

BOHR-ROGOSINSKI TYPE INEQUALITIES FOR CONCAVE UNIVALENT FUNCTIONS

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ABSTRACT. In this paper, we generalize and investigate the Bohr-Rogosinski's inequalities and Bohr-Rogosinski phenomenon for the subfamilies of univalent (i.e., one-to-one) functions defined on unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ which maps to the concave domain, i.e., the domain whose complement is a convex set. All the results are proved to be sharp.

1. Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in \mathbb{C} and $H(\mathbb{D})$ denote the family of analytic functions on \mathbb{D} . Let \mathcal{S} be the class of all univalent functions $f \in H(\mathbb{D})$ satisfying the normalization $f(0) = 0 = f'(0) - 1$. Let \mathcal{B} be the subclass of $H(\mathbb{D})$ consisting of functions that are bounded by 1. In 1914, Harald Bohr [21] proved the following remarkable result.

Theorem A. *Let $f \in \mathcal{B}$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then*

$$(1.1) \quad \sum_{n=1}^{\infty} |a_n| r^n \leq 1 - |f(0)|$$

for $|z| = r \leq 1/3$. The number $1/3$ cannot be improved.

This constant $1/3$ and the inequality (1.1) is known as *Bohr radius* and *classical Bohr inequality*, respectively for the family \mathcal{B} . Bohr originally obtained the inequality (1.1) for $r \leq 1/6$, which was improved to $r \leq 1/3$ by Wiener, Riesz, and Schur, independently. It is worth pointing out that if $|f(0)|$ in the classical Bohr inequality is replaced by $|f(0)|^2$, then the constant $1/3$ could be replaced by $1/2$ proved by Paulsen et al. [43].

There are lots of works about the classical Bohr inequality and its generalized forms in the recent years. For example, the notion of the Bohr radius was generalized by Abu-Muhanna and Ali [1, 2] to include mappings from \mathbb{D} to simply connected domain and to exterior of a unit disk in \mathbb{C} . Moreover, the Bohr phenomenon for shifted disks and simply connected domains are discussed in [7, 23, 24, 34]. Allu and Halder [9], and Bhowmik and Das [17] have considered the Bohr phenomenon for the class of subordinations. In [32], Kayumov et al. have studied the Bohr radius for locally univalent planar harmonic mappings. Kayumov and Ponnusamy have [29, 30] obtained several different improved version of the classical Bohr inequality which are sharp. Bohr-type inequalities for certain integral operators have been obtained by Kayumov et al. [28], and Kumar and Sahoo [33]. For the recent study on the Bohr radius, we refer to [3, 4, 8, 18, 31, 39–41] and the references therein. The recent survey article [5] and references therein may be good sources for this topic.

Similar to the Bohr radius, there is a concept of the Rogosinski radius [44] which is defined as follows: if $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}$ then $|S_M(z)| = |\sum_{n=0}^{M-1} a_n z^n| < 1$ for $|z| < 1/2$, where

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$1/2$ is the best possible quantity (see also [36,45]). The number $r = 1/2$ is called the *Rogosinski Radius* for the family \mathcal{B} . The *Bohr-Rogosinski inequality*, which is considered by Kayumov *et al.* in [27], is given by

$$(1.2) \quad |f(z)| + \sum_{n=N}^{\infty} |a_n| r^n \leq 1 = |f(0)| + d(f(0), \partial f(\mathbb{D})),$$

for $|z| = r \leq R_N$ where R_N is the positive root of the equation $2(1+r)r^N - (1-r)^2 = 0$. Here $d(f(0), \partial f(\mathbb{D}))$ denotes the Euclidean distance from $f(0)$ to the boundary of domain $f(\mathbb{D})$. Note that the right hand side of the inequality (1.2) can be rewritten as

$$1 = 1 - |f(0)| + |f(0)| = |f(0)| + d(f(0), \partial f(\mathbb{D})).$$

To generalize the classical Bohr inequality (1.1) and Bohr-Rogosinski inequality (1.2) to the family of function f defined in \mathbb{D} and takes values in any given domain $f(\mathbb{D})$, the above observation is useful. If we replace $|f(z)|$ by $|f(0)|$ and $N = 1$ in (1.2), then we see the connection between the Bohr-Rogosinski inequality and classical Bohr inequality. Recently, a generalization of the Bohr-Rogosinski inequality (1.2) has been studied by Kumar and Sahoo [35], and Liu *et al.* [37].

2. Preliminaries

To state our main results, we need some preparation. We set

$$\mathcal{B}_0 = \{w \in \mathcal{B} : w(0) = 0\} = \cup_{n=1}^{\infty} \mathcal{B}_n,$$

where

$$\mathcal{B}_n = \{w \in \mathcal{B} : w(0) = \dots = w^{(n-1)}(0) = 0 \text{ and } w^{(n)}(0) \neq 0\} \text{ for } n \in \mathbb{N}.$$

The members of the class \mathcal{B}_0 are called the *Schwarz functions*.

