# BMS charge algebra

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ABSTRACT. The surface charges associated with the symmetries of asymptotically flat four dimensional spacetimes at null infinity are constructed. They realize the symmetry algebra in general only up to a field-dependent central extension that satisfies a suitably generalized cocycle condition. This extension vanishes when using the globally well defined BMS algebra. For the Kerr black hole and the enlarged BMS algebra with both supertranslations and superrotations, some of the supertranslations charges diverge whereas there are no divergences for the superrotation charges. The central extension is proportional to the rotation parameter and involves divergent integrals on the sphere.

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### 1 Introduction

In the study of gravitational radiation in the early sixties [1, 2], it turned out that the asymptotic symmetry group at null infinity in four dimensions is not the Poincaré group, but an enhanced group where translations are replaced by supertranslations. We have recently shown [3, 4] that on the level of the algebra, one can consistently allow for infinitesimal superrotations as well and have worked out the transformation laws of the functions parametrizing solution space. The resulting symmetry algebra  $\mathfrak{bms}_4$  is an extension of the Poincaré algebra that contains two copies of the Virasoro algebra. It thus follows that asymptotically flat general relativity in four dimensions is dual to an extended conformal field theory.

An important element that is missing in our analysis is the construction of surface charges associated to  $\mathfrak{bms}_4$  together with their transformation laws. This is a notoriously difficult task as the surface charges are non-conserved and non-integrable at null infinity [5]. It is the purpose of the present paper to fill this gap. What we are especially interested in are the transformation properties of the surface charges. Indeed, in the anti-de Sitter case in three dimensions, the central extension [6] that appears has been used to argue for a microscopic explanation of the Bekenstein-Hawking entropy of the BTZ black hole [7]. A similar analysis has been applied in the near-horizon limit of an extreme four dimensional Kerr black hole [8, 9].

The main result of our paper is the construction of the field dependent central extension that generically occurs in the charge algebra at null infinity.

When the symmetry algebra is the standard, globally well-defined BMS algebra, we show that the extension vanishes. When using the extended BMS algebra with both supertranslations and superrotations instead and evaluating for a Kerr black hole, some of the supertranslation charges as well as the non-vanishing extension involves divergent integrals on the 2-sphere.

Whether our results can be used in the context of a microscopic derivation of the entropy of a Kerr black hole thus depends on the question of how to regularize the divergent integrals that occur and how to extract meaningful answers. Some comments on this problem are provided at the end of the paper.

A more complete and general theory for surface charges in the non-integrable case, together with a better understanding of how they generate the asymptotic symmetry transformations in a Dirac or Peierls bracket, is also needed. We hope to address some of these issues elsewhere.

## 2 Summary of previous results

# 2.1 General expressions for surface charge one-forms from linearized theory

Our starting point is the covariant approach to surface charges and their algebra developed in [10] (see also [11, 12]). In particular, for pure Einstein gravity with or without a cosmological constant, it has been shown in [13] that for the linearized theory, described by  $h_{\mu\nu}$  around a background  $g_{\mu\nu}$ , the conserved surface charges are completely classified by the Killing vectors  $\xi^{\mu}$  of the metric  $g_{\mu\nu}$ . These charges only depend on the Einstein equations of motion and not on the choice of Lagrangian. They form a representation of the Lie algebra of Killing vectors of  $g_{\mu\nu}$ . Their explicit expression coincides with formulas derived earlier in [14] and is given by

$$\delta Q_{\xi}[h,g] = \frac{1}{16\pi G} \int_{S} (d^{n-2}x)_{\mu\nu} \sqrt{-g} \Big[ \xi^{\nu} D^{\mu} h - \xi^{\nu} D_{\sigma} h^{\mu\sigma} + \xi_{\sigma} D^{\nu} h^{\mu\sigma} + \frac{1}{2} h D^{\nu} \xi^{\mu} + \frac{1}{2} h^{\nu\sigma} (D^{\mu} \xi_{\sigma} - D_{\sigma} \xi^{\mu}) - (\mu \leftrightarrow \nu) \Big], \quad (2.1)$$

where

$$(d^{n-k}x)_{\nu\mu} = \frac{1}{k!(n-k)!} \epsilon_{\nu\mu\alpha_1...\alpha_{n-2}} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_{n-2}}, \quad \epsilon_{01...n-1} = 1.$$

In view of these universal properties of the surface charges in the linearized theory, we use them in the context of asymptotically flat four dimensional spacetimes at null infinity. Whereas there is no issue with integrability in the linearized theory, in the full interacting theory with prescribed asymptotics, the expressions are one-forms on solution space indexed by asymptotic symmetries and one has to face the question whether these one-forms are integrable, i.e., whether one can construct suitable "Hamiltonians" for them [5]. This explains the notation  $\delta$  in (2.1).

More precisely, in the case at hand, S is a spherical cross-section of future or past null infinity, "Scri" denoted by  $\mathscr{I}$ . The metric  $g_{\mu\nu}$  is an asymptotically flat solution to Einstein's equations,  $h_{\mu\nu}$  a solution to the linearized equations at  $g_{\mu\nu}$  and  $\xi^{\mu}$  a space-time vector realizing the  $\mathfrak{bms}_4$  algebra on asymptotically flat spacetimes. Throughout, we will use the conventions of [4] to which we refer for further details. We thus have n=4, the coordinates are u,r and  $x^A=\theta,\phi$ , with S the 2-sphere at  $u=u_0$  and  $r=cst\to\infty$ , i.e., the limits of integration are  $0\leqslant\theta\leqslant\pi$  and  $0\leqslant\phi\leqslant2\pi$ . We will also use the notation  $\int d^2\Omega^\varphi=\int dx^2dx^3\sqrt{\gamma}=\int_0^{2\pi}d\phi\int_0^\pi d\theta\sin\theta\,e^{2\varphi}$  below.

### 2.2 Solution space

Asymptotically flat metrics solving Einstein's equation are of the form

$$ds^{2} = e^{2\beta} \frac{V}{r} du^{2} - 2e^{2\beta} du dr + g_{AB} (dx^{A} - U^{A} du) (dx^{B} - U^{B} du), \qquad (2.2)$$

where

$$g_{AB} = r^2 \bar{\gamma}_{AB} + rC_{AB} + D_{AB} + \frac{1}{4} \bar{\gamma}_{AB} C_D^C C_C^D + o(r^{-\epsilon}).$$
 (2.3)

The background metric is

$$\bar{\gamma}_{AB}dx^{A}dx^{B} = e^{2\varphi}(d\theta^{2} + \sin\theta d\phi^{2}) = e^{2\tilde{\varphi}}d\zeta d\bar{\zeta},$$

$$\zeta = \cot\frac{\theta}{2}e^{i\phi}, \quad \tilde{\varphi} = \varphi - \varphi_{0}, \quad \varphi_{0} = \ln P, \quad P = \frac{1}{2}(1 + \zeta\bar{\zeta}).$$
(2.4)

We assume for simplicity that  $\varphi, \widetilde{\varphi}$  do not depend on  $u, \varphi = \varphi(x^A)$ . Indices on  $C_{AB}, D_{AB}$  are raised with the inverse of  $\overline{\gamma}_{AB}$  and  $C_A^A = 0 = D_A^A$ . In addition  $\partial_u D_{AB} = 0$  and the news tensor is  $N_{AB} = \partial_u C_{AB}$ . Furthermore,

$$\beta = -\frac{1}{32}r^{-2}C_B^A C_A^B - \frac{1}{12}r^{-3}C_B^A D_A^B + o(r^{-3-\epsilon}), \qquad (2.5)$$

$$g_{uA} = \frac{1}{2}\bar{D}_B C_A^B + \frac{2}{3}r^{-1} \left[ (\ln r + \frac{1}{3})\bar{D}_B D_A^B + \frac{1}{4}C_{AB}\bar{D}_C C^{CB} + N_A \right] + o(r^{-1-\varepsilon}), \quad (2.6)$$

where  $\bar{D}_A$  is the covariant derivative associated to  $\bar{\gamma}_{AB}$  and  $N_A(u, x^A)$  is the angular momentum aspect;

$$\frac{V}{r} = -\frac{1}{2}\bar{R} + r^{-1}2M + o(r^{-1-\epsilon}), \tag{2.7}$$

where  $\bar{R}$  is the scalar curvature of  $\bar{D}_A$ ,  $\bar{R}=2e^{-2\varphi}-2\bar{\Delta}\varphi$  with  $\bar{\Delta}$  the Laplacian for  $\bar{\gamma}_{AB}$  and  $M(u,x^A)$  is the mass aspect. Finally, the evolution of the mass and angular momentum aspects in retarded time u is determined by

$$\partial_u M = -\frac{1}{8} N_B^A N_A^B + \frac{1}{8} \bar{\Delta} \bar{R} + \frac{1}{4} \bar{D}_A \bar{D}_C N^{CA}, \tag{2.8}$$

$$\partial_{u}N_{A} = \partial_{A}M + \frac{1}{4}C_{A}^{B}\partial_{B}\bar{R} + \frac{1}{16}\partial_{A}\left[N_{C}^{B}C_{B}^{C}\right] - \frac{1}{4}\bar{D}_{A}C_{B}^{C}N_{C}^{B} - \frac{1}{4}\bar{D}_{B}\left[C_{C}^{B}N_{A}^{C} - N_{C}^{B}C_{A}^{C}\right] - \frac{1}{4}\bar{D}_{B}\left[\bar{D}^{B}\bar{D}_{C}C_{A}^{C} - \bar{D}_{A}\bar{D}_{C}C^{BC}\right]. \tag{2.9}$$

To summarize, coordinates on solution space to the order we need, are given by

$$\mathcal{X}^{\Gamma} \equiv \{C_{AB}, N_{AB}, D_{AB}, M, N_A\}. \tag{2.10}$$

Using the evolution equation equations (2.8),(2.9), the definition of the news and the u independence of  $D_{AB}$ , all these fields can be taken at fixed  $u = u_0$  and thus depend only on  $x^A$ , except for the news which contains an arbitrary u dependence,  $N_{AB} = N_{AB}(u, x^A)$ .

We consider  $\varphi$  to be part of the gauge fixing which we do not vary at this stage. It thus follows that  $h_{\mu\nu}$  is entirely determined to the order we need by  $\delta \mathcal{X}^{\Gamma}$ .

Note in particular that (2.8) controls the mass loss as shown in [1, 2]. By integrating over the sphere, one finds  $\partial_u \int_S d^2\Omega \, M = -\frac{1}{8} \int d^2\Omega \, N_B^A N_A^B$ . By definition, the left hand side is the Bondi mass whereas, in spherical or in stereographic coordinates, the right hand side can easily be seen to be negative and zero if and only if the news tensor vanishes. It follows that the Bondi mass is constant unless the news tensor is non-vanishing in which case the Bondi mass can only decrease in retarded time u.

