (see [Reid87, (5.7)]) which shows the existence of economic resolutions of cyclic quotient terminal singularities. In this paper, we use it in order to estimate the number of prime divisors with small discrepancies.

Proposition 2.9. *Let $X = (x, y, z)/\mathbb{Z}_m(a, -a, 1)$ (a is prime to m) be a germ of a cyclic quotient terminal singularity of index $m \geq 2$. Then there is a projective birational morphism $\nu : Z \to X$ such that*

(i) *Z is non-singular,*

(ii) $K_Z = \nu^*(K_X) + \sum_{i=1}^{m-1} \frac{i}{m} F_i$, *where* $\sum_{i=1}^{m-1} F_i$ *is the exceptional divisor of* ν. *Furthermore, if* D *is a* $\mathbb{Q}$*-Cartier Weil divisor on* X *defined by a* $\mathbb{Z}_m$*-semi-invariant* $f(x,y,z) \in \mathbb{C}\{x,y,z\}$, *then*

(iii) $\nu^*(D) = \nu^{-1}[D] + \sum_{i=1}^{m-1} d_i F_i$, *where* $\nu^{-1}[D]$ *denotes the proper transform of* D *by* ν, *and* $d_i = \sigma_i$*-wt*$(f(x,y,z))$, $\sigma_i = (\langle ai/m \rangle, \langle -ai/m \rangle, i/m)$.

The following is also used to estimate the number of prime divisors with small discrepancies (see [Hay99, 5.3]):

Proposition 2.10. *Let* X *be a germ of a* 3*-dimensional terminal singularity of index* $m \geq 2$. *Let* $\pi : \bar{X} \to X$ *be a divisorial blow up with discrepancy* $1/m$ *and let* E *be the exceptional divisor of* π. *Let* $\nu : Z \to \bar{X}$ *be a projective birational morphism such that* K_Z *is* $\mathbb{Q}$*-Cartier and* $\sum F_i$ *be the exceptional divisor of* ν. *If* $\nu^*(E) = \nu^{-1}[E] + \sum a_i F_i$, *then we have*

$$a(F_i, X) = a(F_i, \bar{X}) + a_i/m$$

for each i. *In particular, if* $Q \in \bar{X}$ *is of index* $\leq m$ *and* $Q \in E$, *then there are no prime divisors over* Q *with discrepancies* $1/m$ *over* X.

§3. Main Results and an Outline of Proofs

The purpose of this paper is to prove the following theorems concerning with terminal singularities of type (cD/2):

Theorem 3.1. *Let* X *be a germ of a* 3*-dimensional terminal singularity of type* (cD/2) *and let* E *be an arbitrary prime divisor over* X *with discrepancy* $1/2$. *Then there exists* $v \in \mathcal{W}_{1/2} \cup \mathcal{W}'_{1/2}$ *such that*

(i) *the* v*-blow up* $\pi : \bar{X} \to X$ *is pre-divisorial with discrepancy* $1/2$, *and*

(ii) E *is the exceptional divisor of* π.

Furthermore, if π *is divisorial with discrepancy* $1/2$, *then we also have*

(iii) $\sum_{Q \in \bar{X}} (\mathrm{aw}(\bar{X}, Q) - 1) < \mathrm{aw}(X) - 1$.

Theorem 3.2. *If* X *be a germ of a* 3*-dimensional terminal singularity of type* (cD/2), *then* X *admits at least one divisorial blow up with discrepancy* $1/2$.

Theorem 3.3. *Let* X *be a germ of a* 3*-dimensional terminal singularity of type* (cD/2). *Let* n *be the number of prime divisors over* X *with discrepancies* $1/2$. *Using the notation* (2.3), *we have the following:*

(cD/2-1)

$$n = \begin{cases} 1 & \textit{if } a = b = 2, \\ 2 & \textit{if } a \geq 3, b = 2 \textit{ or } a = 2, b \geq 3, \\ 3 & \textit{if } a \geq 3, b \geq 3. \end{cases}$$

(cD/2-2) *Let* τ*-wt*$(x) = 1/2$, τ*-wt*$(z) = 2$ (*resp.* τ'*-wt*$(x) = 1/2$, τ'*-wt*$(z) = 1$) *and assume that* τ*-wt*$(g(x,z)) = b$ (*resp.* τ'*-wt*$(g(x,z)) = b'$). *We consider the following conditions,* (*we think* $a = +\infty$ *if* $\lambda = 0$):

(i) $2a < b'$.

(ii) $2a > b'$, b' *is odd and* $g_{\tau'\text{-}wt=b'}(x,z)$ *is a square.*

(iii) $2a > b'$, b' *is even and* $g_{\tau'\text{-}wt=b'}(x,z)z$ *is a square.*

(iv) $2a = b'$ *and* $\frac{\lambda^2}{4}x^{4a+2} - g_{\tau'\text{-}wt=b'}(x,z)z$ *is a square.*

Then we have

$$n = \begin{cases} \lfloor \frac{1}{2}\min\{2a+2, b+3\} \rfloor & \textit{if (i), (ii), (iii) or (iv) holds,} \\ \lfloor \frac{1}{2}\min\{2a, b+1\} \rfloor & \textit{otherwise,} \end{cases}$$

where $\lfloor \alpha \rfloor$ *is the largest integer* $\leq \alpha$.

The method to prove these theorems is essentially the same as [Hay99]. We first find $v \in \mathcal{W}_{1/2}$ and make a v-blow up $\pi : \bar{X} \to X$. If π is divisorial with discrepancy $1/2$, then we further blow up $\bar{X}$ in order to count the number of prime divisors with discrepancies $1/2$ over X. Once we know this number, the rest is only to find other blow ups which give divisorial or pre-divisorial blow ups not isomorphic to π. In this process, we sometimes need an embedding of X into $(x,y,z,u,t)/\mathbb{Z}_2$ and a weight σ. These calculations will be done in Sections 4 and 5.

Combined with the results in [Hay99], we find all prime divisors with discrepancies $1/m$ if X is a germ of a 3-dimensional terminal singularities of indices $m \geq 2$. In particular, by using [Hay99, 4.5], (3.1) and (3.2), we get the following:

Theorem 3.4. *Let* X *be a germ of a* 3*-dimensional terminal singularity of index* $m \geq 2$. *Then there exists at least one divisorial blow up* $\pi : \bar{X} \to X$ *with discrepancy* $1/m$. *Furthermore, we have*

$$\sum_{Q \in \bar{X}} (\mathrm{aw}(\bar{X}, Q) - 1) \leq \mathrm{aw}(X) - 1,$$

and the equality holds only if X *is a cyclic quotient singularity or of type* (cD/3).

We also have the following which will be proved by using (3.4) and the induction on axial weights. The proof will be given in Section 6.

Theorem 3.5. *Let X be a germ of a 3-dimensional terminal singularity of index $m \geq 2$. Then there is a sequence*

$$X_N \xrightarrow{\pi_N} X_{N-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\pi_1} X_0 = X$$

such that

(i) *X_i has only terminal singularities ($i = 0, 1, \ldots, N$) and furthermore X_N has only Gorenstein terminal singularities, and*

(ii) *π_i is a divisorial blow up at $P_{i-1} \in X_{i-1}$ with discrepancy $1/m_i$, where m_i is the index at P_{i-1} ($i = 1, \ldots, N$).*

Let $\pi : X_N \to X$ be the projective birational morphism obtained by composing $\pi_1, \ldots, \pi_N$ in (3.5), and let E be the exceptional divisor of π. Then all divisors with discrepancies < 1 over X appear as the irrreducible component of E. However, some irreducible components of E may have discrepancy ≥ 1 over X.

§4. Terminal Singularities of Type (cD/2-1)

4.1. In this section, X denotes a germ of a 3-dimensional terminal singularity of type (cD/2-1). There is a standard embedding

$$\begin{aligned} j : X \simeq \{u^2 + xyz + x^{2a} + y^{2b} + z^c = 0\}/\mathbb{Z}_2(1,1,0,1) \\ \subseteq (x,y,z,u)/\mathbb{Z}_2(1,1,0,1) \end{aligned}$$

where a, $b \geq 2$ and $c \geq 3$. We have $\mathrm{aw}(X) = c$ in this case. By symmetry of x and y, we may assume that $a \geq b$.

Lemma 4.2. *If a weight $\sigma \in \mathbb{Z}^4 + \frac{1}{2}(1,1,0,1)\mathbb{Z}$ satisfies $(j, \sigma) \in \mathcal{W}_{1/2}$, then $\sigma = \frac{1}{2}(1,1,2,1)$, $\frac{1}{2}(1,1,2,3)$, $\frac{1}{2}(1,3,2,3)$, $\frac{1}{2}(3,1,2,3)$ or $\frac{1}{2}(1,1,4,3)$.*

Proof. Let $\varphi = u^2 + xyz + x^{2a} + y^{2b} + z^c$. We write $w = \sigma\text{-}wt$. Since $xyz \in \varphi$, we see that $w(x) + w(y) + w(z) \geq w(\varphi)$. Thus we have

$$1/2 = w(xyzu) - w(\varphi) - 1 \geq w(u) - 1,$$

and we get $w(u) = 1/2$ or $3/2$. If $w(u) = 1/2$, then $u^2 \in \varphi$ shows that $w(\varphi) = 1$ and $w(xyz) = 2$, thus $\sigma = \frac{1}{2}(1,1,2,1)$. If $w(u) = 3/2$, then $w(\varphi) = 2$ or 3 since a, $b \geq 2$, $c \geq 3$. In this case we have $w(xyz) = 2$ or 3 and know that $\sigma = \frac{1}{2}(1,1,2,3)$, $\frac{1}{2}(1,3,2,3)$, $\frac{1}{2}(3,1,2,3)$ or $\frac{1}{2}(1,1,4,3)$. □

We define

$$\sigma_1 = \tfrac{1}{2}(1,1,2,3),\ \sigma_2 = \tfrac{1}{2}(1,3,2,3),\ \sigma_2' = \tfrac{1}{2}(3,1,2,3) \text{ and } \sigma_3 = \tfrac{1}{2}(1,1,4,3).$$

We also define $v_1 = (j, \sigma_1)$, $v_2 = (j, \sigma_2)$, $v_2' = (j, \sigma_2')$ and $v_3 = (j, \sigma_3)$.

§4-A. Case: $a = b = 2$

4.3. We first assume that $a = b = 2$ and study this case in (4.4). It is easy to see that $v_1 \in \mathcal{W}_{1/2}$ in this case.

Proposition 4.4. *Assume that $a = b = 2$. Let $\pi_1 : \bar{X}_1 \to X$ be the v_1-blow up and let E_1 be the exceptional divisor of π_1. Then π_1 is divisorial with discrepancy $1/2$ and $\sum_{Q \in \bar{X}_1}(\mathrm{aw}(\bar{X}_1, Q) - 1) = c - 3$. Moreover, E_1 is the unique prime divisor with discrepancy $1/2$ over X.*

Proof. Since $E_1 \simeq \{xyz + x^4 + y^4 = 0\} \subseteq \mathbb{P}(1,1,2,3)$, we see that E_1 is Cartier outside $\{Q_3, Q_4\}$ and that $\mathrm{Sing}(E_1) = \{x = y = 0\}$. Hence we need only the z-chart U_3 and the u-chart U_4 in order to study singularities of $\bar{X}_1$:

$$U_3 = \{\bar{z}\bar{u}^2 + \bar{x}\bar{y} + \bar{x}^4 + \bar{y}^4 + \bar{z}^{c-2} = 0\}/\mathbb{Z}_2(1,1,0,1),$$
$$U_4 = \{\bar{u} + \bar{x}\bar{y}\bar{z} + \bar{x}^4 + \bar{y}^4 + \bar{u}^{c-2}\bar{z}^c = 0\}/\mathbb{Z}_3(2,2,1,2).$$

The origin Q_3 of U_3 is terminal of type (cA/2) with axial weight $c-2$, the origin Q_4 of U_4 is isomorphic to $(\bar{x}, \bar{y}, \bar{z})/\mathbb{Z}_3(1,1,2)$, and other singularities on $\bar{X}_1$ are all isolated cDV points. Therefore $\bar{X}_1$ has only terminal singularities and $\sum_{Q \in \bar{X}_1}(\mathrm{aw}(\bar{X}_1, Q) - 1) = c - 3$. Since E_1 is irreducible and $d(v_1) = 1/2$, π_1 is divisorial with discrepancy $1/2$.

We can resolve the origin Q_4 of U_4 by using (2.9) and get a projective birational morphism $\nu : Z \to \bar{X}_1$ such that $K_Z = \nu^*(K_{\bar{X}_1}) + \frac{1}{3}F_1 + \frac{2}{3}F_2$ where $F_1 + F_2$ is the exceptional divisor of ν over Q_4. Since E_1 is defined by $\bar{x}\bar{y}\bar{z} + \bar{x}^4 + \bar{y}^4 = 0$ near Q_4, we see that $\nu^*(E_1) = \nu^{-1}[E_1] + \frac{4}{3}F_1 + \frac{5}{3}F_2$. Therefore $a(F_1, X) = 1/3 + 1/2 \cdot 4/3 = 1$ and $a(F_2, X) = 2/3 + 1/2 \cdot 5/3 = 3/2$. By (2.10), there are no prime divisors over Q_3 with discrepancies $1/2$. Thus E_1 is the unique prime divisor with discrepancy $1/2$ over X. □

4.5. Thus (4.4) completes the proof of (3.1), (3.2) and (3.3) if $a = b = 2$.

§4-B. Case: $a \geq 3$ and $b = 2$

4.6. We next assume that $a \geq 3$ and $b = 2$ and study this case in (4.7) and (4.9). We easily see that $v_2 \in \mathcal{W}_{1/2}$ in this case.

