BOCHNER AND CONFORMAL FLATNESS ON NORMAL COMPLEX CONTACT METRIC MANIFOLDS

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Abstract. We will prove that normal complex contact metric manifolds that are Bochner flat must have constant holomorphic sectional curvature 4 and be Kähler. If they are also complete and simply connected, they must be isometric to the odd-dimensional complex projective space $\mathbb{C}P^{2n+1}(4)$ with the Fubini-Study metric. On the other hand, it is not possible for normal complex contact metric manifolds to be conformally flat.

1. Introduction

Complex contact manifolds were first introduced by Kobayashi in [6]. However, they were not as widely studied as real contact manifolds until recently, when more examples have been published. Ishihara and Konishi introduced in [4] a concept of normality, which forced the structure to be Kählerian and did not include some natural examples like the complex Heisenberg group. This led Korkmaz to define a weaker version of normality in [7], which included these examples as well as the odd-dimensional complex projective spaces and is the notion of normality that we use here.

In Hermitian geometry, the Bochner tensor plays the role of the Weyl conformal curvature tensor in real Riemannian geometry. Thus it is natural to study how normal complex contact metric manifolds are affected by Bochner flatness, i.e. the vanishing of the Bochner conformal tensor of an Hermitian manifold as defined by Tricerri and Vanhecke in [8]. We will prove in Section 3 that this condition means that the manifold must have constant holomorphic sectional curvature 4 and be Kähler. Moreover, if they are also complete and simply connected, they must be isometric to the odd-dimensional complex projective space. In contrast we show in Section 4 that there are no conformally flat normal complex contact metric manifolds.

2. Preliminaries

We will first recall the basic concepts and results on complex contact metric manifolds that we will use throughout this paper. For more background, see [2] or [7] as a general reference.

A complex manifold M with $dim_{\mathbb{C}} M = 2n+1$ and complex structure J is a *complex contact manifold* if there exists an open covering $\mathcal{U} = \{\mathcal{O}_\alpha\}$ of M, such that:

(1) On each $\mathcal{O}_\alpha$, there is a holomorphic 1-form ω_α with $\omega_\alpha \wedge (d\omega_\alpha)^n \neq 0$ everywhere,
(2) If $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq 0$, then there is a non-vanishing holomorphic function $\lambda_{\alpha\beta}$ in $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$ such that

$$\omega_\alpha = \lambda_{\alpha\beta}\omega_\beta \text{ in } \mathcal{O}_\alpha \cap \mathcal{O}_\beta.$$

2000 *Mathematics Subject Classification.* 53C15.
Key words and phrases. Complex contact metric manifold, Bochner tensor, Complex projective space, Weyl tensor, Conformally flat.
The second author is supported by the FPU scholarship program of the Ministerio de Educación (Spain) and the PAI group FQM-327 (Junta de Andalucía, Spain).

On each $\mathcal{O}_\alpha$, we define $\mathcal{H}_\alpha = \{X \in T\mathcal{O}_\alpha \quad / \quad \omega_\alpha(X) = 0\}$. Since the functions $\lambda_{\alpha\beta}$'s are nonvanishing, $\mathcal{H}_\alpha = \mathcal{H}_\beta$ on $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$, so $\mathcal{H} = \cup\mathcal{H}_\alpha$ is a well-defined, holomorphic, non-integrable subbundle on M, called the *horizontal subbundle*.

From now on, we will suppress the subindexes if $\mathcal{O}_\alpha$ is understood.

A complex contact manifold M admits a *complex almost contact metric structure*, i.e. local real 1-forms $u, v = uJ$, $(1,1)$-tensors $G, H = GJ$, unit vector fields U and $V = -JU$ and a Hermitian metric g such that

$$\begin{gathered} H^2 = G^2 = -Id + u \otimes U + v \otimes V, \\ g(GX, Y) = -g(X, GY), \quad g(U, X) = u(X), \\ GJ = -JG, \quad GU = 0, \quad u(U) = 1, \end{gathered}$$

and on the overlaps, the above tensors transform as

$$\begin{gathered} u' = au - bv, \quad v' = bu + av, \\ G' = aG - bH, \quad H' = bG + aH, \end{gathered}$$

for some functions a, b defined on the overlaps with $a^2 + b^2 = 1$. As a result of the above formulas, on a complex almost contact metric manifold M, the following identities also hold (see [4]):

$$\begin{gathered} HG = -GH = J + u \otimes V - v \otimes U, \\ JH = -HJ = G, \quad g(HX, Y) = -g(X, HY), \\ GV = HU = HV = 0, \quad uG = vG = uH = vH = 0, \\ JV = U, \quad g(U, V) = 0. \end{gathered}$$

Moreover, given a complex contact manifold, the complex almost contact metric structure can be chosen such that $du(X, Y) = g(X, GY) + (\sigma \wedge v)(X, Y)$ and $dv(X, Y) = g(X, HY) - (\sigma \wedge u)(X, Y)$ for some 1-form σ; see [3] or [5]. In this case we say that M has a *complex contact metric structure*.