### 2.3 Asymptotic symmetry algebra and its action on solution space

Let  $s=(T,Y)\in\mathfrak{bms}_4$  denote a generic element of the symmetry algebra, which consists of the semi-direct sum of the Lie algebra  $Y^A\partial_A$  of conformal Killing vectors of the 2 sphere, "infinitesimal superrotations", acting in a suitable way on infinitesimal supertranslations which are parametrized by arbitrary functions  $T=T(x^A)$ ,  $[s_1,s_2]\equiv[(T_1,Y_1),(T_2,Y_2)]=(\widehat{T},\widehat{Y})$ , with

$$\hat{Y} = Y_1^B \partial_B Y_2^A - (1 \leftrightarrow 2), \quad \hat{T} = Y_1^A \partial_A T_2 - \frac{1}{2} \bar{D}_A Y_1^A T_2 - (1 \leftrightarrow 2).$$
 (2.11)

In stereographic coordinates  $\zeta, \bar{\zeta}$ , the algebra may be realized through the vector fields  $y=Y(\zeta)\partial, \bar{y}=\bar{Y}(\bar{\zeta})\bar{\partial}$ , with  $\partial=\frac{\partial}{\partial\zeta}, \bar{\partial}=\frac{\partial}{\partial\bar{\zeta}}$ . Let  $T(\zeta,\bar{\zeta})=\tilde{T}(\zeta,\bar{\zeta})e^{\tilde{\varphi}}$ . In the language used in the study of the Virasoro algebra (see e.g. [15]), the conformal Killing vectors act on tensor densities  $\mathcal{F}_{\frac{1}{2},\frac{1}{2}}$  of degree  $(\frac{1}{2},\frac{1}{2}), t=\tilde{T}(\zeta,\bar{\zeta})e^{\tilde{\varphi}}(d\zeta)^{-\frac{1}{2}}(d\bar{\zeta})^{-\frac{1}{2}}$  through

$$\rho(y)t = (Y\partial \widetilde{T} - \frac{1}{2}\partial Y\widetilde{T})e^{\widetilde{\varphi}}(d\zeta)^{-\frac{1}{2}}(d\overline{\zeta})^{-\frac{1}{2}}, \qquad (2.12)$$

$$\rho(\bar{y})t = (\bar{Y}\bar{\partial}\tilde{T} - \frac{1}{2}\bar{\partial}\bar{Y}\tilde{T})e^{\tilde{\varphi}}(d\zeta)^{-\frac{1}{2}}(d\bar{\zeta})^{-\frac{1}{2}}.$$
(2.13)

The algebra  $\mathfrak{bms}_4$  is then the semi-direct sum of the algebra of vector fields  $y, \bar{y}$  with the abelian ideal  $\mathcal{F}_{\frac{1}{2},\frac{1}{2}}$ , the bracket being induced by the module action,  $[y,t]=\rho(y)t$ ,  $[\bar{y},t]=\rho(\bar{y})t$ . When expanding  $y=a^nl_n$ ,  $\bar{y}=\bar{a}^n\bar{l}_n$ ,  $t=b^{m,n}T_{m,n}$ , where

$$l_n = -\zeta^{n+1}\partial, \quad \bar{l}_n = -\bar{\zeta}^{n+1}\bar{\partial}, \quad T_{m,n} = \zeta^m\bar{\zeta}^n e^{\tilde{\varphi}} (d\zeta)^{-\frac{1}{2}} (d\bar{\zeta})^{-\frac{1}{2}},$$
 (2.14)

with  $m, n \cdots \in \mathbb{Z}$ , the enhanced symmetry algebra reads

$$[l_m, l_n] = (m-n)l_{m+n}, \quad [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n}, \quad [l_m, \bar{l}_n] = 0,$$
$$[l_l, T_{m,n}] = (\frac{l+1}{2} - m)T_{m+l,n}, \quad [\bar{l}_l, T_{m,n}] = (\frac{l+1}{2} - n)T_{m,n+l}, \quad [T_{m,n}, T_{o,p}] = 0. (2.15)$$

The Poincaré algebra is the subalgebra spanned by the generators  $T_{0,0}$ ,  $T_{0,1}$ ,  $T_{1,0}$ ,  $T_{1,1}$  for ordinary translations and  $l_{-1}$ ,  $l_0$ ,  $l_1$ ,  $\bar{l}_{-1}$ ,  $\bar{l}_0$ ,  $\bar{l}_1$  for ordinary (Lorentz) rotations.

The space-time vectors  $\xi=\xi[s;g]$  that realize the asymptotic symmetry algebra  $\mathfrak{bms}_4$  in the modified bracket,

$$[\xi[s_1;g],\xi[s_2;g]]_M \equiv [[\xi[s_1;g],\xi[s_2;g]] - \delta^g_{\xi[s_1;g]}\xi[s_2;g] + \delta^g_{\xi[s_2;g]}\xi[s_1;g] =$$

$$= \xi[[s_1,s_2];g], \quad (2.16)$$

with  $\delta_{\xi}^{g} g_{\mu\nu} = \mathcal{L}_{\xi} g_{\mu\nu}$ , are explicitly given by

$$\begin{cases} \xi^{u} = f, \\ \xi^{A} = Y^{A} + I^{A}, \quad I^{A} = -f_{,B} \int_{r}^{\infty} dr'(e^{2\beta}g^{AB}), \\ \xi^{r} = -\frac{1}{2}r(\bar{D}_{A}\xi^{A} - f_{,B}U^{B}), \end{cases}$$
(2.17)

where  $Y^A=Y^A(x^B)$  are conformal Killing vectors of the 2 sphere,  $f=e^{\varphi}T+\frac{1}{2}u\psi$  with  $\psi=\bar{D}_AY^A$ .

Their action on solution space can be worked out to be

$$-\delta_s C_{AB} = \left[ f \partial_u + \mathcal{L}_Y - \frac{1}{2} \psi \right] C_{AB} - 2\bar{D}_A \bar{D}_B f + \bar{\Delta} f \bar{\gamma}_{AB} \,. \tag{2.18}$$

$$-\delta_s N_{AB} = [f\partial_u + \mathcal{L}_Y]N_{AB} - (\bar{D}_A \bar{D}_B \psi - \frac{1}{2}\bar{\Delta}\psi \bar{\gamma}_{AB}), \qquad (2.19)$$

$$-\delta_s D_{AB} = \mathcal{L}_Y D_{AB} \,, \tag{2.20}$$

$$-\delta_{s}M = [f\partial_{u} + Y^{A}\partial_{A} + \frac{3}{2}\psi]M$$
  
+  $\frac{1}{4}\partial_{u}[\bar{D}_{C}\bar{D}_{B}fC^{CB} + 2\bar{D}_{B}f\bar{D}_{C}C^{CB}] - \frac{1}{4}\bar{D}_{A}\psi\bar{D}_{B}C^{BA} + \frac{1}{4}\partial_{A}f\partial^{A}\bar{R}, \quad (2.21)$ 

$$-\delta_{s}N_{A} = [f\partial_{u} + \mathcal{L}_{Y} + \psi]N_{A} - \frac{1}{2}[\bar{D}_{B}\psi + \psi\bar{D}_{B}]D_{A}^{B}$$

$$+3\bar{D}_{A}fM - \frac{3}{16}\bar{D}_{A}fN_{C}^{B}C_{B}^{C} + \frac{1}{2}\bar{D}_{B}fN_{C}^{B}C_{A}^{C} - \frac{1}{32}\bar{D}_{A}\psi(C_{C}^{B}C_{B}^{C})$$

$$+\frac{1}{4}(\bar{D}_{B}f\bar{R} + \bar{D}_{B}\bar{\Delta}f)C_{A}^{B} - \frac{3}{4}\bar{D}_{B}f(\bar{D}^{B}\bar{D}_{C}C_{A}^{C} - \bar{D}_{A}\bar{D}_{C}C^{BC})$$

$$+\frac{1}{2}(\bar{D}_{A}\bar{D}_{B}f - \frac{1}{2}\bar{\Delta}f\bar{\gamma}_{AB})\bar{D}_{C}C^{CB} + \frac{3}{8}\bar{D}_{A}(\bar{D}_{C}\bar{D}_{B}fC^{CB}). \quad (2.22)$$

### 2.4 Globally well-defined symmetry algebra

In the standard approach to the BMS symmetry algebra in general relativity, one restricts oneself to globally well-defined transformations on the sphere. This amounts to considering only  $l_m$ ,  $\bar{l}_n$ , with m,n taking the values -1,0,1. At the same time, the supertranslations are restricted to those that can be expanded into spherical harmonics  $Y_{lm}$ . The

supertranslation generators are then,  $t = c^{lm} \mathcal{Y}_{lm}$  where  $\mathcal{Y}_{lm} = Y_{lm}(\zeta, \bar{\zeta})(d\zeta)^{-\frac{1}{2}}(d\zeta)^{-\frac{1}{2}}$ . The commutation relations  $[l_n, \mathcal{Y}_{lm}]$  have been worked out already in [16]. More general considerations on the transformation properties of (spin weighted) spherical harmonics under Lorentz transformations can be found in [17, 18, 19]. For later use, let us denote the standard, globally well-defined BMS algebra on the sphere by  $\mathfrak{bms}_4^{\mathrm{glob}}$ .

# 3 Charge algebra

### 3.1 Charges for asymptotically flat spacetimes at null infinity

Using the data summarized in the previous section and inserting into (2.1) gives, after a lengthy computation whose main steps are summarized in the appendix,

$$\delta Q_{\xi}[\delta \mathcal{X}, \mathcal{X}] = \delta \left( Q_s[\mathcal{X}] \right) + \Theta_s[\delta \mathcal{X}, \mathcal{X}], \tag{3.1}$$

where the integrable part of the surface charge one-form is given by

$$Q_s[\mathcal{X}] = \frac{1}{16\pi G} \int d^2 \Omega^{\varphi} \left[ 4fM + Y^A \left( 2N_A + \frac{1}{16} \partial_A (C^{CB} C_{CB}) \right) \right], \tag{3.2}$$

and the non-integrable part is due to the news tensor,

$$\Theta_s[\delta \mathcal{X}, \mathcal{X}] = \frac{1}{16\pi G} \int d^2 \Omega^{\varphi} \left[ \frac{f}{2} N_{AB} \delta C^{AB} \right]. \tag{3.3}$$

The separation into an integrable and non-integrable part in (3.1) is not uniquely defined as this equation also holds in terms  $Q_s' = Q_s - N_s$ ,  $\Theta_s' = \Theta_s + \delta N_s$  for some  $N_s[\mathcal{X}]$ .

These charges are very similar and should be compared to those proposed earlier in [5] in the context of a closely related, but slightly different approach to asymptotically flat spacetimes.

