Blowup or No Blowup? The Interplay between Theory and Numerics

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Introduction

- Computing Euler singularities is an extremely challenging task.
- It requires huge computational resource.
- Careful resolution study. It is dangerous to interpret the blowup of an under-resolved computation as the blowup of the Euler equations.
- Validation check: Is the fitting $\|\omega\|_{L^\infty} \approx \frac{C}{(T-t)^\alpha}$ asymptotically valid as $t \to T$ to be used to check if $\int_0^T \|\omega\|_{L^\infty} dt = \infty$?
- Consistency check with other non-blowup criteria. Is there any depletion of vortex stretching? Guidance from the theory is essential.

Numerical evidence of Euler singularity

In 1993 (and 2005), R. Kerr [Phys. Fluids] presented numerical evidence of 3D Euler singularity for two anti-parallel vortex tubes:

- Pseudo-spectral in x and y, Chebyshev in z direction;
- Best resolution: $512 \times 256 \times 192$;
- Predicted singularity time $T = 18.7$, but his numerical solutions became under-resolved after $t = 17$; Note that $\Delta = 1.7$ is not small.
- $\|\boldsymbol{\omega}\|_{L^\infty} \approx (T - t)^{-1}$;
- $\|\mathbf{u}\|_{L^\infty} \approx (T - t)^{-1/2}$;
- Anisotropic scaling: $(T - t) \times \sqrt{T - t} \times \sqrt{T - t}$;
- Vortex lines: relatively straight, $|\nabla \boldsymbol{\xi}| \approx (T - t)^{-1/2}$;

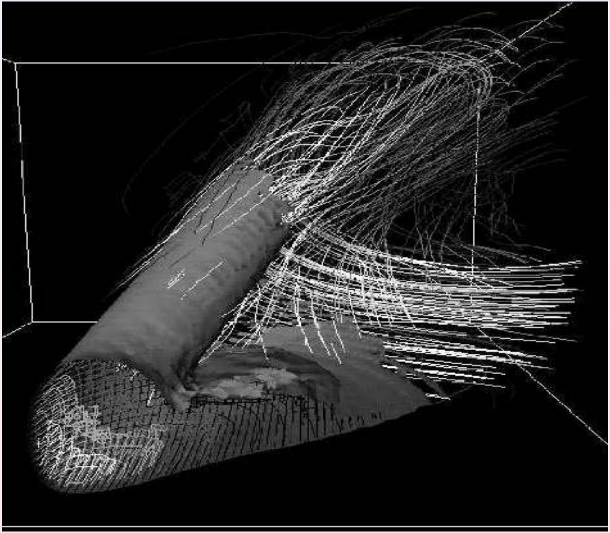


Figure: Isosurface of peak vorrticity at $t = 17$, from R. Kerr, Euler singularities and turbulence, 19th ICTAM Kyoto '96, 1997, pp57-70.

Non-blowup criterion by Constantin-Fefferman-Majda

- Kerr's blowup scenario is consistent with the Beale-Kato-Majda (1984) and the Constantin-Fefferman-Majda criteria (1996).

- Constantin-Fefferman-Majda's non-blowup criterion (1996).

  Let $\boldsymbol{\omega} = |\boldsymbol{\omega}|\boldsymbol{\xi}$, no blow-up if

  (1) (Bounded velocity) $\|\mathbf{u}\|_\infty$ is bounded in a $O(1)$ region of large vorticity;

  (2) (Regular orientedness) $\int_0^t \|\nabla \boldsymbol{\xi}\|_\infty^2 d\tau$ is uniformly bounded;

Local non-blowup criteria by Deng-Hou-Yu

But it falls into the critical case of Deng-Hou-Yu's non-blowup criteria.

Theorem 1 (Deng-Hou-Yu, 2005 and 2006, CPDE)

- Denote by $L(t)$ the arclength of a vortex line segment L_t around the maximum vorticity. If
  1. $\max_{L_t}(|\mathbf{u}\cdot\boldsymbol{\xi}| + |\mathbf{u}\cdot\mathbf{n}|) \leq C_U(T-t)^{-A}$ with $A < 1$;
  2. $C_L(T-t)^B \leq L(t) \leq C_0/\max_{L_t}(|\kappa|, |\nabla\cdot\xi|)$ with $B < 1 - A$;

  then the solution of the 3D Euler equations remains regular at T.