On a complex contact metric manifold M, we can write $TM = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V}$ is the *vertical subbundle* on M, locally spanned by U and $V = -JU$, and is usually assumed to be integrable. In this case $\sigma(X) = g(\nabla_X U, V)$. From now on, we will work with a complex contact metric manifold M with structure tensors (u, v, U, V, G, H, g) and complex structure J. The following identities are established in [7]:

$$\begin{gathered} \nabla_U G = \sigma(U)H, \quad \nabla_V H = -\sigma(V)G, \\ d\sigma(U, X) = v(X)d\sigma(U, V), \quad d\sigma(V, X) = -u(X)d\sigma(U, V). \end{gathered}$$

In real contact geometry, Sasakian manifolds play an important role. As an analogue of Sasakian manifolds, in complex contact geometry we also have the concept of normality. We will use the definition that Korkmaz gave in [7], instead of the stronger notion which Ishihara and Konishi introduced in [4].

A complex contact metric manifold M is *normal* if it satisfies the next two conditions:

(1) $S(X, Y) = T(X, Y) = 0$ for all X, Y in $\mathcal{H}$, and
(2) $S(U, X) = T(V, X) = 0$ for all X,

where S and T are $(1,2)$-tensors on M defined as follows

$$\begin{aligned} S(X,Y) &= [G,G](X,Y) + 2v(Y)HX - 2v(X)HY + 2g(X,GY)U - 2g(X,HY)V \\ &\quad - \sigma(GX)HY + \sigma(GY)HX + \sigma(X)GHY - \sigma(Y)GHX, \\ T(X,Y) &= [H,H](X,Y) + 2u(Y)GX - 2u(X)GY + 2g(X,HY)V - 2g(X,GY)U \\ &\quad + \sigma(HX)GY - \sigma(HY)GX - \sigma(X)HGY + \sigma(Y)HGX. \end{aligned}$$

We recall that

$$[G,G](X,Y) = (\nabla_{GX}G)Y - (\nabla_{GY}G)X - G(\nabla_X G)Y + G(\nabla_Y G)X$$

is the Nijenhuis torsion of G. It was also proved in [7] that:

Proposition 2.0.1. *Let M be a complex contact metric manifold. Then M is normal if and only if*

$$\begin{aligned} g((\nabla_X G)Y,Z) &= \sigma(X)g(HY,Z) + v(X)d\sigma(GZ,GY) - 2v(X)g(HGY,Z) \\ &\quad - u(Y)g(X,Z) - v(Y)g(JX,Z) + u(Z)g(X,Y) - v(Z)g(X,JY), \end{aligned} \tag{1}$$

$$\begin{aligned} g((\nabla_X H)Y,Z) &= -\sigma(X)g(GY,Z) - u(X)d\sigma(HZ,HY) + 2u(X)g(HGY,Z) \\ &\quad + u(Y)g(JX,Z) - v(Y)g(X,Z) + u(Z)g(X,JY) + v(Z)g(X,Y). \end{aligned} \tag{2}$$

As a result of this proposition, on a normal complex contact metric manifold the covariant derivative of J satisfies:

$$g((\nabla_X J)Y,Z) = u(X)(\Omega(Z,GY) - 2g(HY,Z)) + v(X)(\Omega(Z,HY) + 2g(GY,Z)). \tag{3}$$

Also on a normal complex contact metric manifold we have:

$$\nabla_X U = -GX + \sigma(X)V, \quad \nabla_X V = -HX - \sigma(X)U, \tag{4}$$

$$d\sigma(GX,GY) = d\sigma(HX,HY) = d\sigma(Y,X) - 2u \wedge v(Y,X)d\sigma(U,V). \tag{5}$$

Let M be a normal complex contact metric manifold. For any horizontal vector field X, the plane section generated by X and $Y = aGX + bHX$, where $a^2 + b^2 = 1$, is called a *GH-section* and we define the *GH-sectional curvature* $\mathcal{G}H_{a,b}(X)$ as the curvature of the GH-section, i.e. $\mathcal{G}H_{a,b}(X) = K(X, aGX + bHX)$, where $K(X,Y)$ is the sectional curvature of the plane section spanned by X and Y.