Proposition 4.7. *Assume that $a \geq 3$ and $b = 2$. Let $\pi_2 : \bar{X}_2 \to X$ be the v_2-blow up. Then π_2 is divisorial with discrepancy $1/2$ and $\sum_{Q \in \bar{X}_2}(\mathrm{aw}(\bar{X}_2, Q) - 1) = \max\{c - 3, 1\}$. Furthermore, there are exactly two prime divisors with discrepancies $1/2$ over X.*

Proof. Let E_2 be the exceptional divisor of π_2. Since

$$E_2 \simeq \{u^2 + xyz + \delta_{a,3}x^6 + \delta_{c,3}z^3 = 0\} \subseteq \mathbb{P}(1,3,2,3),$$

we see that E_2 is Cartier outside $\{Q_2, Q_3\}$ and that $\mathrm{Sing}(E_2) \subseteq \{x = u = 0\} \cup \{z = u = 0\}$. Since $Q_4 \notin E_2$, $\bar{X}_2$ is covered by three affine open sets as follows:

$$\begin{aligned} U_1 &= \{\bar{u}^2 + \bar{y}\bar{z} + \bar{x}^{a-3} + \bar{x}^{3b-3}\bar{y}^{2b} + \bar{x}^{c-3}\bar{z}^c = 0\} \subseteq \mathbb{C}^4, \\ U_2 &= \{\bar{u}^2 + \bar{x}\bar{z} + \bar{x}^{2a}\bar{y}^{a-3} + \bar{y}^{3b-3} + \bar{y}^{c-3}\bar{z}^c = 0\}/\mathbb{Z}_3(2,2,1,0), \\ U_3 &= \{\bar{u}^2 + \bar{x}\bar{y} + \bar{x}^{2a}\bar{z}^{a-3} + \bar{y}^{2b}\bar{z}^{3b-3} + \bar{z}^{c-3} = 0\}/\mathbb{Z}_2(1,1,0,1). \end{aligned}$$

The origin Q_2 of U_2 is terminal of type (cA/3) with axial weight 2. We see that $Q_3 \in E_2$ if and only if $c \geq 4$. If $Q_3 \in E_2$, then the origin Q_3 of U_3 is terminal of type (cA/2) with axial weight $c - 3$. Other singularities on $\bar{X}_2$ are all isolated cDV points. Hence $\bar{X}_2$ has only terminal singularities and $\sum_{Q \in \bar{X}_2}(\mathrm{aw}(\bar{X}_2, Q) - 1) = \max\{c - 3, 1\}$. Since E_2 is irreducible and $d(v_2) = 1/2$, we see that π_2 is divisorial with discrepancy $1/2$.

By [Hay99, 6.4], there is only one prime divisor with discrepancy $1/3$ over the origin Q_2 of the y-chart U_2 and there is a projective birational morphism $\nu : Z \to \bar{X}_2$ such that $K_Z = \nu^*(K_{\bar{X}_2}) + \frac{1}{3}F$ where F is the exceptional divisor of ν over Q_2. Since ν is obtained by the $\frac{1}{3}(1,1,2,3)$-blow up of U_2 and since E_2 is defined by $\bar{y} = 0$ on U_2, we have $\nu^*(E_2) = \nu^{-1}[E_2] + \frac{1}{3}F$. Hence $a(F, X) = 1/3 + 1/2 \cdot 1/3 = 1/2$. By (2.10), all other prime divisors over X have discrepancies ≥ 1. Therefore E_2 and F are the prime divisors with discrepancies $1/2$ over X. □

4.8. In the case $a \geq 3$ and $b = 2$, it is impossible to obtain a liftable embedding $j : X \hookrightarrow \mathbb{C}^4/\mathbb{Z}_2(1,1,0,1)$ and a weight σ such that the (j, σ)-blow up of X is divisorial with discrepancy $1/2$ and is different from π_2 in (4.7). Instead, we use the generalized liftable embedding $j' : X \hookrightarrow (x, y, z, u, t)/\mathbb{Z}_2(1,1,0,1,1)$ such that

$$\begin{aligned} j' : X \simeq &\left\{ \begin{array}{c} u^2 + yt + x^{2a} + z^c = 0 \\ t = xz + y^3 \end{array} \right\}/\mathbb{Z}_2(1,1,0,1,1) \\ &\subseteq (x, y, z, u, t)/\mathbb{Z}_2(1,1,0,1,1). \end{aligned}$$

We define $\sigma' = \frac{1}{2}(1,1,2,3,5)$, then we have $v' = (j', \sigma') \in \mathcal{W}'_{1/2}$.

Proposition 4.9. *Assume that $a \geq 3$ and $b = 2$. Let $\pi' : \bar{X}' \to X$ be the v'-blow up. Then π' is divisorial with discrepancy $1/2$ and $\sum_{Q \in \bar{X}'}(\mathrm{aw}(\bar{X}', Q) - 1) = \max\{c - 4, 0\}$. Moreover, π' and π_2 in (4.7) are not isomorphic over X.*

Proof. Let E' be the exceptional divisor of π'. Since

$$E' \simeq \left\{ \begin{array}{c} u^2 + yt + \delta_{a,3}x^6 + \delta_{c,3}z^3 = 0 \\ xz + y^3 = 0 \end{array} \right\} \subseteq \mathbb{P}(1,1,2,3,5),$$

we see that E' is Cartier outside $\{Q_3, Q_5\} \cap E'$ and $\operatorname{Sing}(E') \subseteq \{Q_1, Q_3, Q_5\}$. We also see that $E' \cap \{x = z = t = 0\} = \emptyset$, hence $\bar{X}'$ is covered by three affine open sets as follows:

$$\begin{aligned} U_1 &= \{\bar{u}^2 + \bar{y}\bar{t} + \bar{x}^{a-3} + \bar{x}^{c-3}\bar{z}^c = 0,\ \bar{x}\bar{t} - \bar{z} - \bar{y}^3 = 0\} \subseteq \mathbb{C}^5, \\ U_3 &= \{\bar{u}^2 + \bar{y}\bar{t} + \bar{x}^{2a}\bar{z}^{a-3} + \bar{z}^{c-3} = 0,\ \bar{z}\bar{t} - \bar{x} - \bar{y}^3 = 0\}/\mathbb{Z}_2(1,1,0,1,1), \\ U_5 &= \{\bar{u}^2 + \bar{y} + \bar{x}^{2a}\bar{t}^{a-3} + \bar{z}^c\bar{t}^{c-3} = 0,\ \bar{t} - \bar{x}\bar{z} - \bar{y}^3 = 0\}/\mathbb{Z}_5(1,1,2,3,3). \end{aligned}$$

It is easy to see that $Q_3 \in E'$ if and only if $c \geq 4$. If $Q_3 \in E'$, then Q_3 is terminal of type (cA/2) with axial weight $c - 3$. The origin Q_5 of U_5 is isomorphic to $(\bar{x}, \bar{z}, \bar{u})/\mathbb{Z}_5(1,2,3)$, and the origin Q_1 of U_1 is an isolated cDV point. Thus $\bar{X}'$ has only terminal singularities and $\sum_{Q \in \bar{X}'}(\operatorname{aw}(\bar{X}', Q) - 1) = \max\{c-4, 0\}$. Since E' is irreducible and $d(v') = 1/2$, we see that π' is divisorial with discrepancy $1/2$.

Let D be the $\mathbb{Q}$-Cartier Weil divisor on X defined by $y = 0$. Then we have

$$\pi_2^*(D) = \pi_2^{-1}[D] + \frac{3}{2}E_2 \text{ and } \pi'^*(D) = \pi'^{-1}[D] + \frac{1}{2}E'.$$

Thus π_2 and π' are not isomorphic over X. □

4.10. In the case $a \geq 3$, $b = 2$, we saw in (4.7) that there are exactly two prime divisors with discrepancies $1/2$ over X. We also gave two divisorial blow ups of X with discrepancies $1/2$ in (4.7) and (4.9) which are not isomorphic over X. Thus we complete the proof of (3.1), (3.2) and (3.3) if $a \geq 3$ and $b = 2$.

§4-C. Case: $a \geq 3$ and $b \geq 3$

4.11. Lastly we assume that $a \geq 3$ and $b \geq 3$. We shall treat this case in (4.12). We easily see that v_2, v_2', $v_3 \in \mathcal{W}_{1/2}$ in this case.

Proposition 4.12. *Assume that $a \geq 3$ and $b \geq 3$. Let $\pi_2 : \bar{X}_2 \to X$ (resp. $\pi_2' : \bar{X}_2' \to X$, $\pi_3 : \bar{X}_3 \to X$) be the v_2-blow up (resp. v_2'-blow up, v_3-blow up). Then π_2, π_2' and π_3 are all divisorial with discrepancies $1/2$. No two of these three blow ups are isomorphic over X and there are exactly three prime divisors with discrepancies $1/2$ over X. Moreover, we have $\sum_{Q \in \bar{X}_2}(\operatorname{aw}(\bar{X}_2, Q) - 1) = \sum_{Q \in \bar{X}_2'}(\operatorname{aw}(\bar{X}_2', Q) - 1) = \max\{c - 3, 1\}$ and $\sum_{Q \in \bar{X}_3}(\operatorname{aw}(\bar{X}_3, Q) - 1) = c - 2$.*

Proof. The calculations for π_2 and π_2' are almost the same as in (4.7). We shall treat the v_3-blow up π_3 here. Let E_3 be the exceptional divisor of π_3. Since

$$E_3 \simeq \{u^2 + xyz + \delta_{a,3}x^6 + \delta_{b,3}y^6 = 0\} \subseteq \mathbb{P}(1,1,4,3),$$

we see that E_3 is Cartier outside $\{Q_3\}$ and that $\mathrm{Sing}(E_3) \subseteq \{x = u = 0\} \cup \{y = u = 0\}$. Since $Q_4 \notin E_3$, $\bar{X}_3$ is covered by three affine open sets as follows:

$$\begin{aligned} U_1 &= \{\bar{u}^2 + \bar{y}\bar{z} + \bar{x}^{a-3} + \bar{x}^{b-3}\bar{y}^{2b} + \bar{x}^{2c-3}\bar{z}^c = 0\} \subseteq \mathbb{C}^4, \\ U_2 &= \{\bar{u}^2 + \bar{x}\bar{z} + \bar{x}^{2a}\bar{y}^{a-3} + \bar{y}^{b-3} + \bar{y}^{2c-3}\bar{z}^c = 0\} \subseteq \mathbb{C}^4, \\ U_3 &= \{\bar{u}^2 + \bar{x}\bar{y} + \bar{x}^{2a}\bar{z}^{a-3} + \bar{y}^{2b}\bar{z}^{b-3} + \bar{z}^{2c-3} = 0\}/\mathbb{Z}_4(1,1,2,3). \end{aligned}$$

The origin Q_3 of U_3 is terminal of type (cAx/4) with axial weight $c-1$ and other singularities are all isolated cDV points. Hence $\bar{X}_3$ has only terminal singularities and $\sum_{Q\in\bar{X}_3}(\mathrm{aw}(\bar{X}_3, Q) - 1) = c - 2$. Thus we see that π_3 is divisorial with discrepancy $1/2$ since $d(v_3) = 1/2$.

Let D_1 be the $\mathbb{Q}$-Cartier Weil divisor on X defined by $x = 0$. Then we have

$$\pi_2^*(D_1) = \pi_2^{-1}[D_1] + \frac{1}{2}E_2,\ {\pi_2'}^*(D_1) = {\pi_2'}^{-1}[D_1] + \frac{3}{2}E_2'$$
$$\text{and } \pi_3^*(D_1) = \pi_3^{-1}[D_1] + \frac{1}{2}E_3.$$

where E_2 (resp. E_2') is the exceptional divisor of π_2 (resp. π_2'). Therefore π_2' is not isomorphic to π_2 and π_3 over X. By considering the divisors defined by $y = 0$ and $z = 0$, we see that π_2, π_2' and π_3 are not mutually isomorphic over X.

In order to calculate the number of prime divisors with discrepancies $1/2$, we use the v_3-blow up as the first blow up. The origin Q_3 of the z-chart U_3 of $\bar{X}_3$ is the unique non-Gorenstein point of $\bar{X}_3$ and it is terminal of type (cAx/4). By [Hay99, 7.4, 7.9], there are at most two prime divisors with discrepancies $1/4$ over $\bar{X}_3$. Thus, by (2.10), we see that there are at most three prime divisors (including E_3) with discrepancies $1/2$ over X. On the other hand, we already have three prime divisors E_2, E_2', E_3 which have discrepancies $1/2$ over X. Therefore there are exactly three prime divisors with discrepancies $1/2$ over X. □

4.13. In the case a, $b \geq 3$, we saw in (4.12) that there are exactly three prime divisors with discrepancies $1/2$ over X. We also gave three divisorial blow ups of X with discrepancies $1/2$, and they are not mutually isomorphic over X. Thus we complete the proof of (3.1), (3.2) and (3.3) if a, $b \geq 3$.

