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BOND PRICING AND THE TERM STRUCTURE OF INTEREST RATES: A NEW METHODOLOGY FOR CONTINGENT CLAIMS VALUATION[1]

by

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Abstract

This paper presents a unifying theory for valuing contingent claims under a stochastic term structure of interest rates. The methodology, based on the equivalent martingale measure technique, takes as given an initial forward rate curve and a family of potential stochastic processes for its subsequent movements. A no arbitrage condition restricts this family of processes yielding valuation formula for interest rate sensitive contingent claims which are independent of the market prices of risk. Examples are provided to illustrate the key results.

Bond Pricing and the Term Structure of Interest Rates:
A New Methodology for Contingent Claims Valuation

In relation to the term structure of interest rates, arbitrage pricing theory has two related purposes. The first, is to price all zero coupon (default free) bonds of varying maturities from a finite number of economic fundamentals, called state variables. The second, is to price all interest rate sensitive contingent claims, taking as given the prices of the zero coupon bonds. This paper presents a general theory and a unifying framework for understanding arbitrage pricing theory in this context, of which all existing arbitrage pricing models are special cases (in particular, Vasicek [1977], Brennan and Schwartz [1979], Langetieg [1980], Ball and Torous [1983], Ho and Lee [1986], Schaefer and Schwartz [1987], and Artzner and Delbaen [1988]).

The primary contribution of this paper, however, is a new methodology for solving the second problem, i.e., the pricing of interest rate sensitive contingent claims given the prices of all zero coupon bonds. The methodology is new because (i) it is independent of the "market prices of risk," (ii) it has a stochastic spot rate process with multiple stochastic factors influencing the term structure, (iii) it can be used to consistently price (and hedge) all contingent claims (American or European) on the term structure, and (iv) it is derived from necessary and (more importantly) <u>sufficient</u> conditions for the absence of arbitrage. None of the existing models simultaneously satisfy all of these criteria.

Indeed, the arbitrage pricing models of Vasichek [1977], Brennan and Schwartz [1979], Langetieg [1980], and Artzner and Delbaen [1988] all require estimates of the market prices of risk to price contingent claims. These quantities, being stochastic and nonstationary, are difficult to estimate. They appear in the valuation formulae due to the two-step procedure utilized in these papers to price contingent claims. The first step is to price the zero coupon bonds from a finite number of state variables. Given these derived prices, the second step is to value contingent claims. The equilibrium model of Cox, Ingersoll, Ross [1985], when used to value contingent claims, also follows this same two-step procedure. It is in the first step in this procedure that introduces the market prices of risk into the valuation formulae.

In the above models, for parameterized forms of the market prices for risk, it is possible to invert the bond pricing formula after step one, to obtain the market prices for risk as functions of the zero coupon bond prices. This inversion would remove the market prices for risk from contingent claim values. The inversion, however, is problematic. First, it is computationally difficult since the bond pricing formula are highly non-linear. Secondly, as will be shown later, the spot rate and bond price processes parameters are not independent of the market prices for risk. Hence, arbitrarily specifying a parameterized form of the market prices for risk as a function of the state variables can lead to an inconsistent model, i.e., one which admits arbitrage opportunities. This possibility was originally noted by Cox, Ingersoll, and Ross [1985, p. 398].

Another class of arbitrage pricing models, illustrated by Ball and Torous [1983] and Shaefer and Schwartz [1987] avoids this two-step procedure by taking a finite number of initial bond prices and bond price processes as exogenously given. Unfortunately, Schaefer and Schwartz's model requires a constant spot rate process, and as shown by Cheng [1987], Ball and Torous' model is inconsistent with stochastic spot rate processes and the absence of arbitrage. Furthermore, due to their special structure, neither model can be utilized to vàlue American type claims with a continuum of possible early exercise dates.

The model of Ho and Lee [1976] is the closest in spirit to our model. They also avoid the two-step procedure by taking the initial bond prices and bond price processes as exogenously given. Unlike all the previous models, however, they utilize a discrete trading economy. The zero coupon bond price curve, in constrast to a finite number of bond prices, is assumed to fluctuate randomly over time according to a binomial process. Unfortunately, it is only a single factor model, so bonds of all maturities are perfectly correlated. The discrete process chosen for estimation and computation also implies negative interest rates with positive probability. This is another drawback of their model, not shared by ours. Last, to implement their model, they estimate the parameters of the discrete time binomial process. For large step sizes, as shown by Heath, Jarrow, Morton [1988], the parameters are not independent. This makes estimation problematic, as the dependence is usually not explicitly taken into account. The continuous time limit of this model, which is studied below as a special case, is not subject to this same difficulty.

In contrast to the previous class of models which avoid the two-step procedure, our model imposes the exogenous stochastic structure upon forward rates, and not the zero coupon bond prices. This change in perspective facilitates the empirical estimation and practical implementation of the model. Indeed, forward rates as a stochastic process are more stationary than are zero coupon bond prices. Since zero coupon bond prices are a fixed constant at maturity, their "volatilities" must change over time. In contrast, constant forward rate volatilities are consistent with a fixed value for a default free, zero coupon bond at maturity. This stationarity facilitates estimation. Further, the standard properties of the bond price process and the spot rate process are easily deducible from forward rates.

The forward rate perspective is also advantageous for implementing these models. Indeed, traders use the Black-Scholes technology and are accustomed to thinking in terms of a "term structure of volatilities." Our approach requires such a "term structure of volatilities" as the input. Analogous to the Black-Scholes model when applied to price bond options, the inputs to our contingent claim valuation formulae are only the initial forward rate curve, volatilities, and the details of the contract.

As mentioned above, the model in this paper takes as given the initial forward rate curve. We then specify a general (possibly non-Markov) continuous time stochastic process for its evolution across time. To ensure that the process is consistent with an arbitrage free economy (and hence with some equilibrium), we use the insights of Harrison and Kreps

[1979] to characterize the conditions on the forward rate process such that there exists a unique, equivalent martingale probability measure. Under these conditions, contingent claim valuation is then a straightforward application of the methods known from Harrison and Pliska [1981]. We illustrate this approach with multiple examples. The examples themselves should prove useful for practical applications to financial trading.

An outline of this paper is as follows: Section 2 presents the terminology and notation. Section 3 presents the forward rate process. Section 4 characterizes arbitrage free forward rate processes. Section 5 extends the model to price interest rate dependent contingent claims. Sections 6-9 provide examples. Section 10 relates the arbitrage pricing approach to the equilibrium pricing approach, while Section 11 summarizes the paper and discusses generalizations.

2. Terminology and Notation

This section of the paper presents the model's terminology and notation. We consider a continuous trading economy with a trading interval $[0,\tau]$ for a fixed $\tau > 0$. The uncertainty in the economy is characterized by the probability space (Ω, F, Q) where Ω is the state space, F is the σ-algebra representing measurable events, and Q is a probability measure. Information evolves over the trading interval according to the augmented, right continuous, complete filtration[1] $\{F_t: t \in [0,\tau]\}$ generated by two independent Brownian motions $\{W_1(t), W_2(t): t \in [0,\tau]\}$ both initialized at zero. The restriction to two Brownian motions is imposed only for expositional clarity. All the subsequent results are readily extended to the case of a finite number of independent Brownian motions. We let $E(\cdot)$ denote expectation with respect to the probability measure Q.

A continuum of default free discount bonds trade, one for each trading date $T \in [0,\tau]$. The T maturity bond pays a certain dollar at date T. $P(t,T)$ denotes the time t price of the T maturity bond for all $T \in [0,\tau]$ and $t \in [0,T]$. We require that $P(T,T) = 1$ for all $T \in [0,\tau]$, $P(t,T) > 0$ for all $T \in [0,\tau]$ and $t \in [0,T]$, and that $\partial \log P(t,T)/\partial T$ exists for all $T \in [0,\tau]$ and $t \in [0,T]$. The first condition normalizes the bond's payoff to be one dollar at maturity. The second condition excludes the trivial arbitrage opportunity where a certain dollar can be obtained for free. The last condition guarantees that forward rates are well-defined.

The instantaneous <u>forward rate</u> at time t for date $T > t$, $f(t,T)$, is defined by

$$f(t,T) = -\partial \log P(t,T)/\partial T \text{ for all } T \in [0,\tau],\ t \in [0,T]. \tag{1}$$

It corresponds to the rate that one can contract for at time t, on a riskless loan that begins at date T and is returned an instant later. Solving the partial differential equation of expression (1) yields:

$$P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right) \text{ for all } T \in [0,\tau],\ t \in [0,T]. \tag{2}$$

The <u>spot rate</u> at time t, $r(t)$, is the instantaneous forward rate at time t for date t,[2] i.e.,

$$r(t) = f(t,t) \text{ for all } t \in [0,\tau]. \tag{3}$$

For the subsequent analysis, it is convenient to define an accumulation factor, $B(t)$, corresponding to the price of a money market account (rolling over at $r(t)$) initialized at time 0 with a dollar investment, i.e.,

$$B(t) = \exp\left(\int_0^t r(y)dy\right) \text{ for all } t \in [0,\tau]. \tag{4}$$

3. <u>Term Structure Movements</u>

This section of the paper presents the family of stochastic processes representing forward rate movements, condition (C.1). This condition describes forward rates, and uniquely specifies the spot rate process and the bond price process. Additional boundedness conditions, (C.2) and (C.3), are required to guarantee that the spot rate and the bond price process are well-behaved.

(C.1: A Family of Forward Rate Processes)

For fixed, but arbitrary $T \in [0,\tau]$, $f(t,T)$ satisfies the following equation:

$$f(t,T)-f(0,T) = \int_0^t \alpha(v,T,\omega)dv + \sum_{i=1}^{2} \int_0^t \sigma_i(v,T,\omega)dW_i(v) \text{ for all } 0 \leq t \leq T \qquad (5)$$

where

(i) $\{f(0,T): T \in [0,\tau]\}$ is a fixed, non-random initial forward rate curve which is measurable as a mapping $f(0,\cdot)$: $([0,\tau],B[0,\tau]) \to (R,B)$ where $B[0,\tau]$ is the Borel σ-algebra restricted to $[0,\tau]$,

(ii) α: $\{(t,s): 0 \leq t \leq s \leq T\} \times \Omega \to R$ is jointly measurable from $B\{(t,s): 0 \leq t \leq s \leq T\} \times F \to B$, adapted, with

$$\int_0^T |\alpha(t,T,\omega)|dt < +\infty \text{ a.e. } Q, \text{ and}$$

(iii) The volatilities σ_i: $\{(t,s): 0 \leq t \leq s \leq T\} \times \Omega \to R$ are jointly measurable from $B\{(t,s): 0 \leq t \leq s \leq T\} \times F \to B$, adapted, and satisfy

$$\int_0^T \sigma_i^2(t,T,\omega)dt < +\infty \text{ a.e. } Q \text{ for } i = 1, 2.$$

This is a very general stochastic process. Two Brownian motions determine the stochastic fluctuation of the entire forward rate curve starting from a fixed initial curve $\{f(0,T): T \in [0,\tau]\}$. The sensitivity of a particular maturity forward rate's change to each Brownian motion is reflected by differing volatility coefficients. The volatility coefficients, $\{\sigma_i(t,T,\omega): T \in [0,\tau]\}$ for $i = 1, 2$ are left unspecified, except for mild measurability and integrability conditions, and can depend on the entire past of all of the

Brownian motions. Different specifications for these volatility coefficients generate significantly different qualitative characteristics of the forward rate process.

For a fixed maturity T, the forward rate process's drift term, $\alpha(t,T,\omega)$, is non-constant and may depend on the path of the Brownian motion from time 0 to t. The family of drift functions $\{\alpha(\cdot,T): T \in [0,\tau]\}$ is unrestricted (at this point), except for mild measurability and integrability conditions.

It is important to emphasize that the only substantive economic restrictions imposed on the forward rate process are that they have continuous sample paths and that they depend on only a finite number of random shocks (across the entire forward rate curve). The first restriction implies that information flows continuously over time, and can be relaxed by standard techniques involving jump processes (see Merton [1976]). The second restriction can also be relaxed by including a countably infinite number of Brownian motions where all but a finite number are diversifiable. This generalization would proceed along the lines of Ross [1976] or Chamberlain [1988].

Given condition (C.1), we can determine the dynamics of the spot rate process:

$$r(t) = f(0,t) + \int_0^t \alpha(v,t,\omega)dv + \int_0^t \sigma_1(v,t,\omega)dW_1(v) + \int_0^t \sigma_2(v,t,\omega)dW_2(v)$$

$$\text{for all } t \in [0,T]. \qquad (6)$$

The spot rate process is similar to the forward rate process, except that both the time and maturity arguments vary simultaneously.

Given the dynamics of the spot rate process, we need to ensure that the value of the money market account, as defined in expression (4), satisfies:

$$0 < B(t,\omega) < +\infty \quad \text{a.e. } Q \text{ for all } t \in [0,\tau]. \qquad (7)$$

This is guaranteed by condition (C.2).[3]

(C.2: Regularity of the Money Market Account)

$$\int_0^{\tau} |f(0,v)|dv < +\infty,$$

$$\int_0^{\tau} \{\int_0^{t} |\alpha(v,t,\omega)|dv\}dt < +\infty \quad \text{a.e.} \quad Q,$$

$$(t,\omega) \rightarrow \int_0^{T} \sigma_i(v,t,\omega)dW_i(v) \text{ is measurable as a mapping from } B[0,\tau] \times F \rightarrow B$$

for all $T \in [0,t]$, and

$$\int_0^{\tau} | \int_0^{t} \sigma_i(v,t,\omega)dW_i(v)|dt < +\infty \quad \text{a.e.} \quad Q \text{ for } i = 1, 2.$$

Next, we are interested in the dynamics of the bond price process. The following condition imposes sufficient regularity on the coefficients of the forward rate process so that the bond price process is well-behaved.