If the *GH-sectional curvature* $\mathcal{G}H_{a,b}(X)$ is independent of the choice of a and b, then we will denote it by $\mathcal{G}H(X)$. If it is also independent of the choice of the GH-section at each point, then the holomorphic sectional curvature is

$$K(X,JX) = \mathcal{G}H(X) + 3, \tag{6}$$

see Proposition 5.2 of [7].

The odd-dimensional complex projective spaces $\mathbb{C}P^{2n+1}$ with the standard Fubini-Study metric g of constant holomorphic curvature 4 are examples of normal complex contact metric manifold and have constant GH-sectional curvature 1 (see [7]).

The following result of Korkmaz [7] will be important for our work.

Theorem 2.1. *Let M be a normal complex contact metric manifold with constant GH-sectional curvature $+1$ and $d\sigma(U,V) = -2$. Then M is Kähler and has constant holomorphic curvature $+4$. If, in addition, M in complete and simply connected, then M is isometric to the complex projective space $\mathbb{C}P^{2n+1}$ with the Fubini-Study metric.*

Our conventions for the curvature tensor are

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X - \nabla_{[X,Y]}Z, \quad R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

A number of basic curvature properties of normal complex contact metric manifolds were given in [7] and we note the following for our use

$$R(U,V,V,U) = -2d\sigma(U,V). \tag{7}$$

For a horizontal vector field X we have

$$R(X,U)U = X, \quad R(X,V)V = X, \tag{8}$$

$$R(X,U)V = \sigma(U)GX + (\nabla_U H)X - JX, \tag{9}$$

$$\begin{aligned} &R(X,JX,JX,X) + R(X,GX,GX,X) + R(X,HX,HX,X) \\ &\qquad = -6g(X,X)(d\sigma(JX,X) + g(X,X)). \end{aligned} \tag{10}$$

For horizontal vector fields X and Y

$$\begin{aligned} &R(X,JX,JY,Y) = R(X,Y,Y,X) + R(X,JY,JY,X) \\ &+4(g(X,GY)d\sigma(X,HY) - g(X,HY)d\sigma(X,GY) + 2g(X,GY)^2 + 2g(X,HY)^2). \end{aligned} \tag{11}$$

The definition of Bochner tensor of an almost Hermitian manifold was given by Tricerri and Vanhecke in [8]. Define $(0,4)$-tensors π_1, π_2 and L_3R by:

$$\begin{aligned} \pi_1(X,Y,Z,W) &= g(X,Z)g(Y,W) - g(Y,Z)g(X,W), \\ \pi_2(X,Y,Z,W) &= 2g(JX,Y)g(JZ,W) + g(JX,Z)g(JY,W) - g(JY,Z)g(JX,W), \\ L_3R(X,Y,Z,W) &= R(JX,JY,JZ,JW). \end{aligned}$$

Given a $(0,2)$-tensor S, we denote by $\varphi(S)$ and $\psi(S)$:

$$\begin{aligned} \varphi(S)(X,Y,Z,W) &= g(X,Z)S(Y,Z) + g(Y,W)S(X,Z) \\ &\quad - g(X,W)S(Y,Z) - g(Y,Z)S(X,W), \\ \psi(S)(X,Y,Z,W) &= 2g(X,JY)S(Z,JW) + 2g(Z,JW)S(X,JY) \\ &\quad + g(X,JZ)S(Y,JW) + g(Y,JW)S(X,JZ) \\ &\quad - g(X,JW)S(Y,JZ) - g(Y,JZ)S(X,JW). \end{aligned}$$

Given an Hermitian manifold of complex dimension $2n+1$ (which the complex contact metric manifolds are in particular), the *Bochner conformal tensor* is defined as

$$\begin{aligned} B = R &+ \left(\frac{1}{8(n+1)}\psi(\rho^*) + \frac{1}{8n}\varphi(\rho)\right)(R - L_3R) \\ &+ \frac{1}{16(2n+3)}(\varphi+\psi)(\rho+3\rho^*)(R+L_3R) + \frac{1}{16(2n-1)}(3\varphi-\psi)(\rho-\rho^*)(R+L_3R) \\ &- \frac{1}{32(n+1)(2n+3)}(\tau+3\tau^*)(\pi_1+\pi_2) - \frac{1}{32n(2n-1)}(\tau-\tau^*)(3\pi_1-\pi_2), \end{aligned}$$

where $\rho(R)$, $\rho^*(R)$ are the *Ricci tensor* and *Ricci $*$-tensor* associated to the curvature tensor R, which are defined for an arbitrary orthonormal basis $\{e_i\}$ by:

$$\rho(X,Y) = \sum_i R(X,e_i,e_i,Y), \quad \rho^*(X,Y) = \sum_i R(X,e_i,Je_i,JY),$$

and τ, τ^* are their associated *scalar curvature* and *$*$-scalar curvature* respectively.