§5. Terminal Singularities of Type (cD/2-2)

5.1. In this section, X denotes a germ of a 3-dimensional terminal singularity of type (cD/2-2). There is a standard embedding

$$j : X \simeq \{u^2 + y^2z + \lambda yx^{2a+1} + g(x,z) = 0\}/\mathbb{Z}_2(1,1,0,1) \\ \subseteq (x,y,z,u)/\mathbb{Z}_2(1,1,0,1)$$

where $\lambda \in \mathbb{C}$, $a \geq 1$ and $g(x,z) \in (x^4, x^2z^2, z^3)\mathbb{C}\{x^2, z\}$. We shall write $g(x,z) = \sum_{i,j} a_{i,j}x^{2i}z^j$. We have $\mathrm{aw}(X) = \min\{j \mid a_{0,j} \neq 0\}$ in this case.

By considering $a = +\infty$ when $\lambda = 0$, we shall always assume that $\lambda \neq 0$. This will cause no trouble in the following discussion.

Lemma 5.2. *If a weight $\sigma \in \mathbb{Z}^4 + \frac{1}{2}(1,1,0,1)\mathbb{Z}$ satisfies $(j,\sigma) \in \mathcal{W}_{1/2}$, then $\sigma = \frac{1}{2}(1,l,2,l)$, $\frac{1}{2}(1,l,2,l+2)$ or $\frac{1}{2}(1,l,4,l+2)$ for some positive odd integer l.*

Proof. Let $\varphi = u^2 + y^2z + \lambda yx^{2a+1} + g(x,z)$. We write $w = \sigma\text{-}wt$. Since u^2, $y^2z \in \varphi$, we see that $w(u) \geq w(\varphi)/2$ and $2w(y) + w(z) \geq w(\varphi)$. Thus we have

$$1/2 = w(xyzu) - w(\varphi) - 1 \geq w(x) + w(z)/2 - 1,$$

and we see that $w(x) = 1/2$ and $w(z) = 1$ or 2. If $w(x) = 1/2$, $w(z) = 2$, then the above inequalities shows that $w(u) \geq w(\varphi)/2$, $w(y) \geq w(\varphi)/2 - 1$ and $w(y) + w(u) - w(\varphi) \leq 1$. Hence we get $w(u) = w(y) + 1$. In the case $w(x) = 1/2$, $w(z) = 1$, we can argue similarly to get $w(u) = w(y)$ or $w(u) = w(y) + 1$. □

These are the candidates of weights when one wants to blow up X using $(j,\sigma) \in \mathcal{W}_{1/2}$.

5.3. We denote $\tau\text{-}wt(x) = 1/2$, $\tau\text{-}wt(z) = 2$, and assume that $b = \tau\text{-}wt(g(x,z)) = \min\{i + 2j \mid a_{i,j} \neq 0\}$. We denote

$$A = \{l \in \mathbb{Z} \mid 1 \leq l \leq \min\{2a - 3, b - 2\},\ l : \text{odd}\ \},$$

and define $\sigma_l = \frac{1}{2}(1,l,4,l+2)$, then it is easy to see that $v_l = (j,\sigma_l) \in \mathcal{W}_{1/2}$ for $l \in A$. We remark that A has $\lfloor \frac{1}{2}\min\{2a-2, b-1\} \rfloor$ elements.

Furthermore, we denote $\tau'\text{-}wt(x) = 1/2$, $\tau'\text{-}wt(z) = 1$, and assume that $b' = \tau'\text{-}wt(g(x,z)) = \min\{i + j \mid a_{i,j} \neq 0\}$.

Proposition 5.4. *Under the notation (5.3), the v_l-blow up $\pi_l : \bar{X}_l \to X$ are all pre-divisorial with discrepancies $1/2$. No two of these π_l are isomorphic over X. Furthermore, if $l \in A$ is maximal, then π_l is divisorial with discrepancy $1/2$ and $\sum_{Q \in \bar{X}_l}(\mathrm{aw}(\bar{X}_l, Q) - 1) = \mathrm{aw}(X) - (l+3)/2$.*

Proof. Let E_l be the exceptional divisor of π_l. Since

$$E_l \simeq \{u^2 + y^2 z + \lambda\delta_{l,2a-3} y x^{2a+1} + g_{\tau\text{-}wt=l+2}(x,z) = 0\} \subseteq \mathbb{P}(1,l,4,l+2),$$

we see that E_l is Cartier outside $\{Q_2, Q_3\}$ and that $\mathrm{Sing}(E_l) \subseteq \{u=0\}$. We shall study singularities of $\bar{X}_l$. Since $Q_4 \notin E_l$, $\bar{X}_l$ is covered by three affine open sets as follows:

$$\begin{aligned}
U_1 &= \{\bar{u}^2 + \bar{y}^2\bar{z} + \lambda\bar{y}\bar{x}^{a-(l+3)/2} + g(\bar{x}^{1/2}, \bar{x}^2\bar{z})/\bar{x}^{l+2} = 0\} \subseteq \mathbb{C}^4,\\
U_2 &= \{\bar{u}^2 + \bar{z} + \lambda\bar{y}^{a-(l+3)/2}\bar{x}^{2a+1} + g(\bar{x}\bar{y}^{1/2}, \bar{y}^2\bar{z})/\bar{y}^{l+2} = 0\}/\mathbb{Z}_l(1,-2,4,2),\\
U_3 &= \{\bar{u}^2 + \bar{y}^2 + \lambda\bar{y}\bar{x}^{2a+1}\bar{z}^{a-(l+3)/2} + g(\bar{x}\bar{z}^{1/2}, \bar{z}^2)/\bar{z}^{l+2} = 0\}/\mathbb{Z}_4(1,l,2,l+2).
\end{aligned}$$

We first assume that l is not maximal, i.e., $l \le \min\{2a-5, b-4\}$. Then $\bar{X}_l$ is singular along $\bar{z}$-axis of U_1 and $\bar{x}$-axis of U_3 which are all canonical. The origin Q_2 of U_2 is also a singular point of $\bar{X}_l$ and it is isomorphic to $(\bar{x},\bar{y},\bar{u})/\mathbb{Z}_l(1,-2,2)$. Thus we see that $\bar{X}_l$ has only canonical (non-terminal) singularities.

Next, we assume that l is maximal. Then we have $l = \min\{2a-4, b-3\}$ or $l = \min\{2a-3, b-2\}$, and we see that $\bar{X}_l$ has only isolated singularities which are all terminal. In particular the origin Q_3 of U_3 is terminal of type (cAx/4) with axial weight $\mathrm{aw}(X) - (l+1)/2$. Therefore $\bar{X}_l$ has only terminal singularities and we have $\sum_{Q\in\bar{X}_l}(\mathrm{aw}(\bar{X}_l,Q) - 1) = \mathrm{aw}(X) - (l+3)/2$.

Since E_l is irreducible and $d(v_l) = 1/2$, we see that $K_{\bar{X}_l} = \pi_l^*(K_X) + \frac{1}{2}E_l$.

Let D be the $\mathbb{Q}$-Cartier Weil divisor on X defined by $y=0$. Then we have $\pi_l^*(D) = \pi_l^{-1}[D] + \frac{l}{2}E_l$. Hence no two of π_l are isomorphic over X. □

Proposition 5.5. *Let n be the number of prime divisors with discrepancies $1/2$ over X. Under the notation* (5.3), *we consider the following conditions*:

(i) $2a < b'$.

(ii) $2a > b'$, *b' is odd and $g_{\tau'\text{-}wt=b'}(x,z)$ is a square.*

(iii) $2a > b'$, *b' is even and $g_{\tau'\text{-}wt=b'}(x,z)z$ is a square.*

(iv) $2a = b'$ *and* $-\frac{\lambda^2}{4}x^{4a+2} + g_{\tau'\text{-}wt=b'}(x,z)z$ *is a square.*

If we further assume that $A \neq \emptyset$, then we have

$$(5.5.1)\qquad n \le \begin{cases} \lfloor \frac{1}{2}\min\{2a+2, b+3\} \rfloor & \text{if (i), (ii), (iii) or (iv) holds,}\\ \lfloor \frac{1}{2}\min\{2a, b+1\} \rfloor & \text{otherwise.}\end{cases}$$

Proof. We first make a blow up $\pi_l : \bar{X}_l \to X$ as in the proof of (5.4) with $l = \min\{2a-3, b-2\}$ or $l = \min\{2a-4, b-3\}$. By (2.10), we only need to study prime divisors F over $\bar{X}_l$ with $a(F, \bar{X}_l) < 1/2$ in order to get an estimate of n. So we shall study the origin Q_2 of U_2 and Q_3 of U_3.

We can resolve the origin Q_2 of U_2 by using (2.9), and get a projective birational morphism $\nu : Z \to \bar{X}_l$ such that $K_Z = \nu^*(K_{\bar{X}_l}) + \sum_{i=1}^{l-1} \frac{i}{l} F_i$ where $\sum_{i=1}^{l-1} F_i$ is the exceptional divisor of ν over Q_2. Since E_l is defined by $\bar{y} = 0$ near Q_2, we have

$$\nu^*(E_l) = \nu^{-1}[E_l] + \sum_{i=1}^{(l-1)/2} \frac{l-2i}{l} F_i + \sum_{i=(l+1)/2}^{l-1} \frac{2l-2i}{l} F_i.$$

Thus we see that

$$a(F_i, X) = \begin{cases} \dfrac{i}{l} + \dfrac{1}{2} \cdot \dfrac{l-2i}{l} = \dfrac{1}{2} & \text{if } i = 1, 2, \dots, (l-1)/2, \\ \dfrac{i}{l} + \dfrac{1}{2} \cdot \dfrac{2l-2i}{l} = 1 & \text{if } i = (l+1)/2, \dots, l-1. \end{cases}$$

Therefore, among F_i, there are exactly $(l-1)/2 = \lfloor \frac{1}{2} \min\{2a-2, b-1\} \rfloor - 1$ prime divisors with discrepancies $1/2$ over X.

Next we study the singularity at $Q_3 \in U_3$ and prime divisors F over U_3 with $a(F, \bar{X}_l) = 1/4$. We have

$$U_3 = \{\bar{u}^2 + (\bar{y} + \tfrac{\lambda}{2}\bar{x}^{2a+1}\bar{z}^{a-(l+3)/2})^2 - \tfrac{\lambda^2}{4}\bar{x}^{4a+2}\bar{z}^{2a-l-3} + g(\bar{x}\bar{z}^{1/2}, \bar{z}^2)/\bar{z}^{l+2} = 0\} \\ /\mathbb{Z}_4(1, l, 2, l+2),$$

and it is of type (cAx/4). Let $h(\bar{x}, \bar{z}) = -\frac{\lambda^2}{4}\bar{x}^{4a+2}\bar{z}^{2a-l-3} + g(\bar{x}\bar{z}^{1/2}, \bar{z}^2)/\bar{z}^{l+2}$. We denote $\bar{\tau}\text{-}wt(\bar{x}) = 1/4$, $\bar{\tau}\text{-}wt(\bar{z}) = 1/2$, and denote $k = \bar{\tau}\text{-}wt(h(\bar{x}, \bar{z}))$. Then $k = \min\{2a, b'\} - 1 - l/2$. Since

$$h_{\bar{\tau}\text{-}wt=k}(\bar{x}, \bar{z}) = \begin{cases} -\frac{\lambda^2}{4}\bar{x}^{4a+2}\bar{z}^{2a-l-3} & \text{if } 2a < b', \\ \sum_{i+j=b'} a_{ij}\bar{x}^{2i}\bar{z}^j/\bar{z}^{l-b'+2} & \text{if } 2a > b', \\ (-\frac{\lambda^2}{4}\bar{x}^{4a+2} + \sum_{i+j=b'} a_{ij}\bar{x}^{2i}\bar{z}^{j+1})/\bar{z}^{l-b'+3} & \text{if } 2a = b', \end{cases}$$

we see that $h_{\bar{\tau}\text{-}wt=k}(\bar{x}, \bar{z})$ is a square if and only if (i), (ii), (iii) or (iv) holds. By [Hay99, 7.4, 7.9], there are exactly two (resp. one) prime divisors with discrepancies $1/4$ over the origin Q_3 of U_3 if $h_{\bar{\tau}\text{-}wt=k}(\bar{x}, \bar{z})$ is a square (resp. is not a square). By (2.10), over the origin Q_3 of U_3, there are at most two (resp. one) prime divisors F with $a(F, X) = 1/2$ if (i), (ii), (iii) or (iv) holds (resp. (i), (ii), (iii) and (iv) does not hold).

Therefore, including E_l, we see that

$$n \le \begin{cases} \lfloor \frac{1}{2} \min\{2a+2, b+3\} \rfloor & \text{if (i), (ii), (iii) or (iv) holds,} \\ \lfloor \frac{1}{2} \min\{2a, b+1\} \rfloor & \text{otherwise.} \end{cases}$$

□

Remark 5.6. In fact, (5.5.1) holds with equality. Moreover (5.5.1) holds with equality even if $A = \emptyset$. These will be proved by finding one or two more prime divisors with discrepancies $1/2$ over X which are different from E_l. The rest of this section is devoted to finding these prime divisors as the exceptional divisors of divisorial or pre-divisorial blow ups of X. We shall do this by dividing into several cases. We also remark that $A = \emptyset$ if and only if $a = 1$ or $b' = 2$. If $b' = 2$, then $x^4 \in g(x, z)$ and we have $b = b'$.