### 3.2 Charges as representations of the symmetry algebra

In the integrable Hamiltonian case [20, 21, 6], it has been shown that the asymptotic symmetry algebra is represented through the Dirac bracket of the surface charges, up to a central extension,

$$\{Q_{s_1}^H, Q_{s_2}^H\}^* = -\delta_{s_2} Q_{s_1}^H = Q_{[s_1, s_2]}^H + K_{s_1, s_2}^H$$
(3.4)

where  $K^H_{s_1,s_2}$  is a Lie algebra 2-cocycle (with values in the real numbers). In the covariant approach, one can show a similar result [10, 12]. More precisely, when the charges are integrable, one can show that  $-\delta_{s_2}Q_{s_1}=Q_{[s_1,s_2]}+K_{s_1,s_2}$  where  $K_{s_1,s_2}$  is again a Lie

algebra 2-cocycle taking values in the real numbers. When using the equivalence of the Hamiltonian and the covariant approaches, one can infer that this coincides with the Dirac bracket  $\{Q_{s_1}, Q_{s_2}\}^*$  of the charges.

In the non integrable case, we propose as a definition

$$\{Q_{s_1}, Q_{s_2}\}^* [\mathcal{X}] = (-\delta_{s_2})Q_{s_1}[\mathcal{X}] + \Theta_{s_2}[-\delta_{s_1}\mathcal{X}, \mathcal{X}]. \tag{3.5}$$

Whether this definition generically makes sense and defines a Dirac bracket will be addressed elsewhere. The point we want to make is that, in the case at hand, the right hand side can be shown to be given by the charges for the commutators of the symmetries, up to a field dependent central extension. Indeed, we will show in the appendix that

$${Q_{s_1}, Q_{s_2}}^* = Q_{[s_1, s_2]} + K_{s_1, s_2},$$
 (3.6)

where the field dependent central extension is

$$K_{s_1,s_2}[\mathcal{X}] = \frac{1}{32\pi G} \int d^2\Omega^{\varphi} \left[ (f_1 \partial_A f_2 - f_2 \partial_A f_1) \partial^A \bar{R} + C^{BC} (f_1 \bar{D}_B \bar{D}_C \psi_2 - f_2 \bar{D}_B \bar{D}_C \psi_1) \right]. \quad (3.7)$$

This central extension satisfies the suitably generalized cocycle condition

$$K_{[s_1,s_2],s_3} - \delta_{s_3} K_{s_1,s_2} + \text{cyclic}(1,2,3) = 0.$$
 (3.8)

In fact, (3.6) and (3.8) imply the Jacobi identity for the proposed bracket when the algebra element associated to  $K_{s_1,s_2}$  is central and thus generates no transformation. More precisely,  $\{\cdot,\cdot\}^*$  defines a Lie bracket for the elements  $Q_{s_1},K_{s_2,s_3}$  if one defines in addition that  $\{K_{s_1,s_2},Q_{s_3}\}^*=-\delta_{s_3}K_{s_1,s_2}=-\{Q_{s_3},K_{s_1,s_2}\}^*$  and  $\{K_{s_1,s_2},K_{s_3,s_4}\}^*=0$ .

When defining as before,  $\{Q'_{s_1}, Q'_{s_2}\}^*[\mathcal{X}] = (-\delta_{s_2})Q'_{s_1}[\mathcal{X}] + \Theta'_{s_2}[-\delta_{s_1}\mathcal{X}, \mathcal{X}]$ , one gets  $\{Q'_{s_1}, Q'_{s_2}\}^* = Q'_{[s_1, s_2]} + K'_{s_1, s_2}$ , where

$$K'_{s_1,s_2} = K_{s_1,s_2} + \delta_{s_2} N_{s_1} - \delta_{s_1} N_{s_2} + N_{[s_1,s_2]}.$$
(3.9)

Note that  $\delta_{s_2}N_{s_1} - \delta_{s_1}N_{s_2} + N_{[s_1,s_2]}$  is a trivial field dependent 2-cocycle in the sense that it automatically satisfies the cocycle condition (3.8).

#### **Discussion:**

• The proved equality between the right hand sides of (3.5) and (3.6) controls the non-conservation of the charges. Indeed, by taking  $s_2=(T=1,Y^A=0)$  and  $s_1=s$  we find from  $\frac{d}{du}Q_s=\frac{\partial}{\partial u}Q_s-\delta_{1,0}Q_s$  that

$$\frac{d}{du}Q_s = -\frac{1}{32\pi G} \int d^2\Omega^{\varphi} \left[ N^{AB} \left( [f\partial_u + \mathcal{L}_Y - \frac{1}{2}\psi] C_{AB} - 2\bar{D}_A \bar{D}_B f \right) + \partial_A f \partial^A \bar{R} + C^{BC} \bar{D}_B \bar{D}_C \psi \right].$$
(3.10)

The standard result that the mass loss is positive and vanishes only in the absence of news then follows by taking  $s = (T = 1, Y^A = 0)$ .

- It also follows that on the sphere, the standard  $\mathfrak{bms}_4^{\mathrm{glob}}$  charges are all conserved in the absence of news.
- In the case of the standard  $\mathfrak{bms}_4^{\mathrm{glob}}$  algebra on the sphere, there are no divergences provided the asymptotic solutions  $\mathcal{X}$  are well-defined. The central charge  $K_{s_1,s_2}$  vanishes and the representation of the asymptotic symmetry algebra through the charges simplifies to

$${Q_{s_1}, Q_{s_2}}^* = Q_{[s_1, s_2]}$$
 (3.11)

To the best of our knowledge, even in this well-studied case this representation theorem is a new result that does so far not exist in any other formulation of the problem.

## 4 Charges and central extension for the Kerr black hole

We will now take as a background metric the standard metric on the sphere, i.e.,  $\varphi=0$ . By following [22] and choosing the radial coordinate appropriately, one can put the Kerr black hole in BMS coordinates. As shown in the appendix, the Kerr solution  $\mathcal{X}^{Kerr}$  corresponds to  $M(u,\theta,\phi)=M$ , with M the constant mass parameter of the Kerr black hole,  $D_{AB}=0=N_{AB}$  while

$$C_{\theta\theta} = \frac{a}{\sin \theta}, \quad C_{\phi\phi} = -a \sin \theta, \quad C_{\theta\phi} = 0,$$
 (4.1)

$$N_{\theta} = 3Ma\cos\theta + \frac{a^2}{8}\frac{\cos\theta}{\sin^3\theta}, \quad N_{\phi} = -3aM\sin^2\theta. \tag{4.2}$$

Note that in the BMS gauge,  $C_{\theta\theta}$  and  $N_{\theta}$  are singular both on the north and the south pole.

For the supertranslation charges, we find

$$Q_{T_{m,n},0}[\mathcal{X}^{Kerr}] = \frac{2M}{G} I_{m,n}, \quad I_{m,n} = \frac{1}{4\pi} \int d^2\Omega \frac{1}{1+\zeta\bar{\zeta}} \zeta^m \bar{\zeta}^n. \tag{4.3}$$

A direct integration on the sphere gives  $I_{m,n} = \delta_n^m I(m)$ , with

$$I(m) = \frac{1}{4} \int_{-1}^{1} d\mu \frac{(1+\mu)^m}{(1-\mu)^{m-1}}.$$
 (4.4)

We have I(m)=I(1-m). In particular  $I(0)=\frac{1}{2}=I(1)$ , so that the mass, which is associated to the exact Killing vector  $\partial_u$  of the Kerr solution and corresponds to T=1, Y=0 and thus to  $\frac{1}{2}(T_{0,0}+T_{1,1})$ , is given by

$$Q_{T=1,Y=0}[\mathcal{X}^{Kerr}] = \frac{M}{G}, \qquad (4.5)$$

as it should. For m > 1 and m < 0, the charges are not directly well-defined as the integrals diverge. Note that in the case of the globally well-defined BMS algebra, the

supertranslations are expanded in spherical harmonics,  $T = c^{lm}Y_{lm}$ . It follows that the only non vanishing charge is (4.5) while all other supertranslation charges with l > 0 vanish in this case.

The superrotations charges are given by

$$Q_{0,l_m}[\mathcal{X}^{Kerr}] = -\delta_0^m \frac{iaM}{2G}.$$
(4.6)

In particular, the standard angular momentum is associated to the exact Killing vector  $\partial_{\phi}=-i(l_0-\bar{l}_0)$  of the Kerr solution and is thus given by

$$Q_{T=0,Y^{\phi}=1,Y^{\theta}=0}[\mathcal{X}^{Kerr}] = -\frac{Ma}{G}, \qquad (4.7)$$

as it should1

For the central extension, we find

$$K_{(0,l_m),(0,l_n)}[\mathcal{X}^{Kerr}] = 0 = K_{(0,\bar{l}_m),(0,\bar{l}_n)}[\mathcal{X}^{Kerr}] = K_{(0,l_m),(0,\bar{l}_n)}[\mathcal{X}^{Kerr}], \tag{4.8}$$

and

$$K_{(0,l_l),(T_{m,n},0)}[\mathcal{X}^{Kerr}] = \frac{a \, l(l-1)(l+1)}{16G} J_{m+l,n} \,, \tag{4.9}$$

$$K_{(0,\bar{l}_l),(T_{m,n},0)}[\mathcal{X}^{Kerr}] = \frac{a \, l(l-1)(l+1)}{16G} J_{m,n+l}, \qquad (4.10)$$

with

$$J_{m,n} = \frac{1}{4\pi} \int d^2\Omega \frac{(1+\zeta\zeta)^2}{\sqrt{\zeta^3\bar{\zeta}^3}} \zeta^m \bar{\zeta}^n.$$
 (4.11)

The integration gives  $J_{m,n} = \delta_n^m J(m)$  with

$$J(m) = 2 \int_{-1}^{1} d\mu \, \frac{(1+\mu)^{m-\frac{3}{2}}}{(1-\mu)^{m+\frac{1}{2}}},\tag{4.12}$$

and J(m) = J(1-m). These integrals diverge for all integer values of m.

### 5 Discussion

The extended conformal field dual for four dimensional asymptotically flat gravity is non-standard because the generator of time translations is not related to the Virasoro generators  $l_0$  and  $\bar{l}_0$  but to  $\frac{1}{2}(T_{0,0}+T_{1,1})$  instead. At the same time, the non trivial central extension appears between the supertranslation and superrotation generators, and not among the Virasoro generators alone.

<sup>&</sup>lt;sup>1</sup>A discussion of the minus sign can for instance be found in [23], after equation (89).