(C.3: Regularity of the Bond Price Process)

$$\int_0^{t} [\int_v^{t} \sigma_i(v,y,\omega)dy]^2 dv < +\infty \quad \text{a.e.} \quad Q \text{ for all } t \in [0,\tau] \text{ and } i = 1, 2,$$

$$\int_0^{t} [\int_t^{T} \sigma_i(v,y,\omega)dy]^2 dv < +\infty \quad \text{a.e.} \quad Q \text{ for all } t \in [0,T],\ T \in [0,\tau],\ i = 1, 2,$$

and

$$t \rightarrow \int_t^{T} [\int_0^{t} \sigma_i(v,y,\omega)dW_i(v)]dy \text{ is continuous} \quad \text{a.e.} \quad Q \text{ for all } T \in [0,\tau] \text{ and } i = 1,2.$$

It is shown in the appendix that under conditions (C.1) - (C.3), the dynamics of the bond price process (suppressing the notational dependence on ω) is:

$$\ln P(t,T) = \ln P(0,T) + \int_0^t [r(v) + b(v,T)]dv - (1/2) \sum_{i=1}^{2} \int_0^t a_i(v,T)^2 dv$$

$$+ \sum_{i=1}^{2} \int_0^t a_i(v,T)dW_i(v) \quad \text{a.e.} \quad Q \tag{8}$$

where

$$a_i(t,T,\omega) \equiv - \int_t^T \sigma_i(t,v,\omega)dv \text{ for } i = 1,2, \quad \text{and}$$

$$b(t,T,\omega) \equiv - \int_t^T \alpha(t,v,\omega)dv + (1/2) \sum_{i=1}^{2} a_i(t,T,\omega)^2.$$

A straightforward application of Ito's lemma to expression (8) yields $P(t,T)$ as the strong solution to the following stochastic differential equation:

$$dP(t,T) = [r(t) + b(t,T)]P(t,T)dt + \sum_{i=1}^{2} a_i(t,T)P(t,T)dW_i(t) \quad \text{a.e.} \quad Q. \tag{9}$$

In general, the bond price process is non-Markov since the drift term $[r(t,\omega) + b(t,T,\omega)]$ and the volatility coefficients $a_i(t,T,\omega)$ for $i = 1,2$ can depend on the history of the Brownian motions $\{W_1(t), W_2(t): t \in [0,\tau]\}$. The form of the bond price process as given in expression (9) is similar to, but more general than, that appearing in the existing literature, see for example, Brennan and Schwartz [1979] or Langetieg [1980]. The process as given in expression (9) is more general since it requires less regularity assumptions and it need not be Markov.

We define the relative bond price for a T-maturity bond as $Z(t,T) = P(t,T)/B(t)$ for $T \in [0,\tau]$ and $t \in [0,T]$. The relative bond price is the bond's value expressed in units of the accumulation factor, not dollars. This transformation removes the portion of the bond's drift due to the spot rate process. As such, it is particularly useful for analysis. Applying Ito's lemma to the definition of $Z(t,T)$ yields:

$$\ln Z(t,T) = \ln Z(0,T) + \int_0^t b(v,T)dv - (1/2) \sum_{i=1}^{2} \int_0^t a_i(v,T)^2 dv$$

$$+ \sum_{i=1}^{2} \int_0^t a_i(v,T)dW_i(v) \quad \text{a.e. } Q \tag{10}$$

or

$$dZ(t,T) = b(t,T)Z(t,T)dt + a_1(t,T)Z(t,T)dW_1(t) + a_2(t,T)Z(t,T)dW_2(t)$$
$$\text{for all } T \in [0,\tau],\ t \in [0,T]. \tag{11}$$

Again, the relative bond price at date t depends on the path of the Brownian motion through the cumulative forward rate drifts, volatilities, and, in general, it cannot be written as a function of only the current values of the Brownian motions $\{W_1(t), W_2(t): t \in [0,\tau]\}$.

4. Arbitrage Free Bond Pricing and Term Structure Movements

Given conditions (C.1) - (C.3), this section characterizes the conditions on the forward rate process which are necessary and sufficient to guarantee the existence of a unique, equivalent[4] martingale probability measure.

(C.4: Existence of the Market Prices for Risk)

Fix an S, T $\in [0,\tau]$ such that $0 < S < T \leq \tau$.

Assume there exists solutions $\gamma_i(\cdot,\cdot,S,T): \Omega \times [0,S] \to R$ for i = 1, 2 a.e. $Q \times \lambda$ to the following system of equations:

$$\begin{bmatrix} b(t,S) \\ b(t,T) \end{bmatrix} + \begin{bmatrix} a_1(t,S) & a_2(t,S) \\ a_1(t,T) & a_2(t,T) \end{bmatrix} \begin{bmatrix} \gamma_1(t,S,T) \\ \gamma_2(t,S,T) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12}$$

which satisfy

$$\int_0^S \gamma_i(v,S,T)^2 dv < +\infty \text{ a.e. } Q \text{ for } i = 1, 2, \tag{13.a}$$

$$E(\exp\{\sum_{i=1}^{2} \int_0^S \gamma_i(v,S,T)dW_i(v) - (1/2)\sum_{i=1}^{2} \int_0^S \gamma_i(v,S,T)^2 dv\}) = 1, \text{ and} \tag{13.b}$$

$$E(\exp\{\sum_{i=1}^{2} \int_0^S [a_i(v,y) + \gamma_i(v,S,T)]dW_i(v) - (1/2)\sum_{i=1}^{2} \int_0^S [a_i(v,y) + \gamma_i(v,S,T)]^2 dv\}) = 1 \quad \text{for } y \in \{S,T\} \tag{13.c}$$

where λ is Lebesque measure.

The system of equations in expression (12) gives $\gamma_i(t,S,T)$ for i = 1,2 the interpretation of being the <u>market prices for risk</u> associated with the random factors $W_i(t)$ for i = 1,2, respectively. Indeed, to see this, we can rewrite expression (12) for the T-maturity bond as:

$$b(t,T) = \sum_{i=1}^{2} a_i(t,T)(-\gamma_i(t,S,T)) \tag{14}$$

The left side of expression (14) is the instantaneous excess expected return on the T-maturity bond <u>above</u> the risk free rate. The right side is the sum of (minus) the "market price of risk for factor i" times the instantaneous covariance between the T-maturity bond's return and the i-th random factor for i = 1 to 2. It is important to emphasize that the solutions to expression (12) depend, in general, on the pair of bonds {S,T} chosen.

The following proposition shows that condition (C.4) guarantees the existence of an equivalent martingale probability measure.

Proposition 1 (Existence of an Equivalent Martingale Measure)

Fix an $S,T \in [0,\tau]$ such that $0 < S < T \leq \tau$. Given a pair of forward rate drifts $\{\alpha(\cdot,S), \alpha(\cdot,T)\}$ and volatilities $\{\sigma_i(\cdot,S), \sigma_i(\cdot,T)\}$ for i = 1, 2 satisfying conditions (C.1) - (C.3); then, condition (C.4) holds if and only if there exists an equivalent probability measure $\tilde{Q}_{ST}$ such that Z(t,S) and Z(t,T) are both martingales with respect to $\{F_t: t \in [0,S]\}$.

Proof: In the appendix. ///

This proposition asserts that under conditions (C.1) - (C.3), condition (C.4) is both necessary and sufficient for the existence of an equivalent martingale measure $\tilde{Q}_{ST}$. In fact, the proof of proposition 1 identifies this probability measure as

$$d\tilde{Q}_{ST}/dQ = \exp\{\sum_{i=1}^{2} \int_0^S \gamma_i(v,S,T)dW_i(v) - (1/2) \sum_{i=1}^{2} \int_0^S \gamma_i(v,S,T)^2 dv\}. \tag{15}$$

Furthermore, it can also be shown that

$$\tilde{W}_i^{ST}(t) = W_i(t) - \int_0^t \gamma_i(v,S,T)dv \qquad \text{for } i = 1,2 \tag{16}$$

are independent Brownian motions on $[(\Omega,\tilde{Q}_{ST},F), \{F_t: t \in [0,S]\}]$.

Although condition (C.4) guarantees the existence of an equivalent martingale measure, it does not guarantee that it is unique. To obtain uniqueness, we impose:

(C.5: Uniqueness of the Equivalent Martingale Measure)

Fix an S, $T \in [0,\tau]$ such that $0 < S < T \leq \tau$. Assume that

$$\begin{pmatrix} a_1(t,S) & a_2(t,S) \\ a_1(t,T) & a_2(t,T) \end{pmatrix} \quad \text{is nonsingular a.e. } Q \times \lambda.$$

The following proposition demonstrates that condition (C.5) is both necessary and sufficient for the uniqueness of the equivalent martingale measure.[5]

Proposition 2: (Characterization of Uniqueness of the Equivalent Martingale Measure)

Fix an S, T ϵ $[0,\tau]$ such that $0 < S < T \leq \tau$. Given a pair of forward rate drifts $\{\alpha(\cdot,S), \alpha(\cdot,T)\}$ and volatilities $\{\sigma_i(\cdot,S), \sigma_i(\cdot,T)\}$ for i = 1, 2 satisfying conditions (C.1) - (C.4); then, condition (C.5) holds if and only if the martingale measure is unique.

Proof: In the appendix. ///

Conditions (C.1) - (C.5), through the functions $\gamma_i(t,S,T)$ for i = 1, 2, impose restrictions upon the drifts for the forward rate processes $\{\alpha(\cdot,T), \alpha(\cdot,S)\}$. It imposes just enough restrictions so that there is a unique martingale measure for the pair of bonds P(t,S), P(t,T) with $0 < S < T \leq \tau$. Both the market prices for risk and the martingale measure, however, depend on the particular pair of bonds {S,T} chosen. To guarantee that there exists a unique equivalent martingale measure simultaneously making all relative bond prices martingales, we prove the following proposition.

Proposition 3: (Uniqueness of the Martingale Measure Across all Bonds)

Given a family of forward rate drifts $\{\alpha(\cdot,T): T \epsilon [0,\tau]\}$ and a family of volatilities $\{\sigma_i(\cdot,T): T \epsilon [0,\tau]\}$ for i = 1, 2 satisfying conditions (C.1) - (C.5), the following are equivalent:

[$\tilde{Q}$ defined by $\tilde{Q} = \tilde{Q}_{S\tau}$ for any S ϵ $(0,\tau)$ is the unique equivalent probability measure such that Z(t,T) is a martingale for all T ϵ $[0,\tau]$ and t ϵ [0,S]]; (17)

$[\gamma_i(t,S_1,T_1) = \gamma_i(t,S_2,T_2)$ for $i = 1, 2$ for all $S_1,S_2,T_1,T_2 \in [0,\tau]$, (18)
$t \in [0,\tau]$ such that $0 \leq t < S_1 < T_1 \leq \tau$ and $0 \leq t < S_2 < T_2 \leq \tau]$;

$$[\alpha(t,T) = -\sum_{i=1}^{2} \sigma_i(t,T)(\phi_i(t) - \int_t^T \sigma_i(t,v)dv) \text{ for all} \qquad (19)$$

$T \in [0,\tau]$ and $t \in [0,T]$ where for $i = 1, 2$

$\phi_i(t) = \gamma_i(t,S,\tau)$ for any $S \in (t,\tau)$ and $t \in [0,S]]$.

<u>Proof</u>:

From proposition 2, for each pair S,T with $S < T$, $\tilde{Q}_{ST}$ is the unique equivalent probability measure making $Z(t,S)$ a martingale over $t \leq S$. These measures are all equal to $\tilde{Q}$ if and only if

$\gamma_i(t,S_1,T_1) = \gamma_i(t,S_2,T_2)$ for $i = 1, 2$ and all $S_1,S_2,T_1,T_2 \in [0,\tau]$ and
$t \in [0,\tau]$ such that $0 \leq t < S_1 < T_1 \leq \tau$ and $0 \leq t < S_2 < T_2 \leq \tau$.

To obtain the third condition, by expression (14) and the fact that $(\phi_1(t),\phi_2(t))$ is independent of T, one obtains $b(t,T) = -a_1(t,T)\phi_1(t) - a_2(t,T)\phi_2(t)$. Substitution for $b(t,T)$, $a_1(t,T)$, $a_2(t,T)$ and taking the partial derivative with respect to T gives (19). Q.E.D.

This proposition asserts that there is a unique equivalent probability measure, $\tilde{Q}$, such that for all maturity bonds, relative prices are martingales (condition (17)) if and only if the market prices for risk are independent of the particular pairs of bonds {S,T} chosen (condition (18)) if and only if a forward rate drift restriction is satisfied (condition (19)). We discuss each of these conditions in turn.

The martingale condition (17) implies that

$$\tilde{E}(Z(T,T)|F_t) = Z(t,T) \text{ a.e. } Q \text{ for all } t \in [0,T] \qquad (20)$$

where $\tilde{E}(\cdot)$ denotes expectation with respect to the probability measure $\tilde{Q}$. Using the definition of $Z(t,T)$, we obtain

$$P(t,T) = B(t)\tilde{E}(1/B(T)|F_t) \tag{21}$$

or

$$P(t,T) = B(t)E(\exp\{\sum_{i=1}^{2}\int_0^T \phi_i(t)dW_i(t) - (1/2)\sum_{i=1}^{2}\int_0^T \phi_i(t)^2 dt\}/B(T)|F_t). \tag{22}$$

Expression (22) demonstrates that the bond's price depends on the forward rate drifts $\{\alpha(\cdot,T): T \in [0,\tau]\}$, the initial forward rate curve $\{f(0,T): T \in [0,\tau]\}$, and the forward rate volatilities $\{\sigma_i(\cdot,T): T \in [0,\tau]\}$ for $i = 1, 2$. All of these parameters enter into expression (22) implicitly through $\phi_i(t)$ $i = 1, 2$, the market prices for risk and $B(T)$, the money market account. For explicit representations of the forward rate volatility functions, expression (22) simplifies considerably. Examples are provided in subsequent sections.

Condition (18) of proposition 3 is called the standard finance condition for arbitrage free pricing. This is the necessary condition for the absence of arbitrage used in the existing literature to derive the fundamental partial differential equation for pricing contingent claims (see Brennan and Schwartz [1979] or Langetieg [1980]).

Last, for purposes of contingent claim valuation, the final condition contained in expression (19) will be most useful. It is called the forward rate drift restriction. It shows the restriction needed on the family of drift processes $\{\alpha(\cdot,T): T \in [0,\tau]\}$ in order to guarantee the existence of a unique equivalent martingale probability measure. As seen below, not all potential forward rate processes satisfy this restriction.

5. Contingent Claim Valuation

This section serves two purposes. First, it demonstrates how to value contingent claims in the preceding economy. This analysis is a slight extension of the ideas contained in Harrison and Kreps [1979] and Harrison and Pliska [1981], so the presentation will be brief. Second, it provides the unifying framework for categorizing the various arbitrage pricing theories in the literature (i.e., Vasichek [1977], Brennan and Schwartz [1979], Langetieg [1980], Ball and Torous [1983], Ho and Lee [1986], Schaefer and Schwartz [1987], Artzner and Delbaen [1988]) in relation to our own.

