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BMO ON THE BERGMAN SPACES OF THE CLASSICAL DOMAINS

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Let Ω be a bounded symmetric (Cartan) domain with its Harish-Chandra realization in $\mathbf{C}^n$ [**T**]. For dv the usual Euclidean volume measure on $\mathbf{C}^n = \mathbf{R}^{2n}$, normalized so that $v(\Omega) = 1$, we consider the Hilbert space of square-integrable complex-valued functions $L^2 = L^2(\Omega, dv)$ and the Bergman subspace $H^2 = H^2(\Omega)$ of holomorphic functions in L^2. The self-adjoint projection from L^2 onto H^2 is denoted by P. For f, g in L^2, we consider the *multiplication operator* M_f on L^2 given by $M_f g = fg$ and the *Hankel operator* H_f on L^2 given by $H_f = (I - P)M_f P$. For f in L^2, these operators are only densely defined and may be unbounded. The commutator $[M_f, P] = M_f P - PM_f$ is densely defined on L^2 and may also be unbounded. From the equations

$$[M_f, P] = H_f - H_{\bar{f}}^*, \quad (I - P)[M_f, P] = H_f, \quad [M_f, P](I - P) = -H_{\bar{f}}^*,$$

it follows that $[M_f, P]$ is a bounded operator if and only if $H_f, H_{\bar{f}}$ are bounded. Moreover, $[M_f, P]$ is a compact operator if and only if $H_f, H_{\bar{f}}$ are compact.

In earlier work [**BCZ**], it was shown that for f in $L^\infty(\Omega)$, the algebra of bounded measurable functions on Ω, $[M_f, P]$ is compact if and only if f has vanishing mean oscillation at the boundary $\partial\Omega$, where oscillation is defined in terms of the Bergman metric on Ω. In this note, we announce the companion result: *For f in L^2, $[M_f, P]$ is bounded if and only if f is of "bounded mean oscillation on Ω"*, where oscillation is defined as in [**BCZ**]. The space of such functions is denoted by BMO(Ω). We also obtain the expected result that: *For f in L^2, $[M_f, P]$ is compact if and only if f is in the subspace* $\mathrm{VMO}_\partial(\Omega)$ *of functions which have vanishing mean oscillation at the boundary* $\partial\Omega$. Our results are analogous to known results for arc-length measure on the unit circle [**G**, p. 278] and demonstrate the value of the Bergman metric in function-theoretic analysis on the classical domains.

Let $K(\cdot, a)$ be the Bergman reproducing kernel in $H^2(\Omega)$ for evaluation at $a \in \Omega$. For

$$k_a(\cdot) = K(a, a)^{-1/2} K(\cdot, a),$$

we define the *Berezin transform* of f in L^2 [**BCZ**] by

$$\tilde{f}(a) = \langle f k_a, k_a \rangle$$

where $\langle \cdot, \cdot \rangle$ is the usual L^2 inner product. For typographical reasons, we write the Berezin transform of $|f|^2$ as $(|f|^2)^\sim$. It follows from known properties of

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the $\{k_a\}$ that $\tilde{f}$ is defined and smooth (C^∞) everywhere on Ω. The transform $\tilde{f}$ is critical to our analysis as are some previously unremarked properties of the Bergman metric $\beta(\cdot,\cdot)$ on Ω [**K**, p. 45; **H**, p. 298]. In the remainder of this note, we provide enough detail to rigorously state our main results and the main technical lemmas required for the proofs.

We recall that the Bergman metric $\beta(\cdot,\cdot)$ is a complete Riemannian metric on Ω which gives the usual topology on Ω [**H**, p. 52]. Moreover, the closed metric balls

$$E(a,r) = \{z: \beta(a,z) \le r\}$$

are compact [**H**, p. 56]. By definition [**K**, p. 45], β is the "integrated form" of the infinitesimal metric

$$g_{ij}(z) = 1/2 \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K(z,z).$$