§5-A. Case: $2a < b'$

5.7. We first treat the case $2a < b'$. In this case, we define $\sigma = \frac{1}{2}(1, 2a+1, 2, 2a+1)$, then $v = (j, \sigma) \in \mathcal{W}_{1/2}$ where $j : X \hookrightarrow (x, y, z, u)/\mathbb{Z}_2(1, 1, 0, 1)$ is the standard embedding. We also need another embedding $j' : X \hookrightarrow (x, y, z, u, t)/\mathbb{Z}_2(1, 1, 0, 1, 1)$ such that

$$j' : X \simeq \left\{ \begin{array}{c} u^2 + yt + g(x,z) = 0 \\ t = yz + \lambda x^{2a+1} \end{array} \right\} / \mathbb{Z}_2(1, 1, 0, 1, 1)$$
$$\subseteq (x, y, z, u, t)/\mathbb{Z}_2(1, 1, 0, 1, 1).$$

We define $\sigma' = \frac{1}{2}(1, 2a-1, 2, 2a+1, 2a+3)$ and $v' = (j', \sigma')$. It is easy to see that $v' \in \mathcal{W}'_{1/2}$.

Proposition 5.8. *Under the notation and assumptions* (5.7), *the v-blow up $\pi : \bar{X} \to X$ is divisorial with discrepancy $1/2$ and we have $\sum_{Q \in \bar{X}} (\mathrm{aw}(\bar{X}, Q) - 1) = \max\{\mathrm{aw}(X) - 2a - 1, 1\}$.*

Proof. Let E be the exceptional divisor of π. Since

$$E \simeq \{u^2 + \lambda y x^{2a+1} + g_{\tau'\text{-}wt=2a+1}(x, z) = 0\} \subseteq \mathbb{P}(1, 2a+1, 2, 2a+1),$$

we see that E is Cartier outside $\{Q_2, Q_3\} \cap E$ and that $\mathrm{Sing}(E) \subseteq \{x = u = 0\}$. Since $Q_4 \notin E$, $\bar{X}$ is covered by three affine open sets as follows:

$$U_1 = \{\bar{u}^2 + \bar{x}\bar{y}^2\bar{z} + \lambda\bar{y} + g(\bar{x}^{1/2}, \bar{x}\bar{z})/\bar{x}^{2a+1} = 0\} \subseteq \mathbb{C}^4,$$
$$U_2 = \{\bar{u}^2 + \bar{y}\bar{z} + \lambda\bar{x}^{2a+1} + g(\bar{x}\bar{y}^{1/2}, \bar{y}\bar{z})/\bar{y}^{2a+1} = 0\}/\mathbb{Z}_{2a+1}(1, -2, 2, 0),$$
$$U_3 = \{\bar{u}^2 + \bar{y}^2\bar{z} + \lambda\bar{y}\bar{x}^{2a+1} + g(\bar{x}\bar{z}^{1/2}, \bar{z})/\bar{z}^{2a+1} = 0\}/\mathbb{Z}_2(1, 1, 0, 1).$$

We always have $\mathrm{aw}(X) \geq b' \geq 2a+1$ and easily see that $Q_3 \notin E$ if and only if $\mathrm{aw}(X) = b' = 2a+1$. If $Q_3 \in E$, then Q_3 is at worst terminal of type (cD/2) with axial weight $\mathrm{aw}(X) - 2a - 1$. The origin Q_2 of U_2 is terminal of type (cA/$2a+1$) with axial weight 2. Other singularities of $\bar{X}$ are all isolated cDV points. Hence $\bar{X}$ has only terminal singularities and $\sum_{Q \in \bar{X}} (\mathrm{aw}(\bar{X}, Q) - 1) = \max\{\mathrm{aw}(X) - 2a - 1, 1\}$. Since E is irreducible and $d(v) = 1/2$, we see that π is divisorial with discrepancy $1/2$. □

Proposition 5.9. *Under the notation and assumptions* (5.7), *the* v'*-blow up* $\pi' : \bar{X}' \to X$ *is divisorial with discrepancy* $1/2$ *and we have* $\sum_{Q \in \bar{X}'}(\mathrm{aw}(\bar{X}', Q) - 1) = \max\{\mathrm{aw}(X) - 2a - 2, 0\}$.

Proof. Let E' be the exceptional divisor of π'. Since

$$E' \simeq \left\{ \begin{array}{c} u^2 + yt + g_{T'\text{-}wt=2a+1}(x,z) = 0 \\ yz + \lambda x^{2a+1} = 0 \end{array} \right\} \subseteq \mathbb{P}(1, 2a-1, 2, 2a+1, 2a+3),$$

we see that E' is Cartier outside $\{Q_2, Q_3, Q_5\} \cap E'$ and $\mathrm{Sing}(E') \subseteq \{x = y = u = 0\} \cup \{Q_2\}$. Since $\{y = z = t = 0\} \cap E' = \emptyset$, $\bar{X}'$ is covered three affine open sets as follows:

$$U_2 = \left\{ \begin{array}{c} \bar{u}^2 + \bar{t} + g(\bar{x}\bar{y}^{1/2}, \bar{y}\bar{z})/\bar{y}^{2a+1} = 0 \\ \bar{t}\bar{y} - \bar{z} - \lambda\bar{x}^{2a+1} = 0 \end{array} \right\} / \mathbb{Z}_{2a-1}(1, -2, 2, 2, 4),$$

$$U_3 = \left\{ \begin{array}{c} \bar{u}^2 + \bar{y}\bar{t} + g(\bar{x}\bar{z}^{1/2}, \bar{z})/\bar{z}^{2a+1} = 0 \\ \bar{t}\bar{z} - \bar{y} - \lambda\bar{x}^{2a+1} = 0 \end{array} \right\} / \mathbb{Z}_2(1, 1, 0, 1, 1),$$

$$U_5 = \left\{ \begin{array}{c} \bar{u}^2 + \bar{y} + g(\bar{x}\bar{t}^{1/2}, \bar{z}\bar{t})/\bar{t}^{2a+1} = 0 \\ \bar{t} - \bar{y}\bar{z} - \lambda\bar{x}^{2a+1} = 0 \end{array} \right\} / \mathbb{Z}_{2a+3}(1, -4, 2, -2, -2).$$

We have $\mathrm{aw}(X) \geq b' \geq 2a + 1$ and it is easy to see that $Q_3 \notin E'$ if and only if $\mathrm{aw}(X) = 2a + 1$. If $Q_3 \in E'$, then Q_3 is at worst terminal of type (cD/2) with axial weight $\mathrm{aw}(X) - 2a - 1$. The origin Q_2 of U_2 (resp. Q_5 of U_5) is isomorphic to $(\bar{x}, \bar{y}, \bar{u})/\mathbb{Z}_{2a-1}(1, -2, 2)$ (resp. $(\bar{x}, \bar{z}, \bar{u})/\mathbb{Z}_{2a+3}(1, 2, -2)$) and other singularities of $\bar{X}'$ are all isolated cDV points. Thus we see that $\bar{X}'$ has only terminal singularities and that $\sum_{Q \in \bar{X}'}(\mathrm{aw}(\bar{X}', Q) - 1) = \max\{\mathrm{aw}(X) - 2a - 2, 0\}$. Since E' is irreducible, we see that π' is divisorial with discrepancy $1/2$. □

Proposition 5.10. *If* $2a < b'$, *then there are exactly* $a+1$ *prime divisors with discrepancies* $1/2$ *over* X. *Furthermore, there are exactly three* (*resp. two*) *divisorial blow ups of* X *with discrepancies* $1/2$ *if* $a \geq 2$ (*resp.* $a = 1$).

Proof. We remark that $2a + 2 < 2a + 3 < b' + 3 \leq b + 3$ in this case. We first estimate the number of prime divisors with discrepancies $1/2$ in the case $A = \emptyset$. If $A = \emptyset$, then we have $a = 1$ in our case. We use the v-blow up $\pi : \bar{X} \to X$ in (5.8) as the first blow up. The origin Q_2 of U_2 is terminal of type (cA/3) and there is exactly one prime divisor with discrepancy $1/3$ over Q_2 by [Hay99, 6.4]. Using (2.10), we see that there are at most two prime divisors (including E) with discrepancies $1/2$ over X. Thus we know that (5.5) is true even if $A = \emptyset$ in the case $2a < b'$.

We already have $a-1$ prime divisors E_l with discrepancies $1/2$ over X in (5.4) and two more prime divisors E and E' in (5.8) and (5.9) respectively. If D is the Cartier divisor on X defined by $z=0$, then we have

$$\pi_l^*(D) = \pi_l^{-1}[D] + 2E_l,\ \pi^*(D) = \pi^{-1}[D] + E \text{ and } \pi'^*(D) = \pi'^{-1}[D] + E'.$$

By considering the $\mathbb{Q}$-Cartier Weil divisor D' on X defined by $y=0$, we have

$$\pi^*(D') = \pi^{-1}[D'] + \frac{2a+1}{2}E \text{ and } \pi'^*(D') = \pi'^{-1}[D'] + \frac{2a-1}{2}E'.$$

Thus we see that these $a+1$ prime divisors over X are all distinct.

The last part follows from (5.4), (5.8) and (5.9). If $a=1$, then $A=\emptyset$ and there is no π_l in this case. □

5.11. Thus we complete the proof of (3.1), (3.2) and (3.3) if $2a < b'$.

§5-B. Case: $2a > b'$ and b' is odd

5.12. Next we assume that $2a > b'$ and b' is odd. We first treat the case where $g_{\tau'\text{-}wt=b'}(x,z)$ is not a square. We define $\sigma = \frac{1}{2}(1,b',2,b')$, then $v=(j,\sigma)\in \mathcal{W}_{1/2}$ where $j: X \hookrightarrow (x,y,z,u)/\mathbb{Z}_2(1,1,0,1)$ is the standard embedding. We shall consider the following condition:

(5.12.1) $$b' = b \text{ or } b' = b-1 \text{ or } b' = 2a-1,$$

which is equivalent to

(5.12.2) $$x^{2b'} \in g(x,z) \text{ or } x^{2b'-2}z \in g(x,z) \text{ or } x^{2b'+2} \in g(x,z) \text{ or } b' = 2a-1.$$

Proposition 5.13. *Under the notation and assumptions* (5.12), *the v-blow up $\pi: \bar{X} \to X$ is pre-divisorial with discrepancy $1/2$. Furthermore, π is divisorial with discrepancy $1/2$ if and only if* (5.12.1) *holds, and we have $\sum_{Q\in\bar{X}}(\mathrm{aw}(\bar{X},Q)-1) = \max\{\mathrm{aw}(X)-b',1\}$ if* (5.12.1) *holds.*

Proof. Let E be the exceptional divisor of π. Since

$$E \simeq \{u^2 + g_{\tau'\text{-}wt=b'}(x,z) = 0\} \subseteq \mathbb{P}(1,b',2,b'),$$

we see that E is Cartier outside $\{Q_2,Q_3\}\cap E$ and that $\mathrm{Sing}(E) \subseteq \{u=0\}$. Since $Q_4 \notin E$, $\bar{X}$ is covered by three affine open sets as follows:

$$U_1 = \{\bar{u}^2 + \bar{x}\bar{y}^2\bar{z} + \lambda\bar{y}\bar{x}^{a-(b'-1)/2} + g(\bar{x}^{1/2},\bar{x}\bar{z})/\bar{x}^{b'} = 0\} \subseteq \mathbb{C}^4,$$
$$U_2 = \{\bar{u}^2 + \bar{y}\bar{z} + \lambda\bar{x}^{2a+1}\bar{y}^{a-(b'-1)/2} + g(\bar{x}\bar{y}^{1/2},\bar{y}\bar{z})/\bar{y}^{b'} = 0\}/\mathbb{Z}_{b'}(1,-2,2,0),$$
$$U_3 = \{\bar{u}^2 + \bar{y}^2\bar{z} + \lambda\bar{y}\bar{x}^{2a+1}\bar{z}^{a-(b'-1)/2} + g(\bar{x}\bar{z}^{1/2},\bar{z})/\bar{z}^{b'} = 0\}/\mathbb{Z}_2(1,1,0,1).$$

We first assume that (5.12.1) does not hold. Then U_1 (resp. U_2) has singularities along the $\bar{y}$-axis (resp. $\bar{x}$-axis), which are canonical. We always have $\mathrm{aw}(X) \geq b'$ and the equality holds if and only if $Q_3 \notin E$. If $Q_3 \in E$, then Q_3 is at worst terminal of type (cD/2) with axial weight $\mathrm{aw}(X) - b'$. Other singularities of $\bar{X}$ are all isolated cDV points. Thus we see that $\bar{X}$ has only canonical (non-terminal) singularities.

Next we assume that (5.12.1) holds. In this case singularities of $\bar{X}$ are all isolated. The origin Q_2 of U_2 is terminal of type (cA/b') with axial weight 2, the origin Q_3 of U_3 has the same properties as above and other singularities of $\bar{X}$ are all isolated cDV points. Hence we see that $\bar{X}$ has only terminal singularities and that $\sum_{Q\in\bar{X}}(\mathrm{aw}(\bar{X},Q)-1) = \max\{\mathrm{aw}(X)-b', 1\}$.