2.1. Bohr phenomenon and Bohr-Rogosinski phenomenon.

For any two analytic functions f and g in the unit disk \mathbb{D} , we say that the function g is *subordinate* to f , denoted by $g \prec f$ in \mathbb{D} , if there exist an $w \in \mathcal{B}$ with $w(0) = 0$ and $g(z) = f(w(z))$ for $z \in \mathbb{D}$. Moreover, it is well known that if f is univalent in \mathbb{D} , then $g \prec f$ if, and only if, $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. By the Schwarz lemma, it follows that

$$|g'(0)| = |f'(w(0))w'(0)| \leq |f'(0)|.$$

The concept of Bohr phenomenon for the family of functions which are defined by subordination was introduced by Abu-Muhanna [1]. Now for a given analytic function f from \mathbb{D} onto $f(\mathbb{D})$ with the expansion

$$(2.1) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

let $S(f) = \{g : g \prec f\}$. We say that the family $S(f)$ has a *Bohr phenomenon* if there exists an r_f , $0 < r_f \leq 1$, such that whenever

$$(2.2) \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f),$$

then

$$(2.3) \quad \sum_{n=1}^{\infty} |b_n| r^n \leq d(f(0), \partial f(\mathbb{D})) \text{ for } |z| = r \leq r_f.$$

We observe that if $f(z) = (a_0 - z)/(1 - \overline{a_0}z)$ with $|a_0| < 1$, then $f(\mathbb{D}) = \mathbb{D}$, $S(f) = \mathcal{B}$, and $d(f(0), \partial f(\mathbb{D})) = 1 - |f(0)| = 1 - |a_0|$ so that (2.3) holds with $r_f = 1/3$ in view of Theorem A. For univalent functions f , Abu-Muhanna [1] showed that $S(f)$ has a Bohr phenomenon and the Bohr radius is $3 - 2\sqrt{2} \approx 0.17157$.

Similar to Bohr's inequality, the Bohr-Rogosinski inequality can be generalized to the family of analytic functions f in \mathbb{D} which take values in a given domain $f(\mathbb{D})$. We say that the family $S(f)$ has *Bohr-Rogosinski phenomenon* if there exists $r_N^f \in (0, 1]$ such that for any $g \in S(f)$ the inequality:

$$(2.4) \quad |g(z)| + \sum_{n=N}^{\infty} |b_n| r^n \leq |f(0)| + d(f(0), \partial f(\mathbb{D}))$$

holds for $|z| = r \leq r_N^f$. The largest such r_N^f is called the Bohr-Rogosinski radius.

2.2. Bohr-Rogosinski type inequalities involving Schwarz functions.

The Bohr-Rogosinski inequality (1.2) was also generalized by replacing the Taylor coefficients a_n partly or completely by higher order derivative in [35, 42]. In this sequence, Liu [38] has generalized several Bohr-Rogosinski's inequalities by replacing the Taylor coefficient a_n of f by $f^{(n)}(w_n(z))/n!$ and r^m by $|w_m(z)|$ in part or in whole, where both w_n and w_m are some Schwarz functions. In particular, for $N \in \mathbb{N}$, $w_i \in \mathcal{B}_{m_i}$, $m_i \in \mathbb{N}$ ($i = 0, 1, 2$), and $w_n^* \in \mathcal{B}_{h(n)}$, Liu [38] has proved the following Bohr-Rogosinski type inequality:

$$(2.5) \quad |f(w_0(z))| + |f'(w_1(z))| |w_2(z)| + \sum_{n=N}^{\infty} |a_n| |w_n^*(z)| \leq |f(0)| + d(f(0), \partial f(\mathbb{D}))$$

for certain subclasses of analytic functions. In context of the above problem, Liu [38] has also generalized the notion of Bohr-Rogosinski's phenomenon (2.4) in terms of the Schwarz functions. Let $f, g \in H(\mathbb{D})$ be of the form (2.1) and (2.2), respectively. Then the family $S(f)$ has a Bohr-Rogosinski's phenomenon in terms of the Schwarz functions if there exists an r_{f, m_0}^N , $0 < r_{f, m_0}^N \leq 1$, such that whenever $g \prec f$, we have

$$(2.6) \quad |g(w_0(z))| + \sum_{n=N}^{\infty} |b_n| r^n \leq |f(0)| + d(f(0), \partial f(\mathbb{D}))$$

for $w_0 \in \mathcal{B}_{m_0}$, $m_0 \in \mathbb{N}$ and $|z| = r < r_{f, m_0}^N$. Note that, the Bohr-Rogosinski inequality can be deduced from the Bohr-Rogosinski type inequality by choosing proper combination of the Schwarz functions. More precisely, if we put $w_0(z) = z$, $w_n^*(z) = z^n$, and let $m_2 \rightarrow \infty$ in (2.5), then it reduces to (1.2). If we choose $w_0(z) = z$ in (2.6) then it reduces to (2.4). To find the recent developments in this context, we refer to [25, 27, 39].

One of our main concern in this article is to deal with Bohr-Rogosinski's type inequalities of the form (2.5) and (2.6) for concave-wedge domains. For the sake of simplification, we have used the following assumptions throughout this paper:

- I. $w_i \in \mathcal{B}_{m_i}$ for $m_i \in \mathbb{N}$ ($i = 0, 1, 2$).
- II. $w_n^* \in \mathcal{B}_{h(n)}$, for $n \in \mathbb{N}$, where $h(n)$ is some function of n .

