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BOCHNER-RIESZ MEANS ON SYMMETRIC SPACES

CHRISTOPHER MEANEY AND ELENA PRESTINI

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Abstract. We combine results of Giulini and Mauceri and our earlier work to obtain an almost-everywhere convergence result for the Bochner-Riesz means of the inverse spherical transform of bi-invariant L^p functions on a noncompact rank one Riemannian symmetric space. Following a technique of Kanjin, we show that this result is sharp.

1. Notation. Suppose that G/K is a noncompact rank one Riemannian symmetric space of dimension d. Here functions on G/K can be viewed as being right-K-invariant functions on G, and K-invariant functions on G/K are identified with bi-K-invariant functions on G. Denote by $-\Delta_0$ the Laplace-Beltrami operator on G/K, and $-\Delta$ its self-adjoint extension to $L^2(G/K)$. Its spectral resolution is

$$-\Delta=\int_{|\rho|^2}^{\infty} tdE(t)\,,$$

where the constant $|\rho|^2$ depends on the geometry of G/K. For every $z\in \boldsymbol{C}$ with $\Re(z)\geq 0$ there are the Bochner-Riesz mean operators

$$S_R^z f=\int_{|\rho|^2}^{\infty}\left(1-\frac{t}{R}\right)_+^z dE(t)f\,.$$

In fact there is a $C^\infty(K\backslash G/K)$ kernel s_R^z so that

$$S_R^z f=f*s_R^z\,,\qquad \text{for all}\quad f\in C_c^\infty(G/K)\,.$$

The special case $z=0$ amounts to the usual partial sums:

$$S_R^0 f=E_R f\,,\qquad \text{for}\quad R\geq |\rho|^2\,,$$

and these converge in norm for elements $f\in L^2(G/K)$,

$$\|f-S_R^0\|_2\to 0\,,\qquad \text{as}\quad R\to\infty\,.$$

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2. Some structure. We use the paper of Stanton and Tomas [8] as our main source for analysis of bi-K-invariant functions on G. The group G has an Iwasawa decomposition $G=ANK$ where A is isomorphic with the real line and N is a nilpotent group. There is also the Cartan decomposition $G=KAK$. There is a smooth parameterization $t\mapsto a_t$ (taking $\boldsymbol{R}$ to A), and bi-K-invariant functions on G can be identified with even functions on $A\cong\boldsymbol{R}$, via the assignment $t\mapsto f(a_t)$.

The measure on G/K can be described in terms of the Cartan decomposition. For $f\in C_c(K\backslash G/K)$,

$$\int_G f(x)dx=c\int_0^\infty f(a_t)(\sinh(t))^{m_1}(\sinh(2t))^{m_2}dt\,,$$

with $m_1+m_2+1=d=\dim(G/K)$. The integers m_1 and m_2 are the multiplicities of the positive restricted roots α and 2α. The constant ρ is $(m_1+2m_2)/2$.

In general, if $f\in C_c(G/K)$,

$$\int_G f(x)dx=c\int_0^\infty\int_K f(ka_t)(\sinh(t))^{m_1}(\sinh(2t))^{m_2}dkdt\,.$$

The density $D(t)$ of this invariant measure has the property

$$D(t)\approx\begin{cases} ct^{d-1} & \text{for small } t>0\,,\\ ce^{t(m_1+2m_2)}=ce^{2\rho t} & \text{for large values of } t\,.\end{cases}$$

For these formulae see Section 1 of [8].

3. Spherical functions. The bi-K-invariant eigenfunctions of the Laplace-Beltrami operator are given by spherical functions. For each $\lambda\in\boldsymbol{C}$ there is a spherical function φ_λ.

$$-\Delta_0\varphi_\lambda=(\lambda^2+\rho^2)\varphi_\lambda\,.$$

There is the spherical transform

$$\hat{f}(\lambda)=\int_G f(x)\varphi_\lambda(x^{-1})dx \qquad \text{for} \quad f\in C_c(K\backslash G/K)\,,$$

and the Plancherel formula

$$\|f\|_2^2=\int_0^\infty|\hat{f}(\lambda)|^2|c(\lambda)|^{-2}d\lambda \qquad \text{for} \quad f\in L^2(K\backslash G/K)\,.$$

The density $|c(\lambda)|^{-2}$ in this formula is given on [8, p. 266]. The condition that t is an eigenvalue of $-\Delta$ and $t\leq R$ translates into $\lambda^2+\rho^2\leq R$.

4. Maximal functions.

DEFINITION 4.1. For complex z with $\Re(z)\geq 0$ and a test function f define the maximal function

$$S_*^z f(x) = \sup_{R>0} |S_R^z f(x)| .$$

The goal is to use the L^p-mapping properties of these operators to provide results about the a.e. convergence of $S_R^z f(x)$ as $R \to \infty$, for f in some subspace of $L^p(G/K)$. However, Theorem 4.4 in [8] states that for $1 \leq p < 2$ there is a strip containing the real line (and depending on p) in the complex plane such that the spherical transform

$$\lambda \mapsto \hat{f}(\lambda) \text{ is analytic for all } f \in L^p(K \backslash G/K) .$$

Hence, none of the operators S_R^z can be bounded from L^p to L^p. Moreover, in [5] it is shown that if $1 \leq p < 2$ then there exist elements of $L^p(G/K)$ with

$$\{S_R^0 f(x) : R > 0\} \text{ divergent almost everywhere} .$$

Although the partial sums $S_R^z f$ of a nonzero element $f \in L^p(G/K)$ will no longer be in $L^p(G/K)$, they will be in $L^2(G/K)$. The Kunze-Stein phenomenon (see [1]) states that for $1 \leq p < 2$ there is an inclusion

$$L^p(G/K) * L^2(G/K) \subseteq L^2(G/K) .$$

5. Theorem of Giulini and Mauceri. In [2] the main theorem gives conditions on p and z so that there is some boundedness of the maximal operator S_*^z taking $L^p(G/K)$ into a sum of Lebesgue spaces.