For our analysis, the key insight of Harrison and Kreps [1979], as extended by Harrison and Pliska [1981], is the association between the absence of arbitrage opportunities and the existence of equivalent martingale probability measures. To clarify this association (and to use the propositions of the preceding section), we first need to present the detailed definitions of trading strategies and arbitrage opportunities. We do this in the context of our economy.

Let conditions (C.1) - (C.3) hold. These define the forward rate process, money market account dynamics, and the zero coupon bond price processes. Fix two particular zero coupon bonds of maturities $\{S,T\} \in [0,\tau]$ where $0 < S < T \leq \tau$. We consider the subset of our economy consisting of these two bonds and the money market account.

A <u>trading strategy</u> is defined to be a 3-dimensional stochastic process $\{N_0(t), N_S(t), N_T(t): t \in [0,S]\}$ such that

(i) $N_0(t)$, $N_S(t)$, $N_T(t)$ are measurable and adapted, and

(ii) $\int_0^S |N_0(t)|r(t)B(t)dt < +\infty$ a.e. Q,

$$\int_0^S |N_u(t)[r(t) + b(t,u)]|P(t,u)dt < +\infty \quad \text{a.e. } Q \text{ for } u \in \{S,T\}, \text{ and}$$

$$\int_0^S N_u(t)^2 P(t,u)^2 a_i(t,u)^2 dt < +\infty \quad \text{a.e. } Q \quad \text{for } i = 1,2 \text{ and } u \in \{S,T\}.$$

Define the <u>value process</u> $V: [0,S] \times \Omega \to R$ for the trading stragy $\{N_0(t), N_S(t), N_T(t): t \in [0,S]\}$ by

$$V(t,\omega) = N_0(t,\omega)B(t) + N_S(t,\omega)P(t,S) + N_T(t,\omega)P(t,T) \text{ a.e. } Q \quad \text{for all } 0 \leq t \leq S.$$

A trading strategy $\{N_0(t), N_S(t), N_T(t): t \in [0,S]\}$ is said to be <u>self-financing</u> if

$$V(t,\omega) = V(0,\omega) + \int_0^t N_0(t)dB(t) + \int_0^t N_S(t)dP(t,S) + \int_0^t N_T(t)dP(t,T) \text{ a.e. } Q \quad \text{for all } 0 \leq t \leq S.$$

An <u>arbitrage opportunity</u> is any self-financing trading strategy (s.f.t.s.) $\{N_0(t), N_S(t), N_T(t): t \in [0,S]\}$ such that its value process satisfies:

$$V(0) \leq 0, \; Q(V(S) > 0) > 0, \text{ and } Q(V(S) \geq 0) = 1.$$

Assuming condition (C.4) for the bonds $\{S,T\} \in [0,\tau]$ in our economy, proposition 1 guarantees the existence of an equivalent probability measure $\tilde{Q}_{ST}$ on (Ω,F_S) such that $Z(t,S)$ and $Z(t,T)$ are $\tilde{Q}_{ST}$ martingales on $t \in [0,S]$. Let $\tilde{E}_{ST}(\cdot)$ denote expectation with respect to $\tilde{Q}_{ST}$.

An <u>admissible</u> s.f.t.s. is defined to be a s.f.t.s. $\{N_0(t), N_S(t), N_T(t): t \in [0,S]\}$ such that its value process satisfies:

$$V(t) \geq 0 \text{ a.e. } Q \text{ for all } t \in [0,S] \text{ and } \{\frac{V(t)}{B(t)}: t \in [0,S]\} \text{ is a } \tilde{Q}_{ST}\text{-martingale.}$$

If we restrict traders to use only admissible s.f.t.s., then by proposition 2 assuming condition (C.5) gives the existence of a <u>unique</u> martingale measure $\tilde{Q}_{ST}$. This implies that there are no arbitrage opportunities in the economy (nor any suicide strategies), see

Harrison and Pliska [1981; p. 239]. Consequently, we assume for the sequel that all of conditions (C.1) - (C.5) hold.

The uniqueness of $\tilde{Q}_{ST}$ is important. The uniqueness of $\tilde{Q}_{ST}$ implies that the market is also <u>complete</u> (Harrison and Pliska [1981]; Corollary 3.36, p. 241]), i.e., given any random variable $X: \Omega \rightarrow R$ which is nonnegative, F_S measurable with $\tilde{E}_{ST}(X/B(S)) < +\infty$, there exists an admissible s.f.t.s. $\{N_0(t), N_S(t), N_T(t): t \in [0,S]\}$ such that its value process satisfies

$$V(S) = X \quad \text{a.e. } Q.$$

The random variable X is interpreted as the payout to a <u>contingent claim</u> at time S.

For example, consider a European type call option on the bond P(t,T) with maturity date S and exercise price $K > 0$. Its time S cash flow would be $\{\max[P(S,T) - K,0]\}$. This is F_S measurable, non-negative, and satisfies:

$$\tilde{E}_{ST}(\max[P(S,T) - K,0]/B(S)) \leq \tilde{E}_{ST}(Z(S,T)) < +\infty.$$

The admissible s.f.t.s. which attains $\max[P(S,T) - K,0]$ is called the <u>synthetic call</u>. We define the <u>arbitrage free price</u> for the European call at time t to be the value process of the synthetic call, i.e.,

$$V(t) = \tilde{E}_{ST}(V(S)/B(S)|F_t)B(t) = \tilde{E}_{ST}(\max[P(S,T) - K,0]/B(S)|F_t)B(t).$$

By analogy, given any contingent claim $X: \Omega \rightarrow R$ (which is F_S measurable, non-negative with $\tilde{E}_{ST}(X/B(S)) < +\infty$)), its unique arbitrage free price at time t is defined to be:

$$V(t) = \tilde{E}_{ST}(X/B(S)|F_t)B(t), \tag{23}$$

where V(t) is the value process for the admissible s.f.t.s. $\{N_0(t), N_S(t), N_T(t): t \in [0,S]\}$ which duplicates X, i.e., such that

$$X = N_0(S)B(S) + N_S(S)P(S,S) + N_T(S)P(S,T) \quad \text{a.e. } Q.$$

Using this s.f.t.s. and expression (23) we can gain insights into contingent claim valuation under the above structure. Substitution yields:

$$V(t) = \tilde{E}_{ST}(N_0(S) + N_S(S)/B(S) + N_T(S)Z(S,T)|F_t)B(t). \tag{24}$$

To evaluate the arbitrage free price V(t) expression (24) demonstrates that we need to know the distributions of (i) $N_0(S)$, $N_S(S)$, $N_T(S)$ at time S, (ii) $1/B(S) = -\exp\{\int_0^S r(y)dy\}$, and (iii) Z(S,T), all under the martingale measure $\tilde{Q}_{ST}$. The dynamics for r(t), Z(t,S), and Z(t,T) are obtainable from expressions (6), (11), and (16); that is,

$$r(t) = f(0,t) + \int_0^t \alpha(v,t)dv + \sum_{i=1}^{2} \int_0^t \sigma_i(v,t)d\tilde{W}_i^{ST}(v) + \sum_{i=1}^{2} \int_0^t \gamma_i(v,S,T)\sigma_i(v,t)dv \tag{25}$$

a.e. Q

and

$$Z(t,u) = Z(0,u)\exp\{-(1/2) \sum_{i=1}^{2} \int_0^t a_i(v,u)^2 dv + \sum_{i=1}^{2} \int_0^t a_i(v,u)d\tilde{W}_i^{ST}(v)\} \quad \text{a.e. } Q \tag{26}$$

for $u \in \{S,T\}$.

Observe that to evaluate the arbitrage free price of any contingent claim (in general) under the above structure, we need to know $\gamma_i(v,S,T)$ for i = 1,2, the market prices for risk. These enter through the dynamics of the spot rate process in expression (25). This is true even though the evaluation proceeds in the risk neutral economy under the martingale measure $\tilde{Q}_{ST}$.

As the notation also makes explicit, the pricing procedure depends on the particular pair of bonds {S,T} chosen. All other bonds of differing maturities $u \in [0,\tau]$ are assumed to have values at time S, through expression (8), which are F_S measurable. Assuming $\tilde{E}_{ST}(P(S,u)/B(S)) < +\infty$ for all $u \in [0,\tau]$, since the market is complete under conditions (C.1)

- (C.5), every other bond can be duplicated with an admissible s.f.t.s. involving only the two bonds {S,T} and the money market account. Thus, given the two bond price dynamics (as in expression (26)), the spot rate process (as in expression (25)), and the assumptions that Z(S,V) are F_S measurable and $\tilde{Q}_{ST}$ integrable, one can price all the remaining bonds and all contingent claims. These are the two purposes for the arbitrage pricing methodology as stated in the introduction.

As expressions (25) and (26) make clear, the dynamics for the bond price process, spot rate process, and the market prices for risk cannot be chosen independently. Independently specifying the bond price processes parameters, the spot rate processes parameters, and the market prices for risk will in general lead to inconsistent pricing models. This is a serious practical problem when implementing the preceding valuation formula to price traded claims. Indeed, since the market prices of risk must be estimated, a simplified parametric form must be assumed. The existing literature, with the exception of Cox, Ingersoll, Ross [1985], are subject to this difficulty. This is the logic underlying the criticism of the arbitrage pricing methodology presented in Cox, Ingersoll, Ross [1985; p. 398].

The model, as presented above, captures the essence of all the existing arbitrage pricing models. To see this, let us first consider Vasichek [1977], Brennan and Schwartz [1979], and Langetieg [1980]. Since all three models are similar, we focus upon that of Brennan and Schwartz. Brennan and Schwartz's model differs marginally from the above approach. Instead of specifying the two bond processes for {S,T} directly as in expression (26), they derive these expressions from two more fundamental assumptions. First, they exogenously specify a long rate process and a spot rate process. Second, they assume that all bond prices at time t can be written as twice-continuously differentiable functions of the current values of the long rate and short rate. In conjunction, these assumptions (by Ito's lemma) imply condition (26). The analysis could then proceed as above, yielding contingent claim values dependent on the market prices for risk.[6]

Artzner and Delbaen's [1988] analysis differs from Brennan and Schwartz. They exogenously specify only the spot rate process and assume the weaker condition that the bond's price is adapted with respect to the Brownian filtration. In a complete market, they use the martingale approach outlined above to generate bond prices and contingent claim values dependent upon the market price for risk.

Ball and Torous [1983] and Schaefer and Schwartz [1987] exogenously specify the two bond price processes $\{P(t,S),\ P(t,T)\}$ directly. They price contingent claims based on necessary, but not sufficient conditions, for the absence of arbitrage. Unfortunately, both the Ball and Torous model (as shown by Cheng [1987]) and the Schaefer and Schwartz model are inconsistent with stochastic spot rate processes and the absence of arbitrage.

Along with the framework for categorizing the various models, an additional contribution of our approach is to extend the above analysis to eliminate the market prices for risk from the valuation formulas. Intuitively speaking, this is done by utilizing the remaining information contained in the bond price processes to "substitute out" the market prices for risk in the spot rate process (condition (25)). Since the spot rate process is deducible from the bond price (forward rate) processes alone, and the bond price processes are independent of the market price for risk in a risk neutral economy, this should be possible. But first, we need to extend the analysis in the preceding economy. Formally, in the preceding economy, only two bonds $\{S,T\}$ were assumed to trade and the remaining bonds priced by redundancy. This needs to be expanded to allow the simultaneous trading of all available bonds.

Again, we maintain assumptions (C.1) - (C.3) which characterize the forward rate, spot rate, and bond price processes. We now allow trading in the money market account and all bonds of maturities $T \in [0,\tau]$. This is a continuum of tradeable securities. To incorporate this expansion, but still utilize the previous methodology, we define a trading strategy to be any arbitrary, but <u>finite</u> collection of these tradeable securities. Formally, a <u>trading</u>

strategy is defined to be a (n+1)-dimensional stochastic process $\{N_0(t), N_{T_1}(t), \ldots, N_{T_n}(t): t \in [0,\tau]\}$ such that

(i) $n < +\infty$ is F_0-measurable,

(ii) $N_0(t)$, $N_{T_i}(t)$ for $i = 1,\ldots,n$, are measurable, adapted, with $N_{T_i}(t) \equiv 0$ for $t > T_i$, and

(iii) $\int_0^\tau |N_0(t)|r(t)B(t)dt < +\infty$ a.e. Q,

$\int_0^\tau |N_{T_i}(t)[r(t) + b(t,T_i)]|P(t,T_i)dt < +\infty$ a.e. Q for $i = 1,\ldots,n$,

$\int_0^\tau N_{T_i}(t)^2 a_j(t,T_i)^2 P(t,T_i)^2 dt < +\infty$ a.e. Q for $i = 1,\ldots,n$ and $j = 1,2$.

Define the value process V: $[0,\tau] \times \Omega \to R$ for the trading strategy $\{N_0(t), N_{T_1}(t), \ldots, N_{T_n}(t): t \in [0,\tau]\}$ by

$$V(t,\omega) = N_0(t,\omega)B(t) + \sum_{i=1}^{n} N_{T_i}(t,\omega)P(t,T_i) \quad \text{a.e. Q for all } 0 \leq t \leq \tau.$$

The trading strategy is said to be self-financing if

$$V(t) = V(0) + \int_0^t N_0(t)dB(t) + \sum_{i=1}^{n} \int_0^t N_{T_i}(t)dP(t,T_i) \quad \text{a.e. Q}$$

$$\text{for all } 0 \leq t \leq \tau.$$

To be self-financing, by condition (ii) we see that the trading strategy must satisfy $N_0(t) = N_0(S)$ for all $t \geq S$ where $S = \max\{T_1,\ldots,T_n\}$. Thus, the value process at time T relative to the money market account, $V(T)/B(T)$, is F_S measurable where $S = \max\{T_1,T_2,\ldots,T_n\}$.

An arbitrage opportunity is defined to be any s.f.t.s. whose value process $\{V(t): t \in [0,\tau]\}$ satisfies $V(0) \leq 0$, $Q(V(\tau) > 0) > 0$, and $Q(V(\tau) \geq 0) = 1$.