To get to grips with these unusual features it is useful to review the corresponding results for bms<sub>3</sub>: a direct analysis of the Dirac bracket algebra of the charges of asymptotically flat space-times at null infinity in three dimensions [24] gives one noncentrally extended copy of the Virasoro algebra with superrotation charges  $L_m$  that act on the commuting supertranslation charges  $T_m$  with a (field independent) central extension,  $i[L_m,T_n]=(m-n)T_{m+n}+\frac{c'}{12}m(m^2-1)\delta_{m+n}^0$ , where  $c'=\frac{3}{G}$  for the Einstein-Hilbert action. In this case, there is no problem with singularities since the boundary is a cylinder. Furthermore, the relation to the asymptotically  $AdS_3$  case sheds some light: starting from two commuting copies  $L_m^{\pm}$  of the Virasoro algebra with central extensions  $c^{\pm}$ , the redefinition  $L_m = L_m^+ - L_{-m}^-$ ,  $T_m = \frac{1}{l}(L_m^+ + L_{-m}^-)$  implies that the  $L_m$ 's form a copy of the Virasoro algebra with central charge  $c^+ - c^-$ , the same commutation relations between  $L_m$  and  $T_n$  as above with  $c' = (1/l)(c^+ + c^-)$ , while  $i[T_m,T_n]=\frac{1}{l^2}((m-n)L_{m+n}+\frac{c^+-c^-}{12}m(m^2-1)\delta_{m+n}^0)$ . In the case of the Einstein-Hilbert action where  $c^{\pm} = \frac{3l}{2G}$ , one then recovers the  $\mathfrak{bms}_3$  algebra in the limit  $l \to \infty$  with zero central extension for the Virasoro algebra of the  $L_m$ 's and the above value  $c' = \frac{3}{C}$  between the superrotation and supertranslation charges. From this point of view, the reason why the central extensions for  $\mathfrak{bms}_3$  in the pure gravity case have this unusual structure is thus related to the fact that the theory is a contraction of the standard conformal field theory of the anti-Sitter case where left and right movers have the same central charge.

A strategy to get a better understanding of the extended conformal gravity dual in four dimensions is thus to first study the three dimensional case in more detail. In particular, we will discuss elsewhere the relation between the general asymptotically flat and asymptotically anti-de Sitter solutions of three dimensional gravity. The absence of black hole solutions in the purely gravitational case with vanishing cosmological constant then forces one to consider more exotic actions, such as the one for new massive gravity [25] which admit asymptotically flat black holes, to try to see what the analog of a Cardy formula has to look like in order to reproduce the Bekenstein-Hawking entropy. One should also directly study extended conformal field theories with  $\mathfrak{bms}_3$  symmetry by analysing its physically relevant unitary irreducible representations. This has been partly done for the  $\mathfrak{gca}_2$  algebra [26], which is isomorphic to the  $\mathfrak{bms}_3$  algebra. Note however that the main assumption that the energy should be bounded from below implies that such a representation should have a lowest eigenvalue for  $T_0$ . We plan to address some of these questions elsewhere.

In the same way than the modified Lie bracket needed to represent the asymptotic symmetry algebra in the bulk space-time is the bracket of the Lie algebroid naturally associated to gauge systems [27], field dependent central extensions correspond to Lie-algebroid 2-cocycles rather than to Lie algebra 2-cocycles. Note that, besides the standard central extensions in the two Witt subalgebras, the bms<sub>4</sub> algebra does not admit additional non trivial central extensions involving the supertranslation generators, i.e., there

are no additional non trivial Lie algebra 2-cocycles with values in the real numbers (see e.g. [28]). This no-go result is circumvented here because of the presence of the field  $C_{AB}$ .

The charges have been computed with respect to Minkowski space as a background. In the context of the Kerr-CFT correspondence, it might be more appropriate to choose another asymptotically flat solution as a background, such as the extreme Kerr black hole for instance, or to consistently restrict oneself to subclasses of solutions.

The proof that the charges represent the symmetry algebra up to a field dependent central extension relies on the possibility to do integrations by parts on the sphere. This is of course problematic in the case of divergent integrals. Then again, the central extension seems interesting mainly in the case of a symmetry algebra consisting of both supertranslations and superrotations where divergences are unavoidable.

A way to make sense of the divergent integrals could be to use the theory of harmonic variables and distributions on the sphere introduced in the context of harmonic superspace [29] (see also [30] for a review) and applied to local conformal properties of the sphere in [31, 32]. It would mean to probe solution space through objects such as

$$Q_{(T_{m,n},0)}[\mathcal{X}](w^+, w^-) = \frac{1}{G} \int dv \, M P^{-1} (\frac{w^- v^-}{w^- v^+})^m (\frac{w^+ v^+}{w^+ v^-})^n \,,$$

$$Q_{(0,l_m)}[\mathcal{X}](w^-) = \frac{1}{G} \int dv \, (\frac{w^- v^-}{w^- v^+})^{m+1} \left[ \frac{u}{2} \partial (P^{-1} M) - \frac{1}{2} N_\zeta - \frac{1}{64} \partial (C^{BC} C_{BC}) \right] \,.$$

The previous charges are then recovered for  $w_1^-=0, w_2^-=1$  and  $w_1^+=1, w_2^+=0$ .

An alternative to the approach sketched in the previous paragraph consists in mapping the problem from the very beginning from the standard to the Riemann sphere and use more standard conformal field theory techniques. The formulas to do so are well known in the general relativity literature (see e.g. [33, 34]) since finite local conformal transformations of the two dimensional part of metric remain as an ambiguity in Penrose's definition of asymptotically flat spacetimes [35]. In the current set-up, the relevant formulas can be obtained by integrating the infinitesimal transformation properties of the coordinates on solution space under a local shift of the conformal factor  $-\delta\varphi = \omega$  worked out in [4].

### Acknowledgements

This work is supported in part by the Fund for Scientific Research-FNRS (Belgium), by the Belgian Federal Science Policy Office through the Interuniversity Attraction Pole P6/11, by IISN-Belgium, by "Communauté française de Belgique - Actions de Recherche Concertées" and by Fondecyt Projects No. 1085322 and No. 1090753.

## A Evaluation of the surface charge one-forms

When evaluated at a spherical cross-section of  $\mathcal{I}$ , the surface charge one-forms (2.1) become

$$\delta Q_{\xi}[h,g] = \frac{1}{16\pi G} \lim_{r \to \infty} \int d^{2}\Omega^{\varphi} r^{2} e^{2\beta} \Big[ \xi^{r} (D^{u}h - D_{\sigma}h^{u\sigma} + D^{r}h_{r}^{u} - D^{u}h_{r}^{r}) \\
- \xi^{u} (D^{r}h - D_{\sigma}h^{r\sigma} - D^{r}h_{u}^{u} + D^{u}h_{u}^{r}) + \xi^{A} (D^{r}h_{A}^{u} - D^{u}h_{A}^{r}) + \frac{1}{2}h(D^{r}\xi^{u} - D^{u}\xi^{r}) \\
+ \frac{1}{2}h^{r\sigma} (D^{u}\xi_{\sigma} - D_{\sigma}\xi^{u}) - \frac{1}{2}h^{u\sigma} (D^{r}\xi_{\sigma} - D_{\sigma}\xi^{r}) \Big]. \quad (A.1)$$

Using the Christoffel symbols for a metric of the form (2.2), explicitly given in section **4.3** of [4] and the solution to the equations of motion up to the appropriate order as summarized in section **2.2**, we have

$$D^{u}h - D_{\sigma}h^{u\sigma} + D^{r}h_{r}^{u} - D^{u}h_{r}^{r} = g^{ur}g^{AB}(D_{r}h_{AB} - D_{A}h_{rB})$$

$$= -e^{-2\beta} \left(g^{AB}\partial_{r}h_{AB} - k^{AB}h_{AB} + e^{-2\beta}g^{AB}k_{AB}h_{ru}\right)$$

$$= \frac{1}{4r^{3}}C^{AB}\delta C_{AB} + o(r^{-3-\epsilon}),$$

$$-\left(D^{r}h - D_{\sigma}h^{r\sigma} - D^{r}h_{u}^{u} + D^{u}h_{u}^{r}\right) = D^{A}h_{A}^{r} - D^{r}h_{A}^{A}$$

$$= g^{ur}g^{AB}(D_{A}h_{uB} - D_{u}h_{AB}) + O(r^{-3})$$

$$= g^{ur} \left(g^{AB}{}^{(2)}D_{B}h_{uA} - h_{ur}g^{AB}(l_{AB} + k_{AB}\frac{V}{r})\right)$$

$$- kh_{uu} - g^{AB}\partial_{u}h_{AB} + g^{AB}h_{CA}l_{B}^{C}\right) + O(r^{-3})$$

$$= \frac{1}{r^{2}}\left(4\delta M - \frac{1}{2}\bar{D}_{A}\bar{D}_{B}\delta C^{AB} + \frac{1}{2}\delta\partial_{u}(C^{AB}C_{AB})\right)$$

$$- \frac{1}{2}\partial_{u}C_{AB}\delta C^{AB} - C^{AB}\partial_{u}\delta C_{AB}\right) + o(r^{-2-\epsilon}),$$

$$D^{r}h_{A}^{u} - D^{u}h_{A}^{r} = (g^{ur})^{2} \left(\Gamma_{rA}^{C}h_{uC} - \partial_{r}h_{Au}\right) + g^{ur}g^{rB} \left(\Gamma_{rB}^{C}h_{AC} - \partial_{r}h_{AB}\right) + O(r^{-3})$$

$$= \frac{1}{2r}\bar{D}_{B}\delta C_{A}^{B} + \frac{2}{3r^{2}}(2\ln r - \frac{1}{3})\bar{D}_{B}\delta D_{A}^{B} + o(r^{-2-\epsilon})$$

$$+ \frac{1}{r^{2}}\left(\frac{4}{3}\delta N_{A} + \frac{1}{3}\delta(C_{AB}\bar{D}_{C}C^{BC}) - \frac{1}{4}C_{AB}\bar{D}_{C}\delta C^{BC}\right),$$
(A.4)

$$\frac{1}{2}h^{r\sigma}(D^{u}\xi_{\sigma} - D_{\sigma}\xi^{u}) - \frac{1}{2}h^{u\sigma}(D^{r}\xi_{\sigma} - D_{\sigma}\xi^{r}) = 
\frac{1}{2}(h_{u}^{u} + h_{r}^{r})(D^{u}\xi^{r} - D^{r}\xi^{u}) + \frac{1}{2}h_{A}^{r}(D^{u}\xi^{A} - D^{A}\xi^{u}), \quad (A.5)$$

$$\frac{1}{2}(h - h_{u}^{u} - h_{r}^{r})(D^{r}\xi^{u} - D^{u}\xi^{r}) = \frac{1}{2}g^{AB}h_{AB}(D^{r}\xi^{u} - D^{u}\xi^{r}) = 0, \quad (A.6)$$

$$D^{u}\xi^{A} - D^{A}\xi^{u} = g^{ur}\partial_{r}\xi^{A} - g^{AB}\partial_{B}\xi^{u} + (g^{ur}\Gamma_{rC}^{A} - g^{AB}\Gamma_{BC}^{u})\xi^{C} + O(r^{-3})$$

$$= \frac{-2}{r}Y^{A} + \frac{1}{r^{2}}C_{C}^{A}Y^{C} + O(r^{-3}),$$
(A.7)

$$\begin{split} \frac{1}{2}h_A^r &= -\frac{1}{4}\bar{D}_B\delta C_A^B - \frac{1}{3r}(\ln r + \frac{1}{3})\bar{D}_B\delta D_A^B \\ &+ \frac{1}{r}\left(-\frac{1}{3}\delta N_A - \frac{1}{12}\delta(C_{AB}\bar{D}_CC^{BC}) + \frac{1}{4}\delta C_{AB}\bar{D}_CC^{BC}\right) + o(r^{-1-\epsilon}) \,. \quad \text{(A.8)} \end{split}$$