The outline of the paper is as follows. In Section 3, we generalize Bohr-Rogosinski's inequality using subordination and the Schwarz functions when image domain is a concave-wedge domain. Furthermore, we shall state our main results and some of its consequences. In the same spirit, in Section 4, we consider generalized Bohr-Rogosinski's phenomenon in terms of

the Schwarz functions for the family of meromorphic univalent functions which map open unit disk \mathbb{D} into some concave domain. The section 5 contains the proof of our main results.

The following lemma, given by Gangania and Kumar [25], is needed in order to prove our result.

Lemma 2.1. *Let $f, g \in H(\mathbb{D})$ with the Taylor expansion (2.1) and (2.2) respectively. If $g \prec f$, then*

$$\sum_{n=N}^{\infty} |b_n| r^n \leq \sum_{n=N}^{\infty} |a_n| r^n, \quad N \in \mathbb{N}$$

holds for $|z| = r \leq 1/3$.

The case $N = 1$ of Lemma 2.1 has been proved by Bhowmik and Das [17, Lemma 1].

3. The family of concave univalent function with opening angle $\pi\alpha$

For our further discussions we need to introduce the following family of univalent functions: A function $f : \mathbb{D} \rightarrow \mathbb{C}$ is said to belong to the family of *concave univalent functions with opening angle $\pi\alpha$* , $\alpha \in [1, 2]$, at infinity if f satisfies the following conditions:

- (a) $f \in H(\mathbb{D})$ is univalent and $f(1) = \infty$.
- (b) f maps \mathbb{D} conformally onto a set whose complement with respect to \mathbb{C} is convex.
- (c) The opening angle of $f(\mathbb{D})$ at ∞ is less than or equal to $\pi\alpha$, $\alpha \in [1, 2]$.

We denote this family of functions by $\widehat{C}_0(\alpha)$. For such functions, the boundary of $f(\mathbb{D})$ is contained in a wedge shaped region with opening angle $\pi\alpha$ but not in any bigger opening angle. It may be noted that for $f \in \widehat{C}_0(\alpha)$, $\alpha \in [1, 2]$, the closed set $\mathbb{C} \setminus f(\mathbb{D})$ is convex and unbounded. Also, we observe that $\widehat{C}_0(2)$ contains the classes $\widehat{C}_0(\alpha)$, $\alpha \in [1, 2]$ (see [12]). When $\alpha = 1$, the image domain reduces to a convex half plane. Hence we can see that concave univalent functions are related to the convex functions and every $f \in \widehat{C}_0(1)$ is the convex function. The case $\alpha = 2$ yields a slit domain. The family of normalized concave univalent function are denoted by $C_0(\alpha)$, i.e., $C_0(\alpha) := \widehat{C}_0(\alpha) \cap \mathcal{S}$. In 2005, Avkhadiev and Wirths [12] characterized functions in the class $C_0(\alpha)$. For this class Fekete-Szegő problem has been solved by Bhowmik et al. [20]. Results related to Yamashita conjecture on Dirichlet finite integral for $C_0(\alpha)$ has been discussed by Abu-Muhanna and Ponnusamy [6]. For a detailed discussion about concave functions, we refer to [11, 12, 16, 17, 22] and the references therein.

The following lemma is actually contained in the proof of [17, Theorem 1] (see also [12, 16]) as well and thus we are omitting the proof.

Lemma 3.1. *Let $f \in \widehat{C}_0(\alpha)$, $\alpha \in [1, 2]$, have the expansion (2.1). Then we have the following inequalities:*

- (i) $|f'(0)| \leq 2\alpha d(f(0), \partial f(\mathbb{D}))$.
- (ii) $|a_n| \leq A_n |f'(0)|$, for $n \geq 1$.
- (iii) $|f(z) - f(0)| \leq |f'(0)| f_\alpha(r)$, where $f_\alpha(z)$ is define by

$$(3.1) \quad f_\alpha(z) := \frac{1}{2\alpha} \left[\left(\frac{1+z}{1-z} \right)^\alpha - 1 \right] = \sum_{n=1}^{\infty} A_n z^n.$$

All inequalities are sharp for the function $f_\alpha(z)$.

The following is our first main result, which estimate the Bohr-Rogosinski inequality (2.6) for the family $\widehat{C}_0(\alpha)$.

Theorem 3.2. *Let $f, g \in H(\mathbb{D})$, with the Taylor expansion (2.1) and (2.2) respectively, such that $f \in \widehat{C}_0(\alpha)$, $\alpha \in [1, 2]$ and $g \in S(f)$. Then, for each $N \in \mathbb{N}$, the inequality*

$$|g(w_0(z))| + \sum_{n=N}^{\infty} |b_n| r^n \leq |f(0)| + d(f(0), \partial f(\mathbb{D}))$$

holds for $|z| = r < \min\{r_{\alpha, m_0}^N, 1/3\}$, where r_{α, m_0}^N is the positive root of the equation $F_{\alpha, m_0}^N(x) = 0$,

$$(3.2) \quad F_{\alpha, m_0}^N(x) := \sum_{n=N}^{\infty} A_n x^n + f_{\alpha}(x^{m_0}) - \frac{1}{2\alpha},$$

in $(0, 1)$. If $r_{\alpha, m_0}^N \leq 1/3$, the radius r_{α, m_0}^N is sharp for the function f_{α} defined in (3.1).