  When $B = 1 - A$, we can exclude blowup if $f(C_U, C_L, C_0) > 0$. For example, $C_L = 1$, $C_0 = 0.1$, $C_U \leq 0.28$ implies no blowup.

Computation of Hou and Li, J. Nonlinear Science, 2006

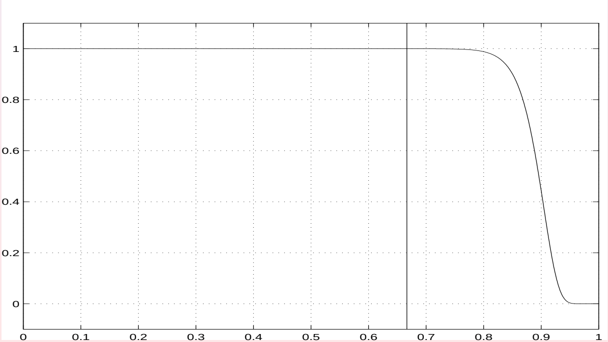
- We repeat Kerr's computations using two pseudo-spectral methods.
- Four step Runge-Kutta scheme for time integration with adaptive time stepping;
- Careful resolution study is performed: $768 \times 512 \times 1536$, $1024 \times 768 \times 2048$ and $1536 \times 1024 \times 3072$.
- We compute the solution up to $t = 19$, beyond the alleged singularity time $T = 18.7$ of Kerr.
- 256 parallel processors with maximal memory comsumption 120Gb.
- The laregest number of grid points is close to 5 billions.

Two spectral methods are used in our computations.

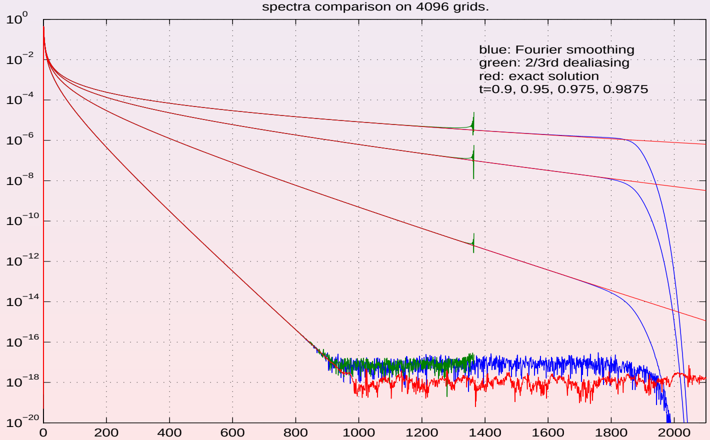
- We use both the 2/3 dealiasing and a 36-order Fourier smoothing to remove aliasing error;
- The Fourier smoother is shaped as along the x_j direction

$$\rho(2k_j/N_j) \equiv \exp(-36(2k_j/N_j)^{36})$$

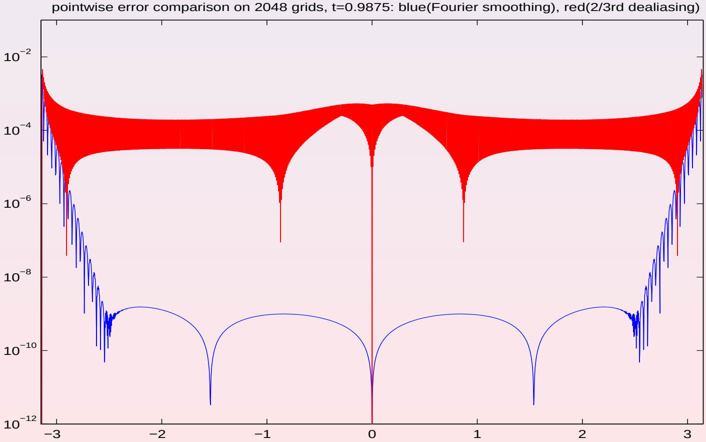
where k_j is the wave number ($|k_j| \leqslant N_j/2$).