THEOREM 5.1. *Let $\Re(z) > (d-1)/2$. There exists $q_0 \geq 2$ such that if*

$$1 < p \leq q_0'$$

then the maximal function S_*^z maps $L^p(G/K)$ continuously into $(L^p + L^r)(G/K)$ for every

$$\frac{q_0 p'}{(p' - q_0)} \leq r \leq \infty .$$

Furthermore, S_*^z maps $L^1(G/K)$ continuously into weak-$(L^1 + L^r)(G/K)$ for every $r \in [q_0, \infty]$. The norm of S_*^z grows at most polynomially in $\Im(z)$ provided $\Re(z)$ is in a bounded subset of $((d-1)/2, \infty)$. The constant q_0 depends on G/K.

THEOREM 5.2. *Let $1 \leq p \leq 2$. If $\Re(z) > (2/p - 1)(d-1)/2$ then for every*

$$r \geq \frac{p q_0}{2 - p + p q_0 - q_0}$$

$$\|S_*^z f\|_{p+r} \leq \kappa(z) \|f\|_p \qquad \textit{for} \quad f \in L^p(G/K) ,$$

where $\kappa(z)$ is independent of f and of admissible growth.

In particular, if $1 \leq p \leq 2$ and $\Re(z) > (2/p - 1)(d-1)/2$ then

$$\lim_{R\to\infty} S_R^{z} f(x) = f(x) \text{ a.e.} \qquad \textit{for} \quad f \in L^p(G/K)\,.$$

This has recently been extended to arbitrary Jacobi function expansions on $(0, \infty)$ by Liu Jianming and Zheng Weixing.

We combine the work of Giulini and Mauceri with our result from [7] about the boundedness of S_*^0 mapping certain $L^p(K\backslash G/K)$ into a sum of Lebesgue spaces.

6. Theorems of Meaney and Prestini.

THEOREM 6.1. *If* $2d/(d+1) < p \le 2$ *then*

$$S_*^0 : L^p(K\backslash G/K) \to (L^p + L^2)(K\backslash G/K)$$

is a bounded operator and so

$$\lim_{R\to\infty} S_R^0 f(x) = f(x) \text{ a.e.} \qquad \textit{for every} \quad f \in L^p(K\backslash G/K)\,.$$

THEOREM 6.2. *For* $p = 2d/(d+1)$ *there is an element of* $f \in L^p(K\backslash G/K)$ *with compact support such that the inverse partial sums*

$$\{S_R^0 f(x) : R > 0\}$$

diverge on a set of positive measure.

7. Complex Interpolation. Since the results cited above deal with sums of Lebesgue spaces, we sketch how complex interpolation can be used in this setting.

Suppose that A and B are reflexive Banach spaces continuously embedded in a locally convex topological vector space $\mathscr{M}$, with the additional property that their dual spaces A' and B' are also continuously embedded in $\mathscr{M}$. Their intersection is a Banach space with norm

$$\|f\|_{A\cap B} = \max(\|f\|_A, \|f\|_B)\,,$$

while their sum $A + B$ has the norm defined by

$$\|f\|_{A+B} = \inf\{\|a\|_A + \|b\|_B : f = a + b, a \in A, b \in B\}\,.$$

LEMMA 7.1. *Suppose* A *and* B *are as above, with* $A \cap B$ *dense in* $\mathscr{M}$. *Then the dual of* $A' \cap B'$ *is isometrically isomorphic with* $A + B$. *Hence, for each* $f \in A + B$, *its norm is*

$$\|f\|_{A+B} = \sup\{|\langle f, x\rangle| : x \in A' \cap B',\ \|x\|_{A'} \le 1, \textit{ and } \|x\|_{B'} \le 1\}\,.$$

Now assume that $1 < p_1 < p_2 \le 2 \le q_2 < q_1 < \infty$, and $\mathscr{M} = L^1_{\mathrm{loc}}(K\backslash G/K)$. Then all of the Lebesgue spaces $L^r(K\backslash G/K)$ (with $1 \le r < \infty$) are continuously embedded in $\mathscr{M}$ and the subspace $S(K\backslash G/K)$ of integrable simple functions is dense in all of these spaces. For the moment we will write L^r in place of $L^r(K\backslash G/K)$ and

$$\|f\|_{p+q} \text{ in place of } \|f\|_{L^p+L^q}.$$

We are interested in the complex interpolation spaces between $L^{p_1}+L^{q_1}$ and $L^{p_2}+L^{q_2}$. As on page 181 of the book of Stein and Weiss [9], let