By $z \to \partial\Omega$, we mean that the usual distance function

$$d(z,\partial\Omega) \equiv \inf\{|z-w|: w \in \partial\Omega\}$$

has the property that $d(z,\partial\Omega) \to 0$. Let $\mathrm{BC}(\Omega)$ denote the algebra of bounded continuous functions on Ω, with $\mathrm{C}_\partial(\Omega)$ the subalgebra of all continuous functions for which $f(z) \to 0$ as $z \to \partial\Omega$. For f continuous on Ω, we define

$$\mathrm{Osc}_z(f) = \sup\{|f(z) - f(w)|: \beta(z,w) \le 1\}.$$

It is not hard to check, using the completeness of β, that $\mathrm{Osc}_z(f)$ is also a continuous function of z. We say f is of *bounded oscillation* ($f \in \mathrm{BO}(\Omega)$) if $\mathrm{Osc}_z(f)$ is in $\mathrm{BC}(\Omega)$ (as a function of z). We say f is of *vanishing oscillation at* $\partial\Omega$ ($f \in \mathrm{VO}_\partial(\Omega)$) if $\mathrm{Osc}_z(f)$ is in $\mathrm{C}_\partial(\Omega)$ (as a function of z) (cf. [**BCZ**]). For f in L^2, the quantity

$$\mathrm{MO}(f,z) = (|f|^2)^\sim(z) - |\tilde{f}(z)|^2$$

is a continuous function on Ω. We say f is of *bounded mean oscillation on* Ω ($f \in \mathrm{BMO}(\Omega)$) if $\mathrm{MO}(f,\cdot)$ is in $\mathrm{BC}(\Omega)$. We say f has *vanishing mean oscillation at* $\partial\Omega$ ($f \in \mathrm{VMO}_\partial(\Omega)$) if $\mathrm{MO}(f,\cdot)$ is in $\mathrm{C}_\partial(\Omega)$.

We also have a more geometric notion of mean oscillation. Let $|E(z,r)| = v(E(z,r))$. For fixed $r > 0$ and f in L^2, the quantities

$$\hat{f}(z,r) = |E(z,r)|^{-1} \int_{E(z,r)} f(w)\,dv(w),$$

$$\begin{aligned}\mathrm{MO}_r(f,z) &= |E(z,r)|^{-1} \int_{E(z,r)} |f(w) - \hat{f}(z,r)|^2\,dv(w)\\ &= \frac{1}{2}|E(z,r)|^{-2} \int_{E(z,r)} \int_{E(z,r)} |f(w)-f(u)|^2\,dv(w)\,dv(u)\end{aligned}$$

are continuous functions on Ω. We say f in L^2 is in $\mathrm{BMO}_r(\Omega)$ if $\mathrm{MO}_r(f,\cdot)$ is in $\mathrm{BC}(\Omega)$. We say f in L^2 is in $\mathrm{VMO}_\partial^r(\Omega)$ if $\mathrm{MO}_r(f,\cdot)$ is in $\mathrm{C}_\partial(\Omega)$.

We require a few additional definitions. We write

$$\mathcal{F} = \{f \in L^2: (|f|^2)^\sim \in \mathrm{BC}(\Omega)\}, \qquad \mathcal{I} = \{f \in L^2: (|f|^2)^\sim \in \mathrm{C}_\partial(\Omega)\}.$$

Clearly, $\mathcal{F} \subset \mathrm{BMO}(\Omega)$ and $\mathcal{I} \subset \mathrm{VMO}_\partial(\Omega)$. For S any subset of Ω, we write

$$\|f\|_{\mathrm{BMO}(S)} \equiv \sup_{z\in S} \mathrm{MO}(f,z)^{1/2}$$

and

$$\|f\|_{\mathrm{BMO}} \equiv \|f\|_{\mathrm{BMO}(\Omega)}, \qquad \|f\|_r \equiv \sup_{z\in\Omega} \mathrm{MO}_r(f,z)^{1/2}.$$

Our main result is

THEOREM A. *For f in L^2, the following are equivalent:*
(i) *$H_f, H_{\bar{f}}$ are bounded.*
(ii) $f \in \mathrm{BMO}(\Omega)$.
(iii) $f \in \mathrm{BMO}_r(\Omega)$ *for all* $r > 0$.
(iv) $f \in \mathrm{BMO}_r(\Omega)$ *for some* $r > 0$.
(v) $f \in \mathrm{BO} + \mathcal{F}$.
Moreover, the quantities $\max\{\|H_f\|, \|H_{\bar{f}}\|\}$, $\|[M_f, P]\|$, $\|f\|_{\mathrm{BMO}}$, *and* $\|f\|_r$ *are equivalent and $\tilde{f}$ is in* BO *with $f - \tilde{f}$ in $\mathcal{F}$ whenever any of* (i)–(v) *hold.*