An almost Hermitian manifold is said to be Bochner flat if B is identically zero. The complex projective spaces $\mathbb{C}P^{2n+1}(4)$ mentioned before are Bochner flat because they have constant holomorphic curvature (Theorem 6 of [9]). We will also recall the definition of the Weyl conformal tensor. Given a Riemannian manifold M of real dimension m, the *Weyl conformal tensor* is defined by

$$W(X,Y)Z = R(X,Y)Z + \frac{\tau}{(m-1)(m-2)}(g(Y,Z)X - g(X,Z)Y) + \frac{1}{m-2}(g(X,Z)QY - g(Y,Z)QX + \rho(X,Z)Y - \rho(Y,Z)X), \tag{12}$$

for X, Y, Z vector fields on M.

A Riemannian manifold M is said to be *conformally flat* if it is locally conformally related to the Euclidian metric. The following characterizations are well known. If $m > 3$, M is conformally flat if and only if $W = 0$. If $m = 3$, the Weyl conformal tensor is identically zero and the manifold is conformally flat if and only if the Schouten tensor is a Codazzi tensor (see e.g. [1] for a discussion).

3. Normal complex contact metric manifolds which are Bochner flat

On an almost Hermitian manifold we have the following relation between the Ricci tensor and $*$-Ricci tensor, [10] (p. 195)

$$\nabla_t \nabla_j J^t{}_i - \nabla_j \nabla_t J^t{}_i = (\rho_{jt} - \rho^*_{jt}) J^t{}_i. \tag{13}$$

Also in [10] (p. 195) one has the result that on a semi-Kähler manifold (i.e., Ω is coclosed),

$$\tau - \tau^* = (\nabla^h J^{ji})(\nabla_j J_{ih}). \tag{14}$$

In the following, we will always assume $\{E_1, \dots, E_{4n+2}\}$ to be an orthonormal basis of M such that $E_{4n+1} = U$ and $E_{4n+2} = V$.

Lemma 3.1. *On a normal complex contact metric manifold $\tau = \tau^*$.*

Proof. First note that from (3) we easily have that

$$\sum_{i=1}^{4n+2} g((\nabla_{E_i} J)E_i, Z) = 0 \tag{15}$$

and hence that a normal complex contact metric manifold is semi-Kähler. Now rewrite (3.2) as

$$\tau - \tau^* = \sum_{h,i,j=1}^{4n+2} g((\nabla_{E_h} J)E_j, E_i) g((\nabla_{E_j} J)E_i, E_h),$$

which vanishes by virtue of (3). □

Theorem 3.2. *If a normal complex contact metric manifold M is Bochner flat, then it has constant holomorphic sectional curvature 4 and is Kähler. If, in addition, M is complete and simply connected, then it is isometric to the odd-dimensional complex projective space $\mathbb{C}P^{2n+1}(4)$ with the Fubini-Study metric.*

Proof. By virtue of Theorem 2.1 it is enough to prove that M has constant GH-curvature 1 and satisfies $d\sigma(U,V) = -2$.

We first compute $\rho(U,U)$ and $\rho^*(U,U)$. We have from formulas (7) and (8) that

$$\rho(U,U) = \sum_{i=1}^{4n+2} R(U, E_i, E_i, U) = \sum_{i=1}^{4n} R(E_i, U, UE_i) + R(U,V,V,U) = 4n - 2d\sigma(U,V). \tag{16}$$