$$\alpha(z)=\frac{1-z}{p_1}+\frac{z}{p_2}, \quad \alpha(0)=\frac{1}{p_1}, \quad \alpha(1)=\frac{1}{p_2},$$

$$\beta(z)=\frac{1-z}{q_1}+\frac{z}{q_2}, \quad \beta(0)=\frac{1}{q_1}, \quad \beta(1)=\frac{1}{q_2},$$

$$\gamma(z)=\frac{1-z}{p_1'}+\frac{z}{p_2'}, \quad \gamma(0)=\frac{1}{p_1'}, \quad \gamma(1)=\frac{1}{p_2'},$$

$$\delta(z)=\frac{1-z}{q_1'}+\frac{z}{q_2'}, \quad \delta(0)=\frac{1}{q_1'}, \quad \delta(1)=\frac{1}{q_2'}.$$

Let $\boldsymbol{S}$ denote the closed vertical strip $\{z\in C: 0\le\Re(z)\le 1\}$. In the Calderón theory of complex interpolation (cf. [9, pp. 210–211]) the space $\mathscr{F}$ is defined to consist of all functions

$$f: \boldsymbol{S}\to L^{p_1}+L^{q_1}+L^{p_2}+L^{q_2}$$

with the properties:

1. f is analytic in the interior of $\boldsymbol{S}$;
2. f is continuous and bounded on $\boldsymbol{S}$;
3. $t\mapsto f(it)\in L^{p_1}+L^{q_1}$ is bounded and continuous as a function from $\boldsymbol{R}$ to $L^{p_1}+L^{q_1}$;
4. $t\mapsto f(1+it)\in L^{p_2}+L^{q_2}$ is bounded and continuous as a function from $\boldsymbol{R}$ to $L^{p_2}+L^{q_2}$.

The norm on this space is

$$\|f\|_{\mathscr{F}}=\max\left(\sup_{t\in\boldsymbol{R}}\|f(it)\|_{p_1+q_1}, \sup_{t\in\boldsymbol{R}}\|f(1+it)\|_{p_2+q_2}\right).$$

For each $t\in(0,1)$ there is a subspace $\mathscr{N}_t=\{f\in\mathscr{F}: f(t)=0\}$ and the intermediate space

$$\mathscr{A}_t=\mathscr{F}/\mathscr{N}_t.$$

This is identified with the subspace

$$A_t=\{f(t): f\in\mathscr{F}\}\subseteq L^{p_1}+L^{q_1}+L^{p_2}+L^{q_2},$$

and equipped with the norm

$$\|a\|_{A_t}=\inf\{\|f\|_{\mathscr{F}}: f\in\mathscr{F}, a=f(t)\}.$$

We wish to show that this space is $L^{1/\alpha(t)}+L^{1/\beta(t)}$. Because of the lemma above, we must show that every element of A_t corresponds to a bounded linear functional on $L^{1/\gamma(t)} \cap L^{1/\delta(t)}$. Now fix $0<t<1$ and let $1/p=\alpha(t)$ and $1/q=\beta(t)$. Then exponents and their complementary exponents are arranged so that

$$1<p_1<p<p_2\le 2\le q_2<q<q_1<\infty\,,$$
$$1<q_1'<q'<q_2'\le 2\le p_2'<p'<p_1'<\infty\,.$$

Fix $a\in A_t$ and for $\varepsilon>0$ let $f\in\mathscr{F}$ have

$$f(t)=a \quad\text{and}\quad \|f\|_{\mathscr{F}}\le\|a\|_{A_t}+\varepsilon\,.$$

Take a step function $\psi\in S(K\backslash G/K)$, with

$$\|\psi\|_{p'}\le 1 \quad\text{and}\quad \|\psi\|_{q'}\le 1\,,$$

so that it is in the unit ball of $L^{p'}\cap L^{q'}$. Next, let $E=\{x\in G: |\psi(x)|<1\}$ and define an analytic family of functions

$$\psi_z=\frac{\psi}{|\psi|}\left((1-\chi_E)|\psi|^{\gamma(z)p'}+\chi_E|\psi|^{\delta(z)q'}\right), \qquad \text{for}\quad z\in\boldsymbol{S}\,. \tag{1}$$

When y is real, the real part of $\gamma(iy)$ is $1/p_1'$ and the real part of $\delta(iy)$ is $1/q_1'$. Hence, for all $y\in\boldsymbol{R}$,

$$|\psi_{iy}|^{p_1'}=\left((1-\chi_E)|\psi|^{p'}+\chi_E|\psi|^{p_1'q'/q_1'}\right)\le|\psi|^{p'}\,,$$

since $|\psi|<1$ on E and $p_1'q'/q_1'>p_1'>p'$. Similarly,

$$|\psi_{iy}|^{q_1'}=\left((1-\chi_E)|\psi|^{q_1'p'/p_1'}+\chi_E|\psi|^{q'}\right)\le|\psi|^{q'}\,,$$

since $|\psi|\ge 1$ off E and $q_1'p'/p_1'<q_1'<q'$.