Inviscid Burgers equation: spectra comparison with $N = 4096, u_0(x) = sin(x)$, $T_{\text{shock}} = 1$.



Inviscid Burgers equation: the pointwise error comparison with $N = 2048$, $u_0(x) = sin(x)$, $T_{shock} = 1$.



Resolution study of 3D Euler Equations. Enstrophy spectra: $768 \times 512 \times 1024$ vs $1024 \times 768 \times 1536$

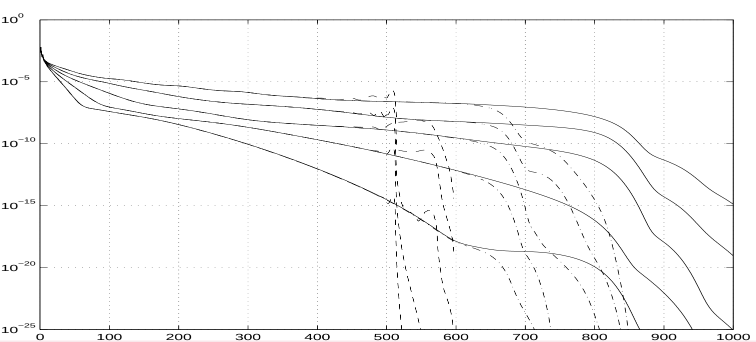
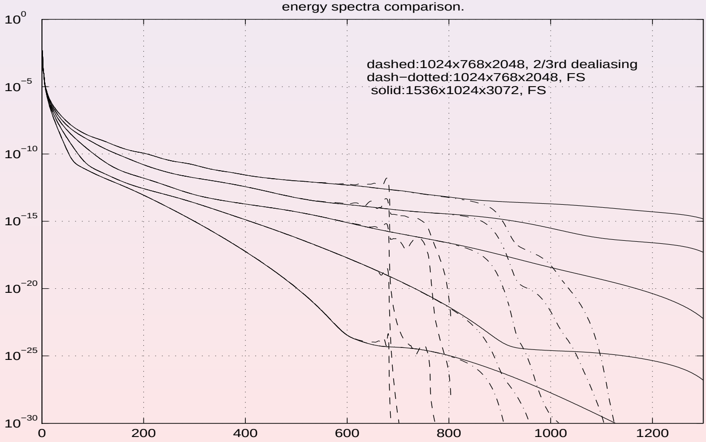


Figure: The enstrophy spectra versus wave numbers. The dashed lines and dashed-dotted lines are solutions with $768 \times 512 \times 1024$ using the 2/3 dealiasing rule and the Fourier smoothing, respectively. The times for the spectra lines are at $t = 15, 16, 17, 18, 19$ respectively.

Resolution study of 3D Euler Equations. Energy spectra: $1024 \times 768 \times 2048$ vs $1536 \times 1024 \times 3072$



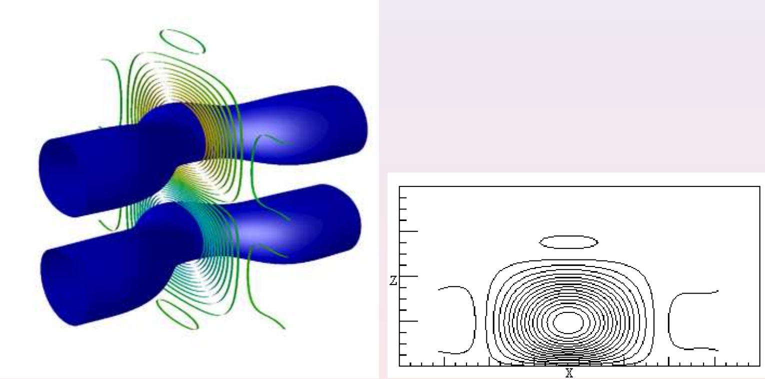


Figure: The 3D vortex tube and axial vorticity on the symmetry plane for initial value.