Since $g_{\tau'\text{-}wt=b'}(x,z)$ is not a square, E is irreducible. We also have $d(v) = 1/2$. Thus we see that $K_{\bar{X}} = \pi^*(K_X) + \frac{1}{2}E$. □

Proposition 5.14. *If $2a > b'$, b' is odd and $g_{\tau'\text{-}wt=b'}(x,z)$ is not a square, then there are exactly $\lfloor \frac{1}{2}\min\{2a, b+1\}\rfloor$ prime divisors with discrepancies 1/2 over X. Furthermore, there are exactly two (resp. one) divisorial blow ups of X with discrepancies 1/2 if $b' \geq \min\{2a-1, b-1\}$ (resp. $b' < \min\{2a-1, b-1\}$).*

Proof. We always have $A \neq \emptyset$ in this case since $b' \geq 3$. The upper bound of the number of prime divisors with discrepancies 1/2 is given in (5.5). On the other hand, as in the proof of (5.10), E in (5.13) is different from E_l in (5.4). Thus we have exactly $\lfloor \frac{1}{2}\min\{2a, b+1\}\rfloor$ prime divisors with discrepancies 1/2 over X. The last part follows from (5.4) and (5.13). □

5.15. The rest of this subsection is devoted to the case where $2a > b'$, b' is odd and $g_{\tau'\text{-}wt=b'}(x,z)$ is a square. We shall write $g_{\tau'\text{-}wt=b'}(x,z) = -h(x,z)^2$. Let $\chi_\pm : (x,y,z,u)/\mathbb{Z}_2 \to (x_1,y_1,z_1,u_1)/\mathbb{Z}_2$ be the automorphisms defined by

$$\chi_\pm^*(x_1) = x,\ \chi_\pm^*(y_1) = y,\ \chi_\pm^*(z_1) = z \text{ and } \chi_\pm^*(u_1) = u \pm h(x,z),$$

and let $j_\pm = \chi_\pm \circ j : X \hookrightarrow (x_1,y_1,z_1,u_1)/\mathbb{Z}_2(1,1,0,1)$ be the embeddings. Then we have

$$\begin{aligned} j_\pm : X \simeq \{u_1^2 \mp 2u_1 h(x_1,z_1) + y_1^2 z_1 + \lambda y_1 x_1^{2a+1} + g_1(x_1,z_1) = 0\}/\mathbb{Z}_2(1,1,0,1) \\ \subseteq (x_1,y_1,z_1,u_1)/\mathbb{Z}_2(1,1,0,1), \end{aligned}$$

where $g_1(x,z) = g_{\tau'\text{-}wt\geq b'+1}(x,z)$. We define $\sigma' = \frac{1}{2}(1,b',2,b'+2)$, then we have $v_\pm = (j_\pm, \sigma') \in \mathcal{W}_{1/2}$. We again consider the condition (5.12.1) and divide the cases whether (5.12.1) holds or not. In our situation, (5.12.1) is equivalent to

(5.15.1) $\qquad x^{b'} \in h(x,z)$ or $x^{2b'+2} \in g_1(x,z)$ or $b' = 2a-1$.

Proposition 5.16. *Under the notation and assumptions* (5.15), *let* $\pi_\pm : \bar{X}_\pm \to X$ *be the* $v_\pm$*-blow up. If* (5.12.1) *holds, then* $\pi_\pm$ *are both divisorial with discrepancies* $1/2$ *and we have* $\sum_{Q\in\bar{X}_\pm}(\mathrm{aw}(\bar{X}_\pm, Q) - 1) = \max\{\mathrm{aw}(X) - b' - 2, 0\}$.

Proof. Let $E_\pm$ be the exceptional divisor of $\pi_\pm$. Since

$$E_\pm \simeq \{\mp 2uh(x,z) + y^2z + \lambda\delta_{b',2a-1}yx^{2a+1} + g_{1,\tau'\text{-}wt=b'+1}(x,z) = 0\} \subseteq \mathbb{P}(1,b',2,b'+2),$$

we see that $E_\pm$ is Cartier outside $\{Q_2, Q_3, Q_4\} \cap E_\pm$. Then $\bar{X}_\pm$ is covered by four affine open sets as follows:

$$U_1 = \{\bar{u}^2\bar{x} \mp 2\bar{u}h(1,\bar{z}) + \bar{y}^2\bar{z} + \lambda\bar{y}\bar{x}^{a-(b'+1)/2} + g_1(\bar{x}^{1/2}, \bar{x}\bar{z})/\bar{x}^{b'+1} = 0\} \subseteq \mathbb{C}^4,$$

$$U_2 = \{\bar{u}^2\bar{y} \mp 2\bar{u}h(\bar{x},\bar{z}) + \bar{z} + \lambda\bar{x}^{2a+1}\bar{y}^{a-(b'+1)/2} + g_1(\bar{x}\bar{y}^{1/2}, \bar{y}\bar{z})/\bar{y}^{b'+1} = 0\} /\mathbb{Z}_{b'}(1,-2,2,2),$$

$$U_3 = \{\bar{u}^2\bar{z} \mp 2\bar{u}h(\bar{x},1) + \bar{y}^2 + \lambda\bar{y}\bar{x}^{2a+1}\bar{z}^{a-(b'+1)/2} + g_1(\bar{x}\bar{z}^{1/2}, \bar{z})/\bar{z}^{b'+1} = 0\} /\mathbb{Z}_2(1,1,0,1),$$

$$U_4 = \{\bar{u} \mp 2h(\bar{x},\bar{z}) + \bar{y}^2\bar{z} + \lambda\bar{y}\bar{x}^{2a+1}\bar{u}^{a-(b'+1)/2} + g_1(\bar{x}\bar{u}^{1/2}, \bar{z}\bar{u})/\bar{u}^{b'+1} = 0\} /\mathbb{Z}_{b'+2}(-1,2,-2,2).$$

The origin Q_2 of U_2 is isomorphic to $(\bar{x},\bar{y},\bar{u})/\mathbb{Z}_{b'}(1,-2,2)$ and the origin Q_4 of U_4 is isomorphic to $(\bar{x},\bar{y},\bar{z})/\mathbb{Z}_{b'+2}(1,-2,2)$. Since b' is odd and $g_{\tau'\text{-}wt=b'}(x,z)$ is a square, we see that $\mathrm{aw}(X) \geq b'+1$ and the equality holds if and only if $Q_3 \notin E_\pm$. If $Q_3 \in E_\pm$, then Q_3 is at worst terminal of type (cD/2) with axial weight $\mathrm{aw}(X) - b' - 1$. Other singularities of $\bar{X}_\pm$ are all isolated cDV points. Hence $\bar{X}_\pm$ has only terminal singularities and $\sum_{Q\in\bar{X}_\pm}(\mathrm{aw}(\bar{X}_\pm,Q) - 1) = \max\{\mathrm{aw}(X) - b' - 2, 0\}$. Since $E_\pm$ is irreducible and $d(v_\pm) = 1/2$, we see that $\pi_\pm$ is divisorial with discrepancy $1/2$. □

5.17. We shall continue our study on the case $2a > b'$, b' is odd and $g_{\tau'\text{-}wt=b'}(x,z)$ is a square. Here we assume that (5.12.1) does not hold. Then we can write

$$h(x,z) = zh_1(x,z) \text{ and } g_1(x,z) = p(x,z)z + q(x)$$

where $\tau'\text{-}wt(h_1(x,z)) = b'/2 - 1$, $\tau'\text{-}wt(p(x,z)) \geq b'$ and $\tau'\text{-}wt(q(x)) \geq b'+2$. Thus there are embeddings $j'_\pm : X \hookrightarrow (x_1,y_1,z_1,u_1,t_1)/\mathbb{Z}_2(1,1,0,1,0)$ such that

$$j'_\pm : X \simeq \left\{ \begin{array}{l} u_1^2 + z_1t_1 + \lambda y_1x_1^{2a+1} + q(x_1) = 0 \\ t_1 = y_1^2 \mp 2u_1h_1(x_1,z_1) + p(x_1,z_1) \end{array} \right\} /\mathbb{Z}_2(1,1,0,1,0) \subseteq (x_1,y_1,z_1,u_1,t_1)/\mathbb{Z}_2(1,1,0,1,0).$$

We define $\sigma'' = \frac{1}{2}(1, b', 2, b'+2, 2b'+2)$, then we have $v'_\pm = (j'_\pm, \sigma'') \in \mathcal{W}'_{1/2}$. We shall consider the condition:

$$b' = b-2 \text{ or } b' = b-3 \text{ or } b' = 2a-3. \tag{5.17.1}$$

This is equivalent to one of the following 8 conditions:

(5.17.2)

$$x^{2b'+4} \in g(x,z),\ x^{2b'}z \in g(x,z),\ x^{2b'-4}z^2 \in g(x,z),\ x^{2b'+6} \in g(x,z),$$
$$x^{2b'+2}z \in g(x,z),\ x^{2b'-2}z^2 \in g(x,z),\ x^{2b'-6}z^3 \in g(x,z) \text{ or } b' = 2a-3.$$

In our situation, this is also equivalent to one of the following 7 conditions:

$$\begin{array}{c} x^{b'-2} \in h_1(x,z),\ x^{2b'+4} \in q(x),\ x^{2b'+6} \in q(x), \\ x^{2b'} \in p(x,z),\ x^{2b'+2} \in p(x,z),\ x^{2b'-2}z \in p(x,z) \text{ or } b' = 2a-3. \end{array} \tag{5.17.3}$$

Proposition 5.18. *Under the notation and assumptions* (5.17), *the $v'_\pm$-blow up $\pi'_\pm : \bar{X}'_\pm \to X$ are both pre-divisorial with discrepancies* $1/2$. *Furthermore, $\pi'_\pm$ are both divisorial with discrepancies* $1/2$ *if and only if* (5.17.1) *holds, and we have $\sum_{Q\in \bar{X}'_\pm}(\mathrm{aw}(\bar{X}'_\pm, Q) - 1) = \max\{\mathrm{aw}(X) - b' - 2, 0\}$ if* (5.17.1) *holds.*

Proof. Let $E'_\pm$ be the exceptional divisor of $\pi'_\pm$. Since

$$E'_\pm \simeq \left\{ \begin{array}{c} u_1^2 + z_1t_1 + \lambda\delta_{b',2a-3}y_1x_1^{2a+1} + q_{\tau'\text{-}wt=b'+2}(x_1) = 0 \\ y_1^2 \mp 2u_1h_1(x_1,z_1) + p_{\tau'\text{-}wt=b'}(x_1,z_1) = 0 \end{array} \right\}$$
$$\subseteq \mathbb{P}(1, b', 2, b'+2, 2b'+2),$$

we see that $E'_\pm$ is Cartier outside $\{Q_3, Q_5\} \cap E'_\pm$. Since $\{x = z = t = 0\} \cap E'_\pm = \emptyset$, $\bar{X}'_\pm$ is covered by three affine open sets as follows:

$$U_1 = \left\{ \begin{array}{c} \bar{u}^2 + \bar{z}\bar{t} + \lambda\bar{y}\bar{x}^{a-(b'+3)/2} + q(\bar{x}^{1/2})/\bar{x}^{b'+2} = 0 \\ \bar{t}\bar{x} - \bar{y}^2 \pm 2\bar{u}h_1(1,\bar{z}) - p(\bar{x}^{1/2}, \bar{x}\bar{z})/\bar{x}^{b'} = 0 \end{array} \right\} \subseteq \mathbb{C}^5,$$

$$U_3 = \left\{ \begin{array}{c} \bar{u}^2 + \bar{t} + \lambda\bar{y}\bar{x}^{2a+1}\bar{z}^{a-(b'+3)/2} + q(\bar{x}\bar{z}^{1/2})/\bar{z}^{b'+2} = 0 \\ \bar{t}\bar{z} - \bar{y}^2 \pm 2\bar{u}h_1(\bar{x},1) - p(\bar{x}\bar{z}^{1/2}, \bar{z})/\bar{z}^{b'} = 0 \end{array} \right\}$$
$$/\mathbb{Z}_2(1,1,0,1,0),$$

$$U_5 = \left\{ \begin{array}{c} \bar{u}^2 + \bar{z} + \lambda\bar{y}\bar{x}^{2a+1}\bar{t}^{a-(b'+3)/2} + q(\bar{x}\bar{t}^{1/2})/\bar{t}^{b'+2} = 0 \\ \bar{t} - \bar{y}^2 \pm 2\bar{u}h_1(\bar{x},\bar{z}) - p(\bar{x}\bar{t}^{1/2}, \bar{z}\bar{t})/\bar{t}^{b'} = 0 \end{array} \right\}$$
$$/\mathbb{Z}_{2b'+2}(1, b', 2, b'+2, -2).$$

We see that the origin Q_1 of the x-chart U_1 is at worst canonical. If (5.17.1) holds, then at least one of the defining equations has non-zero constant or linear terms, and we see that $Q_1 \notin E'_\pm$ or $Q_1 \in \bar{X}'_\pm$ is an isolated cDV point. However, if (5.17.1) does not hold, then $Q_1 \in \bar{X}'_\pm$ is non-terminal.