When y is real, the real part of $\gamma(1+iy)$ is $1/p_2'$ and the real part of $\delta(1+iy)$ is $1/q_2'$. Hence, for all $y\in\boldsymbol{R}$,

$$|\psi_{1+iy}|^{p_2'}=\left((1-\chi_E)|\psi|^{p'}+\chi_E|\psi|^{p_2'q'/q_2'}\right)\le|\psi|^{p'}+|\psi|^{q'}\,,$$

since $|\psi|<1$ on E and $p_2'q'/q_2'\ge q'$. Similarly,

$$|\psi_{1+iy}|^{q_2'}=\left((1-\chi_E)|\psi|^{q_2'p'/p_2'}+\chi_E|\psi|^{q'}\right)\le|\psi|^{p'}+|\psi|^{q'}\,,$$

since $|\psi|\ge 1$ off E and $q_2'p'/p_2'<p'$.

Integrating these over G we have

$$\|\psi_{iy}\|_{p_1}\le 1,\ \|\psi_{iy}\|_{q_1}\le 1,\ \|\psi_{1+iy}\|_{p_2}\le 2^{1/p_2},\ \|\psi_{1+iy}\|_{q_2}\le 2^{1/q_2}\,. \tag{2}$$

Now consider the analytic function

$$F(z)=\int_G f(z)\psi_z\,d\mu\,, \qquad \text{for}\quad z\in\boldsymbol{S}\,.$$

This satisfies the inequalities:

$$|F(iy)| \leq \|f(iy)\|_{p_1+q_1} \leq \|f\|_{\mathscr{F}} \leq \|a\|_{A_t} + \varepsilon\,,$$

and

$$|F(1+iy)| \leq 2\|f(1+iy)\|_{p_2+q_2} \leq 2\|f\|_{\mathscr{F}} \leq 2(\|a\|_{A_t} + \varepsilon)\,.$$

The three lines theorem then shows that

$$|F(t)| = \left| \int_G a\psi \, d\mu \right| \leq 2^t(\|a\|_{A_t} + \varepsilon)\,, \qquad \text{for all} \quad \varepsilon > 0\,.$$

This proves that $a \in (L^{p'} \cap L^{q'})' = L^p + L^q$ and

$$\|a\|_{p+q} \leq 2^t \|a\|_{A_t}\,.$$

LEMMA 7.2. *For* $1 < p_1 < p_2 \leq 2 \leq q_2 < q_1 < \infty$, $0 < t < 1$, $1/p = ((1-t)/p_1) + (t/p_2)$, *and* $1/q = ((1-t)/q_1) + (t/q_2)$, *the Banach space* $L^p(K\backslash G/K) + L^q(K\backslash G/K)$ *is the complex interpolation intermediate space*

$$[L^{p_1}(K\backslash G/K) + L^{q_1}(K\backslash G/K),\, L^{p_2}(K\backslash G/K) + L^{q_2}(K\backslash G/K)]_t\,.$$

Now consider a family of linear operators T_z for $z \in \boldsymbol{S}$, which take integrable simple bi-K-invariant functions to elements of $L^1_{\text{loc}}(K\backslash G/K)$. In addition, suppose that p_1, p_2, q_1, and q_2 are as in the lemma and that for some $1 \leq r_1 \leq r_2 < \infty$,

(3) $$\|T_{iy}f\|_{p_1+q_1} \leq M_1(y)\|f\|_{r_1}\,, \qquad \text{for all} \quad y \in \boldsymbol{R}\,, \quad f \in S(K\backslash G/K)\,,$$

and

(4) $$\|T_{1+iy}f\|_{p_2+q_2} \leq M_2(y)\|f\|_{r_2}\,, \qquad \text{for all} \quad y \in \boldsymbol{R}\,, \quad f \in S(K\backslash G/K)\,,$$

where the functions M_1 and M_2 satisfy the growth condition in [9, p. 205, Theorem 4.1]. Finally, assume that

$$z \mapsto \int_G (T_z f)\psi \, d\mu$$

is analytic in the interior of the strip $\boldsymbol{S}$ and continuous on $\boldsymbol{S}$, for all f and ψ in $S(K\backslash G/K)$. Using the same argument as in Section V.4 of [9], take an $f \in S(K\backslash G/K)$ and define

$$f_z = \frac{f}{|f|}|f|^{h(z)}\,, \qquad \text{where} \quad h(z) = \frac{1-z}{r_1} + \frac{z}{r_2}\,, \quad \text{for} \quad z \in \boldsymbol{S}\,.$$

Take $\psi \in S(K\backslash G/K)$ with $\|\psi\|_p \leq 1$ and $\|\psi\|_q \leq 1$ and form the analytic family $z \mapsto \psi_z$ as in equation (1). The function

$$F(z) = \int_G (T_z f)\psi_z \, d\mu$$

is analytic on the strip $\boldsymbol{S}$ and satisfies:

$$|F(iy)| \leq M_1(y)\|f\|_{r_1} \quad \text{and} \quad |F(1+iy)| \leq 2M_2(y)\|f\|_{r_2}$$

because of the inequalities (2), (3), and (4). An application of [9, p. 206, Lemma 4.2] and the density of the subspace $S(K\backslash G/K)$ shows that

$$T_t : L^r(K\backslash G/K) \to L^p(K\backslash G/K) + L^q(K\backslash G/K) \quad \text{is bounded},$$

where $0<t<1$ and $1/r=((1-t)/r_1)+(t/r_2)$, $1/p=((1-t)/p_1)+(t/p_2)$, and $1/q=((1-t)/q_1)+(t/q_2)$.