Remark 3.3. Choosing $m_0 \rightarrow \infty$ and $N = 1$ in Theorem 3.2, we obtain the Bohr-Rogosinski radius

$$r_{\alpha, \infty}^1 = \frac{2^{1/\alpha} - 1}{2^{1/\alpha} + 1}$$

and hence Theorem 3.2 coincides with [17, Theorem 1]. Moreover, for $\alpha = 1$, it readily follows that the Bohr radius for the class $\widehat{C}_0(1)$, which is the class of convex functions, is $1/3$ and for $\alpha = 2$ the Bohr radius for the class $\widehat{C}_0(2)$ is $3 - 2\sqrt{2}$.

Since $\widehat{C}_0(1)$ contains in the family of univalent convex functions, the following corollary, for $\alpha = 1$, covers the particular case of [38, Corollary 1].

Corollary 3.4. *Let $f, g \in H(\mathbb{D})$, with the Taylor expansion (2.1) and (2.2) respectively, such that $f \in \widehat{C}_0(1)$ and $g \in S(f)$. Then, for each $N \in \mathbb{N}$, the inequality*

$$|g(w_0(z))| + \sum_{n=N}^{\infty} |b_n| r^n \leq |f(0)| + d(f(0), \partial f(\mathbb{D}))$$

holds for $|z| = r < \min\{r_{1, m_0}^N, 1/3\}$, where r_{1, m_0}^N is the positive root of the equation

$$F_{1, m_0}^N(x) = \sum_{n=N}^{\infty} x^n + \frac{r^{m_0}}{1 - r^{m_0}} - \frac{1}{2} = 0,$$

in $(0, 1)$. If $r_{1, m_0}^N \leq 1/3$, the radius r_{1, m_0}^N is sharp for the function f_1 defined in (3.1).

We observe that the case $w_0(z) = z$ in Theorem 3.2 gives the following corollary.

Corollary 3.5. *Let $f, g \in H(\mathbb{D})$, with the Taylor expansion (2.1) and (2.2) respectively, such that $f \in \widehat{C}_0(\alpha)$, $\alpha \in [1, 2]$, $g \in S(f)$. Then for each $N \in \mathbb{N}$ the inequality*

$$|g(z)| + \sum_{n=N}^{\infty} |b_n| r^n \leq |f(0)| + d(f(0), \partial f(\mathbb{D}))$$

holds for $|z| = r < \min\{r_{\alpha, 1}^N, 1/3\}$, where $r_{\alpha, 1}^N$ is the positive root of the equation $F_{\alpha, 1}^N = 0$ is defined in (3.2). If $r_{\alpha, 1}^N \leq 1/3$, the radius $r_{\alpha, 1}^N$ is sharp for the function f_{α} given by (3.1).

Theorem 3.6. *Let $f \in \widehat{C}_0(\alpha)$, $\alpha \in [1, 2]$, with the Taylor expansion (2.1). Then for each $N \in \mathbb{N}$, we have*

$$|f(w_0(z))| + |f'(w_1(z))||w_2(z)| + \sum_{n=N}^{\infty} |a_n||w_n^*(z)| \leq |f(0)| + d(f(0), \partial f(\mathbb{D}))$$

for $|z| = r \leq r_{\alpha, m_0, m_1, m_2}^N$, where $r_{\alpha, m_0, m_1, m_2}^N \in (0, 1)$ is the unique positive root of the equation $K_{\alpha, m_0, m_1, m_2}^N(x) = 0$,

$$(3.3) \quad K_{\alpha, m_0, m_1, m_2}^N(x) := \sum_{n=N}^{\infty} A_n x^{h(n)} + f_{\alpha}(x^{m_0}) + x^{m_2} \frac{(1+x^{m_1})^{\alpha-1}}{(1-x^{m_1})^{\alpha+1}} - \frac{1}{2\alpha}$$

and A_n is given in (3.1). The radius $r_{\alpha, m_0, m_1, m_2}^N$ cannot be improved.

If we allow m_2 tends to infinity in Theorem 3.6, then it leads to the following result.

Corollary 3.7. *For $\alpha \in [1, 2]$, let $f \in \widehat{C}_0(\alpha)$ with the Taylor expansion (2.1). Then for each $N \in \mathbb{N}$, we have*

$$|f(w_0(z))| + \sum_{n=N}^{\infty} |a_n||w_n^*(z)| \leq |f(0)| + d(f(0), \partial f(\mathbb{D}))$$

for $|z| = r \leq r_{\alpha, m_0, m_1, \infty}^N$, where $r_{\alpha, m_0, m_1, \infty}^N \in (0, 1)$ is the unique positive root of the equation $K_{\alpha, m_0, m_1, \infty}^N(x) = 0$ defined in (3.3). The radius $r_{\alpha, m_0, m_1, \infty}^N$ cannot be improved.