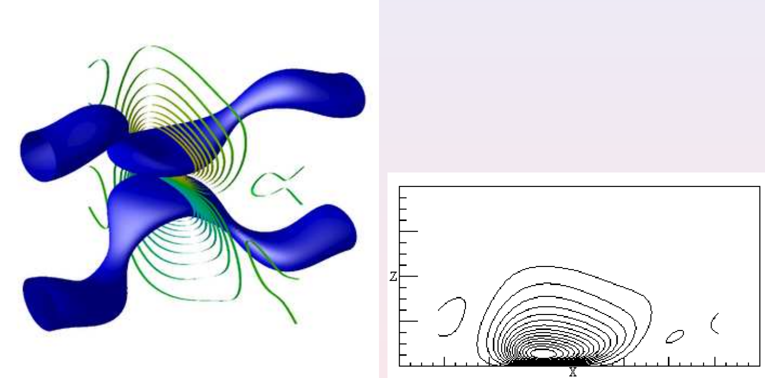


Figure: The 3D vortex tube and axial vorticity on the symmetry plane when $t = 6$.

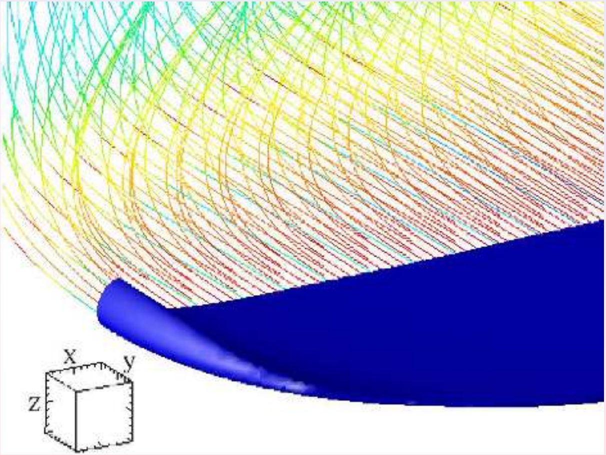


Figure: The local 3D vortex structures and vortex lines around the maximum vorticity at $t = 17$.

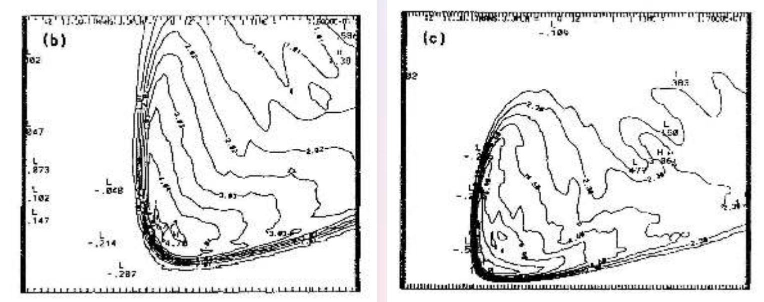


Figure: From: Kerr, Phys. Fluids A 5(7), 1993, pp1725-1746. $t = 15$(left) and $t = 17$(right).

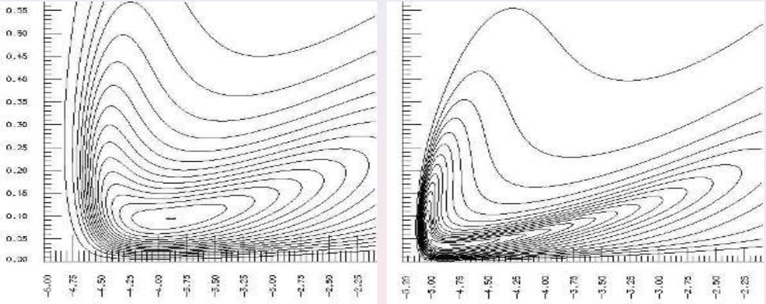


Figure: The contour of axial vorticity around the maximum vorticity on the symmetry plane at $t = 15, 17$.