LEMMA 7.3. *Suppose that $\{T_z : z \in \boldsymbol{S}\}$ is an admissible family of linear operators satisfying*

$$\|T_{iy}f\|_{p_1+q_1} \leq M_1(y)\|f\|_{r_1}, \qquad \text{for all} \quad y \in \boldsymbol{R}, \quad f \in S(K\backslash G/K),$$

and

$$\|T_{1+iy}f\|_{p_2+q_2} \leq M_2(y)\|f\|_{r_2}, \qquad \text{for all} \quad y \in \boldsymbol{R}, \quad f \in S(K\backslash G/K),$$

where $1<p_1<p_2\leq 2\leq q_2<q_1<\infty$, and $1\leq r_1\leq r_2<\infty$. In addition, assume that the functions M_1 and M_2 are independent of f and satisfy

$$\sup_{y\in\boldsymbol{R}} e^{-b|y|}\log(M_1(y)) < \infty \quad \text{and} \quad \sup_{y\in\boldsymbol{R}} e^{-b|y|}\log(M_2(y)) < \infty,$$

for some constant $b<\pi$. Then, if $0<t<1$, there is a constant M_t such that

$$\|T_t f\|_{p+q} \leq M_t\|f\|_r, \qquad \text{for all} \quad f \in S(K\backslash G/K),$$

provided $1/r=((1-t)/r_1)+(t/r_2)$, $1/p=((1-t)/p_1)+(t/p_2)$, and $1/q=((1-t)/q_1)+(t/q_2)$.

A standard linearization argument shows that this result can also be applied to maximal functions. Our result [7] deals with S_*^0 acting on some L^p spaces but we need control on S_*^{iy} in order to use complex interpolation. Use integration by parts:

$$\int_0^R F(s)\left(1-\frac{s}{R}\right)^\alpha ds = \frac{\alpha}{R}\int_0^R \left(\int_0^s F(t)dt\right)\left(1-\frac{s}{R}\right)^{\alpha-1} ds$$

and

$$\left|\int_0^R F(s)\left(1-\frac{s}{R}\right)^\alpha ds\right| \leq \frac{|\alpha|}{R}\int_0^R \left|\int_0^s F(t)dt\right|\left(1-\frac{s}{R}\right)^{\Re(\alpha)-1} ds.$$

It then follows that if we have some control on S_*^0 then we also get control on S_*^z on a vertical line $\Re(z)=\varepsilon>0$.

LEMMA 7.4. *If $\varepsilon>0$ and $\Re(z)=\varepsilon$ then for all $f\in S(K\backslash G/K)$,*

$$S_*^z f(x) \leq |z| C_\varepsilon S_*^0 f(x), \qquad \text{for all} \quad x \in G.$$

8. Main results.

THEOREM 8.1. *Let* $0<\alpha<(d-1)/2$ *and*

$$\frac{2d}{d+2\alpha+1}<p\leq 2\,.$$

Then for every $f\in L^p(K\backslash G/K)$

$$\lim_{R\to\infty} S_R^\alpha f(x)=f(x)\,,\quad a.e.$$

PROOF. When $2d/(d+2\alpha+1)<p$ and $0<\alpha<(d-1)/2$, there are $\delta>0$ and $\varepsilon>0$ so that

$$\varepsilon<\alpha<\frac{d-1}{2}-\varepsilon \quad\text{and}\quad p=\frac{2d+\delta}{d+2\alpha-2\varepsilon+1}\,. \tag{5}$$

Let $t=1-(2\alpha-2\varepsilon)/(d-1)$, so that $0<t<1$. Taking reciprocals in the equation (5) and using p and t, it becomes

$$\frac{1}{p}=\frac{2d}{2d+\delta}+\frac{t}{2d+\delta}-\frac{dt}{2d+\delta}=\frac{(1-t)2d}{2d+\delta}+t\left(\frac{1+d}{2d+\delta}\right).$$

Write $p_1=(2d+\delta)/(2d)>1$ and $p_2=(2d+\delta)/(1+d)>2d/(1+d)$. To take advantage of the complex interpolation theorem, define an analytic family of operators

$$T_R^s=S_R^{\varepsilon+(1-s)(d-1)/2}\,,\qquad \text{for}\quad R>0\,,\quad s\in\boldsymbol{S}\,.$$

When $\Re(s)=0$ this is the family of Bochner-Riesz mean operators with index $\varepsilon+(d-1)/2+iy$, while for $\Re(s)=1$ the operators $S_R^{\varepsilon+iy}$ are estimated in terms of the partial sum operators by Lemma 7.4 are the partial sum operators $S_R^{\varepsilon+iy}$. Let T_*^s denote the corresponding maximal operator.