Furthermore, the following result can be derived from Corollary 3.7 by letting $m_0, m_2 \rightarrow \infty$.

Corollary 3.8. *Let $f \in \widehat{C}_0(\alpha)$, $\alpha \in [1, 2]$, with the Taylor expansion (2.1). Then for each $N \in \mathbb{D}$, we have*

$$\sum_{n=N}^{\infty} |a_n||w_n^*(z)| \leq d(f(0), \partial f(\mathbb{D})), \quad |z| = r \leq r_{\alpha, \infty, m_1, \infty}^N,$$

where $r_{\alpha, \infty, m_1, \infty}^N$ is the unique positive root of the equation $K_{\alpha, \infty, m_1, \infty}^N(x) = 0$ defined in (3.3). The radius $r_{\alpha, \infty, m_1, \infty}^N$ cannot be improved.

4. The family of concave univalent functions with pole p

Motivated by the work of Bhowmik and Das [17], we generalize Bohr-Rogosinski's phenomenon for functions that are subordinate to a meromorphic function defined in the unit disk \mathbb{D} . Recall from [15], the definition of subordination for meromorphic functions is same as for analytic functions.

Analogous to the family of convex analytic functions, it is interesting to consider the family of meromorphic concave functions. Let $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ be the extended complex plane. We denote by \widehat{C}_p the class of meromorphic univalent functions $f : \mathbb{D} \rightarrow \widehat{\mathbb{C}}$ which satisfies the following conditions:

- (i) f is analytic in $\mathbb{D} \setminus \{p\}$ and $\widehat{\mathbb{C}} \setminus f(\mathbb{D})$ is convex domain.
- (ii) f has a simple pole at the point p .

By a suitable rotation, without loss of generality we can assume that $0 < p < 1$. Each function in \widehat{C}_p is called a *concave univalent function with a pole* $p \in (0, 1)$ and has a Taylor series expansion (2.1) in the disk $\mathbb{D}_p := \{z \in \mathbb{D} : |z| < p\}$. Let $C_p := \{f \in \widehat{C}_p : f(0) = f'(0) - 1 = 0\}$. In 2006, Wirths [47] established the representation formula for functions in C_p . Using this representation, the discussion based on the Laurent series expansion about the pole p was started by Bhowmik et al. [19] in 2007, where the authors obtained some coefficient estimates for the family \widehat{C}_p . Such functions are intensively studied by many authors, we refer to the papers [10, 13, 14, 17, 46] for a detailed discussion about this class.

Using the definition of Bohr-Rogosinski's phenomenon for analytic functions, we introduce the notion of Bohr-Rogosinski's phenomenon for the family $S(f)$, where f is meromorphic with pole p . We say that $S(f)$ has the Bohr-Rogosinski phenomenon if for any $g \in S(f)$ where f and g have Taylor expansion (2.1) and (2.2) in \mathbb{D}_p respectively, if there exists r_f^N , $0 < r_f^N \leq p$ such that the inequality (2.4) holds. Similar to the analytic case, this can be generalized to (2.5) in terms of the Schwarz function for further investigation.

In view of [14, Theorem 1] (see also [15, Theorem 8.4]) for normalized functions $f \in C_p$, we can easily obtain the following result.

Lemma 4.1. *Let $p \in (0, 1)$. If $f \in \widehat{C}_p$ and $g = \sum_{n=0}^{\infty} b_n z^n \prec f(z)$, then*

$$|b_n| \leq |f'(0)| \frac{1}{p^{n-1}} \sum_{k=0}^{n-1} p^{2k}, \quad n \in \mathbb{N}.$$

The inequality is sharp for

$$(4.1) \quad k_p(z) = \frac{pz}{(p-z)(1-pz)} = \sum_{n=1}^{\infty} \frac{1-p^{2n}}{(1-p^2)p^{n-1}} z^n = \sum_{n=1}^{\infty} c_n(p) z^n.$$

Proof. For $f \in \widehat{C}_p$, the function defined by $F(z) = (f(z) - f(0))/f'(0)$ belongs to the class C_p . Note that $f(0) = g(0)$ and $f \prec g$ if, and only if,

$$\frac{1}{f'(0)} \sum_{n=1}^{\infty} b_n z^n = \frac{g(z) - g(0)}{f'(0)} \prec \frac{f(z) - f(0)}{f'(0)} = F(z), \quad z \in \mathbb{D}.$$

Therefore, by [14, Theorem 1], we easily get

$$\frac{b_n}{f'(0)} \leq \frac{1}{p^{n-1}} \sum_{k=0}^{n-1} p^{2k}.$$

□

We now state our next result, which deals with the Bohr-Rogosinski phenomenon for the family \widehat{C}_p .

Theorem 4.2. *If $f \in \widehat{C}_p$, $p \in (0, 1)$, and $g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f)$. Then for each $N \in \mathbb{N}$ the inequality*

$$(4.2) \quad |g(w_0(z))| + \sum_{n=N}^{\infty} |b_n| z^n \leq |f(0)| + d(f(0), \partial f(\mathbb{D}))$$

holds for $|z| = r \leq r_{p,m_0}^N < p$, where r_{p,m_0}^N is the positive root of the equation $G_{p,m_0}^N(x) = 0$,

$$(4.3) \quad G_{p,m_0}^N(x) := \sum_{n=N}^{\infty} \frac{1-p^{2n}}{(1-p^2)p^{n-1}} x^n + k_p(x^{m_0}) - \frac{p}{(1+p)^2}.$$

The radius is sharp for the function $f = k_p$ given by (4.1).