Next, we assume that conditions (C.4) and (C.5) simultaneously hold for all bond pairs $\{S,T\} \in [0,\tau]$ with $0 < S < T \leq \tau$. By proposition 2, we know that for any arbitrarily selected pair, $\{S,T\}$, there exists a unique equivalent martingale measure $\tilde{Q}_{ST}$ making both $Z(t,S)$ and $Z(t,T)$ $\tilde{Q}_{ST}$-martingales on $[0,S]$. Proposition 3 concerning the equality of these measures, motivates the following condition which we assume to hold.

(C.6: Common Equivalent Martingale Measures)

Given conditions (C.1) - (C.3), let (C.4) and (C.5) hold for all bond pairs $\{S,T\} \in [0,\tau]$ with $0 < S < T \leq \tau$. Further, let $\tilde{Q}_{S_1,T_1} = \tilde{Q}_{S_2,T_2}$ (on their common domain) for all pairs $0 < S_1 < T_1 \leq \tau$ and $0 < S_2 < T_2 \leq \tau$.

Define a finitely additive measure $\tilde{Q}: \bigcup_{T<\tau} F_T \to [0,1]$ by the following construction. Given $A \in \bigcup_{T<\tau} F_T$, then $A \in F_{S_0}$ for some $S_0 < \tau$. Define $\tilde{Q}$ by $\tilde{Q}(A) = \tilde{Q}_{ST}(A)$ for any pair $\{S,T\} \in [0,\tau]$ such that $S_0 \leq S < T \leq \tau$. Under condition (C.6), this is well-defined. Furthermore, under condition (C.6) by proposition 3, the measure $\tilde{Q}$ when restricted to F_S is a probability measure which is equivalent to Q and for which $Z(t,T)$ is a $\tilde{Q}$-martingale for all $T \in [0,\tau]$ and $t \in [0,S]$. Denote expectation with respect to $\tilde{Q}$ by $\tilde{E}(\cdot)$.

We define an admissible s.f.t.s. to be any s.f.t.s. such that its value process $\{V(t): t \in [0,\tau]\}$ satisfies $V(t) \geq 0$ a.e. Q for all $t \in [0,\tau]$, $\frac{V(\tau)}{B(\tau)}$ is F_S measurable for some $S < \tau$, and $\{\frac{V(t)}{B(t)}: t \in [0,\tau]\}$ is a $\tilde{Q}$-martingale.

We can now characterize the minimal conditions for the absence of arbitrage opportunities in the expanded economy, which justifies the imposition of condition (C.6).

Proposition 4 (Characterization of the Absence of Arbitrage)

Given conditions (C.1) - (C.3), let (C.4) and (C.5) hold for all bond pairs $\{S,T\} \in [0,\tau]$ with $0 < S < T \leq \tau$.

(a) No arbitrage opportunities implies (C.6).

(b) (C.6) implies there are no arbitrage opportunities in the class of admissible self-financing trading strategies.

Proof: In the appendix. ///

This proposition states that modulo inadmissible trading strategies, the absence of arbitrage in the expanded economy is equivalent to condition (C.6). The logic of the proof is straightforward. Given condition (C.6), the standard martingale argument implies that no arbitrage opportunities exist. Conversely, suppose there existed a quadruple of bonds $\{S_1,T_1,S_2,T_2\}$ and an event $A \in F_{S_1}$ such that $\tilde{Q}_{S_1T_1}(A) \neq \tilde{Q}_{S_2T_2}(A)$. Since markets are complete, there exists the contingent claim 1_A with two different "implicit prices", $\tilde{Q}_{S_1T_1}(A)$ and $\tilde{Q}_{S_2T_2}(A)$. Thus, an arbitrage opportunity can be constructed by buying and selling the different duplicating s.f.t.s.'s involving $\{S_1,T_1\}$ and $\{S_2,T_2\}$. This contradiction yields the result.

Thus, to value contingent claims under the absence of arbitrage in the expanded economy we assume that conditions (C.1) - (C.6) hold for all bond pairs $\{S,T\}$. A contingent claim is defined to be any random variable $X: \Omega \to R$ which is F_S measurable for some $S < \tau$, non-negative, and with $\tilde{E}(X/B(\tau)) < +\infty$. Choose a pair of bonds $\{S,T\} \in [0,\tau]$ such that . $0 < S < T \leq \tau$, then by condition (C.6) and proposition 3,

$$\tilde{E}(X/B(S)) = \tilde{E}_{ST}(X/B(S)). \tag{27}$$

The analysis preceding expression (23) now applies. Since the market is complete, there exists an admissible s.f.t.s. involving $B(t)$, $P(t,S)$, and $P(t,T)$ such that expression (24) is true. Furthermore, expressions (25) and (26) hold as written. To remove the market prices for risk from expression (25), we utilize condition (C.6). By proposition 3 (expression (18)), we have that $\gamma_i(v,S,T)$ are independent of S,T for i = 1,2, denoted by $\phi_i(v) = \gamma_i(v,S,T)$. Equivalently, the no arbitrage condition (expression (19)) is:

$$\int_0^t \alpha(v,t)dv = -\sum_{i=1}^{2}\int_0^t \sigma_i(v,t)\phi_i(v)dv + \sum_{i=1}^{2}\int_0^t \sigma_i(v,t)\int_v^t \sigma_i(v,y)dydv. \tag{28}$$

Substitution of this expression into expression (25) for the spot rate yields:

$$r(t) = f(0,t) + \sum_{i=1}^{2}\int_0^t \sigma_i(v,t)\int_v^t \sigma_i(v,y)dydv + \sum_{i=1}^{2}\int_0^t \sigma_i(v,t)d\tilde{W}_i^{ST}(v). \tag{29}$$

The market prices for risk <u>drop out</u> of expression (25) and they are replaced with an expression involving the volatilities across different maturities of the forward rates, i.e., a "term structure of volatilities." Thus, contingent claim values, expression (27), can be calculated independently of the market prices for risk $\{\phi_1(t),\phi_2(t)\}$. We further illustrate this abstract procedure with concrete examples in the next two sections. Before these examples, however, to complete our discussion of the literature, we point out that the analysis of Ho and Lee [1976] can be viewed as a special case of the construction leading to expression (29), see Heath, Jarrow, Morton [1988].

6. An Example (Constant Volatility)

This section presents an example to illustrate and to clarify the analysis in section 5. This example is interesting because it may prove useful in practical applications due to its computational simplicity. It is also a continuous time limit of Ho and Lee's [1986] model, see Heath, Jarrow, and Morton [1988].

We assume that forward rates satisfy the stochastic process from condition (C.1) with a single Brownian motion and the volatility $\sigma_1(t,T,\omega) \equiv \bar{\sigma} > 0$, a positive constant, i.e.,

$$df(t,T) = \alpha(t,T)dt + \bar{\sigma}dW(t) \text{ for all } T \in [0,\tau] \text{ and } t \in [0,T]. \tag{30}$$

We let the initial forward rate curve $\{f(0,T): T \in [0,\tau]\}$ be measurable and absolutely integrable (as in condition (C.2)). Given a particular, but arbitrary stochastic process for the market price of risk, $\phi: [0,\tau] \times \Omega \to R$ which is predictable and bounded, we assume that the forward rate drift condition (19) is satisfied:

$$\alpha(t,T) = -\bar{\sigma}\phi(t) + \bar{\sigma}^2(T-t) \text{ for all } T \in [0,\tau] \text{ and } t \in [0,T]. \tag{31}$$

It is easy to verify that conditions (C.1) - (C.5) are satisfied. Of course, condition (C.6) is satisfied due to expression (31). This implies, therefore, that contingent claim valuation can proceed as in section 5. Before that, however, we analyze the forward rate process in more detail.

To calculate the solution to expression (30) with the drift (31), substitution and integration yield:

$$f(t,T) = f(0,T) + \bar{\sigma}^2 t(T-t/2) - \bar{\sigma}\int_0^t \phi(v)dv + \bar{\sigma}W(t). \tag{32}$$

By Girsanov's theorem, there exists a unique martingale probability measure $\tilde{Q}$ (defined by expression (15)) such that

$$\tilde{W}(t) = W(t) - \int_0^t \phi(y)dy \tag{33}$$

is a Brownian motion. In terms of this Brownian motion, the stochastic process for the forward rate is:

$$f(t,T) = f(0,T) + \bar{\sigma}^2 t(T-t/2) + \bar{\sigma}\tilde{W}(t). \tag{34}$$

This expression demonstrates that under condition (30) forward rates can be negative with positive probability.

To calculate the stochastic spot rate process under the equivalent martingale measure, set T = t in expression (34):

$$r(t) = f(0,t) + \bar{\sigma}\tilde{W}(t) + \bar{\sigma}^2 t^2/2. \tag{35}$$

The spot rate at time t equals the forward rate at time t (projected from time 0), a random deviation, and an adjustment factor. Spot rates can also be negative with positive probability.

The dynamics of the bond price process over time is given by substituting expression (34) into expression (2).

The solution is:

$$P(t,T) = e^{-\int_t^T f(0,y)dy \; - \; (\bar{\sigma}^2/2)Tt(T-t) \; - \; \bar{\sigma}(T-t)\tilde{W}(t)} \quad \text{or} \tag{36.a}$$

$$P(t,T) = [P(0,T)/P(0,t)]e^{-(\bar{\sigma}^2/2)Tt(T-t)-\bar{\sigma}(T-t)\tilde{W}(t)}. \tag{36.b}$$

The bond's price process depends on the initial forward rate curve from time t to time T, $\{f(0,y): y \in [t,T]\}$, the volatility parameter $\bar{\sigma}$, the maturity date T, and the current value of the Brownian motion $\tilde{W}(t)$. Since $\tilde{W}(t)$ is not directly observable, the spot rate process in (35) can be used to provide an observable quantity. Using this approach, the bond's price can be written as:

$$P(t,T) = e^{-\int_t^T [f(0,y) \; - \; f(0,t)]dy \; - \; (\bar{\sigma}^2/2)t(T-t)^2 \; - \; (T-t)r(t)} \; . \tag{37}$$

Given the bond price process as in (36), it is easy to price options on bonds using the analysis given in section 5. Indeed, consider a European call option on the bond P(t,T) with an exercise price of K and a maturity date t^* where $0 \leq t \leq t^* \leq T$. Let C(t) denote the value of this call option at time t. By definition, the cash flow to the call option at maturity is:

$$C(t^*) = \max[P(t^*,T) - K,\ 0]. \tag{38}$$

Using the valuation procedure in section 5 for pricing contingent claims, the value of the call at time t can be written as:

$$C(t) = \tilde{E}(\max[P(t^*,T) - K,\ 0]B(t)/B(t^*)\,|F_t). \tag{39}$$

An explicit calculation, contained in the appendix, shows that expression (39) simplifies to:

$$C(t) = P(t,T)\Phi(h) - KP(t,t^*)\Phi(h-\bar{\sigma}(T-t^*)\sqrt{(t^*-t)}) \tag{40}$$

where $h = [\log(P(t,T)/KP(t,t^*)) + (1/2)\bar{\sigma}(T-t^*)^2(t^*-t)]/\bar{\sigma}(T-t^*)\sqrt{(t^*-t)}$ and

$\Phi(\cdot)$ is the cumulative normal distribution.

The value of the bond option is given by the Black-Scholes formula but in which the bond price, $P(t,T)$, replaces the "stock price," $P(t,t^*)$ replaces the "discount factor," and $\bar{\sigma}(T-t^*)$ replaces the "volatility of the stock." The parameter, $\bar{\sigma}(T-t^*)$, is not equal to the variance of the instantaneous return on the T-maturity bond,

$$\bar{\sigma}(T-t)dt = \sqrt{var}(dP(t,T)/P(t,T)). \tag{41}$$

Rather, it is equivalent to the variance of the instantaneous return on the forward price (at time t^*) of a T-maturity bond, i.e.,

$$\bar{\sigma}(T-t^*)dt = \sqrt{var}(d[P(t,T)/P(t,t^*)]/[P(t,T)/P(t,t^*)]). \tag{42}$$

This distinction is crucial for parameter estimation procedures involving expression (40).

The valuation formula (40) is further clarified by studying an alternative derivation. An easy calculation involving expressions (9) and (31) shows that the T-maturity bond process satisfies:[7]

$$dP(t,T)/P(t,T) = [r(t) + \phi(t)\bar{\sigma}(T-t)]dt - \bar{\sigma}(T-t)dW(t) \text{ for all } T \in [t,\tau]. \quad (43)$$

Expression (43) has the appropriate form to apply Merton's [1973] stochastic interest rate option pricing model. This model generates expression (40) with the following identifications: the T-bond price P(t,T) represents the "stock," the t^*-bond represents the "bond," and the correlation coefficient between the two processes is unity since they are generated by the same Brownian motion.

7. Example with Multiple Brownian Motions

The previous example treated a forward rate process based on a single Brownian motion. This section presents an example based on two independent Brownian motions which is easy to compute and allows different maturity bonds to have imperfectly correlated (instantaneous) returns. Unfortunately, this model still has negative forward rates with positive probability.

Assume that forward rates satisfy condition (C.1) with the volatilities $\sigma_1(t,T,\omega) \equiv \bar{\sigma}_1 > 0$ and $\sigma_2(t,T,\omega) \equiv \bar{\sigma}_2 e^{-(\lambda/2)(T-t)} > 0$ where $\bar{\sigma}_1$, $\bar{\sigma}_2$, λ are strictly positive constants, i.e.,

$$df(t,T) = \alpha(t,T)dt + \bar{\sigma}_1 dW_1(t) + \bar{\sigma}_2 e^{-(\lambda/2)(T-t)} dW_2(t) \quad (44)$$

for all $T \in [0,\tau]$ and $t \in [0,T]$.

Expression (44) indicates that instantaneous changes in forward rates are caused by two sources of randomness $\{W_1(t), W_2(t): t \in [0,\tau]\}$. The first, $\{W_1(t): t \in [0,\tau]\}$, can be interpreted as a "long-run factor" since it uniformly shifts all maturity forward rates equally. The second, $\{W_2(t): t \in [0,\tau]\}$, however, affects the short maturity forward rates significantly more than it does long term rates. Consequently, it can be interpreted as a spread between a "short" and "long term factor."