Giulini and Mauceri's result states that for $\Re(s)=0$ the maximal operator T_*^s maps $L^{p_1}(K\backslash G/K)$ into $L^{p_1}(K\backslash G/K)+L^{q_1}(K\backslash G/K)$ for some $q_1>2$. Our result shows that if $\Re(s)=0$ then T_*^s maps $L^{p_2}(K\backslash G/K)$ into $L^{p_2}(K\backslash G/K)+L^2(K\backslash G/K)$. Now take $s=t$ and

$$\frac{1}{q}=\frac{1-t}{q_1}+\frac{t}{2}\,.$$

Notice that $\varepsilon+(1-t)(d-1)/2=\alpha$. A standard linearization argument and the interpolation theorem proved above shows that the maximal function $T_*^t=S_*^{\varepsilon+(1-t)/(d-1)/2}=S_*^\alpha$ is bounded from $L^p(K\backslash G/K)$ into $L^p(K\backslash G/K)+L^q(K\backslash G/K)$. We have shown that if $0<\alpha<(d-1)/2$ and $2d/(d+2\alpha+1)<p\leq 2$ then the maximal function S_*^α is bounded from $L^p(K\backslash G/K)$ into $L^p(K\backslash G/K)+L^q(K\backslash G/K)$, for some $q\geq 2$ depending on p and α. The almost everywhere convergence statement then follows from this boundedness

of the maximal function. □

This was proved by Kanjin [4] in the case of Bochner-Riesz means of radial functions on Euclidean space. We will follow Kanjin's technique to show that this range of indices is sharp.

9. Sharpness. As in Kanjin's paper [4], this involves several steps. First use a result of Hardy and Riesz to show that convergence of $S_R^\alpha f$ implies some growth condition on $S_R^0 f$. Then use a Cantor-Lebesgue argument to show that the growth condition on $S_R^0 f$ implies the boundedness of some functionals acting on $L^p(K\backslash G/K)$. Finally, find lower bounds on the norms of these functionals and use a uniform boundedness argument to disprove the growth conditions on $S_R^0 f$.

Hardy and Riesz (see [3, p. 38]) showed that if $\alpha>0$ and $S_R^\alpha f(x)$ converges then

$$\int_0^R \hat{f}(\lambda)\varphi_\lambda(x)|c(\lambda)|^{-2}d\lambda = O(R^\alpha) \qquad \text{as} \quad R\to\infty .$$

As in Section 5 of [7] let $L_\alpha^s(a, b)$ denote the weighted Lebesgue space

$$L_\gamma^s(a, b)=\left\{g : \|g\|_{\gamma,s}=\left(\int_a^b |g(t)|^s t^{2\gamma+1}dt\right)^{1/s}<\infty\right\},$$

where $\gamma\geq -1/2$, $1\leq s<\infty$, and $0\leq a<b\leq\infty$.

The Hausdorff-Young theorem states that the spherical transform maps $L^p(K\backslash G/K)$ into $L^{p'}([0, \infty), |c(\lambda)|^{-2}d\lambda)$ for each $1\leq p\leq 2$. The density of the Plancherel measure satisfies

$$|c(\lambda)|^{-2}=O(\lambda^{d-1}) \qquad \text{as} \quad \lambda\to\infty .$$

We then have that $f\in L^p(K\backslash G/K)$ implies that the restriction

$$\hat{f}|_{[1,\infty)}\in L_\gamma^{p'}(1, \infty) \qquad \text{with} \quad \gamma=\frac{d-2}{2} . \tag{6}$$

For small t and large λ the spherical functions behave like Bessel functions. Theorem 2.1 in [8] shows that for $\lambda>1/\varepsilon$,

$$\varphi_\lambda(t)=c_0\left(\frac{t^{d-1}}{D(t)}\right)^{1/2}\left(\frac{J_{(d-2)/2}(t\lambda)}{(t\lambda)^{(d-2)/2}}+t^2a_1(t)\frac{J_{d/2}(t\lambda)}{(t\lambda)^{d/2}}\right)+E_2(\lambda, t)$$

where

$$E_2(\lambda, t)=O(t^{2-(d-1)/2}\lambda^{-(d+3)/2}) \qquad \text{for} \quad t\lambda>1 .$$

For $R>0$ and $h>0$ we then have

$$\int_R^{R+h}\hat{f}(\lambda)\varphi_\lambda(t)|c(\lambda)|^{-2}d\lambda-c_0\left(\frac{t^{d-1}}{D(t)}\right)^{1/2}\int_R^{R+h}\hat{f}(\lambda)\left(\frac{|c(\lambda)|^{-2}}{\lambda^{d-1}}\right)\frac{J_{(d-2)/2}(t\lambda)}{(t\lambda)^{(d-2)/2}}\lambda^{d-1}d\lambda$$

$$=c_0\left(\frac{t^{d-1}}{D(t)}\right)^{1/2}t^2a_1(t)\int_R^{R+h}\hat{f}(\lambda)\left(\frac{|c(\lambda)|^{-2}}{\lambda^{d-1}}\right)\frac{J_{d/2}(t\lambda)}{(t\lambda)^{d/2}}\lambda^{d-1}d\lambda$$

$$+\int_R^{R+h}\hat{f}(\lambda)\left(\frac{|c(\lambda)|^{-2}}{\lambda^{d-1}}\right)E_2(\lambda,t)\lambda^{d-1}d\lambda\,.$$