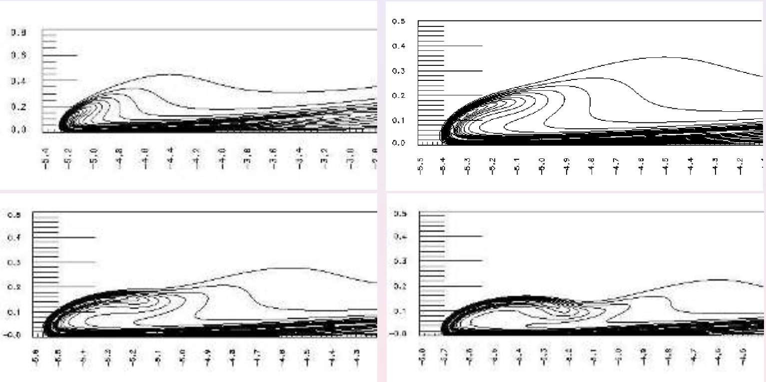


Figure: The contour of axial vorticity around the maximum vorticity on the symmetry plane (the *xz*-plane) at $t = 17.5, 18, 18.5, 19$.

Maximum vorticity in time

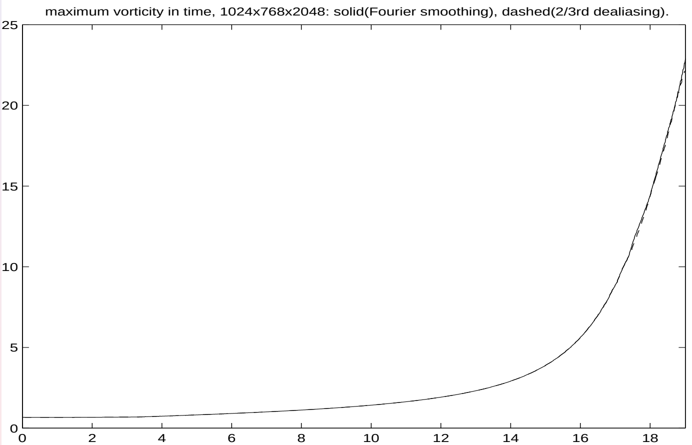


Figure: The maximum vorticity $\|\omega\|_\infty$ in time, $1024 \times 768 \times 2048$, computed by two spectral methods.

Inverse of maximum vorticity in time

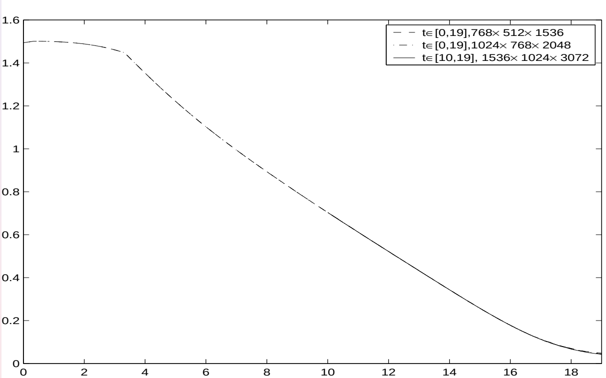


Figure: The inverse of maximum vorticity $\|\boldsymbol{\omega}\|_\infty$ in time using different resolutions.

Dynamic depletion of vortex stretching

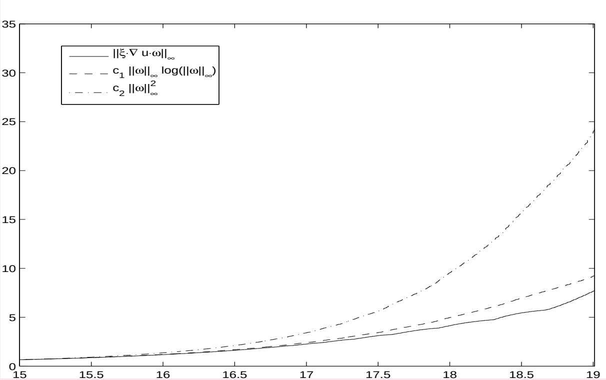


Figure: Study of the vortex stretching term in time, resolution $1536 \times 1024 \times 3072$. The fact $|\boldsymbol{\xi} \cdot \nabla \mathbf{u} \cdot \boldsymbol{\omega}| \leq c_1 |\boldsymbol{\omega}| log |\boldsymbol{\omega}|$ plus $\frac{D}{Dt}|\omega| = \boldsymbol{\xi} \cdot \nabla \mathbf{u} \cdot \boldsymbol{\omega}$ implies $|\boldsymbol{\omega}|$ bounded by doubly exponential.