The volatility functions are strictly positive and bounded. Furthermore, the matrix

$$\begin{pmatrix} a_1(t,S) & a_2(t,S) \\ a_1(t,T) & a_2(t,T) \end{pmatrix} = \begin{pmatrix} -\bar{\sigma}_1(S-t) & + 2\bar{\sigma}_2(e^{-(\lambda/2)(S-t)} - 1)/\lambda \\ -\bar{\sigma}_1(T-t) & + 2\bar{\sigma}_2(e^{-(\lambda/2)(T-t)} - 1)/\lambda \end{pmatrix} \tag{45}$$

is non-singular for all $t,S,T \in [0,\tau]$ such that $t \leq S < T$.

We arbitrarily fix two bounded, predictable processes for the market prices of risk, $\phi_i: [0,\tau] \times \Omega \to R$ for $i = 1,2$. To guarantee the absence of arbitrage (by proposition 4), we set

$$\alpha(t,T) = -\bar{\sigma}_1\phi_1(t)-\bar{\sigma}_2e^{-(\lambda/2)(T-t)}\phi_2(t)+\bar{\sigma}_1^2(T-t)-2(\bar{\sigma}_2^2/\lambda)e^{-(\lambda/2)(T-t)}(e^{-(\lambda/2)(T-t)}-1). \tag{46}$$

Substitution into expression (44) yields:

$$\begin{aligned} df(t,T) = {} & [-\bar{\sigma}_1\phi_1(t)-\bar{\sigma}_2e^{-(\lambda/2)(T-t)}\phi_2(t)+\bar{\sigma}_1^2(T-t)-2(\bar{\sigma}_2^2/\lambda)e^{-(\lambda/2)(T-t)}(e^{-(\lambda/2)(T-t)}-1)]dt \\ & + \bar{\sigma}_1dW_1(t) + \bar{\sigma}_2e^{-(\lambda/2)(T-t)}dW_2(t). \end{aligned} \tag{47}$$

A direct calculation generates the solution:

$$\begin{aligned} f(t,T) = {} & f(0,T) - \bar{\sigma}_1 \int_0^t \phi_1(v)dv - \bar{\sigma}_2 \int_0^t e^{-(\lambda/2)(T-v)} \phi_2(v)dv \\ & + \bar{\sigma}_1^2t(T-t/2) - 2(\bar{\sigma}_2/\lambda)^2[e^{-\lambda T}(e^{\lambda t} - 1) - 2e^{-(\lambda/2)T}(e^{(\lambda/2)t} - 1)] \\ & + \bar{\sigma}_1W_1(t) + \bar{\sigma}_2 \int_0^t e^{-(\lambda/2)(T-v)} dW_2(v). \end{aligned} \tag{48}$$

Proposition 3 implies that there exists a unique equivalent martingale measure $\tilde{Q}$ making $Z(t,T)$ martingales for all $T \in [0,\tau]$. The definitions of the Brownian motions $\{\tilde{W}_1(t),\tilde{W}_2(t): t \in [0,\tau]\}$ under the new measure (expression (15)), are:

$$\tilde{W}_i(t) = W_i(t) - \int_0^t \phi_i(v)dv \text{ for } i = 1,2. \tag{49}$$

Substitution of these quantities into expression (48) simplifies it further,

$$f(t,T) = f(0,T) + \bar{\sigma}_1^2 t(T-t/2) - 2(\bar{\sigma}_2/\lambda)^2[e^{-\lambda T}(e^{\lambda t} - 1) - 2e^{-(\lambda/2)T}(e^{(\lambda/2)t} - 1)]$$
$$+ \bar{\sigma}_1\tilde{W}_1(t) + \bar{\sigma}_2 \int_0^t e^{-(\lambda/2)(T-v)}\, d\tilde{W}_2(v). \tag{50}$$

This expression shows that forward rates can be negative with positive probability.

The spot rate follows the simpler process:

$$r(t) = f(0,t) + \bar{\sigma}_1^2 t^2/2 - 2(\bar{\sigma}_2/\lambda)^2[(1-e^{-\lambda t}) - 2(1-e^{-(\lambda/2)t})]$$
$$+ \bar{\sigma}_1\tilde{W}_1(t) + \bar{\sigma}_2 \int_0^t e^{-(\lambda/2)(t-v)}\, d\tilde{W}_2(v). \tag{51}$$

At any time $t \in [0,\tau]$, spot rates are normally distributed under this risk neutral measure with a mean $(f(0,t) + \bar{\sigma}_1^2 t^2/2 - 2(\bar{\sigma}_2/\lambda)^2[(1-e^{-\lambda t}) - 2(1-e^{-(\lambda/2)t})]$ and variance $(\bar{\sigma}_1^2 t + (\bar{\sigma}_2^2/\lambda)(1-e^{-\lambda t}))$. Both parameters are increasing in t.

The dynamics for the bond price are:

$$P(t,T) = \exp\{-\int_t^T f(0,y)dy - (\bar{\sigma}_1^2/2)Tt(T-t) - (2\bar{\sigma}_2^2/\lambda^3)[(e^{-\lambda t}-1)(e^{-\lambda T}-e^{-\lambda t})$$
$$- 4(e^{(\lambda/2)t}-1)(e^{-(\lambda/2)T}-e^{-(\lambda/2)t})]$$
$$- \bar{\sigma}_1(T-t)\tilde{W}_1(t) + (2\bar{\sigma}_2/\lambda)(e^{-(\lambda/2)T}-e^{-(\lambda/2)t})\int_0^t e^{(\lambda/2)v} d\tilde{W}_2(v)\}. \tag{52}$$

Using the spot rate and any other forward rate, both $\tilde{W}_1(t)$ and $\tilde{W}_2(t)$ could be substituted out of expression (52), yielding an equivalent expression based on only observables.

As in section 6, we can calculate the value of a European call option on the bond $P(t,T)$ with an exercise price of K and a maturity date t^* where $0 \leq t \leq t^* \leq T$. Let $C(t)$ denote the value of this call option at time t. By definition, the cash flow to the call option at maturity is:

$$C(t^*) = \max[P(t^*,T)-K,0].$$

Using the valuation procedure of section 5, the call's value at time t is:

$$C(t) = \tilde{E}(\max[P(t^*,T)-K,0]B(t)/B(t^*)|F_t). \tag{53}$$

An explicit calculation[8] gives:

$$C(t) = P(t,T)\Phi(h) - KP(t,t^*)\Phi(h-q)$$

where

$$h = [\log(P(t,T)/KP(t,t^*)) + (1/2)q^2]/q,$$

$$q^2 = \bar{\sigma}_1^2(T-t^*)^2(t^*-t) + (4\bar{\sigma}_2^2/\lambda^3)(e^{-(\lambda/2)T}-e^{-(\lambda/2)t^*})^2(e^{\lambda t^*}-e^{\lambda t}), \text{ and} \tag{54}$$

$\Phi(\cdot)$ is the cumulative normal distribution.

Again, the value of the bond option is given by the Black-Scholes formula in which the bond price, $P(t,T)$, replaces the "stock price," $P(t,t^*)$ replaces the "discount factor," and $q^2 = \text{var}(d[P(t,T)/P(t,t^*)]/[P(t,T)/P(t,t^*)])$ replaces the "volatility of the stock." The volatility parameter q^2 is the instantaneous variance of the forward price (at time t^*) of the T-maturity bond which depends on the forward rate process's parameters $\bar{\sigma}_1$, $\bar{\sigma}_2$, and λ.

8. A Class of Stochastic Differential Equations

The previous two sections provide examples of forward rate processes satisfying conditions (C.1) - (C.6). These processes all have deterministic volatilities which are independent of the state of nature $\omega \in \Omega$. This section provides a class of processes which allow the volatilities to depend on $\omega \in \Omega$. This class of processes is interesting from a practical perspective since they are estimable, yet consistent with a wide range of potential stochastic processes for forward rates.

This class of processes can be described as the solutions (if they exist) to the following stochastic integral equation with restricted drift.

$$f(t,T) - f(0,T) = \int_0^t \alpha(v,T,\omega)dv + \sum_{i=1}^{2} \int_0^t \sigma_i(v,T,f(v,T))dW_i(v) \qquad (55)$$

$$\text{for all } 0 \leq t \leq T$$

where

$$\alpha(v,T,\omega) \equiv - \sum_{i=1}^{2} \sigma_i(v,T,f(v,T))[\phi_i(v) - \int_v^T \sigma_i(v,y,f(t,y))dy]$$

$$\text{for all } T \in [0,\tau],$$

σ_i: $\{t,S): 0 \leq t \leq S \leq T\} \times R \to R$ is jointly measurable and satisfies

$\int_0^T \sigma_i(t,T,f(t,T))^2 dt < +\infty$ a.e. Q for i = 1,2, and

ϕ_i: $\Omega \times [0,\tau] \to R$ is a bounded predictable process for i = 1,2.

We now study sufficient conditions on the volatility functions such that strong solutions to this class of stochastic differential equations exist. A continuous time analogue of Ho and Lee's [1986] model as given in expression (32) is seen to be a special case of this theorem. A surprising example is also provided below to show that additional hypotheses are needed. This example concerns the forward rate process arising from a proportional volatility function.

A key step in proving the existence theorem is the following lemma, which asserts that the existence of a class of forward rate processes in the initial economy is guaranteed if and only if it can be guaranteed in an "equivalent risk neutral economy."

Lemma 1: (Existence in an Equivalent Risk Neutral Economy)

[The processes $\{f(t,T): T \in [0,\tau]\}$ satisfy (55) with $\gamma_i(t,S,T) = \phi_i(t)$

for all $0 \leq t < S < T \leq \tau$ and $i = 1, 2$] (56)

if and only if

[The process $\{\tilde{\alpha}(\cdot,T): T \in [0,\tau]\}$ defined by

$$\tilde{\alpha}(t,T) = \sum_{i=1}^{2} \sigma_i(t,T,f(t,T)) \int_t^T \sigma_i(t,v,f(t,v))dv \text{ for all } T \in [0,\tau] \tag{57}$$

satisfies (55) with $\tilde{\alpha}(t,T)$ replacing $\alpha(t,T)$, $\tilde{W}_i(t)$ replacing $W_i(t)$ where $\tilde{W}_i(t) \equiv W_i(t) - \int_0^t \phi_i(y)dy$ is a Brownian motion with respect to $[(\Omega,F,\tilde{Q}),\{F_t: t \in [0,\tau]\}]$, and $\tilde{Q}$ replacing Q where $d\tilde{Q}/dQ = \exp\{\sum_{i=1}^{2} \int_0^T \phi_i(t)dW_i(t) - (1/2) \sum_{i=1}^{2} \int_0^T \phi_i(t)^2 dt\}]$.

Proof: In the appendix. ///

This lemma shows that there are "two degrees of freedom" in obtaining drift processes consistent with the arbitrage free pricing condition of proposition 4. These two degrees of freedom are the market prices for risk. Once the market prices for risk are specified, the forward rate drift process is uniquely determined (if it exists).

Lemma 2: (Existence of Forward Rate Processes)

Let σ_i: $\{(t,s): 0 \leq t \leq s \leq T\} \times R \to R$ for $i = 1, 2$ be Lipschitz continuous in the last argument,[9] non-negative and bounded.

Let $(\Omega,F,\tilde{Q})$ be any equivalent probability space with $\{\tilde{W}_1(t),\tilde{W}_2(t): t \in [0,\tau]\}$ independent Brownian motions;

Then, there exists a jointly continuous $f(\cdot,\cdot)$ satisfying (55) with $\tilde{W}(t)$ replacing $W(t)$ and

$$\tilde{\alpha}(t,T) = \sum_{i=1}^{2} \sigma_i(t,T,f(t,T)) \int_t^T \sigma_i(t,v,f(t,v))dv \text{ for all } T \in [0,\tau] \text{ replacing } \alpha(t,T).$$

Proof: Morton [1988]. ///

The proof of this lemma is contained in Morton [1988]. The hypotheses of lemma 2 differ from the standard hypotheses guaranteeing the existence of strong solutions to stochastic differential equations in the boundedness condition on the volatility functions. Lemmas 1 and 2 combined generate the major proposition of this section.

Proposition 5: (Existence of Arbitrage-Free Forward Rate Drift Processes)

Let $\phi_i: [0,\tau] \times \Omega \to R$ be a bounded predictable process.

Let $\sigma_i: \{(t,s): 0 \leq t \leq s \leq T\} \times R \to R$ for i = 1, 2 be Lipschitz continuous in the last argument, non-negative, and bounded; then, there exists a jointly continuous forward rate process satisfying condition (55).

Proof: In the appendix. ///

By appending the nonsingularity condition (C.5), this proposition provides sufficient conditions guaranteeing the existence of a class of forward rate processes satisfying conditions (C.1) - (C.6). This set of sufficient conditions is easily verified in applications.

To show that this boundedness condition in proposition 5 cannot be substantially weakened, we consider the special case of a single Brownian motion where $\sigma(t,T,f(t,T)) \equiv \bar{\sigma} \cdot f(t,T)$ for a fixed constant $\bar{\sigma} > 0$. This volatility function is positive and Lipschitz continuous, but not bounded. The example is the simplest forward rate process consistent with nonnegative forward rates.

For this volatility function, the no arbitrage condition of proposition 5 with $\phi_i(t) \equiv 0$ implies that the forward rate process must satisfy:

$$df(t,T) = [\bar{\sigma}f(t,T)\int_t^T \bar{\sigma}f(t,v)dv]dt + \bar{\sigma}f(t,T)dW(t) \text{ for all } T \in [0,\tau] \text{ and } t \in [0,T]. \quad (58)$$

In integral form expression (58) can be written as:

$$f(t,T) = f(0,T)\exp\{\int_0^t \int_u^T f(u,v)dvdu\}\exp\{-\bar{\sigma}^2 t/2 + \bar{\sigma}W(t)\} \text{ for all } T \in [0,\tau] \text{ and } t \in [0,T]. \quad (59)$$

Unfortunately, it can be shown (see Morton [1988]) that there is no finite valued solution to expression (59).

By proposition 4, this forward rate process is therefore inconsistent with arbitrage free pricing. In fact, it can be shown that under (58), in finite time, forward rates explode with positive probability for the martingale measure, and hence for any equivalent probability measure. Infinite forward rates generate zero bond prices and hence arbitrage opportunities.

The forward rate process given in (58) is in some ways the simplest model consistent with nonnegative forward rates. The incompatibility of this process with arbitrage free bond prices raises the issue as to the general existence of a drift process $\{\alpha(\cdot,T): T \in [0,\tau]\}$ satisfying conditions (C.1) - (C.6), and with nonnegative forward rates. This existence issue is resolved in the next section through an example.