For large values of $x>0$ the Bessel functions satisfy $J_\mu(x)=O(1/\sqrt{x})$. This means that

$$\left|\int_R^{R+h}\hat{f}(\lambda)\left(\frac{|c(\lambda)|^{-2}}{\lambda^{d-1}}\right)\frac{J_{d/2}(t\lambda)}{(t\lambda)^{d/2}}\lambda^{d-1}d\lambda\right| \tag{7}$$

$$\le ct^{-(d-1)/2}\int_R^{R+h}|\hat{f}(\lambda)|\lambda^{-(d+1)/2}\lambda^{d-1}d\lambda \tag{8}$$

$$\le ct^{-(d-1)/2}\left(\int_R^{R+h}|\hat{f}(\lambda)|^{p'}\lambda^{d-1}d\lambda\right)^{1/p'}\left(\int_R^{R+h}\lambda^{-(d+1)p/2}\lambda^{d-1}d\lambda\right)^{1/p}.$$

For $1\le p\le 2$ the exponents in the last integral satisfy

$$-2\le d-1-\frac{p(d+1)}{2}\le\frac{d-3}{2}\,.$$

We are assuming that $0<h\le 1$ and the second mean-value theorem shows that

$$\left(\int_R^{R+h}\lambda^{-(d+1)p/2}\lambda^{d-1}d\lambda\right)^{1/p}=O(R^{(d-1)/p-(d+1)/2})\qquad\text{as}\quad R\to\infty\,.$$

The integral (7) will be $o(1)$ as $R\to\infty$ provided $p\ge(2d-2)/(d+1)$.

Similarly,

$$\left|\int_R^{R+h}\hat{f}(\lambda)\left(\frac{|c(\lambda)|^{-2}}{\lambda^{d-1}}\right)E_2(\lambda,t)\lambda^{d-1}d\lambda\right|$$

$$\le ct^{2-(d-1)/2}\left(\int_R^{R+h}|\hat{f}(\lambda)|^{p'}\lambda^{d-1}d\lambda\right)^{1/p'}\left(\int_R^{R+h}\lambda^{-(d+3)p/2}\lambda^{d-1}d\lambda\right)^{1/p}$$

$$\le ct^{2-(d-1)/2}\cdot o(R^{(d-1)/p-(d+3)/2})$$

and this will be $o(1)$ as $R\to\infty$ provided $p\ge(2d-2)/(d+3)$. Combining these estimates, we find that if t is in a compact subset of $(0,\infty)$ then

$$\int_R^{R+h}\hat{f}(\lambda)\varphi_\lambda(t)|c(\lambda)|^{-2}d\lambda-c_0\left(\frac{t^{d-1}}{D(t)}\right)^{1/2}\int_R^{R+h}\hat{f}(\lambda)\left(\frac{|c(\lambda)|^{-2}}{\lambda^{d-1}}\right)\frac{J_{(d-2)/2}(t\lambda)}{(t\lambda)^{(d-2)/2}}\lambda^{d-1}d\lambda$$

$$=o(R^{(d-1)/p-(d+1)/2})\qquad\text{as}\quad R\to\infty\,,$$

uniformly in t and $0<h\leq 1$.

We wish to examine the behavior of Bochner-Riesz means below the critical index

$$1<p<\frac{2d}{d+2\alpha+1}, \qquad \text{where} \quad 0<\alpha<\frac{d}{p}-\frac{(d+1)}{2}.$$

From Hardy and Riesz, if $S_R^\alpha f(a_t)$ converges uniformly on a set E of positive measure in $(0, \infty)$ then

$$\int_0^R \hat{f}(\lambda)\varphi_\lambda(t)|c(\lambda)|^{-2}d\lambda = O(R^\alpha)$$

uniformly in $t\in E$. When

$$\frac{(d-1)}{p}-\frac{(d+1)}{2}<\alpha<\frac{d}{p}-\frac{(d+1)}{2}$$

our calculations above show that

$$\int_R^{R+h} \hat{f}(\lambda)\left(\frac{|c(\lambda)|^{-2}}{\lambda^{d-1}}\right)\frac{J_{(d-2)/2}(t\lambda)}{(t\lambda)^{(d-2)/2}}\lambda^{d-1}d\lambda = O(R^\alpha)$$

uniformly in $t\in E$ and $0<h\leq 1$. Arguing as on pages 13 and 14 of [4] gives the following lemma.

LEMMA 9.1. *Suppose* $2(d-1)/(d+2\alpha+1)<p<2d/(d+2\alpha+1)$ *and* $0<\alpha<(d-1)/2$, *if a function* $f\in L^p(K\backslash G/K)$ *has* $S_R^\alpha f(a_t)$ *converging uniformly on a set* E *of positive measure in* $(0, \infty)$ *then there is a positive constant* C, *which depends on* f, *such that*

$$\limsup_{R\to\infty}\left|\int_R^{R+h}\hat{f}(\lambda)|c(\lambda)|^{-1}(|c(\lambda)|^{-1}\lambda^{-(d-1)/2})\lambda^{-\alpha}d\lambda\right|\leq C$$

uniformly in $0\leq h\leq 1$.