Putting everything together, we get

$$\delta Q_{\xi}[\delta \mathcal{X}, \mathcal{X}] = \frac{1}{16\pi G} \lim_{r \to \infty} \int d^{2}\Omega^{\varphi} \left[ r \left( Y^{A} \frac{1}{2} \bar{D}_{B} \delta C_{A}^{B} + Y^{A} \frac{1}{2} \bar{D}_{B} \delta C_{A}^{B} \right) \right. \\
+ Y^{A} \bar{D}_{B} \delta D_{A}^{B} \left( \frac{4}{3} \ln r - \frac{2}{9} + \frac{2}{3} \ln r + \frac{2}{9} \right) - \frac{\psi}{8} C^{AB} \delta C_{AB} \\
+ f \left( 4\delta M - \frac{1}{2} \bar{D}_{A} \bar{D}_{B} \delta C^{AB} + \frac{1}{2} \delta \partial_{u} (C^{AB} C_{AB}) - \frac{1}{2} \partial_{u} C_{AB} \delta C^{AB} - C^{AB} \partial_{u} \delta C_{AB} \right) \\
+ Y^{A} \left( \frac{4}{3} \delta N_{A} + \frac{1}{3} \delta (C_{AB} \bar{D}_{C} C^{BC}) - \frac{1}{4} C_{AB} \bar{D}_{C} \delta C^{BC} \right) \\
- 2Y^{A} \left( -\frac{1}{3} \delta N_{A} - \frac{1}{12} \delta (C_{AB} \bar{D}_{C} C^{BC}) + \frac{1}{4} \delta C_{AB} \bar{D}_{C} C^{BC} \right) \\
- \frac{1}{2} \bar{D}_{A} f \bar{D}_{B} \delta C^{AB} - \frac{1}{4} C_{AB} Y^{A} \bar{D}_{C} \delta C^{BC} \right]. \tag{A.9}$$

Using integrations by parts and the conformal Killing equation for the  $Y^A$ , this can be simplified to

$$\delta Q_{\xi}[\delta \mathcal{X}, \mathcal{X}] = \frac{1}{16\pi G} \int d^{2}\Omega^{\varphi} \left[ -\frac{\psi}{8} C^{AB} \delta C_{AB} + Y^{A} 2\delta N_{A} - \frac{1}{2} \bar{D}_{A} f \bar{D}_{B} \delta C^{AB} \right. \\
+ f \left( 4\delta M - \frac{1}{2} \bar{D}_{A} \bar{D}_{B} \delta C^{AB} + \frac{1}{2} \delta \partial_{u} (C^{AB} C_{AB}) \right. \\
\left. -\frac{1}{2} \partial_{u} C_{AB} \delta C^{AB} - C^{AB} \partial_{u} \delta C_{AB} \right) \right]$$

$$= \frac{1}{16\pi G} \delta \int d^{2}\Omega^{\varphi} \left[ -\frac{\psi}{16} C^{AB} C_{AB} + 2Y^{A} N_{A} + 4fM \right] \\
+ \frac{1}{16\pi G} \int d^{2}\Omega \left[ \frac{f}{2} \partial_{u} C_{AB} \delta C^{AB} \right].$$
(A.10)

### **B** Computation of the charge algebra

We will start by computing the usual factor,

$$-\delta_{s_2}Q_{s_1}[\mathcal{X}] = \frac{1}{16\pi G} \int d^2\Omega^{\varphi} \left[ Y_1^A \left( 2(-\delta_{s_2})N_A + \frac{1}{16}\partial_A(-\delta_{s_2})(C^{CB}C_{CB}) \right) + 4f_1(-\delta_{s_2})M \right], \quad (\mathbf{B}.1)$$

and organize according to the different types of terms that appear:

• terms containing M

$$-\delta_{s_{2}}Q_{s_{1}}[\mathcal{X}]_{M} = \frac{1}{16\pi G} \int d^{2}\Omega^{\varphi} \left[ Y_{1}^{A} 2(f_{2}\partial_{A}M + 3\partial_{A}f_{2}M) + 4f_{1}(Y_{2}^{A}\partial_{A}M + \frac{3}{2}\psi_{2}M) \right]$$

$$= \frac{1}{16\pi G} \int d^{2}\Omega^{\varphi} 4M \left[ -\frac{1}{2}\bar{D}_{A}(Y_{1}^{A}f_{2}) + \frac{3}{2}Y_{1}^{A}\partial_{A}f_{2} - \bar{D}_{A}(f_{1}Y_{2}^{A}) + \frac{3}{2}f_{1}\psi_{2}) \right]$$

$$= \frac{1}{16\pi G} \int d^{2}\Omega^{\varphi} 4M \left[ Y_{1}^{A}\partial_{A}f_{2} - \frac{1}{2}\psi_{1}f_{2} - Y_{2}^{A}\partial_{A}f_{1} + \frac{1}{2}f_{1}\psi_{2}) \right]$$

$$= \frac{1}{16\pi G} \int d^{2}\Omega^{\varphi} 4M f_{[s_{1},s_{2}]}, \qquad (B.2)$$

• terms containing  $N_A$ 

$$-\delta_{s_2}Q_{s_1}[\mathcal{X}]_N = \frac{1}{16\pi G} \int d^2\Omega^{\varphi} \left[ 2Y_1^A (\mathcal{L}_{Y_2} + \psi_2) N_A \right]$$

$$= \frac{1}{16\pi G} \int d^2\Omega^{\varphi} \left[ 2Y_1^A (Y_2^B \bar{D}_B + \psi_2) N_A + 2Y_1^A \bar{D}_A Y_2^B N_B \right]$$

$$= \frac{1}{16\pi G} \int d^2\Omega^{\varphi} 2N_A \left[ -Y_2^B \bar{D}_B Y_1^A + Y_1^B \bar{D}_B Y_2^A \right]$$

$$= \frac{1}{16\pi G} \int d^2\Omega^{\varphi} 2N_A Y_{[s_1, s_2]}^A, \tag{B.3}$$

• terms containing  $D_{AB}$ 

$$-\delta_{s_2}Q_{s_1}[\mathcal{X}]_D = \frac{1}{16\pi G} \int d^2\Omega^{\varphi} \, 2Y_1^A \Big[ -\frac{1}{2} [\bar{D}_B \psi_2 + \psi_2 \bar{D}_B] D_A^B \Big]$$

$$= \frac{1}{16\pi G} \int d^2\Omega^{\varphi} \, 2Y_1^A \Big[ -\frac{1}{2} \bar{D}_B(\psi_2 D_A^B) \Big]$$

$$= \frac{1}{16\pi G} \int d^2\Omega^{\varphi} \, \bar{D}^B Y_1^A \psi_2 D_{AB} = 0 \,, \tag{B.4}$$

• terms containing the news

$$-\delta_{s_{2}}Q_{s_{1}}[\mathcal{X}]_{news} = \frac{1}{16\pi G} \int d^{2}\Omega^{\varphi} \left[ 2Y_{1}^{A} \left( -\frac{3}{16}\bar{D}_{A}f_{2}N_{C}^{B}C_{B}^{C} + \frac{1}{2}\bar{D}_{B}f_{2}N_{C}^{B}C_{A}^{C} \right) \right.$$

$$+2Y_{1}^{A}f_{2} \left( \frac{1}{16}\partial_{A} \left[ N_{C}^{B}C_{B}^{C} \right] - \frac{1}{4}\bar{D}_{A}C_{B}^{C}N_{C}^{B} - \frac{1}{4}\bar{D}_{B} \left[ C_{C}^{B}N_{A}^{C} - N_{C}^{B}C_{A}^{C} \right] \right)$$

$$-\psi_{1}\frac{1}{8}C^{AB}f_{2}N_{AB} + 4f_{1} \left( \frac{1}{4}\bar{D}_{B}\bar{D}_{C}f_{2}N^{BC} + \frac{1}{2}\bar{D}_{B}f_{2}\bar{D}_{C}N^{BC} \right)$$

$$+4f_{1}f_{2} \left( -\frac{1}{8}N_{B}^{A}N_{A}^{B} + \frac{1}{4}\bar{D}_{A}\bar{D}_{C}N^{CA} \right) \right]$$

$$= \frac{1}{16\pi G} \int d^{2}\Omega^{\varphi} \frac{-1}{2}N^{BC}f_{2} \left[ f_{1}N_{BC} + \mathcal{L}_{Y_{1}}C_{BC} - \frac{1}{2}\psi_{1}C_{BC} - 2\bar{D}_{B}\bar{D}_{C}f_{1} \right]$$

$$+ \frac{1}{16\pi G} \int d^{2}\Omega^{\varphi} \frac{1}{2}N_{B}^{C}C_{CA} \left[ Y_{1}^{A}\bar{D}^{B}f_{2} + Y_{1}^{B}\bar{D}^{A}f_{2} - \bar{\gamma}^{AB}Y_{1}^{D}\bar{D}_{D}f_{2} \right]. \quad (B.5)$$

The second line is zero. This is coming from the following identity for the symmetrized product of two traceless matrices in 2 dimensions,

$$\frac{1}{2}(C_B^A K_C^B + K_B^A C_C^B) = \frac{1}{2} \delta_C^A C_D^B K_B^D,$$
 (B.6)

and the conformal Killing equation for the  $Y^A$ . The first line can be recognized as,

$$-\delta_{s_2}Q_{s_1}[\mathcal{X}]_{news} = \frac{1}{16\pi G} \int d^2\Omega^{\varphi} \frac{-1}{2} N^{BC} f_2 \Big[ -\delta_{s_1}C_{BC} \Big]$$
$$= -\Theta_{s_2}[-\delta_{s_1}\mathcal{X}, \mathcal{X}], \qquad (B.7)$$