9. An Example with Nonnegative Interest Rates

The constant volatility example yields a consistent forward rate process (in section 6), but it has the possibility of negative forward rates. The proportional volatility example (in section 8) has nonnegative forward rates, but they explode with positive probability. This section provides an example of a forward rate process consistent with conditions (C.1) - (C.6) and with nonnegative forward rates.

This example can be thought of as a combination of the two previous examples. When forward rates are "small" the process has a proportional volatility, and when forward rates are "large" it has a constant volatility. Intuitively, as shown below, rates cannot fall below zero nor explode. Formally, consider a single Brownian motion process with $\sigma(t,T,f(t,T)) = \bar{\sigma}\min(f(t,T),\lambda)$ for $\bar{\sigma},\lambda > 0$ positive constants. This volatility function is positive, Lipschitz continuous, and bounded, thus, for an arbitrary initial forward rate curve proposition 5 guarantees the existence of a jointly continuous f(t,T) which solves

$$df(t,T) = \bar{\sigma}\min(f(t,T),\lambda)\left(\int_t^T \bar{\sigma}\min(f(t,s),\lambda)ds\right)dt + \bar{\sigma}\min(f(t,T),\lambda)dW(t). \tag{60}$$

The following proposition guarantees that this forward rate process remains positive for any strictly positive initial forward rate curve.

<u>Proposition 6</u> (Nonnegative Forward Rate Process)

Given f(t,T) solves expression (60) and given an arbitrary initial forward rate curve

$$f(0,t) = I(t) > 0 \text{ for all } t \in [0,\tau],$$

then with probability one, $f(t,T) \geq 0$ for all $T \in [0,\tau]$ and $t \in [0,T]$.

<u>Proof</u>: In the appendix. ///

The martingale measure for this forward rate process is given in expression (15). Since the forward rate process is a mixture of the constant volatility and proportional volatility models, it is easy to see (using expression (60)) that the forward rate drifts $\{\alpha(\cdot,T): T \in [0,\tau]\}$ will be dependent upon the path of the Brownian motion. This path dependent property of the forward rate process makes computation more difficult for this model than for the constant volatility example. Nonetheless, we conjecture that the multiple Brownian motion analogue of expression (60) will prove to be the most useful from an empirical perspective. Another forward rate process consistent with non-negative forward rates is provided in the next section (expression (68)).

10. The Equilibrium Pricing versus the Arbitrage Pricing Methodology

The crucial difference between our methodology for pricing contingent claims on the term structure of interest rates and that of Cox, Ingersoll, and Ross [1985] is the difference between the arbitrage free pricing methodology and that of equilibrium pricing, respectively. The arbitrage pricing approach is more robust than the equilibrium pricing approach because it requires less structure on preferences, endowments, and the trading mechanism. Indeed, equilibrium requires an exogenous specification of preferences, beliefs, and a specific trading mechanism to generate a valuation model. Observable prices are then tested against the valuation model. In contrast, arbitrage requires only a specific trading mechanism defined on a subset of exogenously given prices to generate a valuation model. This valuation model is then tested against the remaining observable prices. The arbitrage free pricing approach is potentially consistent with many alternative equilibrium based models.

To clarify the relationship between our approach to pricing contingent claims and the equilibrium approach, we illustrate how to describe (or model) the equilibrium determined Cox, Ingersoll, Ross [1985] square root model (CIR) in our framework. The CIR model is based on a single state variable, represented by the spot interest rate $r(t)$ for $t \in [0,\tau]$. The spot rate is assumed to follow a square root process

$$dr(t) = K(\theta(t)-r(t))dt + \sigma\sqrt{r(t)}\, dW(t) \tag{61}$$

where $r(0)$, K, σ are strictly positive constants,

θ: $[0,\tau] \to (0,+\infty)$ is a continuous function of time,

$\{W(t);\ t \in [0,\tau]\}$ is a standard Wiener process initialized at zero, and

$2K\theta(t) \geq \sigma^2$ for all $t \in [0,\tau]$.[10]

Although this stochastic differential equation has a solution (see Feller [1951]), an explicit representation is unavailable. In equilibrium, CIR show that the equilibrium bond dynamics are:

$$dP(t,T) = r(t)[1-\lambda\bar{B}(t,T)]P(t,T)dt - \bar{B}(t,T)P(t,T)\sigma\sqrt{r(t)}\, dW(t) \tag{62}$$

where λ is a constant,

$\bar{B}(t,T) = 2(e^{\gamma(T-t)}-1)/[(\gamma+K+\lambda)(e^{\gamma(T-t)}-1) + 2\gamma]$,

and $\gamma = ((K+\lambda)^2 + 2\sigma^2)^{1/2}$.

The parameter λ is related to the market price of risk, $\phi(t)$, as follows:[11]

$$\begin{aligned}\phi(t) &\equiv [E(dP(t,T)/P(t,T)) - r(t)dt]/\sqrt{\mathrm{Var}}\,(dP(t,T)/P(t,T)) \\ &= -\lambda\sqrt{r(t)}/\sigma.\end{aligned} \tag{63}$$

Although stochastic, the market price of risk is restricted in equilibrium to be of this particular functional form. CIR solve for the bond price process

$$P(t,T) = \hat{A}(t,T)e^{-\bar{B}(t,T)r(t)} \tag{64}$$

where $\hat{A}(t,T) = \exp(-K \int_t^T \theta(s)\bar{B}(s,T)ds)$.

Under this process, the forward rate is:

$$f(t,T) = r(t)(\partial\bar{B}(t,T)/\partial T) + K \int_t^T \theta(s)(\partial\bar{B}(s,T)/\partial T)ds. \tag{65}$$

By Ito's lemma, the stochastic differential equation satisfied by f(t,T) is:

$$df(t,T) = r(t)(\partial^2\bar{B}(t,T)/\partial t\partial T - \partial\bar{B}(t,T)/\partial T)dt + (\partial\bar{B}(t,T)/\partial T)\sigma\sqrt{r(t)}\, dW(t). \tag{66}$$

A solution to this equation, in terms of $r(t)$, $\theta(t)$, λ, K, σ is given in equation (65). An explicit solution to (65) in terms of the original parameters for the stochastic process for $r(t)$ (i.e., $r(0)$, $\theta(t)$, K, σ) is unavailable.

In contrast, our approach starts with an exogeneous forward rate process, initialized at an arbitrary, but fixed initial forward rate curve $\{f(0,T): T \in [0,\tau]\}$. Given its parameters, CIR's model has a predetermined functional form for the forward rate process at time 0 given by expression (65) and repeated below for convenience:

$$f(0,T) = r(0)(\partial\bar{B}(0,T)/\partial T) + K\int_0^T \theta(s)(\partial\bar{B}(s,0)/\partial T)ds \text{ for all } T \in [0,\tau]. \tag{67}$$

To match any arbitrary, but given initial forward rate curve, CIR suggest that one "inverts" expression (67) for $\{\theta(t): t \in [0,\tau]\}$ to make the spot rate process's parameters implicitly determined by the initial forward rate curve, see CIR [p. 395]. CIR never prove that such an inversion is possible, i.e., that a "solution" $\{\theta(t): t \in [0,\tau]\}$ exists to expression (67). We show in the appendix that if $\{\partial f(0,T)/\partial T: T \in [0,\tau]\}$ exists and is continuous, then there is a unique continuous solution to equation (67). In fact, using standard procedures, the solution $\{\theta(s): s \in [0,\tau]\}$ to equation (67) can be approximated to any order of accuracy desired (see Taylor and Lay [1980; pp. 196-201]).

Hence, in our framework we have that CIR's term structure model can be written as:

$$df(t,T) = r(t)(\partial^2\bar{B}(t,T)/\partial t\partial T - \partial\bar{B}(t,T)/\partial T)dt + (\partial\bar{B}(t,T)/\partial T)\sigma\sqrt{r(t)}\, dW(t) \tag{68}$$

where $r(t) = [f(t,T) - K \int_t^T \theta(s)(\partial\bar{B}(s,T)/\partial T)ds]/(\partial\bar{B}(t,T)/\partial T)$,

$\{f(0,T): T \in [0,\tau]\}$ is a continuously differentiable fixed, initial forward rate curve, and

$\theta: [0,\tau] \to (0,+\infty)$ is the unique continuous solution to expression (67).

To apply our analysis based on expression (68), we need to guarantee that conditions (C.1) - (C.6) are satisfied. Recall that conditions (C.1) - (C.3) guarantee that the bond price process satisfies expression (8). Next, given expression (8), conditions (C.4) and (C.5) guarantee that for any pair of bonds $\{S,T\}$ an equivalent martingale measure $\tilde{Q}_{ST}$ exists and is unique. Finally, condition (C.6) ensures that the martingale measure is identical across all pairs of bonds. These conditions are sufficient to price all contingent claims when starting from forward rates.

Alternatively, CIR exogenously specify the spot rate process. Consequently, using different methods, they are able to guarantee that the bond price process satisfies expression (8), (see equation (62)). Hence, we don't need to check sufficient conditions (C.1) - (C.3), since expression (8) is the starting point of our analysis. Next, given that the bond prices are generated by an equilibrium with a single Brownian motion, conditions (C.4), (C.5), and (C.6) are easily verified. In fact, to check condition (C.6) one only needs to verify that expression (19) is satisfied. From proposition 3, this is:

$$\begin{aligned} E(df(t,T))/dt = & -\phi(t)\sigma\sqrt{r(t)}\,(\partial\bar{B}(t,T)/\partial T) \\ & + \sigma^2(\partial\bar{B}(t,T)/\partial T)r(t)\int_t^T(\partial\bar{B}(t,s)/\partial T)ds \end{aligned} \tag{69}$$

for all $0 \leq t \leq T \leq \tau$.

It is shown in the appendix that $\phi(t)$ from expression (63) and $E(df(t,T))$ from expression (68) satisfy this expression.

Given the form of the CIR model as in expression (68), we can now proceed directly as in section 5 to price contingent claims. This analysis will generate the identical contingent claim values as in CIR subject to the determination of $\{\theta(s): s \in [0,\tau]\}$. Note that the forward rate's quadratic variation

$$\langle f(t,T)\rangle_t = \int_0^t [(\partial\bar{B}(s,T)/\partial T)\sigma\sqrt{r(s)}]^2 \, ds$$

depends on the parameters λ, σ, K, r(0), and $\{f(0,T): T \in [0,\tau]\}$. The parameter λ, however, is functionally related to the market price of risk (see expression (63)). This makes contingent claim valuation explicitly dependent on this parameter as well (e.g., see CIR [expression (32), p. 396]).

With this analysis behind us, we can now discuss some differences between the two pricing approaches. First, CIR's model fixes a particular market price for risk (condition (63)) and endogenously derives the stochastic process for forward rates (expression (66)). In contrast, our approach takes the stochastic process for forward rates as a given (it could be from an equilibrium model) and prices contingent claims from them. The contingent claim values, therefore, only depend on the parameters of the forward rate processes quadratic variation. These values are independent of the market prices for risk to the extent that it is possible for the forward rate process to be generated by different economies and therefore different market prices for risk.

11. Summary

This paper presents a new methodology for pricing the term structure of interest rates. Given an initial forward rate curve and a mechanism which describes how it fluctuates, we develop an arbitrage pricing model which yields contingent claim valuations which are independent of the market prices for risk. This is in contrast to the existing methodology which generates pricing models whose values require estimation of these quantities.

For practical applications, we specialize our abstract economy and study particular examples. For these examples, closed form solutions are obtained for some contingent claims depending only upon observables and the forward rate volatilities. These models are testable and their empirical verification awaits subsequent research.

Although our model for the forward rate process is generated by two Brownian motions, this simplication was imposed for expositional purposes. The general theory of this paper readily generalizes to forward rate processes generated by a finite number of independent Brownian motions. Given that a continuum of bonds trade, sufficient discount bonds exist to complete the analysis. Generalized versions of propositions 1, 2, 3, 4, 5 and 6 are easily obtained, using the multivariate versions of Girsanov's theorem and the techniques for proving strong solutions to stochastic differential equations.

The paper can be generalized in another fashion. Our term structure model can be imbedded into the larger economy of Harrison and Pliska [1981] which includes trading in alternative risky assets (e.g., stocks) generated by additional (perhaps distinct) independent Brownian motions. Our model provides a consistent structure for the interest rate process employed therein. This merging of the two analyses can be found in Amin and Jarrow [1989].

FOOTNOTES

1. The text assumes familiarity with the standard terminology of continuous time stochastic processes. For the appropriate definitions, we refer the reader to Elliott [1982].

2. This is equivalent to $r(t) = \lim_{h \to 0} [1-P(t,t+h)]/P(t,t+h)h = f(t,t)$.

3. To see this, note that $0 < B(t,\omega) < +\infty$ a.e. Q for all $t \in [0,\tau]$ if

$$\int_0^{\tau} |r(t,\omega)|dt < +\infty \text{ a.e. } Q.$$

But

$$\int_0^{\tau} |r(t,\omega)|dt \leq \int_0^{\tau} |f(0,t)|dt + \int_0^{\tau}\int_0^{t} |\alpha(v,t)|dvdt + \sum_{i=1}^{2} \int_0^{\tau} \left|\int_0^{t} \sigma_i(v,t,\omega)dW_i(v)\right|dt$$

by expression (6). This is finite a.e. by condition (C.2).

4. Two measures P, Q on (Ω,F) are said to be <u>equivalent</u> if $P(A) = 0$ if and only if $Q(A) = 0$.

5. For the case of a single Brownian motion, condition (C.5) simplifies to the statement that $\sigma_1(t,T,f(t,T)) > 0$ a.e. $Q \times \lambda$. This case will be utilized in subsequent examples.

6. Brennan and Schwartz [1979], however, didn't use this martingale approach. Instead, they priced based on necessary conditions given by the partial differential equation satisfied by a contingent claim's value under condition (18).