Analogously, we can calculate

$$\rho(V,V) = 4n - 2d\sigma(U,V).$$

Now recalling that $JU = -V$ and using equations (9), (2) and (5) (in that order), we obtain

$$\begin{aligned}
\rho^*(U,U) &= \sum_{i=1}^{4n+2} R(U,E_i,JE_i,JU) = -\sum_{i=1}^{4n+2} R(E_i,U,V,JE_i) \\
&= -\sum_{i=1}^{4n} g(\sigma(U)Ge_i + (\nabla_U H)E_i - JE_i, JE_i) + R(U,V,V,U) \\
&= \sum_{i=1}^{4n} (g(d\sigma(E_i,JE_i) - 2 + 1) - 2d\sigma(U,V) \\
&= -4n - 2d\sigma(U,V) + \sum_{i=1}^{4n} d\sigma(E_i,JE_i). \qquad (17)
\end{aligned}$$

Analogously, we also have

$$\rho^*(V,V) = -4n - 2d\sigma(U,V) + \sum_{i=1}^{4n} d\sigma(E_i,JE_i).$$

As we have seen the structure is semi-Kähler so we only have the first term on the left hand side of (13) which we write in the form $\sum_{i=1}^{4n+2} g((\nabla_{E_i}\nabla_X J)E_i, Y)$. Setting $X = Y = U$ there is no contribution for $E_i = U$ or $E_i = V$ and we have

$$\sum_{i=1}^{4n+2} g((\nabla_{E_i}\nabla_U J)E_i, U) = \sum_{i=1}^{4n} g(\nabla_{E_i}((\nabla_U J)E_i) - (\nabla_U J)\nabla_{E_i}E_i, U),$$

which using (3) and (4) vanishes. Therefore $\rho(U,U) = \rho^*(U,U)$ and using equations (16) and (17) we have

$$\sum_{i=1}^{4n} d\sigma(E_i,JE_i) = 8n \qquad (18)$$

and in turn

$$\rho(U,U) = \rho(V,V) = \rho^*(U,U) = \rho^*(V,V) = 4n - 2d\sigma(U,V). \qquad (19)$$

Turning to the Bochner tensor, we first have

$$B(U,V,V,U) = R(U,V,V,U) + \frac{\tau + 3\tau^*}{8(n+1)(2n+3)} \qquad (20)$$

$$+ \frac{1}{2(2n+3)}\big(-\rho(V,V) - \rho(U,U) - 3\rho^*(V,V) - 3\rho^*(U,U)\big). \qquad (21)$$

Now with $B = 0$, (7), (19) and $\tau = \tau^*$, we have

$$(4n-2)d\sigma(U,V) = -16n + \frac{\tau}{2(n+1)}. \qquad (22)$$

As $B(X,U,U,X) = 0$ for every X unit and horizontal vector field, using the fact that $\rho^*(JX,JX) = \rho^*(X,X)$ we deduce:

$$\begin{aligned}
0 &= 6(2n-1) + 2(2n-1)d\sigma(U,V) + \frac{2n-1}{4(n+1)}\tau \qquad (23)\\
&- \frac{8n^2+8n-3}{4n}\rho(X,X) - \frac{3}{4n}\rho(JX,JX) + 3\rho^*(X,X).
\end{aligned}$$

By the first Bianchi Identity and formulas (2) and (9), we know that

$$\begin{aligned}\sum_{i=1}^{4n} R(X,JX,JE_i,E_i) &= -\sum_{i=1}^{4n} R(X,JE_i,E_i,JX) - \sum_{i=1}^{4n} R(X,E_i,JX,JE_i)\\ &= 2\sum_{i=1}^{4n+2} R(X,E_i,JE_i,JX) + 2R(X,U,V,JX) - 2R(X,V,U,JX)\\ &= 2\rho^*(X,X) + 4 - 4d\sigma(X,JX). \qquad (24)\end{aligned}$$

Using (11) we can also prove that

$$\begin{aligned}\sum_{i=1}^{4n} R(X,JX,JE_i,E_i) &= 2\rho(X,X) - 4\\ &\quad + 4\sum_{i=1}^{4n+2}(g(X,GE_i)d\sigma(X,HE_i) - g(X,HE_i)d\sigma(X,GE_i)) + 16\\ &= 2\rho(X,X) + 12 - 8d\sigma(X,JX). \qquad (25)\end{aligned}$$

Comparing equations (24) and (25), we have

$$\rho^*(X,X) = \rho(X,X) - 2d\sigma(X,JX) + 4$$

and

$$\rho(JX,JX) = \rho(X,X).$$

Using these two equations and (22), formula (23) simplifies to

$$0 = -2(2n-3) + \frac{2n+1}{4(n+1)}\tau - 6d\sigma(X,JX) - (2n-1)\rho(X,X), \qquad (26)$$

for every X unit and horizontal.