We also have a corresponding extension of the main result of [**BCZ**]:

THEOREM B. *For f in L^2, the following are equivalent:*
(i) *$H_f, H_{\bar{f}}$ are compact.*
(ii) $f \in \mathrm{VMO}_\partial(\Omega)$.
(iii) $f \in \mathrm{VMO}_\partial^r(\Omega)$ *for all* $r > 0$.
(iv) $f \in \mathrm{VMO}_\partial^r(\Omega)$ *for some* $r > 0$.
(v) $f \in \mathrm{VO}_\partial(\Omega) + \mathcal{I}$.
Moreover, $\tilde{f}$ is in $\mathrm{VO}_\partial(\Omega)$ *and $f - \tilde{f}$ is in $\mathcal{I}$ whenever any of* (i)-(v) *hold.*

For x in $\mathbf{C}^n$ and z in Ω we define

$$H(z,x) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i \bar{x}_j g_{ij}(z).$$

For f in H^2 and z in Ω we have the analytic gradient

$$\nabla_z f = \left(\frac{\partial f}{\partial z_1}(z), \frac{\partial f}{\partial z_2}(z), \ldots, \frac{\partial f}{\partial z_n}(z)\right)$$

and we can define, as in [**T**],

$$Q_f(z) = \sup\{|\langle \nabla_z f, \bar{x}\rangle| H(z,x)^{-1/2} : 0 \neq x \in \mathbf{C}^n\}.$$

Following [**T**], we say f *is in the Bloch space* $B(\Omega)$ if

$$\sup_{z\in\Omega} Q_f(z) \equiv \|f\|_B < \infty.$$

We say f *is in the little Bloch space* $B_0(\Omega)$ if

$$\lim_{z\to\partial\Omega} Q_f(z) = 0.$$

For f in H^2, it is easy to see that $H_f = 0$. In this case, we obtain some additional information:

THEOREM C. *For f in H^2, $H_{\bar{f}}$ is bounded if and only if f is in $B(\Omega)$. Moreover, $\|H_{\bar{f}}\|$ and $\|f\|_B$ are equivalent quantities.*

THEOREM D. *For f in $H^2, H_{\bar{f}}$ is compact if and only if f is in $B_0(\Omega)$. For rank $(\Omega) \neq 1, B_0(\Omega)$ consists of just the constant functions.*

The proofs of Theorems A and B depend on two key results about the Bergman metric and the Berezin transform.

THEOREM E. *The function $\beta(0, \cdot)$ is in $L^p(\Omega, dv)$ for all $p > 0$.*

THEOREM F. *For any smooth curve $\gamma: I \to \Omega$ ($I = [0,1]$) with $s = s(t)$ the arc-length of γ with respect to the Bergman metric $(g_{ij}(z))$, and for any f in* BMO(Ω), *we have*

$$\left|\frac{d}{dt}\tilde{f}(\gamma(t))\right| \leq 2\sqrt{2}\left(\frac{ds}{dt}\right)\|f\|_{\mathrm{BMO}(\gamma(I))}.$$

We also use a recent result of [**FK**] and some estimates from [**BCZ**]. Proofs will appear elsewhere.

It should be recalled that, as $z \to \partial\Omega$,

$$\beta(0, z) \to +\infty, \qquad K(z, z) \to +\infty.$$

The point of Theorem E is that, for bounded symmetric domains Ω, $\beta(0, z)$ does not blow up too badly near $\partial\Omega$.

REMARK. The function $\beta(0, \cdot)$ is the prototype of BO(Ω). Using the invariance of the metric $\beta(\cdot, \cdot)$ under automorphisms of Ω and Theorem E, it is easy to check that $\beta(0, \cdot)$ is in BMO(Ω). The function $\exp\{i\beta(0, \cdot)^{1/2}\}$ is in VMO$_\partial(\Omega)$, as noted in [**BCZ**].

The measure $dv(z)$ may be replaced by Harish-Chandra measures of the form $C_t K(z, z)^t \, dv(z)$ with $t < t_0(\Omega)$ and Theorems A and B remain true. We conjecture that Theorems A and B hold, with different proofs, for domains considerably more general than the bounded symmetric (Cartan) domains.

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