Remark 4.3. We first observe that for $N = 1$ and $m_0 \rightarrow \infty$, we have

$$r_{p,m_0}^N = (p + 1/p + 1) - (\sqrt{p} + 1/\sqrt{p})\sqrt{p + 1/p}$$

as the root of the equation $pr^2 - 2(p^2 + 1 + p)r + p = 0$ and thus Theorem 4.2 contains the result of [17, Corollary 1] as a special case.

Furthermore, the substitution $w_0(z) = z$ bring Theorem 4.2 back into the following form.

Corollary 4.4. *If $f \in \widehat{C}_p$, $p \in (0, 1)$, with the Taylor series expansion (2.1) and $g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f)$. Then for each $N \in \mathbb{N}$, the inequality*

$$|g(z)| + \sum_{n=N}^{\infty} |b_n| z^n \leq |f(0)| + d(f(0), \partial f(\mathbb{D}))$$

holds for $|z| = r \leq r_{p,1}^N < p$, where $r_{p,1}^N$ is the positive root of the equation $G_{p,1}^N(r) = 0$ given by (4.3). The radius is sharp for the function $f(z) = k_p(z)$ as defined in (4.1).

5. Proof of the main results

In order to prove our main results, we frequently use a consequence of the Schwarz lemma which says that “if $w \in \mathcal{B}_m$ for some $m \in \mathbb{N}$, then $|w(z)| \leq |z|^m$ for all $z \in \mathbb{D}$.”

5.1. Proof of Theorem 3.2. Let $g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f)$, where $f \in \widehat{C}_0(\alpha)$, $\alpha \in [1, 2]$. Then the condition $g \prec f$ and the growth theorem [16, Corollary 2.4] lead to the fact that

$$|g(z) - g(0)| \leq |f'(0)| f_{\alpha}(r) \leq 2\alpha d(f(0), \partial f(\mathbb{D})) f_{\alpha}(r)$$

so that (because $f(0) = g(0)$) for $m_0 \in \mathbb{N}$, we have

$$(5.1) \quad |g(w_0(z))| \leq |f(0)| + 2\alpha d(f(0), \partial f(\mathbb{D})) f_{\alpha}(r^{m_0}).$$

By using Lemma 2.1, we obtain the following inequality

$$(5.2) \quad \sum_{n=N}^{\infty} |b_n| r^n \leq \sum_{n=N}^{\infty} |a_n| r^n = |f'(0)| \sum_{n=N}^{\infty} A_n r^n \leq 2\alpha d(f(0), \partial f(\mathbb{D})) \sum_{n=N}^{\infty} A_n r^n$$

for $|z| = r \leq 1/3$. Clearly from (5.1) and (5.2), we deduce that

$$\begin{aligned} |g(w_0(z))| + \sum_{n=N}^{\infty} |b_n| r^n &\leq |f(0)| + 2\alpha d(f(0), \partial f(\mathbb{D})) \left(\sum_{n=N}^{\infty} A_n r^n + f_{\alpha}(r^{m_0}) \right) \\ &= |f(0)| + 2\alpha d(f(0), \partial f(\mathbb{D})) \left(F_{\alpha, m_0}^N(r) + \frac{1}{2\alpha} \right) \\ &:= \Phi_{f, \alpha, m_0}^N(r), \end{aligned}$$

where $F_{\alpha, m_0}^N(r)$ is given by (3.2). Obviously, $F_{\alpha, m_0}^N(r)$ is an increasing function of r on $[0, 1)$. Moreover, $F_{\alpha, m_0}^N(0) < 0$ and $\lim_{r \rightarrow 1} F_{\alpha, m_0}^N(r) = +\infty$ and hence the equation $F_{\alpha, m_0}^N(r) = 0$ has the unique positive root r_{α, m_0}^N in $(0, 1)$. In order to complete the proof, it suffices to show that $\Phi_{f, \alpha, m_0}^N(r) \leq |f(0)| + d(f(0), \partial f(\mathbb{D}))$ holds for $|z| = r \leq \{r_{\alpha, m_0}^N, 1/3\}$.