Log log plot of maximum vorticity in time

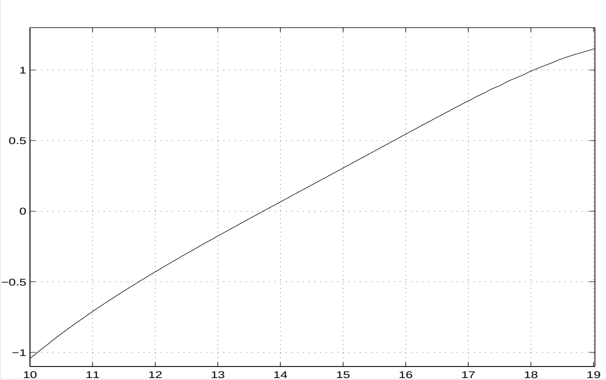
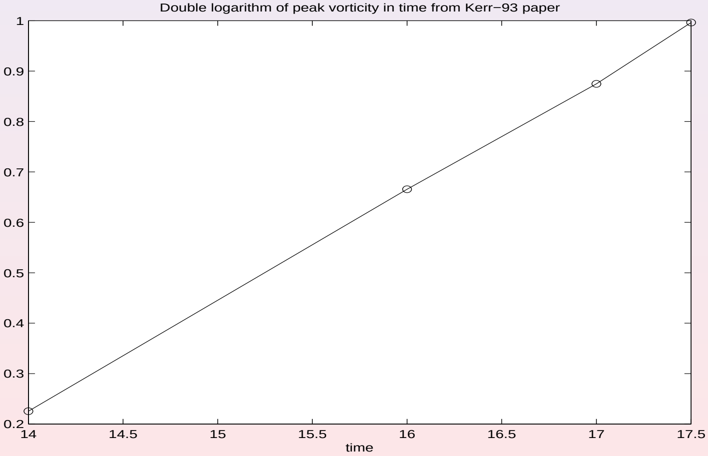


Figure: The plot of $\log\log\|\omega\|_\infty$ vs time, resolution $1536 \times 1024 \times 3072$.

Log log plot of peak vorticity in time from Kerr's 93 paper



Maximum velocity in time

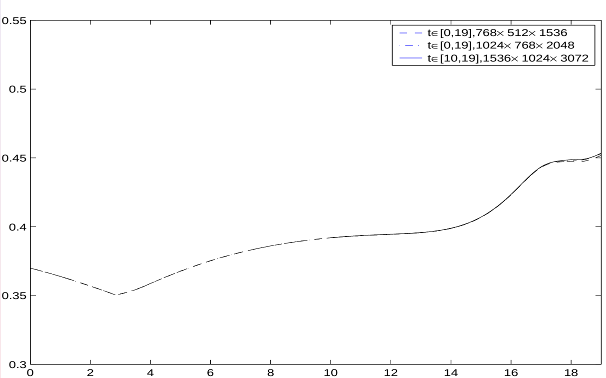


Figure: Maximum velocity $\|\mathbf{u}\|_\infty$ in time using different resolutions.

The local geometric criteria applies

Recall the local geometric criteria by Deng-Hou-Yu:

1. $\max_{L_t}(|\mathbf{u} \cdot \boldsymbol{\xi}| + |\mathbf{u} \cdot \mathbf{n}|) \leq C_U(T - t)^{-A}$ for some $A < 1$;
2. $C_L(T - t)^B \leq L(t) \leq C_0/\max_{L_t}(|\kappa|, |\nabla \cdot \xi|)$ for some $B < 1 - A$,

then the solution of the 3D Euler equations remains regular up to T.

- Since $\|\mathbf{u}\|_{L^\infty}$ is bounded, we have $A = 0$ so our local non-blowup theory applies since $B = 1/2 < 1 - A$. This rules out a singularity up to $T = 19$.