The origin Q_5 of U_5 is isomorphic to $(\bar{x}, \bar{y}, \bar{u})/\mathbb{Z}_{2b'+2}(1, b', b'+2)$. We always have $\mathrm{aw}(X) \geq b'+1$ and the equality holds if and only if $Q_3 \notin E'_\pm$. If $Q_3 \in E'_\pm$, then Q_3 is at worst terminal of type (cD/2) with axial weight $\mathrm{aw}(X) - b' - 1$. We also see that the singularities of $\bar{X}'_\pm$ other than Q_1, Q_3 and Q_5 are all isolated cDV points.

Since $E'_\pm$ is irreducible and $d(v'_\pm) = 1/2$, we see that $K_{\bar{X}'_\pm} = \pi'^*_\pm(K_X) + \frac{1}{2}E'_\pm$. □

Proposition 5.19. *If $2a > b'$, b' is odd and $g_{\tau'\text{-}wt=b'}(x, z)$ is a square, then there are exactly $\lfloor \frac{1}{2}\min\{2a+2, b+3\}\rfloor$ prime divisors with discrepancies $1/2$ over X. Furthermore, there are exactly three (resp. one) divisorial blow ups of X with discrepancies $1/2$ if $b' \geq \min\{2a-3, b-3\}$ (resp. $b' < \min\{2a-3, b-3\}$).*

Proof. In this case, we always have $A \neq \emptyset$ and the upper bound of the number of prime divisors with discrepancies $1/2$ over X is given in (5.5). On the other hand, as in the proof of (5.10), $E_\pm$ in (5.16) and $E'_\pm$ in (5.18) are all different from E_l in (5.4). By considering the $\mathbb{Q}$-Cartier Weil divisor D_+ on X defined by $u + h(x, z) = 0$, we see that

$$\pi_+^*(D_+) = \pi_+^{-1}[D_+] + \frac{b'+2}{2}E_+ \text{ and } \pi_-^*(D_+) = \pi_-^{-1}[D_+] + \frac{b'}{2}E_-.$$

Hence E_+ and E_- in (5.16) are distinct prime divisors. Similarly, E'_+ and E'_- are also distinct. Thus in any case, we have exactly $\lfloor \frac{1}{2}\min\{2a+2, b+3\}\rfloor$ prime divisors with discrepancies $1/2$ over X. The last part follows from (5.4), (5.16) and (5.18). □

5.20. By using propositions above, we complete the proof of (3.1), (3.2) and (3.3) if $2a > b'$ and b' is odd.

§5-C. Case: $2a > b'$ and b' is even

5.21. In the case $2a > b'$ and b' is even, we first assume that $b' = b$. Then it is easy to see that $x^{2b'} \in g(x, z)$ and $g_{\tau'\text{-}wt=b'}(x, z)z$ is not a square. We define $\sigma = \frac{1}{2}(1, b'-1, 2, b'+1)$, then we have $v = (j, \sigma) \in \mathcal{W}_{1/2}$ where $j : X \hookrightarrow (x, y, z, u)/\mathbb{Z}_2(1, 1, 0, 1)$ is the standard embedding.

Proposition 5.22. *Under the notation and assumptions (5.21), let $\pi : \bar{X} \to X$ be the v-blow up. If $b' = b$, then π is divisorial with discrepancy $1/2$ and we have $\sum_{Q \in \bar{X}}(\mathrm{aw}(\bar{X}, Q) - 1) = \max\{\mathrm{aw}(X) - b' - 1, 0\}$.*

Proof. Let E be the exceptional divisor of π. Since

$$E \simeq \{y^2 z + g_{\tau'\text{-}wt=b'}(x,z) = 0\} \subseteq \mathbb{P}(1, b'-1, 2, b'+1),$$

we see that E is Cartier outside $\{Q_2, Q_3, Q_4\} \cap E$ and $\mathrm{Sing}(E) \subseteq \{y=0\} \cup \{x = z = 0\}$. Since $Q_1 \notin E$, $\bar{X}$ is covered by three affine open sets as follows:

$$U_2 = \{\bar{u}^2\bar{y} + \bar{z} + \lambda \bar{y}^{a-b'/2}\bar{x}^{2a+1} + g(\bar{x}\bar{y}^{1/2}, \bar{y}\bar{z})/\bar{y}^{b'} = 0\}/\mathbb{Z}_{b'-1}(1,-2,2,2),$$
$$U_3 = \{\bar{u}^2\bar{z} + \bar{y}^2 + \lambda \bar{y}\bar{x}^{2a+1}\bar{z}^{a-b'/2} + g(\bar{x}\bar{z}^{1/2}, \bar{z})/\bar{z}^{b'} = 0\}/\mathbb{Z}_2(1,1,0,1),$$
$$U_4 = \{\bar{u} + \bar{y}^2\bar{z} + \lambda \bar{y}\bar{x}^{2a+1}\bar{u}^{a-b'/2} + g(\bar{x}\bar{u}^{1/2}, \bar{z}\bar{u})/\bar{u}^{b'} = 0\}/\mathbb{Z}_{b'+1}(1,-2,2,-2).$$

The origin Q_2 of U_2 is isomorphic to $(\bar{x},\bar{y},\bar{u})/\mathbb{Z}_{b'-1}(1,-2,2)$ and the origin Q_4 of U_4 is isomorphic to $(\bar{x},\bar{y},\bar{z})/\mathbb{Z}_{b'+1}(1,-2,2)$. We always have $\mathrm{aw}(X) \geq b'$ and the equality holds if and only if $Q_3 \notin E$. If $Q_3 \in E$, then Q_3 is at worst terminal of type (cD/2) with axial weight $\mathrm{aw}(X) - b'$. Other singularities of $\bar{X}$ are all isolated cDV points. Hence $\bar{X}$ has only terminal singularities and $\sum_{Q\in\bar{X}}(\mathrm{aw}(\bar{X},Q) - 1) = \max\{\mathrm{aw}(X) - b' - 1, 0\}$. Since E is irreducible and $d(v) = 1/2$, we see that π is divisorial with discrepancy $1/2$. □

5.23. In the case $2a > b'$ and b' is even, we next assume that $b' \leq b-1$. Then $x^{2b'} \notin g(x,z)$ and we can write $g(x,z) = p(x,z)z + q(x)$ where $\tau'\text{-}wt(p(x,z)) = b'-1$ and $\tau'\text{-}wt(q(x)) \geq b'+1$.

5.24. Under the situation (5.23), we first assume that $g_{\tau'\text{-}wt=b'}(x,z)z$ is not a square. Then $p_{\tau'\text{-}wt=b'-1}(x,z)$ is not a square. In this case there is an embedding $j' : X \hookrightarrow (x,y,z,u,t)/\mathbb{Z}_2(1,1,0,1,0)$ such that

$$j' : X \simeq \left\{ \begin{array}{c} u^2 + zt + \lambda y x^{2a+1} + q(x) = 0 \\ t = y^2 + p(x,z) \end{array} \right\}/\mathbb{Z}_2(1,1,0,1,0)$$
$$\subseteq (x,y,z,u,t)/\mathbb{Z}_2(1,1,0,1,0).$$

We define $\sigma' = \frac{1}{2}(1, b'-1, 2, b'+1, 2b')$, then we have $v' = (j', \sigma') \in \mathcal{W}'_{1/2}$. We shall consider the condition:

$$b' = b-1 \text{ or } b' = b-2 \text{ or } b' = 2a-2, \tag{5.24.1}$$

which is equivalent to one of the following 6 conditions:

$$\begin{array}{c} x^{2b'+2} \in g(x,z),\ x^{2b'-2}z \in g(x,z),\ x^{2b'+4} \in g(x,z), \\ x^{2b'}z \in g(x,z),\ x^{2b'-4}z^2 \in g(x,z) \text{ or } b' = 2a-2. \end{array} \tag{5.24.2}$$

In the above situation, this is also equivalent to one of the following 6 conditions:

$$\begin{array}{c} x^{2b'+2} \in q(x),\ x^{2b'+4} \in q(x),\ x^{2b'-2} \in p(x,z), \\ x^{2b'} \in p(x,z),\ x^{2b'-4}z \in p(x,z) \text{ or } b' = 2a-2. \end{array} \tag{5.24.3}$$

Proposition 5.25. *Under the notation and assumptions* (5.24), *the* v'-*blow up* $\pi' : \bar{X}' \to X$ *is pre-divisorial with discrepancy* $1/2$. *Furthermore,* π' *is divisorial with discrepancy* $1/2$ *if and only if* (5.24.1) *holds, and we have* $\sum_{Q\in \bar{X}'}(\mathrm{aw}(\bar{X}', Q) - 1) = \max\{\mathrm{aw}(X) - b' - 1, 0\}$ *if* (5.24.1) *holds.*

Proof. Let E' be the exceptional divisor of π'. Since

$$E' \simeq \left\{ \begin{array}{c} u^2 + zt + \lambda\delta_{b',2a-2} y x^{2a+1} + q_{\tau'\text{-}wt=b'+1}(x) = 0 \\ y^2 + p_{\tau'\text{-}wt=b'-1}(x,z) = 0 \end{array} \right\} \subseteq \mathbb{P}(1, b'-1, 2, b'+1, 2b'),$$

we see that E' is Cartier outside $\{Q_3, Q_5\} \cap E'$ and that $\mathrm{Sing}(E') \subseteq \{y = 0\} \cup \{z = u = 0\}$. Since $\{x = z = t = 0\} \cap E' = \emptyset$, $\bar{X}'$ is covered by three affine open sets as follows:

$$U_1 = \left\{ \begin{array}{c} u^2 + zt + \lambda y x^{a-1-b'/2} + q(x^{1/2})/x^{b'+1} = 0 \\ tx - y^2 - p(x^{1/2}, xz)/x^{b'-1} = 0 \end{array} \right\} \subseteq \mathbb{C}^5,$$

$$U_3 = \left\{ \begin{array}{c} u^2 + t + \lambda y x^{2a+1} z^{a-1-b'/2} + q(xz^{1/2})/z^{b'+1} = 0 \\ tz - y^2 - p(xz^{1/2}, z)/z^{b'-1} = 0 \end{array} \right\} /\mathbb{Z}_2(1,1,0,1,0),$$

$$U_5 = \left\{ \begin{array}{c} u^2 + z + \lambda y x^{2a+1} t^{a-1-b'/2} + q(xt^{1/2})/t^{b'+1} = 0 \\ t - y^2 - p(xt^{1/2}, zt)/t^{b'-1} = 0 \end{array} \right\} /\mathbb{Z}_{2b'}(1, b'-1, 2, b'+1, -2),$$

We see that the origin Q_1 of U_1 is at worst canonical. If (5.24.1) holds, then at least one of the defining equations of U_1 has non-zero constant or linear terms, and we see that $Q_1 \notin E'$ or $Q_1 \in \bar{X}'$ is an isolated cDV point. Otherwise $Q_1 \in \bar{X}'$ is non-terminal.

The origin Q_5 of U_5 is isomorphic to $(\bar{x}, \bar{y}, \bar{u})/\mathbb{Z}_{2b'}(1, b'-1, b'+1)$. We always have $\mathrm{aw}(X) \geq b'$ and the equality holds if and only if $Q_3 \notin E'$. If $Q_3 \in E'$, then Q_3 is at worst terminal of type (cD/2) with axial weight $\mathrm{aw}(X) - b'$. Singularities of $\bar{X}'$ other than Q_1, Q_3 and Q_5 are all isolated cDV points.

Since E' is irreducible and $d(v') = 1/2$, we see that $K_{\bar{X}'} = \pi'^*(K_X) + \frac{1}{2}E'$. □

Proposition 5.26. *If* $2a > b'$, b' *is even and* $g_{\tau'\text{-}wt=b'}(x,z)z$ *is not a square, then there are exactly* $\lfloor \frac{1}{2}\min\{2a, b+1\} \rfloor$ *prime divisors with discrepancies* $1/2$ *over* X. *Furthermore, there are exactly two* (*resp. one*) *divisorial blow ups of* X *with discrepancies* $1/2$ *if* $b' \geq \min\{2a-2, b-2\}$ *and* $b \geq 3$ (*resp.* $b' < \min\{2a-2, b-2\}$ *or* $b = 2$).

Proof. In the case $A = \emptyset$, we have to estimate the number of prime divisors with discrepancies $1/2$ over X. Since $b' = 2$, we have $x^4 \in g(x,z)$ so that $b = b'$. We can use the v-blow up $\pi : \bar{X} \to X$ in (5.22) as the first blow up. The origin Q_4 of U_4 is isomorphic to $(\bar{x}, \bar{y}, \bar{z})/\mathbb{Z}_3(1,1,2)$. By (2.9), there is a projective birational morphism $\nu : Z \to \bar{X}$ such that $K_Z = \nu^*(K_{\bar{X}}) + \frac{1}{3}F_1 + \frac{2}{3}F_2$ where $F_1 + F_2$ is the exceptional divisor of ν. Since E is defined by $\bar{y}^2\bar{z} + g_{\tau'\text{-}wt=2}(\bar{x}, \bar{z}) = 0$ near Q_4, we see that $\nu^*(E) = \nu^{-1}[E] + \frac{4}{3}F_1 + \frac{5}{3}F_2$. Hence we have $a(F_1, X) = 1/3 + 1/2 \cdot 4/3 = 1$, $a(F_2, X) = 2/3 + 1/2 \cdot 5/3 = 3/2$ by using (2.10). Thus we see that E is the unique prime divisor with discrepancy $1/2$ over X. Thus we know that (5.5) is true even if $A = \emptyset$.