• the rest

$$-\delta_{s_2}Q_{s_1}[\mathcal{X}]_R = \frac{1}{16\pi G} \int d^2\Omega^{\varphi} \left[ 2Y_1^A \left( -\frac{1}{32}\bar{D}_A\psi_2C_C^BC_C^C + f_2\frac{1}{4}C_A^B\partial_B\bar{R} \right) \right. \\ \left. -\frac{1}{4}f_2\bar{D}_B \left( \bar{D}^B\bar{D}_CC_A^C - \bar{D}_A\bar{D}_CC^{BC} \right) \right. \\ \left. +\frac{1}{4}(\bar{D}_Bf_2\bar{R} + \bar{D}_B\bar{\Delta}f_2)C_A^B - \frac{3}{4}\bar{D}_Bf_2(\bar{D}^B\bar{D}_CC_A^C - \bar{D}_A\bar{D}_CC^{BC}) \right. \\ \left. +\frac{1}{2}(\bar{D}_A\bar{D}_Bf_2 - \frac{1}{2}\bar{\Delta}f_2\bar{\gamma}_{AB})\bar{D}_CC^{CB} + \frac{3}{8}\bar{D}_A(\bar{D}_C\bar{D}_Bf_2C^{CB}) \right) \\ \left. -\psi_1\frac{1}{8}C^{CB} \left( [\mathcal{L}_{Y_2} - \frac{1}{2}\psi_2]C_{CB} - 2\bar{D}_C\bar{D}_Bf_2 + \bar{\Delta}f_2\bar{\gamma}_{CB} \right) \right. \\ \left. +4f_1 \left( f_2\frac{1}{8}\bar{\Delta}\bar{R} + \frac{1}{4}\partial_Af_2\partial^A\bar{R} + \frac{1}{8}\bar{D}_C\bar{D}_B\psi_2C^{CB} \right) \right] \\ = \frac{1}{16\pi G} \int d^2\Omega^{\varphi} \left[ -Y_1^A\frac{1}{16}\bar{D}_A\psi_2C_C^BC_B^C - \psi_1\frac{1}{8}C^{CB} \left( [\mathcal{L}_{Y_2} - \frac{1}{2}\psi_2]C_{CB} \right) \right. \\ \left. +C^{BC} \left( \frac{1}{2}f_1\bar{D}_B\bar{D}_C\psi_2 + \psi_1\frac{1}{4}\bar{D}_C\bar{D}_Bf_2 + \frac{1}{2}f_2Y_{1B}\partial_C\bar{R} \right. \\ \left. -\frac{3}{4}\psi_1\bar{D}_B\bar{D}_Cf_2 - \bar{D}_C(Y_1^A\bar{D}_A\bar{D}_Bf_2) + \frac{1}{2}\bar{D}_C(Y_{1B}\bar{\Delta}f_2) \right. \\ \left. +\frac{1}{2}Y_{1C}(\bar{D}_Bf_2\bar{R} + \bar{D}_B\bar{\Delta}f_2) + \frac{1}{2}\bar{D}_C\bar{\Delta}(Y_{1B}f_2) - \frac{1}{2}\bar{D}_C\bar{D}_A\bar{D}_B(Y_1^Af_2) \right. \\ \left. -\frac{3}{2}\bar{D}_C\bar{D}_A(Y_{1B}\bar{D}^Af_2) + \frac{3}{2}\bar{D}_C\bar{D}_A(Y_1^AD_Bf_2) \right) \\ \left. +\frac{1}{2}(f_1\partial_Af_2 - f_2\partial_Af_1)\partial^A\bar{R} \right]. \tag{B.8}$$

Using the commutation rule for covariant derivatives, this gives

$$-\delta_{s_{2}}Q_{s_{1}}[\mathcal{X}]_{C} = \frac{1}{16\pi G} \int d^{2}\Omega^{\varphi} \left[ -\frac{1}{16} (Y_{1}^{A}\bar{D}_{A}\psi_{2} - Y_{2}^{A}\bar{D}_{A}\psi_{1})C_{C}^{B}C_{B}^{C} + \frac{1}{2} (f_{1}\partial_{A}f_{2} - f_{2}\partial_{A}f_{1})\partial^{A}\bar{R} + C^{BC} \left( \frac{1}{2} (f_{1}\bar{D}_{B}\bar{D}_{C}\psi_{2} - f_{2}\bar{D}_{B}\bar{D}_{C}\psi_{1}) + \frac{1}{4} f_{2}Y_{1B}\partial_{C}\bar{R} + \frac{1}{2} f_{2}\bar{D}_{C}\bar{\Delta}Y_{1B} + \frac{1}{4}\bar{D}_{C}f_{2}Y_{1B}\bar{R} + \frac{1}{2}\bar{D}_{C}f_{2}\bar{\Delta}Y_{1B} \right) \right]$$

$$= \frac{1}{16\pi G} \int d^{2}\Omega^{\varphi} \left[ -\frac{1}{16}\psi_{[s_{1},s_{2}]}C_{C}^{B}C_{B}^{C} + \frac{1}{2} (f_{1}\partial_{A}f_{2} - f_{2}\partial_{A}f_{1})\partial^{A}\bar{R} + C^{BC} \frac{1}{2} (f_{1}\bar{D}_{B}\bar{D}_{C}\psi_{2} - f_{2}\bar{D}_{B}\bar{D}_{C}\psi_{1}) \right], \qquad (B.9)$$

where in the last line we have used the identity  $\bar{\Delta}Y^A = -\frac{1}{2}\bar{R}Y^A$  satisfied by conformal Killing vectors.

Summing everything, we obtain

$$-\delta_{s_{2}}Q_{s_{1}}[\mathcal{X}] = \frac{1}{16\pi G} \int d^{2}\Omega^{\varphi} \left[ -\frac{1}{16}\psi_{[s_{1},s_{2}]}C_{C}^{B}C_{B}^{C} + \frac{1}{2}(f_{1}\partial_{A}f_{2} - f_{2}\partial_{A}f_{1})\partial^{A}\bar{R} \right.$$

$$+C^{BC}\frac{1}{2}(f_{1}\bar{D}_{B}\bar{D}_{C}\psi_{2} - f_{2}\bar{D}_{B}\bar{D}_{C}\psi_{1}) + 4Mf_{[s_{1},s_{2}]} + 2N_{A}Y_{[s_{1},s_{2}]}^{A} \right]$$

$$-\Theta_{2}[-\delta_{1}\mathcal{X},\mathcal{X}]$$

$$= Q_{[s_{1},s_{2}]} - \Theta_{2}[-\delta_{1}\mathcal{X},\mathcal{X}] + K_{s_{1},s_{2}}[\mathcal{X}], \qquad (B.10)$$

with  $K_{s_1,s_2}[\mathcal{X}]$  defined in (3.7).

# C Checking the cocyle condition

Let us treat the two parts of  $K_{s_1,s_2}[\mathcal{X}]$  separately:

• for the second part  $\hat{K}_{s_1,s_2}=\frac{1}{16\pi G}\int d^2\Omega^{\varphi}\,C^{BC}\frac{1}{2}(f_1\bar{D}_B\bar{D}_C\psi_2-f_2\bar{D}_B\bar{D}_C\psi_1)$ , we have

$$A = \int d^{2}\Omega^{\varphi} \left[ (-\delta_{s_{3}}C^{BC})(f_{1}\bar{D}_{B}\bar{D}_{C}\psi_{2} - f_{2}\bar{D}_{B}\bar{D}_{C}\psi_{1}) + \operatorname{cyclic}(1,2,3) \right]$$

$$= \int d^{2}\Omega^{\varphi} \left[ ([f_{3}\partial_{u} + \mathcal{L}_{Y_{3}} - \frac{1}{2}\psi_{3}]C_{AB} - 2\bar{D}_{A}\bar{D}_{B}f_{3} + \Delta f_{3}\bar{\gamma}_{AB}) \right]$$

$$(f_{1}\bar{D}^{B}\bar{D}^{A}\psi_{2} - f_{2}\bar{D}^{B}\bar{D}^{A}\psi_{1}) + \operatorname{cyclic}(1,2,3)$$

$$= \int d^{2}\Omega^{\varphi} \left[ -C_{BC}\bar{D}_{A} \left( (Y_{1}^{A}f_{2} - Y_{2}^{A}f_{1})\bar{D}^{B}\bar{D}^{C}\psi_{3} \right) \right]$$

$$+2C_{BC} \left( \bar{D}_{A}Y_{1}^{B}f_{2} - \bar{D}_{A}Y_{2}^{B}f_{1} \right) \bar{D}^{A}\bar{D}^{C}\psi_{3} - \frac{1}{2}C_{BC} \left( \psi_{1}f_{2} - \psi_{2}f_{1} \right) \bar{D}^{B}\bar{D}^{C}\psi_{3}$$

$$+2\left( \bar{D}_{C}f_{1}f_{2} - \bar{D}_{C}f_{2}f_{1} \right) \left( \bar{\Delta}\bar{D}^{C}\psi_{3} - \frac{1}{2}\bar{D}^{C}\bar{\Delta}\psi_{3} \right) + \operatorname{cyclic}(1,2,3)$$

$$+2\left( \bar{D}_{C}f_{1}f_{2} - \bar{D}_{C}f_{2}f_{1} \right) \left( \bar{\Delta}\bar{D}^{C}\psi_{3} - \frac{1}{2}\bar{D}^{C}\bar{\Delta}\psi_{3} \right) + \operatorname{cyclic}(1,2,3)$$

$$+2\left( \bar{D}_{C}f_{1}f_{2} - \bar{D}_{C}f_{2}f_{1} \right) \left( \bar{\Delta}\bar{D}^{C}\psi_{3} - \frac{1}{2}\bar{D}^{C}\bar{\Delta}\psi_{3} \right) + \operatorname{cyclic}(1,2,3)$$

$$+2\left( \bar{D}_{C}f_{1}f_{2} - \bar{D}_{C}f_{2}f_{1} \right) \left( \bar{\Delta}\bar{D}^{C}\psi_{3} - \frac{1}{2}\bar{D}^{C}\bar{\Delta}\psi_{3} \right) + \operatorname{cyclic}(1,2,3)$$

$$+2\left( \bar{D}_{C}f_{1}f_{2} - \bar{D}_{C}f_{2}f_{1} \right) \left( \bar{\Delta}\bar{D}^{C}\psi_{3} - \frac{1}{2}\bar{D}^{C}\bar{\Delta}\psi_{3} \right) + \operatorname{cyclic}(1,2,3)$$