On the other hand, we can deduce from $B(X,Y,Y,X) = 0$, that

$$\begin{aligned}R(X,JX,JX,X) &= \rho(X,X) + 2 - \frac{\tau}{4(n+1)},\\ R(X,Y,Y,X) &= -\frac{2}{2n-1} + \frac{\tau}{8(n+1)(2n-1)},\end{aligned}$$

for all X, Y are unit, horizontal vector fields such that X is orthogonal to Y and JY. The latter equation gives an expression for the GH-curvature:

$$\mathcal{G}H_{a,b}(X) = K(X, aGX + bGH) = -\frac{2}{2n-1} + \frac{\tau}{8(n+1)(2n-1)}, \qquad (27)$$

for every X unit and horizontal.

Therefore, equation (10) gives:

$$6d\sigma(X,JX) = \rho(X,X) + \frac{4n-6}{2n-1} + 6 - \frac{n-1}{2(n+1)(2n-1)}\tau. \qquad (28)$$

Comparing (26) and (28) we get that

$$\rho(X,X) = -\frac{4n^2-3}{n(2n-1)} + \frac{4n^2+2n-3}{8n(n+1)(2n-1)}\tau, \qquad (29)$$

for every X unit and horizontal. This shows that $\rho(X,X)$ is independent of the vector field X, so $d\sigma(X,JX)$ does not depend on X either because of (26) or (28). Thus from equation (18) we obtain that $d\sigma(X, JX) = 2$. Substituting in (28), we get another expression for $\rho(X, X)$:

$$\rho(X, X) = -\frac{2(2n+3)}{2n-1} + \frac{2n+1}{4(n+1)(2n-1)}\tau, \tag{30}$$

for every X unit and horizontal.

The formulas (29) and (30) must coincide, which means that $\tau = 8(n+1)(2n+1)$ (and therefore $\rho(X, X) = 4(n+1)$). Substituting the expression of the scalar curvature in (22) and (27) we get that $d\sigma(U, V) = -2$ and that the GH-sectional curvature is constant and equal to 1. The result now follows from Theorem 2.1. □

4. Non-existence of normal complex contact metric manifolds which are conformally flat

Theorem 4.1. *There exist no normal complex contact metric manifolds which are conformally flat.*

Proof. The proof is by contradiction. Suppose that M^{2n+1} is a normal complex contact metric manifold which is also conformally flat. Then, as $dim_{\mathbb{R}} M^{2n+1} = 4n + 2$, we get by (12) that

$$\begin{aligned} R(X,Y,Z,W) = &-\frac{\tau}{4n(4n+1)}(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)) \\ &- \frac{1}{4n}(g(X,Z)\rho(Y,W) - g(Y,Z)\rho(X,W) + g(Y,W)\rho(X,Z) - g(X,W)\rho(Y,Z)), \end{aligned} \tag{31}$$

for all vector fields X, Y, Z, W on M.

In particular, we have that

$$R(U, V, V, U) = \frac{1}{4n}(\rho(U, U) + \rho(V, V)) - \frac{\tau}{4n(4n+1)},$$

and by virtue of equation (7) and the fact that

$$\rho(U, U) = \rho(V, V) = 2(2n - d\sigma(U, V)), \tag{32}$$

we obtain

$$d\sigma(U, V) = -\frac{2n}{2n-1} + \frac{\tau}{4(2n-1)(4n+1)}.$$

Substituting this into (32), we get that

$$\rho(U, U) = \rho(V, V) = \frac{8n^2}{2n-1} - \frac{\tau}{2(2n-1)(4n+1)}. \tag{33}$$

On the other hand, we also have from (31) that, for every unit horizontal vector field X on M:

$$R(X, U, U, X) = \frac{1}{4n}(\rho(X, X) + \rho(U, U)) - \frac{\tau}{4n(4n+1)}.$$

Applying equations (8) and (33), we obtain that the Ricci tensor satisfies

$$\rho(X, X) = -\frac{4n}{2n-1} + \frac{(4n-1)\tau}{2(2n-1)(4n+1)}, \tag{34}$$

for every X unit, horizontal vector field.

Finally, if we choose X, Y unit, mutually orthogonal horizontal vector fields in (31), and use formula (34), we have that

$$R(X, Y, Y, X) = -\frac{2}{2n-1} + \frac{\tau}{2(2n-1)(4n+1)}.$$

In particular, the holomorphic sectional curvature and the GH-sectional curvature of an arbitrary horizontal vector field X are equal and (6) yields $0 = 3$, a contradiction. □

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