On the other hand, as in the proof of (5.14), E in (5.22) and E' in (5.25) are both different from E_l in (5.4). Hence there are exactly $\lfloor \frac{1}{2}\min\{2a, b+1\} \rfloor$ prime divisors with discrepancies $1/2$ over X.

The last stetement follows from (5.4), (5.22) and (5.25). □

5.27. We shall continue our study on the case $2a > b'$ and b' is even. The rest is devoted to the case where $b' \le b - 1$ and $g_{\tau'\text{-}wt=b'}(x, z)z$ is a square. We can write $g_{\tau'\text{-}wt=b'}(x, z) = -h(x, z)^2 z$. Let $\chi_\pm : (x, y, z, u)/\mathbb{Z}_2 \to (x_1, y_1, z_1, u_1)/\mathbb{Z}_2$ be the automorphisms defined by

$$\chi^*(x_1) = x,\ \chi^*(y_1) = y \pm h(x, z),\ \chi^*(z_1) = z \text{ and } \chi^*(u_1) = u,$$

and let $j_\pm = \chi_\pm \circ j : X \hookrightarrow (x_1, y_1, z_1, u_1)/\mathbb{Z}_2(1,1,0,1)$ be the embeddings. Then we have

$$\begin{aligned} j_\pm : X \simeq \{u_1^2 + y_1^2 z_1 \mp 2y_1 z_1 h(x_1, z_1) + \lambda y_1 x_1^{2a+1} + g_1(x_1, z_1) = 0\}/\mathbb{Z}_2(1,1,0,1) \\ \subseteq (x_1, y_1, z_1, u_1)/\mathbb{Z}_2(1,1,0,1) \end{aligned}$$

where $g_1(x, z) = g_{\tau'\text{-}wt\ge b'+1}(x, z)$. We define $\sigma_1 = \frac{1}{2}(1, b'+1, 2, b'+1)$, then we have $v_\pm = (j_\pm, \sigma_1) \in \mathcal{W}_{1/2}$. We again consider the condition (5.24.1) which is now equivalent to one of the following 5 conditions:

$$\begin{aligned} x^{b'-1} \in h(x, z),\ x^{2b'+2} \in g_1(x, z),\ x^{2b'+4} \in g_1(x, z), \\ x^{2b'} z \in g_1(x, z) \text{ or } b' = 2a - 2. \end{aligned} \tag{5.27.1}$$

Proposition 5.28. *Under the notation and assumptions* (5.27), *the* $v_\pm$-*blow up* $\pi_\pm : \bar{X}_\pm \to X$ *are both pre-divisorial with discrepancies* $1/2$. *Furthermore,* $\pi_\pm$ *are both divisorial with discrepancies* $1/2$ *if and only if* (5.24.1) *holds, and we have* $\sum_{Q \in \bar{X}_\pm}(\mathrm{aw}(\bar{X}_\pm, Q) - 1) = \max\{\mathrm{aw}(X) - b' - 1, 1\}$ *if* (5.24.1) *holds.*

Proof. Since

$$\begin{aligned} E_\pm \simeq \{u_1^2 \mp 2y_1 z_1 h(x_1, z_1) + g_{1,\tau'\text{-}wt=b'+1}(x_1, z_1) = 0\} \\ \subseteq \mathbb{P}(1, b'+1, 2, b'+1), \end{aligned}$$

we see that $E_\pm$ is Cartier outside $\{Q_2, Q_3\} \cap E_\pm$ and that $\mathrm{Sing}(E_\pm) \subseteq \{u_1 = 0\}$. Since $Q_4 \notin E_\pm$, $\bar{X}_\pm$ is covered by three affine open sets as follows:

$$\begin{aligned}
U_1 &= \{\bar{u}^2 + \bar{x}\bar{y}^2\bar{z} \mp 2\bar{y}\bar{z}h(1,\bar{z}) + \lambda\bar{y}\bar{x}^{a-b'/2} + g_1(\bar{x}^{1/2}, \bar{x}\bar{z})/\bar{x}^{b'+1} = 0\} \subseteq \mathbb{C}^4,\\
U_2 &= \{\bar{u}^2 + \bar{y}\bar{z} \mp 2\bar{z}h(\bar{x},\bar{z}) + \lambda\bar{y}^{a-b'/2}\bar{x}^{2a+1} + g_1(\bar{x}\bar{y}^{1/2}, \bar{y}\bar{z})/\bar{y}^{b'+1} = 0\}\\
&\qquad /\mathbb{Z}_{b'+1}(1,-2,2,0),\\
U_3 &= \{\bar{u}^2 + \bar{y}^2\bar{z} \mp 2\bar{y}h(\bar{x},1) + \lambda\bar{y}\bar{x}^{2a+1}\bar{z}^{a-b'/2} + g_1(\bar{x}\bar{z}^{1/2}, \bar{z})/\bar{z}^{b'+1} = 0\}\\
&\qquad /\mathbb{Z}_2(1,1,0,1).
\end{aligned}$$

We first assume that (5.24.1) does not hold. Then U_1 (resp. U_2) has singularities along $\bar{y}$-axis (resp. $\bar{x}$-axis), which are canonical. We always have $\mathrm{aw}(X) \geq b'+1$ and the equality holds if and only if $Q_3 \notin E_\pm$. If $Q_3 \in E_\pm$, then Q_3 is at worst terminal of type (cD/2) with axial multuplicity $\mathrm{aw}(X) - b' - 1$. Other singularities of $\bar{X}_\pm$ are all isolated cDV points. Thus we see that $\bar{X}_\pm$ has only canonical (non-terminal) singularities.

Next we assume that (5.24.1) holds. In this case, singularities of $\bar{X}_\pm$ are all islated. The origin Q_2 of U_2 is terminal of type $(\mathrm{cA}/b'+1)$ with axial weight 2, the origin Q_3 of U_3 has the same properties as above, and other singular points of $\bar{X}_\pm$ are all isolated cDV points. Hence we see that $\bar{X}_\pm$ has only terminal singularities and that $\sum_{Q\in \bar{X}_\pm}(\mathrm{aw}(\bar{X}_\pm, Q) - 1) = \max\{\mathrm{aw}(X) - b' - 1, 1\}$.

Since $E_\pm$ is irreducible and $d(v_\pm) = 1/2$, we have $K_{\bar{X}_\pm} = \pi_\pm{}^*(K_X) + \frac{1}{2}E_\pm$. □

Proposition 5.29. *If $2a > b'$, b' is even and $g_{\tau'\text{-}wt=b'}(x,z)z$ is a square, then there are exactly $\lfloor \frac{1}{2}\min\{2a+2, b+3\}\rfloor$ prime divisors with discrepancies $1/2$ over X. Furthermore, there are exactly three (resp. one) divisorial blow ups of X with discrepancies $1/2$ if $b' \geq \min\{2a-2, b-2\}$ (resp. $b' < \min\{2a-2, b-2\}$).*

Proof. If $b' = 2$, then $g_{\tau'\text{-}wt=2}(x,z)z = a_{2,0}x^4z$ which is not a square. Hence $A \neq \emptyset$ in this case. As in the proof of (5.14), $E_\pm$ in (5.28) are both different from E_l in (5.4). We also see that E_+ and E_- are distinct prime divisor over X. Therefore there are exactly $\lfloor \frac{1}{2}\min\{2a+2, b+3\}\rfloor$ prime divisors with discrepancies $1/2$ over X. The last part follows from (5.4) and (5.28). □

5.30. By using propositions above, we complete the proof of (3.1), (3.2) and (3.3) if $2a > b'$ and b' is even.

§5-D. Case: $2a = b'$

5.31. In the case $2a = b'$, we first assume that $\frac{\lambda^2}{4}x^{4a+2} - g_{\tau'\text{-}wt=b'}(x,z)z$ is not a square. In this situation, we define $\sigma = \frac{1}{2}(1, b'-1, 2, b'+1)$, then $v = (j, \sigma) \in \mathcal{W}_{1/2}$ where $j : X \hookrightarrow (x,y,z,u)/\mathbb{Z}_2(1,1,0,1)$ is the standard embedding.

Proposition 5.32. *Under the notation and assumptions* (5.31), *the v-blow up $\pi : \bar{X} \dashrightarrow X$ is divisorial with discrepancy $1/2$ and we have $\sum_{Q \in \bar{X}}(\mathrm{aw}(\bar{X}, Q) - 1) = \max\{\mathrm{aw}(X) - b' - 1, 0\}$.*

Proof. Let E be the exceptional divisor of π. Since

$$E \simeq \{y^2 z + \lambda y x^{2a+1} + g_{\tau'\text{-}wt=b'}(x,z) = 0\} \subseteq \mathbb{P}(1, b'-1, 2, b'+1),$$

we see that E is Cartier outside $\{Q_2, Q_3, Q_4\} \cap E$. Then $\bar{X}$ is covered by four affine open sets as follows:

$$\begin{aligned}
U_1 &= \{\bar{u}^2\bar{x} + \bar{y}^2\bar{z} + \lambda\bar{y} + g(\bar{x}^{1/2}, \bar{x}\bar{z})/\bar{x}^{b'} = 0\} \subseteq \mathbb{C}^4,\\
U_2 &= \{\bar{u}^2\bar{y} + \bar{z} + \lambda\bar{x}^{2a+1} + g(\bar{x}\bar{y}^{1/2}, \bar{y}\bar{z})/\bar{y}^{b'} = 0\}/\mathbb{Z}_{b'-1}(1,-2,2,2),\\
U_3 &= \{\bar{u}^2\bar{z} + \bar{y}^2 + \lambda\bar{y}\bar{x}^{2a+1} + g(\bar{x}\bar{z}^{1/2}, \bar{z})/\bar{z}^{b'} = 0\}/\mathbb{Z}_2(1,1,0,1),\\
U_4 &= \{\bar{u} + \bar{y}^2\bar{z} + \lambda\bar{y}\bar{x}^{2a+1} + g(\bar{x}\bar{u}^{1/2}, \bar{z}\bar{u})/\bar{u}^{b'} = 0\}/\mathbb{Z}_{b'+1}(1,-2,2,-2).
\end{aligned}$$

The origin Q_2 of U_2 is isomorphic to $(\bar{x}, \bar{y}, \bar{u})/\mathbb{Z}_{b'-1}(1,-2,2)$ and the origin Q_4 of U_4 is isomorphic to $(\bar{x}, \bar{y}, \bar{z})/\mathbb{Z}_{b'+1}(1,-2,2)$. We always have $\mathrm{aw}(X) \geq b'$ and easily see that $Q_3 \notin E$ if and only if $\mathrm{aw}(X) = b'$. If $Q_3 \in E$, then Q_3 is at worst terminal of type (cD/2) with axial weight $\mathrm{aw}(X) - b'$. Other singularities of $\bar{X}$ are all isolated cDV points. Hence we see that $\bar{X}$ has only terminal singularities and that $\sum_{Q \in \bar{X}}(\mathrm{aw}(\bar{X}, Q) - 1) = \max\{\mathrm{aw}(X) - b' - 1, 0\}$. Since $\frac{\lambda^2}{4}x^{4a+2} - g_{\tau'\text{-}wt=b'}(x,z)z$ is not a square, we see that E is irreducible. Thus π is divisorial with discrepancy $1/2$. □

Proposition 5.33. *If $2a = b'$ and $\frac{\lambda^2}{4}x^{4a+2} - g_{\tau'\text{-}wt=b'}(x,z)z$ is not a square, then there are exactly a prime divisors with discrepancies $1/2$ over X. Furthermore, there are exactly two* (*resp. one*) *divisorial blow ups of X with discrepancies $1/2$ if $a \geq 2$* (*resp. $a = 1$*).

Proof. In the case $A = \emptyset$, we have to estimate the number of prime divisors with discrepancies $1/2$ over X. This can be done exactly the same way as in the proof of (5.26) and we know that E in (5.32) is the unique prime divisor with discrepancy 1/2 over X. Thus (5.5) is true even in the case $A = \emptyset$.

On the other hand, as in the proof of (5.14), E in (5.32) is different from E_l in (5.4). Hence we have exactly a prime divisors with discrepancies 1/2 over X.