If $r_{\alpha, m_0}^N \leq 1/3$, then we show that r_{α, m_0}^N cannot be improved. For $\alpha \in [1, 2]$, consider $g = f = f_\alpha$. Then, for $z = r$,

$$\begin{aligned} |f(w_0(r))| + \sum_{n=N}^{\infty} |a_n| r^n &= |f_\alpha(0)| + 2\alpha d(f_\alpha(0), \partial f_\alpha(\mathbb{D})) \left(f_\alpha(r^{m_0}) + \sum_{n=N}^{\infty} A_n r^n \right) \\ &> d(f_\alpha(0), \partial f_\alpha(\mathbb{D})) = \frac{1}{2\alpha} \end{aligned}$$

holds for $r > r_{\alpha, m_0}^N$ and hence the Bohr radius r_{α, m_0}^N is sharp. \square

5.2. Proof of Theorem 3.6. Let $f \in C_0(\alpha)$ and $w_0 \in \mathcal{B}_0$. Then it follows, for $|z| = r < 1$, from Lemma 3.1 and the classical the Schwarz lemma that

$$\begin{aligned} |f(w_0(z))| &\leq |f(0)| + |f'(0)| \sum_{n=1}^{\infty} A_n |w_0(z)|^n \\ (5.3) \quad &= |f(0)| + 2\alpha d(f(0), \partial f(\mathbb{D})) f_\alpha(r^{m_0}). \end{aligned}$$

On the other hand, the Distortion theorem [16, Corollary 2.3] leads to

$$\begin{aligned} |f'(w_1(z))| |w_2(z)| &\leq |f'(0)| \frac{(1 + |w_1(z)|)^{\alpha-1}}{(1 - |w_1(z)|)^{\alpha+1}} r^{m_2} \\ (5.4) \quad &\leq 2\alpha d(f(0), \partial f(\mathbb{D})) \frac{(1 + r^{m_1})^{\alpha-1}}{(1 - r^{m_1})^{\alpha+1}} r^{m_2}. \end{aligned}$$

The last inequality follows because $u(x) = (1+x)^{\alpha-1}/(1-x)^{\alpha+1}$ is an increasing function on $[0, 1]$. Also, from Lemma 3.1(ii) we can write

$$\begin{aligned} \sum_{n=N}^{\infty} |a_n| |w_n^*(z)| &\leq |f'(0)| \sum_{n=N}^{\infty} A_n |w_n^*(z)| \\ (5.5) \quad &\leq 2\alpha d(f(0), \partial f(\mathbb{D})) \sum_{n=N}^{\infty} A_n r^{b_n}. \end{aligned}$$

Thus combining the equations (5.3), (5.4), and (5.5) we obtain

$$\begin{aligned} |f(w_0(z))| + |f'(w_1(z))| |w_2(z)| + \sum_{n=N}^{\infty} |a_n| |w_n^*(z)| \\ \leq |f(0)| + 2\alpha d(f(0), \partial f(\mathbb{D})) \left(f_\alpha(r^{m_0}) + \sum_{n=N}^{\infty} A_n r^{b_n} + \frac{(1 + r^{m_1})^{\alpha-1}}{(1 - r^{m_1})^{\alpha+1}} r^{m_2} \right) \\ (5.6) \quad = |f(0)| + 2\alpha d(f(0), \partial f(\mathbb{D})) \left(K_{\alpha, m_0, m_1, m_2}^N(r) + \frac{1}{2\alpha} \right), \end{aligned}$$

where

$$K_{\alpha, m_0, m_1, m_2}^N(r) = f_\alpha(r^{m_0}) + \sum_{n=N}^{\infty} A_n r^{b_n} + \frac{(1 + r^{m_1})^{\alpha-1}}{(1 - r^{m_1})^{\alpha+1}} r^{m_2} - \frac{1}{2\alpha}.$$

Note that the function $K_{\alpha, m_0, m_1, m_2}^N(r)$ is a strictly increasing function of r in $(0, 1)$, $K_{\alpha, m_0, m_1, m_2}^N(0)$ is negative, and $\lim_{r \rightarrow 1} K_{\alpha, m_0, m_1, m_2}^N(r) = \infty$ and hence there exists the unique positive root say $r_{\alpha, m_0, m_1, m_2}^N$ of the equation (3.3) in $(0, 1)$. Thus the last quantity of the inequality (5.6) is less than or equal to $|f(0)| + d(f(0), \partial f(\mathbb{D}))$ for $r \leq r_{\alpha, m_0, m_1, m_2}^N$ and this completes the first part of the proof.

Next we show that the radius $r_{\alpha, m_0, m_1, m_2}^N$ is sharp. Let $w_i(z) = z^{m_i}$ ($i = 0, 1, 2$), $w_n^*(z) = z^{h(n)}$, and $f = f_\alpha$. Simple computation shows that

$$|f_\alpha(w_0(z))| + |f'_\alpha(w_1(z))||w_2(z)| + \sum_{n=N}^{\infty} A_n |w_n^*(z)| = |f_\alpha(z^{m_0})| + |f'_\alpha(z^{m_1})| r^{m_2} + \sum_{n=N}^{\infty} A_n r^{h(n)}.$$

After substituting $z = r$ in the above equation, we obtain

$$\begin{aligned} & |f_\alpha(w_0(r))| + |f'_\alpha(w_1(r))||w_2(r)| + \sum_{n=N}^{\infty} A_n |w_n^*(r)| \\ &= |f_\alpha(0)| + 2\alpha d(f_\alpha(0), \partial f_\alpha(\mathbb{D})) \left(K_{\alpha, m_0, m_1, m_2}^N(r) + \frac{1}{2\alpha} \right) \\ &> d(f_\alpha(0), \partial f_\alpha(\mathbb{D})) = \frac{1}{2\alpha} \end{aligned}$$

which holds if, and only if, $r > r_{\alpha, m_0, m_1, m_2}^N$. This completes the proof. \square