7. Expression (9) and (19) yield

$$dP(t,T)/P(t,T) = [r(t) - \int_t^T \alpha(t,v)dv + 1/2 \sum_{i=1}^{2} [\int_t^T \sigma_i(t,v)dv]^2]dt$$

$$+ \sum_{i=1}^{2} a_i(t,T)dW_i(t).$$

with

$$- \int_t^T \alpha(t,v)dv = + \sum_{i=1}^{2} [\int_t^T \sigma_i(t,v)dv]\phi_i(t) - \sum_{i=1}^{2} \int_t^T \sigma_i(t,v)[\int_t^v \sigma_i(t,y)dy]dv.$$

But

$$(1/2) [\int_t^T \sigma_i(t,v)dv]^2 = \int_t^T \sigma_i(t,v)[\int_t^v \sigma_i(t,y)dy]dv.$$

Substitution yields,

$$dP(t,T)/P(t,T) = [r(t) + \sum_{i=1}^{2} [\int_t^T \sigma_i(t,v)dv]\phi(t)]dt + \sum_{i=1}^{2} a_i(t,T)dW_i(t).$$

Using $a_i(t,T) = - \int_t^T \sigma_i(t,v)dv$ we get:

$$dP(t,T)/P(t,T) = [r(t) - \sum_{i=1}^{2} a_i(t,T)\phi(t)]dt + \sum_{i=1}^{2} a_i(t,T)dW_i(t).$$

8. This calculation can be found in Brenner and Jarrow [1988]. One method is to write the dynamics for the bond price process as in expression (43) and apply Merton's [1973] formula.

9. <u>Lipschitz continuous</u> means that there exists a positive constant K such that $|\sigma(t,T,x) - \sigma(t,T,y)| \leq K|x-y|$ for all $t,T, \epsilon [0,\tau]$ and $x,y \epsilon R$.

10. This last condition guarantees that zero is an inaccessible boundary for spot rates (see CIR [p. 391]).

11. This is most easily seen from footnote 7.

APPENDIX

Proof of Expression (8):

Before proving expression (8), we need to prove a generalized form of Fubini's theorem for stochastic integrals. This proof of Lemma 0.1 follows Ikeda and Watanabe [1981; p. 116] very closely, and is consequently omitted.

Lemma 0.1

Let (Ω,F,Q) be a probability space, (F_t) a reference family.

$M \in \mathcal{M}_2^c$, i.e., a continuous square-integrable martingale such that $M_0 = 0$ a.s.

Let $\{\Phi(t,a,\omega): (t,a) \in [0,\tau] \times [0,\tau]\}$ be a family of real random variables such that

(i) $((t,\omega),a) \in \{([0,\tau] \times \Omega) \times [0,\tau]\} \to \Phi(t,a,\omega)$ is $L \times B[0,\tau]$ measurable where L is the smallest σ-field on $[0,\tau] \times \Omega$ such that all left-continuous (F_t) adapted processes are measurable, and $B[0,\tau]$ is the Borel σ-field on $[0,\tau]$.

(ii) $\int_0^t \Phi(s,a,\omega)dM_s \in \mathcal{M}_2^c$. Since $\Phi(s,a,\omega)$ is predictable for all $a \in [0,\tau]$ (this follows by (i) above), this condition is that $E(\int_0^t \Phi^2(s,a,\omega)d\langle M\rangle_s) < +\infty$ for all $t \in [0,\tau]$.

(iii) $(a,\omega) \to \int_0^t \Phi(s,a,\omega)dM_s$ is $B[0,\tau] \times F$ measurable for all $0 \leq t \leq \tau$.

Let $\mu(da)$ be a non-negative Borel measure on R.

(iv) $E(\int_0^t \{\int_0^\tau \Phi(s,a,\omega)\mu(da)\}^2 d\langle M\rangle_s) < +\infty$ for all $t \in [0,\tau]$.

[Note that $(s,\omega) \to \int_0^\tau \Phi(s,a,\omega)\mu(da)$ is predictable since $(s,\omega) \to \Phi(s,a,\omega)$ is predictable for all $a \in [0,\tau]$.

Thus $t \to \int_0^{\tau} \Phi(t,a,\omega)\mu(da) \in L_2(\langle M \rangle)$ by (iv), and

$$\int_0^t \{\int_0^{\tau} \Phi(s,a,\omega)\mu(da)\}dM_s \in \mathcal{M}_2^c \,].$$

Given (i) - (iv), if

(A) $E\{[\int_0^{\tau} (\int_0^t \Phi(s,a,\omega)dM_s)\mu(da)]^2\} < +\infty$ for all $t \in [0,\tau]$

(B) $\int_0^{\tau} E\{\int_0^t \Phi^2(s,a,\omega)d\langle M \rangle_s\}^{1/2}\mu(da) < +\infty$ for all $t \in [0,\tau]$

then

(4.5) $\int_0^t \{\int_0^{\tau} \Phi(s,a,\omega)\mu(da)\}dM_s = \int_0^{\tau} \{\int_0^t \Phi(s,a,\omega)dM_s\}\mu(da)$ for all $t \in [0,\tau]$.

Lemma 0.2

Let (Ω,F,Q) be a probability space. (F_t) a reference family.

$M \in \mathcal{M}_2^{c,loc}$, i.e., a continuous local square-integrable martingale such that

$M_0 = 0$ a.s.

Let $\{\Phi(t,a,\omega): (t,a) \in [0,\tau] \times [0,\tau]\}$ be a family of real random variables such that

(i) $((t,\omega),a) \in \{([0,\tau] \times \Omega) \times [0,\tau]\} \to \Phi(t,a,\omega)$ is $L \times B[0,\tau]$ measurable.

(ii) $\int_0^t \Phi(s,a,\omega)dM_s \in \mathcal{M}_2^{c,loc}$. Since $\Phi(s,a,\omega)$ is predictable for all $a \in [0,\tau]$ (this follows by (i) above) this condition is that $\int_0^t \Phi^2(s,a,\omega)d\langle M \rangle_s < +\infty$ a.e. for all $t \in [0,\tau]$.

(iii) $(a,\omega) \to \int_0^t \Phi(s,a,\omega)dM_s$ is $B[0,\tau] \times F$ measurable for all $0 \leq t \leq \tau$.

Let $\mu(da)$ be a non-negative Borel measure on R.

(iv) $\int_0^t \{\int_0^\tau \Phi(s,a,\omega)\mu(da)\}^2 d\langle M\rangle_s < +\infty$ a.e. for all $t \in [0,\tau]$.

(Note that (iii) and (iv) imply

$$\int_0^t \{\int_0^\tau \Phi(s,a,\omega)\mu(da)\}dM_s \in \mathcal{M}_2^{c,loc})$$

Given (i) - (iv),

(4.5) $$\int_0^t \{\int_0^\tau \Phi(s,a,\omega)\mu(da)\}dM_s = \int_0^\tau \{\int_0^t \Phi(s,a,\omega)dM_s\}\mu(da) \text{ for all } t \in [0,\tau]$$

if and only if

(A) $t \to \int_0^\tau \{\int_0^t \Phi(s,a,\omega)dM_s\}\mu(da)$ is continuous a.e. and

(B) there exists a sequence of stopping times $\{\tau_n\}_{n=1}^\infty$ such that $\tau_n \to \tau$ a.s. and

$\int_0^\tau E\{[\int_0^{t\Lambda\tau_n}\Phi(s,a,\omega)dM_s]^2\}^{1/2} \mu(da) < +\infty$ for all $n = 1,2,\ldots$ where $t \Lambda T_n = \min(t,\tau_n)$.

Proof: Given (i) - (iv), if (4.5) holds then the right side is in $\mathcal{M}_2^{c,loc}$. This implies (A) and (B).

Conversely, given (i) - (iv), if (A) and (B) hold then define

$$y_n = \inf\{t \in [0,\tau]: |\int_0^\tau\{\int_0^t \Phi(s,a,\omega)dM_s\}\mu(da)| \geq n\}.$$

Since $\int_0^{\tau}\{\int_0^{t} \Phi(s,a,\omega)dM_s\}\mu(da)$ is

continuous in t on each sample path, it is bounded on $[0,\tau]$.

So, as $n\to\infty$, $y_n \to \tau$ a.s. Define $t_n \equiv y_n \wedge \tau_n$ for n = 1,2,... .

Note that $E\{\int_0^{\tau} (\int_0^{t\wedge t_n} \Phi(s,a,\omega)dM_s)\mu(da)\}^2 \leq n^2$ for all $0 \leq t \leq \tau$

and

$$\int_0^{\tau} E\{[\int_0^{t\wedge t_n} \Phi(s,a,\omega)dM_s]^2\}^{1/2}\mu(da) < +\infty$$

Thus, by lemma 0.1 for $t \in [0,\tau]$,

$$\int_0^{t\wedge t_n} \{\int_0^{\tau} \Phi(s,a,\omega)\mu(da)\}dM_s = \int_0^{\tau} \{\int_0^{t\wedge t_n} \Phi(s,a,\omega)dM_s\}\mu(da) \text{ for all } 0 \leq t \leq \tau.$$

since $t_n \to \tau$ a.s. this gives us our result.

Q.E.D.

<u>Lemma 0.3</u>

Let the hypotheses (i) - (iv) of lemma 0.2 hold, and let $\mu(a)$ be Lebesgue measure on R, and $d\langle M\rangle_s$ be absolutely continuous with respect to Lebesgue measure, then if

$$t \to \int_0^{t} \{\int_0^{t} \Phi(s,a,\omega)dM_s\}\mu(da) \text{ is continuous a.e.}$$

then $$\int_0^{t} \{\int_0^{\tau} \Phi(s,a,\omega)\mu(da)\}dM_s = \int_0^{\tau} \{\int_0^{t} \Phi(s,a,\omega)dM_s\}\mu(da) \text{ for all } t \in [0,\tau].$$

Proof:

We show that (A) and (B) of lemma 0.2 hold. First, (A) holds as part of the hypothesis. Next, consider

$$\left(E\left\{\left[\int_0^t \Phi(s,a,\omega)dM_s\right]^2\right\}\right)^{1/2}$$

$$= \left(\int_0^t \Phi^2(s,a,\omega)d\langle M\rangle_s\right)^{1/2} \leq 1 + \int_0^t \Phi^2(s,a,\omega)d\langle M\rangle_s.$$

Integrating over $[0,\tau]$ yields,

$$\int_0^\tau \left(E\left\{\left[\int_0^t \Phi(s,a,\omega)dM_s\right]^2\right\}\right)^{1/2} \mu(da) \leq \tau + \int_0^\tau \left[\int_0^t \Phi^2(s,a,\omega)d\langle M\rangle_s\right]d\mu(a)$$

By the standard Fubini's theorem on the last integral (along with condition (ii)) since the integrand is positive, we get

$$= \tau + \int_0^t \left[\int_0^\tau \Phi^2(s,a,\omega)d\mu(a)\right]d\langle M\rangle_s.$$

The last integral is continuous in t, hence it is bounded on $[0,\tau]$ for a fixed sample path $\omega \in \Omega$.

Define $\tau_n = \inf\{t \in [0,\tau]: \int_0^t \left[\int_0^\tau \Phi^2(s,a,\omega)d\mu(a)\right]d\langle M\rangle_s \geq n\}$.

Then $\tau_n \to \tau$ a.e. as $n \to \infty$ and

$$\int_0^\tau \left(E\left\{\left[\int_0^{t\wedge\tau_n} \Phi(s,a,\omega)dM_s\right]^2\right\}\right)^{1/2} \mu(da) < +\infty \text{ for each } n$$

This completes the proof.

Q.E.D.

Lemma 0.4

Let the hypotheses of lemma 0.3 hold.

Let $M_s \equiv W_i(s)$ for $s \in [0,\tau]$ and fix $t \in [0,\tau]$.

Define

$$\Phi(s,a,\omega) = \begin{cases} 0 & \text{if} \quad (s,a) \notin [0,t] \times [t,\tau] \\ \sigma_i(s,a,\omega) & \text{if} \quad (s,a) \in [0,t] \times [t,\tau]. \end{cases}$$

then

$$\int_0^y \{\int_t^T \sigma_i(s,a,\omega)da\}dW_i(s) = \int_t^T \{\int_0^y \sigma_i(s,a,\omega)dW_i(s)\}da \text{ for all } y \in [0,t].$$

Proof:

Direct from lemma 0.3.

Q.E.D.

Lemma 0.5

Let the hypotheses of lemma 0.3 hold.

Let $M_s \equiv W_i(s)$ for $s \in [0,\tau]$ and fix $t \in [0,\tau]$.

Define

$$\Phi(s,a,\omega) = \begin{cases} 0 & \text{if} \quad (s,a) \notin [0,t] \times [0,t] \\ \sigma_i(s,a,\omega)1_{s \leq a} & \text{if} \quad (s,a) \in [0,t] \times [0,t]. \end{cases}$$

then

$$\int_0^y [\int_s^t \sigma_i(s,a,\omega)da]dW_i(s) = \int_0^t [\int_0^{a \wedge y} \sigma_i(s,a,\omega)dW_i(s)]da \text{ for all } y \in [0,t].$$

Proof:

Direct from lemma 0.3.

Q.E.D.

Now we can proceed with the proof of expression (8).

$$\ln P(t,T) = -\int_t^T f(t,y)dy$$

$$= -\int_t^T f(0,y)dy - \int_t^T\left[\int_0^t \alpha(v,y)dv\right]dy - \sum_{i=1}^{2} \int_t^T\left[\int_0^t \sigma_i(v,y)dW_i(v)\right]dy$$

Note that the integrals are well-defined by conditions (C.1), (C.2).

By condition (C.2), we can apply the standard Fubini's theorem to get

$$\int_t^T\left[\int_0^t \alpha(v,y)dv\right]dy = \int_0^t\left[\int_t^T \alpha(v,y)dv\right]dy.$$

By conditions (C.1) - (C.3) we can apply lemma 0.4 with y = t to get

$$\int_t^T\left[\int_0^t \sigma_i(v,y)dW_i(v)\right]dy = \int_0^t\left[\int_t^T \sigma_i(v,y)dy\right]dW_i(v).$$

Substitution yields

$$\ln P(t,T) = -\int_t^T f(0,y)dy - \int_0^t\left[\int_t^T \alpha(v,y)dv\right]dy - \sum_{i=1}^{2} \int_0^t\left[\int_t^T \sigma_i(v,y)dy\right]dW_i(v).$$

Adding and subtracting the same terms yields

$$= -\int_0^T f(0,y)dy - \int_0^t[\int_v^T \alpha(v,y)dy]dv - \sum_{i=1}^{2} \int_0^t[\int_v^T \sigma_i(v,y)dy]dW_i(v)$$

$$+ \int_0^T f(0,y)dy + \int_0^t[\int_v^t \alpha(v,y)dy]dv + \sum_{i=1}^{2} \int_0^t[\int_v^t \sigma_i(v,y)dy]dW_i(v).$$

But, recall that

$$\int_0^t r(y)dy = \int_0^t f(0,y)dy + \int_0^t[\int_0^y \alpha(v,y)dv]dy + \sum_{i=1}^{2} \int_0^t[\int_0^y \sigma_i(v,y)dW_i(v)]dy$$

which is obtained by utilizing expression (6).