Next, we want to use a uniform boundedness argument to see that there are elements of $L^p(K\backslash G/K)$ for which this cannot happen. Let $\boldsymbol{E}_p$ be the subspace

$$\boldsymbol{E}_p=\{f\in L^p(K\backslash G/K): t\mapsto f(a_t) \text{ is supported in } [0, 1]\}.$$

For each $R>0$ consider the functional on $\boldsymbol{E}_p$ given by

$$\begin{aligned}\boldsymbol{F}_R(f)&=\int_R^{R+1}\hat{f}(\lambda)|c(\lambda)|^{-1}(|c(\lambda)|^{-1}\lambda^{-(d-1)/2})\lambda^{-\alpha}d\lambda\\ &=\int_R^{R+1}\int_0^1 f(a_t)|c(\lambda)|^{-1}(|c(\lambda)|^{-1}\lambda^{-(d-1)/2})\lambda^{-\alpha}\varphi_\lambda(t)D(t)dtd\lambda.\end{aligned}$$

To understand the norm of this functional, we need the $L^{p'}([0, 1], D(t)dt)$ norm of the function

$$t \mapsto \int_R^{R+1} |c(\lambda)|^{-1}(|c(\lambda)|^{-1}\lambda^{-(d-1)/2})\lambda^{-\alpha}\varphi_\lambda(t)d\lambda .$$

From Theorem 2.1 in Stanton and Tomas [8], we know that for $0 \le t \le 1$ and $t\lambda > 1$ the spherical function satisfies

$$\varphi_\lambda(t) = c_0\left(\frac{t^{n-1}}{D(t)}\right)^{1/2} \frac{J_{(d-2)/2}(t\lambda)}{(t\lambda)^{(d-2)/2}} + E_1(\lambda, t)$$

with $|E_1(\lambda, t)| \le c_1 t^2 (t\lambda)^{-(d+1)/2}$. For $2(d-1)/(d+2\alpha+1) < p < 2d/(d+2\alpha+1)$, its dual exponent is

$$\frac{2(d-1)}{d-2\alpha-3} > p' > \frac{2d}{d-1-2\alpha} .$$

For large R we estimate the norm of the error term as

$$\text{(9)} \qquad \int_{1/R}^1 \left| \int_R^{R+1} E_1(\lambda, t)|c(\lambda)|^{-1}(|c(\lambda)|^{-1}\lambda^{-(d-1)/2})\lambda^{-\alpha}d\lambda \right|^{p'} D(t)dt$$

$$\le c_1 \int_{1/R}^1 \left| \int_R^{R+1} \lambda^{-(d+1)/2+(d-1)/2-\alpha}d\lambda \right|^{p'} t^{p'(2-(d+1)/2)}t^{d-1}dt$$

$$\le c_1 \int_{1/R}^1 \left| \int_R^{R+1} \lambda^{-1-\alpha}d\lambda \right|^{p'} t^{d-1+p'(3-d)/2}dt .$$

The exponent of t will be greater than -1 when $\alpha < 1$, is -1 when $\alpha = 1$, and is less than -1 when $\alpha > 1$. Hence, the integral in (9) is less than or equal to

$$\begin{cases} cR^{-p'(1+\alpha)} & \text{when } \alpha < 1 \\ cR^{-2p'}\log R & \text{when } \alpha = 1 \\ cR^{-p'(1+\alpha)-d+p'(d-3)/2} & \text{when } \alpha > 1 . \end{cases}$$

In the last term,

$$-p'(1+\alpha) - d + p'(3-d)/2 = p'\left(\frac{1-d-2\alpha}{2}\right) - d .$$

In all cases, the integral (9) tends to zero as $R \to \infty$.

For the main part we follow the proof of Lemma 1 in Kanjin's paper [4], that is, the sequence

$$\left(\int_0^1 \left| \int_k^{k+1} \frac{J_{(d-2)/2}(t\lambda)}{(t\lambda)^{(d-2)/2}} \lambda^{(d-1)/2-\alpha} d\lambda \right|^{p'} D(t)dt \right)^{1/p'} \qquad \text{for all integers } k \ge 1 ,$$

is unbounded. The factor $|c(\lambda)|^{-1}\lambda^{-(d-1)/2}$ in the integrals above is handled by noting that there is a polynomial $P(\lambda)$ of degree $d-1$ such that

$$|c(\lambda)|^{-2}=P(\lambda)\cdot\begin{cases}1 & \text{when } m_\alpha \text{ is even and } m_{2\alpha}=0\,,\\ \coth(\pi\lambda/2) & \text{when } m_\alpha\equiv 2 \bmod 4 \text{ and } m_{2\alpha} \text{ is odd}\,,\\ \tanh(\pi\lambda/2) & \text{in all other cases}\,.\end{cases}$$

This is stated in [7, Lemma 3] and follows from [8, p. 266].

COROLLARY 9.2. *For* $2(d-1)/(d+1)\le p\le 2d/(d+1)$ *and* $0\le\alpha<(d/p)-((d+1)/2)$ *there is an element* $f\in L^p(K\backslash G/K)$ *with* $\{S_R^\alpha f(x):\ R>0\}$ *divergent on a set of positive measure.*

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DEPARTMENT OF MATHEMATICS
MACQUARIE UNIVERSITY
NEW SOUTH WALES 2109
AUSTRALIA

E-mail address: chrism@macadam.mpce.mq.edu.au

DIPARTIMENTO DI MATEMATICA
IL UNIVERSITA DEGLI STUDI DI ROMA
VIA FONTANILE DI CARCARICOLA
00133 TOR VERGATA
ROMA
ITALY

E-mail address: prestini@utovrm.it