5.3. Proof of Theorem 4.2. For a given $f \in \widehat{C}_p$, let $g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f)$. Let the Koebe transform of f ,

$$F(z) = \frac{f\left(\frac{z+a}{1+\bar{a}z}\right) - f(a)}{(1-|a|^2)f'(a)}$$

for any $a \in \mathbb{D} \setminus \{p\}$. Since for some $t \in \mathbb{R}$, $e^{-it}F(ze^{it}) \in C_{|\frac{p-a}{1-\bar{a}p}|}$. Therefore, from [46], we have

$$(5.7) \quad d(f(0), \partial f(\mathbb{D})) \geq (p/(1+p^2))|f'(0)|.$$

As $F(z) = (f(z) - f(0))/f'(0) \in C_p$, it follows that (see [26])

$$(5.8) \quad |a_n| \leq |f'(0)| \frac{1-p^{2n}}{(1-p^2)p^{n-1}}, \text{ for all } n \geq 1.$$

In view of $g \prec f$ and (5.8) we have the following inequality

$$|g(z) - g(0)| \leq |f(z) - f(0)| \leq \sum_{n=1}^{\infty} |a_n| r^n \leq k_p(r) = |f'(0)| \sum_{n=1}^{\infty} \frac{1-p^{2n}}{(1-p^2)p^{n-1}} r^n.$$

Equivalently, for $w_0 \in \mathcal{B}_{m_0}$, we obtain

$$(5.9) \quad |g(w_0(z))| \leq |f(0)| + |f'(0)| \sum_{n=1}^{\infty} \frac{1-p^{2n}}{(1-p^2)p^{n-1}} r^{m_0 n} = |f(0)| + |f'(0)| k_p(r^{m_0}).$$

Also, Lemma 4.1 gives that

$$(5.10) \quad \sum_{n=N}^{\infty} |b_n| r^n \leq |f'(0)| \sum_{n=N}^{\infty} \frac{1-p^{2n}}{(1-p^2)p^{n-1}} r^n.$$

Thus, using (5.9) and (5.10) together with inequality (5.7), we obtain

$$\begin{aligned} & |g(w_0(z))| + \sum_{n=N}^{\infty} |b_n| r^n \leq |f(0)| + \frac{(1+p)^2}{p} d(f(0), \partial f(\mathbb{D})) \left(\sum_{n=N}^{\infty} \frac{1-p^{2n}}{(1-p^2)p^{n-1}} r^n + k_p(r^{m_0}) \right) \\ &= |f(0)| + \frac{(1+p)^2}{p} d(f(0), \partial f(\mathbb{D})) \left(G_{p, m_0}^N(r) + \frac{p}{(1+p)^2} \right) \\ &:= T_{f, p, m_0}^N(r), \end{aligned}$$

where $G_{p,m_0}^N(r)$ is given by (4.3). Note that $G_{p,m_0}^N(0) < 0$ and since

$$\sum_{n=N}^{\infty} \frac{1-p^{2n}}{(1-p^2)p^{n-1}} p^n = \frac{p(1-p^{2N})}{(1-p^2)} + \sum_{n=N+1}^{\infty} \frac{p(1-p^n)(1+p^n)}{(1-p^2)} \geq \sum_{n=N+1}^{\infty} p(1+p^n)$$

diverges to infinity, we find that $G_{p,m_0}^N(p) > 0$ while $G_{p,m_0}^N(r)$ is strictly increasing in $(0, 1)$. Therefore we conclude that the equation (4.3) has the unique positive root say r_{p,m_0}^N , $0 < r_{p,m_0}^N < p$ in $(0, 1)$. This shows that $T_{f,p,m_0}^N(r) \leq |f(0)| + d(f(0), \partial f(\mathbb{D}))$, for $r \leq r_{p,m_0}^N$.

For the equality, we consider the function $w_0(z) = z^{m_0}$ and $f = g = k_p$ given by (4.1). For this function it is well known that $\widehat{\mathbb{C}} \setminus k_p(\mathbb{D}) = [-p/(1-p)^2, -p/(1+p)^2]$ (see [15, p. 137]) and hence we obtain $d(k_p(0), \partial k_p(\mathbb{D})) = p/(1+p)^2$. Taking $z = r$, then the left side of the inequality (4.2) reduces to

$$\begin{aligned} |k_p(r^{m_0})| + \sum_{n=N}^{\infty} \frac{1-p^{2n}}{(1-p^2)p^{n-1}} r^n \\ = |k_p(0)| + \frac{(1+p)^2}{p} d(k_p(0), \partial k_p(\mathbb{D})) \left(k_p(r^{m_0}) + \sum_{n=N}^{\infty} \frac{1-p^{2n}}{(1-p^2)p^{n-1}} r^n \right) \\ > d(k_p(0), \partial k_p(\mathbb{D})) = \frac{p}{(1+p)^2}, \end{aligned}$$

which holds if, and only if, $r > r_{p,m_0}^N$, which means that the number r_{p,m_0}^N cannot be improved. This shows the sharpness. \square

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