The second term is given by

$$B = \int d^{2}\Omega^{\varphi} \left[ C^{BC} (f_{[s_{1},s_{2}]} \bar{D}_{B} \bar{D}_{C} \psi_{3} - f_{3} \bar{D}_{B} \bar{D}_{C} \psi_{[s_{1},s_{2}]}) + \text{cyclic } (1,2,3) \right]$$

$$= \int d^{2}\Omega^{\varphi} C^{BC} \left[ \bar{D}_{A} \left( (Y_{1}^{A} f_{2} - Y_{2}^{A} f_{1}) \bar{D}_{B} \bar{D}_{C} \psi_{3} \right) - \frac{3}{2} (\psi_{1} f_{2} - \psi_{2} f_{1}) \bar{D}_{B} \bar{D}_{C} \psi_{3} \right.$$

$$- \left( Y_{1}^{A} f_{2} - Y_{2}^{A} f_{1} \right) \bar{D}_{A} \bar{D}_{B} \bar{D}_{C} \psi_{3} - f_{3} \bar{D}_{B} \bar{D}_{C} \left( Y_{1}^{A} \bar{D}_{A} \psi_{2} - Y_{2}^{A} \bar{D}_{A} \psi_{1} \right) + \text{cyclic } (1,2,3) \right]$$

$$= \int d^{2}\Omega^{\varphi} C^{BC} \left[ \bar{D}_{A} \left( (Y_{1}^{A} f_{2} - Y_{2}^{A} f_{1}) \bar{D}_{B} \bar{D}_{C} \psi_{3} \right) - \frac{3}{2} (\psi_{1} f_{2} - \psi_{2} f_{1}) \bar{D}_{B} \bar{D}_{C} \psi_{3} \right.$$

$$- 2 \left( f_{1} \bar{D}_{B} Y_{2}^{A} - f_{2} \bar{D}_{B} Y_{1}^{A} \right) \bar{D}_{C} \bar{D}_{A} \psi_{3} + \text{cyclic } (1,2,3) \right]. \tag{C.2}$$

Summing the two, we get

$$A + B = \int d^{2}\Omega^{\varphi} \left\{ C^{BC} \left[ -2 \left( f_{1}(\bar{D}_{B}Y_{2}^{A} + \bar{D}^{A}Y_{2B}) - f_{2}(\bar{D}_{B}Y_{1}^{A} + \bar{D}^{A}Y_{1B}) \right) \bar{D}_{C}\bar{D}_{A}\psi_{3} \right. \\ \left. -2 \left( \psi_{1}f_{2} - \psi_{2}f_{1} \right) \bar{D}_{B}\bar{D}_{C}\psi_{3} \right] + 2 \left( \bar{D}_{C}f_{1}f_{2} - \bar{D}_{C}f_{2}f_{1} \right) \left( \bar{\Delta}\bar{D}^{C}\psi_{3} - \frac{1}{2}\bar{D}^{C}\bar{\Delta}\psi_{3} \right) \\ + \text{cyclic} \left( 1, 2, 3 \right) \right\} \\ = \int d^{2}\Omega^{\varphi} \left[ 2 \left( \bar{D}_{C}f_{1}f_{2} - \bar{D}_{C}f_{2}f_{1} \right) \left( \bar{\Delta}\bar{D}^{C}\psi_{3} - \frac{1}{2}\bar{D}^{C}\bar{\Delta}\psi_{3} \right) + \text{cyclic} \left( 1, 2, 3 \right) \right].$$
 (C.3)

We can then use the following identities  $\bar{\Delta}\psi = -\bar{D}_A(\bar{R}Y^A)$  and  $\Delta\bar{D}^C\psi = \bar{D}^C\Delta\psi + \frac{1}{2}\bar{R}\bar{D}^C\psi$  that can be deduced from the identity (4.59) for covariant derivatives of conformal Killing vectors in [4]<sup>2</sup> to simplify the above to

$$A + B = \int d^{2}\Omega^{\varphi} \left[ 2 \left( \bar{D}_{C} f_{1} f_{2} - \bar{D}_{C} f_{2} f_{1} \right) \left( -\frac{1}{2} \bar{D}^{C} (Y_{3}^{A} \bar{D}_{A} R) - \psi_{3} \frac{1}{2} \bar{D}^{C} \bar{R} \right) + \text{cyclic } (1, 2, 3) \right]$$

$$= \int d^{2}\Omega^{\varphi} \left[ 2 \left( \bar{D}_{C} f_{1} f_{2} - \bar{D}_{C} f_{2} f_{1} \right) \left( -\frac{1}{2} \mathcal{L}_{Y_{3}} \bar{D}^{C} \bar{R} - \psi_{3} \bar{D}^{C} \bar{R} \right) + \text{cyclic } (1, 2, 3) \right]. \tag{C.4}$$

• for the first part  $\widetilde{K}_{s_1,s_2}=\frac{1}{16\pi G}\int d^2\Omega^{\varphi}\,\frac{1}{2}(f_1\partial_A f_2-f_2\partial_A f_1)\partial^A\bar{R}$ , condition (3.8) leads to

$$C = \int d^{2}\Omega^{\varphi} \left[ f_{[s_{1},s_{2}]} \partial_{A} f_{3} - f_{3} \partial_{A} f_{[s_{1},s_{2}]} \partial^{A} \bar{R} + \text{cyclic} (1,2,3) \right]$$

$$= \int d^{2}\Omega^{\varphi} \left[ \mathcal{L}_{Y_{1}} (f_{2} \partial_{A} f_{3} - f_{3} \partial_{A} f_{2}) - \psi_{1} (f_{2} \partial_{A} f_{3} - f_{3} \partial_{A} f_{2}) \partial^{A} \bar{R} + \text{cyclic} (1,2,3) \right]$$

$$= \int d^{2}\Omega^{\varphi} \left[ (f_{2} \partial_{A} f_{3} - f_{3} \partial_{A} f_{2}) (-\mathcal{L}_{Y_{1}} - 2\psi_{1}) \partial^{A} \bar{R} + \text{cyclic} (1,2,3) \right]. \tag{C.5}$$

The different contributions then sum up to zero, A + B + C = 0.

### D Kerr solution in BMS gauge

We start from equation (48) of [22] giving the Kerr metric in generalized Bond-Metzner-Sachs coordinates, that is to say in a coordinate system  $u, \tilde{r}, \theta, \phi$  such that  $g_{\tilde{r}\tilde{r}} = g_{\tilde{r}A} = 0$ . When changing the signature to (-, +, +, +) and expanding in  $\tilde{r}$ , one finds

<sup>&</sup>lt;sup>2</sup>Note that the first identity corrects the corresponding identity of [4] in the case of non constant curvature. Note also that the second relation after (4.57) in [4] should be replaced by  $\bar{D}_A f \bar{D}_C C_B^C + \bar{D}_B f \bar{D}_C C_A^C + \bar{D}_C f \bar{D}_A C_B^C + \bar{D}_C f \bar{D}_B C_A^C - 2\bar{D}^C f \bar{D}_C C_{AB} - 2\bar{\gamma}_{AB} \bar{D}_C f \bar{D}_D C^{CD} = 0$ .

$$g_{uu} = -1 + 2M \, \tilde{r}^{-1} + O(\tilde{r}^{-2}) \,,$$
 (D.1)

$$g_{u\tilde{r}} = -1 + a^2 (\frac{1}{2} - \cos^2 \theta) \tilde{r}^{-2} + O(\tilde{r}^{-3}),$$
 (D.2)

$$g_{u\theta} = -a\cos\theta + 2a\cos\theta(M - a\sin\theta)\,\widetilde{r}^{-1} + O(\widetilde{r}^{-2})\,,\tag{D.3}$$

$$g_{u\phi} = -2aM\sin^2\theta \,\widetilde{r}^{-1} + O(\widetilde{r}^{-2}), \qquad (D.4)$$

$$g_{\theta\theta} = \tilde{r}^2 + 2a\sin\theta \, \tilde{r} + a^2(3\sin^2\theta - 1) + O(\tilde{r}^{-1}),$$
 (D.5)

$$g_{\phi\phi} = \widetilde{r}^2 \sin^2 \theta - 2a \sin \theta \cos^2 \theta \, \widetilde{r} + a^2 (1 - 3\sin^2 \theta \cos^2 \theta) + O(\widetilde{r}^{-1}) \quad (D.6)$$

$$g_{\theta\phi} = O(\widetilde{r}^{-1}). \tag{D.7}$$

The Bondi-Metzner-Sachs gauge is reached by defining r through  $\det g_{AB} = r^4 \sin^2 \theta$ , which implies that

$$\widetilde{r} = r + \frac{a\cos(2\theta)}{2\sin\theta} + \frac{a^2}{8}(4\cos(2\theta) + \frac{1}{\sin^2\theta})r^{-1} + O(r^{-2}).$$
 (D.8)

In the coordinates  $u, r, \theta, \phi$ , the metric components  $g_{uu}, g_{ur}, g_{u\phi}$  are simply obtained from the above expressions by replacing  $\tilde{r}$  by r, while

$$g_{\theta\theta} = r^2 + \frac{a}{\sin\theta} r + \frac{a^2}{2\sin^2\theta} + O(r^{-1}),$$
 (D.9)

$$g_{\phi\phi} = r^2 \sin^2 \theta - a \sin \theta \, r + \frac{a^2}{2} + O(r^{-1}),$$
 (D.10)

$$g_{\theta\phi} = O(r^{-1}), \tag{D.11}$$

$$g_{u\theta} = \frac{a \cos \theta}{2 \sin^2 \theta} + \frac{a \cos \theta}{4} (8M + \frac{a}{\sin^3 \theta}) r^{-1} + O(r^{-2}).$$
 (D.12)

When comparing with section **2.2**, one can read off  $\mathcal{X}^{Kerr}$  as described at the beginning of section 4.