The last part follows from (5.4) and (5.32). □

5.34. In the case $2a = b'$, we next assume that $\frac{\lambda^2}{4}x^{4a+2} - g_{\tau'\text{-}wt=b'}(x,z)z$ is a square. Let

$$\tfrac{\lambda^2}{4}x^{4a+2} - g_{\tau'\text{-}wt=b'}(x,z)z = (\tfrac{\lambda}{2}x^{2a+1} + h(x,z)z)^2.$$

Then we have $g_{\tau'\text{-}wt=b'}(x,z) = -\lambda x^{2a+1}h(x,z) - h(x,z)^2 z$ and

$$\begin{aligned} j : X \simeq \{u^2 + (y - h(x,z))(yz + h(x,z)z + \lambda x^{2a+1}) + g_1(x,z) = 0\}/\mathbb{Z}_2(1,1,0,1) \\ \subseteq (x,y,z,u)/\mathbb{Z}_2(1,1,0,1) \end{aligned}$$

where $g_1(x,z) = g_{\tau'\text{-}wt\geq b'+1}(x,z)$. Let $\chi : (x,y,z,u)/\mathbb{Z}_2 \to (x_1,y_1,z_1,u_1)/\mathbb{Z}_2$ be the automorphism defined by

$$\chi^*(x_1) = x,\ \chi^*(y_1) = y - h(x,z),\ \chi^*(z_1) = z \text{ and } \chi^*(u_1) = u.$$

Let $j_1 = \chi \circ j : X \hookrightarrow (x_1,y_1,z_1,u_1)/\mathbb{Z}_2(1,1,0,1)$ be the embedding. Then we have

$$\begin{aligned} j_1 : X \simeq \{u_1^2 + y_1^2 z_1 + 2y_1 z_1 h(x_1,z_1) + \lambda y_1 x_1^{2a+1} + g_1(x_1,z_1) = 0\}/\mathbb{Z}_2(1,1,0,1) \\ \subseteq (x_1,y_1,z_1,u_1)/\mathbb{Z}_2(1,1,0,1). \end{aligned}$$

We define $\sigma_1 = \frac{1}{2}(1, b'+1, 2, b'+1)$, then it is easy to see that $v_1 = (j_1, \sigma_1) \in \mathcal{W}_{1/2}$. We also need another embedding $j_2 : X \hookrightarrow (x_1,y_1,z_1,u_1,t_1)/\mathbb{Z}_2(1,1,0,1,1)$ such that

$$\begin{aligned} j_2 : X \simeq \left\{ \begin{array}{c} u_1^2 + y_1 t_1 + g_1(x_1,z_1) = 0 \\ t_1 = z_1(y_1 + 2h(x_1,z_1)) + \lambda x_1^{2a+1} \end{array} \right\}/\mathbb{Z}_2(1,1,0,1,1) \\ \subseteq (x_1,y_1,z_1,u_1,t_1)/\mathbb{Z}_2(1,1,0,1,1). \end{aligned}$$

We define $\sigma_2 = \frac{1}{2}(1, b'-1, 2, b'+1, b'+3)$, then we see that $v_2 = (j_2, \sigma_2) \in \mathcal{W}'_{1/2}$.

Proposition 5.35. *Under the notation and assumptions* (5.34), *the v_1-blow up $\pi_1 : \bar{X}_1 \to X$ is divisorial with discrepancy* 1/2 *and we have* $\sum_{Q\in\bar{X}_1}(\mathrm{aw}(\bar{X}_1,Q) - 1) = \max\{\mathrm{aw}(X) - b' - 1, 1\}$.

Proof. Let E_1 be the exceptional divisor of π_1. Since

$$\begin{aligned} E_1 \simeq \{u^2 + 2yzh(x,z) + \lambda yx^{2a+1} + g_{1,\tau'\text{-}wt=b'+1}(x,z) = 0\} \\ \subseteq \mathbb{P}(1, b'+1, 2, b'+1), \end{aligned}$$

we see that E_1 is Cartier outside $\{Q_2, Q_3\} \cap E_1$. Since $Q_4 \notin E_1$, $\bar{X}_1$ is covered by three affine open sets as follows:

$$U_1 = \{\bar{u}^2 + \bar{x}\bar{y}^2\bar{z} + 2\bar{y}\bar{z}h(1,\bar{z}) + \lambda\bar{y} + g_1(\bar{x}^{1/2}, \bar{x}\bar{z})/\bar{x}^{b'+1} = 0\} \subseteq \mathbb{C}^4,$$
$$U_2 = \{\bar{u}^2 + \bar{y}\bar{z} + 2\bar{z}h(\bar{x},\bar{z}) + \lambda\bar{x}^{2a+1} + g_1(\bar{x}\bar{y}^{1/2}, \bar{y}\bar{z})/\bar{y}^{b'+1} = 0\}/\mathbb{Z}_{b'+1}(1,-2,2,0),$$
$$U_3 = \{\bar{u}^2 + \bar{y}^2\bar{z} + 2\bar{y}h(\bar{x},1) + \lambda\bar{y}\bar{x}^{2a+1} + g_1(\bar{x}\bar{z}^{1/2}, \bar{z})/\bar{z}^{b'+1} = 0\}/\mathbb{Z}_2(1,1,0,1).$$

The origin Q_2 of U_2 is terminal of type $(\mathrm{cA}/b'+1)$ with axial weight 2. We always have $\mathrm{aw}(X) \geq b'+1$ and the equality holds if and only if $Q_3 \notin E_1$. If $Q_3 \in E_1$, then Q_3 is at worst terminal of type $(\mathrm{cD}/2)$ with axial weight $\mathrm{aw}(X) - b' - 1$. Hence $\bar{X}_1$ has only terminal singularities and $\sum_{Q \in \bar{X}_1}(\mathrm{aw}(\bar{X}_1, Q) - 1) = \max\{\mathrm{aw}(X) - b' - 1, 1\}$. Since E_1 is irreducible and $d(v_1) = 1/2$, we see that π_1 is divisorial with discrepancy $1/2$. □

Proposition 5.36. *Under the notation and assumptions* (5.34), *the* v_2-*blow up* $\pi_2 : \bar{X}_2 \to X$ *is divisorial with discrepancy* $1/2$ *and we have* $\sum_{Q \in \bar{X}_2}(\mathrm{aw}(\bar{X}_2, Q) - 1) = \max\{\mathrm{aw}(X) - b' - 2, 0\}$.

Proof. Let E_2 be the exceptional divisor of π_2. Since

$$E_2 \simeq \left\{ \begin{array}{c} u^2 + yt + g_{1,\tau'\text{-}wt=b'+1}(x,z) = 0 \\ yz + 2zh(x,z) + \lambda x^{2a+1} = 0 \end{array} \right\} \subseteq \mathbb{P}(1, b'-1, 2, b'+1, b'+3),$$

we see that E_2 is Cartier outside $\{Q_2, Q_3, Q_5\} \cap E_2$. Since $Q_4 \notin E_2$, $\bar{X}_2$ is covered by three affine open sets as follows:

$$U_1 = \left\{ \begin{array}{c} \bar{u}^2 + \bar{y}\bar{t} + g_1(\bar{x}^{1/2}, \bar{x}\bar{z})/\bar{x}^{b'+1} = 0 \\ \bar{t}\bar{x} - \bar{y}\bar{z} - 2\bar{z}h(1,\bar{z}) - \lambda = 0 \end{array} \right\} \subseteq \mathbb{C}^5,$$
$$U_2 = \left\{ \begin{array}{c} \bar{u}^2 + \bar{t} + g_1(\bar{x}\bar{y}^{1/2}, \bar{y}\bar{z})/\bar{y}^{b'+1} = 0 \\ \bar{t}\bar{y} - \bar{z} - 2\bar{z}h(\bar{x},\bar{z}) - \lambda\bar{x}^{2a+1} = 0 \end{array} \right\}/\mathbb{Z}_{b'-1}(1,-2,2,2,4),$$
$$U_3 = \left\{ \begin{array}{c} \bar{u}^2 + \bar{y}\bar{t} + g_1(\bar{x}\bar{z}^{1/2}, \bar{z})/\bar{z}^{b'+1} = 0 \\ \bar{t}\bar{z} - \bar{y} - 2h(\bar{x},1) - \lambda\bar{x}^{2a+1} = 0 \end{array} \right\}/\mathbb{Z}_2(1,1,0,1,1),$$
$$U_5 = \left\{ \begin{array}{c} \bar{u}^2 + \bar{y} + g_1(\bar{x}\bar{t}^{1/2}, \bar{z}\bar{t})/\bar{t}^{b'+1} = 0 \\ \bar{t} - \bar{y}\bar{z} - 2\bar{z}h(\bar{x},\bar{z}) - \lambda\bar{x}^{2a+1} = 0 \end{array} \right\}/\mathbb{Z}_{b'+3}(-1,2,-2,2,2).$$

The origin Q_2 of U_2 (resp. Q_5 of U_5) is isomorphic to $(\bar{x},\bar{y},\bar{u})/\mathbb{Z}_{b'-1}(1,-2,2)$ (resp. $(\bar{x},\bar{z},\bar{u})/\mathbb{Z}_{b'+1}(1,-2,2)$). We always have $\mathrm{aw}(X) \geq b'+1$ and the equality holds if and only if $Q_3 \notin E_2$. If $Q_3 \in E_2$, then Q_3 is at worst terminal of type $(\mathrm{cD}/2)$ with axial weight $\mathrm{aw}(X) - b' - 1$. Other singularities of $\bar{X}_2$ are all isolated cDV points. Hence $\bar{X}_2$ has only terminal singularities. Since E_2 is irreducible, we see that π_2 is divisorial with discrepancy $1/2$. □

Proposition 5.37. *If $2a = b'$ and $\frac{\lambda^2}{4}x^{4a+2} - g_{\tau'\text{-}wt=b'}(x,z)z$ is a square, then there are exactly $a+1$ prime divisors with discrepancies $1/2$ over X. Furthermore, there are exactly three divisorial blow ups of X with discrepancies $1/2$.*

Proof. If $2a = b' = 2$, then $\frac{\lambda^2}{4}x^6 - g_{\tau'\text{-}wt=2}(x,z)z = \frac{\lambda^2}{4}x^6 - a_{2,0}x^4z$ is not a square. Hence $A \neq \emptyset$ in this case. As in the proof of (5.14), we see that E_1 in (5.35) and E_2 in (5.36) are both different from E_l in (5.4). By considering the $\mathbb{Q}$-Cartier Weil divisor D on X defined by $y = 0$, we see that

$$\pi_1^*(D) = \pi_1^{-1}[D] + \frac{b'+1}{2}E_1 \text{ and } \pi_2^*(D) = \pi_2^{-1}[D] + \frac{b'-1}{2}E_2.$$

Thus D_1 and D_2 are distinct prime divisors over X. Therefore there are exactly $a+1$ prime divisors with discrepancies $1/2$ over X. The last part follows from (5.4), (5.35) and (5.36). □

5.38. By using propositions above, we complete the proof of (3.1), (3.2) and (3.3) if $2a = b'$.

§6. Gorenstein Resolutions of Terminal Singularities

In this section, we shall give a proof of the following theorem, which we already stated in (3.5).

Theorem 6.1. *Let X be a germ of a 3-dimensional terminal singularity of index $m \geq 2$. Then there is a sequence*

$$X_N \xrightarrow{\pi_N} X_{N-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\pi_1} X_0 = X$$

such that

(i) *X_i has only terminal singularities ($i = 0, 1, \ldots, N$) and furthermore X_N has only Gorenstein terminal singularities, and*

(ii) *π_i is a divisorial blow up at $P_{i-1} \in X_{i-1}$ with discrepancy $1/m_i$, where m_i is the index at P_{i-1} ($i = 1, \ldots, N$).*

The proof will be done by induction on axial weights. The following lemma treats the case where the axial multiplicity is one and this is easily deduced from [Kaw96] or [Reid87]:

Lemma 6.2. *Let $X = (x,y,z)/\mathbb{Z}_m(a,-a,1)$ $(0 < a < m,\ (a,m) = 1)$ be a germ of a cyclic quotient terminal singularity of index $m \geq 2$ and let $\pi : \bar{X} \to X$ be the divisorial blow up with discrepancy $1/m$. Then singularities of $\bar{X}$ are all cyclic quotient terminal singularities of index $< m$.*

Proof. Since π is given by the $\frac{1}{m}(a, m-a, 1)$-blow up of X, we easily see that there are at most two singular points on $\bar{X}$. These are both cyclic quotient terminal singularities. One of them is of index a, and the other is of index $m - a$. □

Proof of (6.1). Let X be a germ of a 3-dimensional terminal singularity of index $m \geq 2$. We shall prove this by induction on $\mathrm{aw}(X)$.

If $\mathrm{aw}(X) = 1$, then X is a cyclic quotient terminal singularity. By using (6.2) and induction on the index, we can construct a sequence of blow ups as required.

If $\mathrm{aw}(X) > 1$, then there is a divisorial blow up $\pi_1 : X_1 \to X$ with discrepancy $1/m$ and we have

$$(*) \qquad \sum_{Q \in X_1} (\mathrm{aw}(X_1, Q) - 1) \leq \mathrm{aw}(X) - 1$$

by (3.4). If the inequality $(*)$ is strict, we have $\mathrm{aw}(X_1, Q) < \mathrm{aw}(X)$ for each $Q \in X_1$. By induction on $\mathrm{aw}(X)$, we can construct a sequence of blow ups for X_1. Thus we complete the proof when the inequality $(*)$ is strict. If $(*)$ holds with equality, then we shall study π_1 more closely. In this case, X is of type (cD/3) and there is a unique non-Gorenstein point $Q_1 \in X_1$ which is of type (cAx/4) by [Hay99, 4.5, 9.4]. There is a divisorial blow up $\pi_2 : X_2 \to X_1$ with discrepancy $1/4$ over X_1 and this satisfies $\sum_{R \in X_2} (\mathrm{aw}(X_2, R) - 1) < \mathrm{aw}(X_1, Q_1) - 1$. Thus we have $\sum_{R \in X_2} (\mathrm{aw}(X_2, R) - 1) < \mathrm{aw}(X) - 1$. We again use the induction hypothesis on X_2 and get a sequence of blow ups as required. □

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