Vorticity vector alignment

Recall that

$$\frac{\partial}{\partial t}\boldsymbol{\omega} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = S\cdot\boldsymbol{\omega}, \quad S = \frac{1}{2}(\nabla u + \nabla^T u).$$

Let $\lambda_1 < \lambda_2 < \lambda_3$ be the three eigenvalues of S, $\lambda_1 + \lambda_2 + \lambda_3 = 0$.

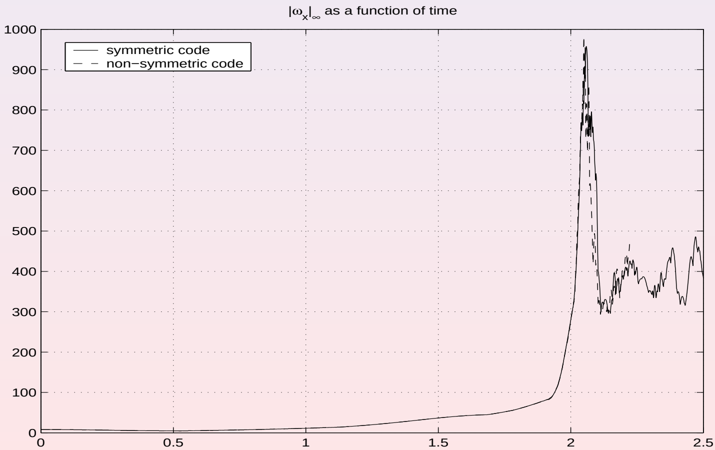
time	$\|\boldsymbol{\omega}\|$	λ_1	θ_1	λ_2	θ_2	λ_3	θ_3
16.012	5.628	-1.508	89.992	0.206	0.007	1.302	89.998
16.515	7.016	-1.864	89.995	0.232	0.010	1.631	89.990
17.013	8.910	-2.322	89.998	0.254	0.006	2.066	89.993
17.515	11.430	-2.630	89.969	0.224	0.085	2.415	89.920
18.011	14.890	-3.625	89.969	0.257	0.036	3.378	89.979
18.516	19.130	-4.501	89.966	0.246	0.036	4.274	89.984
19.014	23.590	-5.477	89.966	0.247	0.034	5.258	89.994

Table: The alignment of the vorticity vector and the eigenvectors of S around the point of maximum vorticity with resolution $1536 \times 1024 \times 3072$. Here, θ_i is the angle between the i-th eigenvector of S and the vorticity vector.

Kida-Pelz's high symmetry initial data

- We have also repeated Pelz's computations, and found no evidence of a finite time singularity.
- Pelz's filament model indeed leads to a finite time blowup [PRE, 97]. But when we use the same high symmetry initial condition to solve the full 3D Euler equations, the solution remains regular.
- Boratav and Pelz's Navier-Stokes computations [Phys Fluid,94] suggested a potential singularity around $t = 2.06$ as $Re \to \infty$.
- Our resolution study shows that their computations are resolved only up to $t = 1.6$ when the growth is only exponential in time. The rapid growth around $t = 2.06$ seems due to under-resolution.
- We have used two codes to compute the high symmetry solution, one code built in the high symmetry explicitly, the other did not. The symmetry is preserved by the second code to many digits.

Maximum vorticity of the high symmetry data in time, one code built in high symmetry explicitly, the other did not.



Concluding Remarks

- Our analysis and computations reveal a subtle dynamic depletion of vortex stretching. Sufficient numerical resolution is essential in capturing this dynamic depletion.
- Our computations show that the velocity is bounded and that $\|\xi \cdot \nabla \mathbf{u} \cdot \omega\|_{L^\infty} = O(\|\omega\|_{L^\infty} log(\|\omega\|_{L^\infty}))$, instead of $\|\omega\|_{L^\infty}^2$.
- It is natural to ask what is the driving mechanism for this dynamic depletion of vortex stretching? Is this scaling generic?
- The geometric regularity of local vortex lines and the anisotropic scaling of the support of maximum vorticity seem to play an important role in the dynamic depletion of vortex stretching.
- New analytic tools that exploit the local geometric structure of the solution near a potential singularity are needed.

References

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