## **E** Integration on the sphere

Consider stereographic coordinates  $\zeta = e^{i\phi}\cot\frac{\theta}{2}$  and let  $\mu = \cos\theta$ ,  $P = \frac{1}{2}(1+\zeta\bar{\zeta})$ . We have

$$\int_{0}^{2\pi} d\phi \int_{0}^{\pi} \sin\theta d\theta \, \zeta^{m} \zeta^{n} = 4\pi \delta_{m+n}^{0}, \quad \int_{0}^{2\pi} d\phi \int_{0}^{\pi} \sin\theta d\theta \, \bar{\zeta}^{m} \bar{\zeta}^{n} = 4\pi \delta_{m+n}^{0},$$

$$\int_{0}^{2\pi} d\phi \int_{0}^{\pi} \sin\theta d\theta \, \zeta^{m} \bar{\zeta}^{n} = 2\pi \delta_{m}^{n} \int_{-1}^{1} d\mu (\frac{1+\mu}{1-\mu})^{m},$$
(E.1)

$$d^2\Omega = \sin\theta d\theta \wedge d\phi = \frac{d\zeta \wedge d\bar{\zeta}}{2iP^2},$$

$$\cos \theta = -\frac{1 - \zeta \bar{\zeta}}{1 + \zeta \bar{\zeta}} = \mu, \quad \sin \theta = P^{-1} \sqrt{\zeta \bar{\zeta}},$$

$$2 \sin^2 \frac{\theta}{2} = P^{-1} = 1 - \mu, \quad \zeta \bar{\zeta} = \frac{1 + \mu}{1 - \mu},$$
(E.2)

$$\frac{\partial(\zeta,\bar{\zeta})}{\partial(\theta,\phi)} = \begin{pmatrix} -P\sqrt{\frac{\zeta}{\zeta}} & i\zeta \\ -P\sqrt{\frac{\zeta}{\zeta}} & -i\bar{\zeta} \end{pmatrix}, \qquad \frac{\partial(\theta,\phi)}{\partial(\zeta,\bar{\zeta})} = \begin{pmatrix} -\frac{1}{2P}\sqrt{\frac{\zeta}{\zeta}} & -\frac{1}{2P}\sqrt{\frac{\zeta}{\zeta}} \\ -\frac{i}{2\zeta} & \frac{i}{2\zeta} \end{pmatrix}. \quad (E.3)$$

# **F** Computations for the Kerr black hole

$$\bar{D}_{B}C^{B\theta} = \frac{a\cos\theta}{\sin^{2}\theta}, \quad \bar{D}_{B}C^{B\phi} = 0, \quad C^{\zeta\zeta} = \frac{a}{8} \frac{(1+\zeta\bar{\zeta})^{3}2\zeta^{2}}{\sqrt{\zeta^{3}\bar{\zeta}^{3}}}, 
D_{\zeta}C^{\zeta\zeta} = \frac{a}{8} \frac{(1+\zeta\bar{\zeta})^{2}(\zeta-\zeta^{2}\bar{\zeta})}{\sqrt{\zeta^{3}\bar{\zeta}^{3}}}, \quad D_{\zeta}D_{\zeta}C^{\zeta\zeta} = -\frac{a}{16} \frac{(1+\zeta\bar{\zeta})^{3}}{\sqrt{\zeta^{3}\bar{\zeta}^{3}}}, 
C^{AB}C_{AB} = \frac{2a^{2}}{\sin^{2}\theta} = \frac{a^{2}}{2} \frac{(1+\zeta\bar{\zeta})^{2}}{\zeta\bar{\zeta}}.$$
(F.1)

For  $Y_m = -\zeta^{m+1}$ ,

$$\psi_{m} = \frac{-(m+1)\zeta^{m} + (1-m)\zeta^{m+1}\zeta}{1+\zeta\bar{\zeta}},$$

$$\partial\psi_{m} = \frac{-m(m+1)\zeta^{m-1} + 2(1-m^{2})\zeta^{m}\bar{\zeta} + m(1-m)\zeta^{m+1}\bar{\zeta}^{2}}{(1+\zeta\bar{\zeta})^{2}},$$

$$\bar{\partial}\psi_{m} = \frac{2\zeta^{m+1}}{(1+\zeta\bar{\zeta})^{2}}.$$
(F.2)

$$N_{\zeta} = \frac{4}{(1+\zeta\bar{\zeta})^{2}} \left[ \frac{3aM}{2} \left( \frac{1}{2} \sqrt{\frac{\bar{\zeta}}{\zeta}} (1-\zeta\bar{\zeta}) + i\bar{\zeta} \right) + \frac{a^{2}}{256} (1-\zeta\bar{\zeta}) (1+\zeta\bar{\zeta})^{3} \frac{1}{\zeta^{2}\bar{\zeta}} \right],$$

$$N_{\zeta} + \frac{1}{32} \partial (C^{AB}C_{AB}) = \frac{4}{(1+\zeta\bar{\zeta})^{2}} \left[ \frac{3aM}{2} \left( \frac{1}{2} \sqrt{\frac{\bar{\zeta}}{\zeta}} (1-\zeta\bar{\zeta}) + i\bar{\zeta} \right) \right].$$

$$C^{AB}D_{A}D_{B}\psi = C^{\zeta\zeta} (\partial\partial\psi - \Gamma\partial\psi) + C^{\bar{\zeta}\bar{\zeta}} (\bar{\partial}\bar{\partial}\psi - \bar{\Gamma}\bar{\partial}\psi),$$
(F.4)

$$\partial \partial \psi_{m} - \Gamma \partial \psi_{m} = \partial \left( \frac{-m(m+1)\zeta^{m-1} + 2(1-m^{2})\zeta^{m}\bar{\zeta} + m(1-m)\zeta^{m+1}\bar{\zeta}^{2}}{(1+\zeta\bar{\zeta})^{2}} \right) + \frac{2\bar{\zeta}}{1+\zeta\bar{\zeta}} \frac{-m(m+1)\zeta^{m-1} + 2(1-m^{2})\zeta^{m}\bar{\zeta} + m(1-lm\zeta^{m+1}\bar{\zeta}^{2})}{(1+\zeta\bar{\zeta})^{2}}$$

$$= \frac{\partial \left[ -m(m+1)\zeta^{m-1} + 2(1-m^{2})\zeta^{m}\bar{\zeta} + m(1-m)\zeta^{m+1}\bar{\zeta}^{2} \right]}{(1+\zeta\bar{\zeta})^{2}}$$

$$= m(1-m^{2})\frac{\zeta^{m-2} + 2\zeta^{m-1}\bar{\zeta} + \zeta^{m}\bar{\zeta}^{2}}{(1+\zeta\bar{\zeta})^{2}}$$

$$= m(1-m^{2})\zeta^{m-2}, \qquad (F.5)$$

$$\bar{\partial}\bar{\partial}\psi_m - \bar{\Gamma}\bar{\partial}\psi_m = \bar{\partial}\left(\frac{2\zeta^{m+1}}{(1+\zeta\bar{\zeta})^2}\right) + \frac{2\zeta}{1+\zeta\bar{\zeta}}\frac{2\zeta^{m+1}}{(1+\zeta\bar{\zeta})^2} = 0.$$
 (F.6)

If T=0 then  $4fM=D_A(2uMY^A)$  and the associated term in the charge vanishes. More directly, for  $Y_m$ , we get  $\frac{Mu}{2\pi G}\int d^2\Omega\,\psi_m=\frac{Mu}{G}\int_{-1}^1d\mu\,\mu=0$ . It follows that

$$Q_{0,Y}[\mathcal{X}^{Kerr}] = \frac{1}{8\pi G} \int d^2\Omega Y^A \left( N_A + \frac{1}{32} \partial_A (C^{BC} C_{BC}) \right). \tag{F.7}$$

$$\begin{split} Q_{0,l_m}[\mathcal{X}^{Kerr}] &= -\frac{1}{8\pi G} \int d^2\Omega \frac{4}{(1+\zeta\bar{\zeta})^2} \zeta^{m+1} \frac{3aM}{2} \left(\frac{1}{2} \sqrt{\frac{\zeta}{\zeta}} (1-\zeta\bar{\zeta}) + i\bar{\zeta}\right) \\ &= -\frac{3aM}{8G} \delta_0^m \int_{-1}^1 d\mu \frac{4}{(1+\zeta\bar{\zeta})^2} \left(\frac{1}{2} \sqrt{\zeta\bar{\zeta}} (1-\zeta\bar{\zeta}) + i\zeta\bar{\zeta}\right) \\ &= -\frac{3aM}{8G} \delta_0^m \int_{-1}^1 d\mu \left[ -\mu (1+\mu)^{1/2} (1-\mu)^{1/2} + i(1+\mu)(1-\mu) \right] \\ &= -\delta_0^m \frac{iaM}{2G}. \end{split} \tag{F.8}$$

$$\begin{split} K_{(0,l_m),(0,l_n)}[\mathcal{X}^{Kerr}] &= \frac{1}{32\pi G} \int d^2\Omega \left\{ \frac{u}{2} \psi_m C^{\zeta\zeta} n (1-n^2) \zeta^{n-2} - (m\leftrightarrow n) \right\} \\ &= \frac{u}{64\pi G} \int d^2\Omega \frac{a}{4} \frac{(1+\zeta\bar{\zeta})^3 \zeta^2}{\sqrt{\zeta^3\bar{\zeta}^3}} \left\{ \frac{-(m+1)\zeta^m + (1-m)\zeta^{m+1}\bar{\zeta}}{1+\zeta\bar{\zeta}} n (1-n^2) \zeta^{n-2} - (m\leftrightarrow n) \right\} \\ &= \frac{ua}{256\pi G} \int d^2\Omega \frac{(1+\zeta\bar{\zeta})^2}{\sqrt{\zeta^3\bar{\zeta}^3}} \left\{ -(m+1)n(1-n^2)\zeta^{m+n} + (1-m)n(1-n^2)\zeta^{m+n+1}\bar{\zeta} - (m\leftrightarrow n) \right\} \\ &= \frac{ua\, m (1-m^2)\delta^0_{m+n}}{128\pi G} \int d^2\Omega \frac{(1+\zeta\bar{\zeta})^2}{\sqrt{\zeta^3\bar{\zeta}^3}} (1-\zeta\bar{\zeta}) \\ &= \frac{ua\, m (m^2-1)\delta_{m+n,0}}{8G} \int_{-1}^1 d\mu \frac{\mu}{\sqrt{(1+\mu)^3(1-\mu)^3}} \\ &= 0 \, . \end{split}$$

(F.9)

$$\begin{split} K_{(0,l_m),(0,\bar{l}_n)}[\mathcal{X}^{Kerr}] &= \frac{1}{32\pi G} \int d^2\Omega \left\{ \frac{u}{2} \psi_m C^{\bar{\zeta}\bar{\zeta}} n (1-n^2) \bar{\zeta}^{n-2} - \frac{u}{2} \psi_{\bar{n}} C^{\zeta\zeta} m (1-m^2) \zeta^{m-2} \right\} \\ &= \frac{u}{64\pi G} \int d^2\Omega \frac{a}{4} \frac{(1+\zeta\bar{\zeta})^2}{\sqrt{\zeta^3\bar{\zeta}^3}} \left\{ -(m+1)n(1-n^2)\bar{\zeta}^n \zeta^m + (1-m)\zeta^{m+1} n (1-n^2)\bar{\zeta}^{n+1} \right. \\ &\qquad \qquad + (n+1)m(1-m^2)\zeta^m \bar{\zeta}^n - (1-n)\bar{\zeta}^{n+1} m (1-m^2)\zeta^{m+1} \right\} \\ &= 0 \, . \end{split}$$

 $K_{(T_{m,n},0),(0,l_{l})}[\mathcal{X}^{Kerr}] = \frac{1}{32\pi G} \int d^{2}\Omega T_{m,n} C^{\zeta\zeta} l(1-l^{2}) \zeta^{l-2}$   $= \frac{a l(1-l^{2})}{64\pi G} \int d^{2}\Omega \frac{(1+\zeta\bar{\zeta})^{2}}{\sqrt{\zeta^{3}\bar{\zeta}^{3}}} \zeta^{m+l}\bar{\zeta}^{n}.$ (F.10)

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