Using the standard Fubini's theorem, and since $t \to \int_0^t[\int_0^y \sigma_i(v,y)dW_i(v)]dy$ is continuous a.s. (by the properties of Lebesgue integrals), conditions (C.1) - (C.3) with lemma 0.5 for y = t imply:

$$\int_0^t r(y)dy = \int_0^t f(0,y)dy + \int_0^t[\int_v^t \alpha(v,y)dy]dv + \sum_{i=1}^{2} \int_0^t[\int_v^t \sigma_i(v,y)dy]dW_i(v).$$

Substitution yields:

$$\ln P(t,T) = \ln P(0,T) + \int_0^t r(y)dy - \int_0^t[\int_v^T \alpha(v,y)dv]dy - \sum_{i=1}^{2} \int_0^t[\int_v^T \sigma_i(v,y)dy]dW_i(v).$$

This completes the proof. Q.E.D.

<u>Proof of Proposition 1</u>:

This proposition is proved in a sequence of two lemmas.

<u>Lemma 1.1</u> Assume (C.1) - (C.3) hold for fixed $S,T \in [0,\tau]$ such that $0 < S < T \leq \tau$.

Define $X(t,y) = \int_0^t b(v,y)dv + \sum_{i=1}^{2} \int_0^t a_i(v,y)dW_i(v)$ for all $t \in [0,y]$ and $y \in \{S,T\}$.

Then, $\gamma_i: \Omega \times [0,\tau] \to R$ for $i = 1, 2$ satisfies

(i) $$\begin{pmatrix} b(t,S) \\ b(t,T) \end{pmatrix} + \begin{pmatrix} a_1(t,S) & a_2(t,S) \\ a_1(t,T) & a_2(t,T) \end{pmatrix} \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ a.e. } \lambda \times Q,$$

(ii) $\int_0^S \gamma_i(v)^2 dv < +\infty$ a.e. Q for $i = 1, 2$,

(iii) $E(\exp\{\sum_{i=1}^{2} \int_0^S \gamma_i(v)dW_i(v) - (1/2) \sum_{i=1}^{2} \int_0^S \gamma_i(v)^2 dv\}) = 1$, and

(iv) $E(\exp\{\sum_{i=1}^{2} \int_0^S [a_i(v,y) + \gamma_i(v)]dW_i(v) - (1/2) \sum_{i=1}^{2} \int_0^S [a_i(v,y) + \gamma_i(v)]^2 dv\}) = 1$

for $y \in \{S,T\}$.

if and only if

there exists a probability measure $\tilde{Q}_{ST}$ such that

(a) $d\tilde{Q}_{ST}/dQ = \exp\{\sum_{i=1}^{2} \int_0^S \gamma_i(v)dW_i(v) - (1/2) \sum_{i=1}^{2} \int_0^S \gamma_i(v)^2 dv\}$

(b) $\tilde{W}_i^{ST}(t) = W_i(t) - \int_0^t \gamma_i(v)dv$ are Brownian motions on $\{(\Omega,F,\tilde{Q}_{ST}), \{F_t: t \in [0,S]\}\}$ for $i = 1, 2$

(c) $$\begin{pmatrix} dX(t,S) \\ dX(t,T) \end{pmatrix} = \begin{pmatrix} a_1(t,S) & a_2(t,S) \\ a_1(t,T) & a_2(t,T) \end{pmatrix} \begin{pmatrix} d\tilde{W}_1^{ST}(t) \\ d\tilde{W}_2^{ST}(t) \end{pmatrix} \text{ for } t \in [0,S], \text{ and}$$

(d) $Z(t,S)$ and $Z(t,T)$ are martingales on $\{(\Omega,F,\tilde{Q}_{ST}), \{F_t: t \in [0,S]\}\}$.

Proof:

Suppose (i), (ii) and (iii) hold.

By Girsanov's theorem (Elliott [1982; p. 169]), the existence of a $\tilde{Q}_{ST}$ satisfying (a) and (b) follows. Condition (c) follows by a substitution of (b) into the definition of $X(t,y)$. This implies that $X(t,y)$ is a local martingale for $y \in \{S,T\}$. From expression (11) of the text, $dZ(t,y) = Z(t,y)dX(t,y)$. This stochastic differential equation has the unique solution

$$Z(t,y) = Z(0,y)\ \exp\{X(t,y) - (1/2)\sum_{i=1}^{2}\int_0^t a_i(v,y)^2 dv\},$$

see Elliott [1982, Theorem 13.5, p. 156]. This is a martingale with respect to $\tilde{Q}_{ST}$ if and only if $\tilde{E}_{ST}(Z(t,y)) = Z(0,y)$ for all $t \in [0,S]$. This holds if and only if

$$E(\exp\{\int_0^t b(v,y)dv + \sum_{i=1}^{2} [a_i(v,y) + \gamma_i(v)]dW_i(v) - (1/2)\sum_{i=1}^{2}\int_0^t [a_i(v,y)^2 + \gamma_i(v)^2]dv\}) = 1$$

for all $t \in [0,S]$. But, $b(v,y) = -\sum_{i=1}^{2} a_i(v,y)\gamma_i(v)$ from condition (i). Substitution yields

$$E(\exp\{\sum_{i=1}^{2}\int_0^t [a_i(v,y) + \gamma_i(v)]dW_i(v) - (1/2)\sum_{i=1}^{2}\int_0^t (a_i(v,y) + \gamma_i(v))^2 dv\}) = 1$$

for all $t \in [0,S]$. Since $Z(t,y)$ is a supermartingale, this equation holds if and only if

$$E(\exp\{\sum_{i=1}^{2}\int_0^S [a_i(v,y) + \gamma_i(v)]dW_i(v) - (1/2)\sum_{i=1}^{2}\int_0^S [a_i(v,y) + \gamma_i(v)]^2 dv\}) = 1.$$

Hence, $Z(t,y)$ is a martingale with respect to $\tilde{Q}_{ST}$ if and only if (iv) holds. This completes the proof in one direction. Note that we have just proved: given (i), (ii), and (iii), [(iv) if and only if (d)].

Suppose (a), (b), (c), and (d) hold.

Conditions (ii) and (iii) follow since $\tilde{Q}_{ST}$ is a probability measure satisfying (a). Substitution of (b) into the definition of X(t,y) yields

$$\begin{pmatrix} dX(t,S) \\ dX(t,T) \end{pmatrix} = \left\{ \begin{pmatrix} b(t,S) \\ b(t,T) \end{pmatrix} + \begin{pmatrix} a_1(t,S) & a_2(t,S) \\ a_1(t,T) & a_2(t,T) \end{pmatrix} \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix} \right\} dt$$

$$+ \begin{pmatrix} a_1(t,S) & a_2(t,S) \\ a_1(t,T) & a_2(t,T) \end{pmatrix} \begin{pmatrix} d\tilde{W}_1^{ST}(t) \\ d\tilde{W}_2^{ST}(t) \end{pmatrix} .$$

The difference between a process satisfying this expression and one satisfying (c) must be of bounded variation and also a $\tilde{Q}_{ST}$ local martingale. By Elliott [1982; p. 121] this implies (i). Hence, (a), (b), and (c) imply (i), (ii), and (iii). By the remark above, (d) implies (iv).

Q.E.D.

Lemma 1.2 Assume (C.1) - (C.3) hold for fixed $S,T \in [0,\tau]$ such that $0 < S < T \leq \tau$.

Define $X(t,y) = \int_0^t b(v,y)dv + \sum_{i=1}^{2} \int_0^t a_i(v,y)dW_i(v)$ for all $t \in [0,y]$ and $y \in \{S,T\}$.

There exists a probability measure $\bar{Q}$ equivalent to Q such that Z(t,S) and Z(t,T) are martingales on $\{(\Omega,F,\bar{Q}), \{F_t: t \in [0,S]\}\}$.

if and only if

there exists $\gamma_i: \Omega \times [0,\tau] \to R$ for i = 1, 2 and a probability measure $\tilde{Q}_{ST}$ such that (a), (b), (c), and (d) of Lemma 1.1 hold.

Proof:

Suppose γ_i for i = 1, 2 and $\tilde{Q}_{ST}$ exist as stated. Set $\bar{Q} = \tilde{Q}_{ST}$.

Conversely, suppose there exists $\bar{Q}$ with the properties stated.

Define $M(t) \equiv \bar{E}(dQ/d\bar{Q}|F_t)$. This is a Q-martingale.

Since $W_i(t)$ is a martingale under Q, by Elliott [1982, p. 162]

$$\bar{W}_i(t) = W_i(t) - \int_0^t M(v)^{-1} d\langle W_i, M\rangle_v \text{ is a local martingale under } \bar{Q} \text{ for } i = 1, 2.$$

Note that $\langle \bar{W}_i, \bar{W}_i\rangle_t = t$ for $i = 1, 2$ and

$$\langle \bar{W}_1, \bar{W}_2\rangle_t = 0.$$

Since $\bar{W}_i(t)$ have continuous sample paths, by Levy's theorem (Durrett [1984; p. 78]) $\bar{W}_i(t)$ are independent Brownian motions with respect to $\bar{Q}$.

By the martingale representation theorem, Ikeda and Watanabe [1981; p. 80], there exist $h_i: \Omega \times [0,T] \to R$ for $i = 1,2$ such that $M(t) = \sum_{i=1}^{2} \int_0^t h_i(v) dW_i(v)$ and $\int_0^t h_i^2(s)ds < +\infty$ a.e. Q. Note that $d\langle W_i, M\rangle_v = h_i(v)dv$, so define $\gamma_i(v) \equiv M(v)^{-1} h_i(v)$, then $\bar{W}_i(t) = W_i(t) - \int_0^t \gamma_i(v)dv$ as required. Finally, substitution of $W_i(t) = \bar{W}_i(t) + \int_0^t \gamma_i(v)dv$ for $i = 1, 2$ into the definition of $X(t,y)$ yields:

$$\begin{pmatrix} dX(t,S) \\ dX(t,T) \end{pmatrix} = \left\{ \begin{pmatrix} b(t,S) \\ b(t,T) \end{pmatrix} + \begin{pmatrix} a_1(t,S) & a_2(t,S) \\ a_1(t,T) & a_2(t,T) \end{pmatrix} \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix} \right\} dt$$

$$+ \begin{pmatrix} a_1(t,S) & a_2(t,S) \\ a_1(t,T) & a_2(t,T) \end{pmatrix} \begin{pmatrix} d\bar{W}_1(t) \\ d\bar{W}_2(t) \end{pmatrix}.$$

But, by definition $dX(t,y) = dZ(t,y)/Z(t,y)$. Since $Z(t,y)$ is a $\bar{Q}$-martingale, $X(t,y)$ is a $\bar{Q}$ local martingale, (see Elliott [1982; Theorem 13.22, p. 167]). The contribution of the drift term, a process of bounded variation, is also a local martingale and hence by Elliott [1982, p. 121] equal to zero a.e. $\lambda \times \bar{Q}$.

Q.E.D.

Proof of Proposition 2:

The proof of this proposition requires the following two lemmas.

Lemma 2.1 Fix $S < \tau$.

Let $\beta_i: \Omega \times [0,\tau] \to R$ for i = 1, 2 be such that

$$\int_0^S \beta_i^2(v)dv < +\infty \text{ a.e. } Q.$$

Define $$T_n \equiv \inf\{t \in [0,S]: E(\exp\{(1/2) \sum_{i=1}^{2} \int_0^t \beta_i(v)^2 dv\}) \geq n\}$$

$$M^n(t) \equiv \exp\{\sum_{i=1}^{2} \int_0^{\min(T_n,t)} \beta_i(v)dW_i(v) - (1/2) \sum_{i=1}^{2} \int_0^{\min(T_n,t)} \beta_i(v)^2 dv\}.$$

$$E(\exp\{\sum_{i=1}^{2} \int_0^S \beta_i(v)dW_i(v) - (1/2) \sum_{i=1}^{2} \int_0^S \beta_i(v)^2 dv\}) = 1 \text{ if and only if}$$

$\{M^n(S)\}_{n=1}^{\infty}$ are uniformly integrable.

Proof:

Define $\beta_i^n(v) \equiv \beta_i(v)1_{\{v \leq T_n\}}$, then by Elliott [1982; p. 165],

$M^n(t) = \exp\{\sum_{i=1}^{2} \int_0^S \beta_i^n(v)dW_i(v) - (1/2) \sum_{i=1}^{2} \int_0^S \beta_i^n(v)^2 dv\}$ is a supermartingale.

Since $E(\exp\{(1/2) \sum_{i=1}^{2} \int_0^{T_n} \beta_i(v)^2 dv\}) = E(\exp\{(1/2) \sum_{i=1}^{2} \int_0^S \beta_i^n(v)^2 dv\}) \leq n$,

by Elliott [1982; p. 178] $E(M^n(S)) = 1$. Hence,

$M^n(t)$ is a martingale. Note $\lim_{n\to\infty} M^n(S) = \exp\{\sum_{i=1}^{2} \int_0^S \beta_i(v)dW_i(v) - (1/2) \sum_{i=1}^{2} \int_0^S \beta_i^2(v)dv\}$

with probability one since $T_n \to S$ with probability one. Observe that $\{M^n(S)\}_{n=1}^{\infty}$ is a martingale with respect to n= 1, 2, ... because $\sup_n E(M^n(S)) = 1 < +\infty$ and

$E(M^{n+1}(S)|F_{\min(S,T_n)}) = M^{n+1}(\min(S,T_n))$ by the Optional Stopping Theorem

(since $T_n \leq S$, see Elliott [1982; p. 17]) = $M^n(S)$ by the definition of M^n.

(STEP 1) Suppose $\{M^n(S)\}_{n=1}^{\infty}$ are uniformly integrable, then

$\lim_{n\to\infty} M^n(S) = \exp\{\sum_{i=1}^{2} \int_0^S \beta_i(v)dW_i(v) - (1/2) \sum_{i=1}^{2} \int_0^S \beta_i(v)^2 dv\}$ in L^1 (see Elliott [1982; p. 22]) and thus $E(\exp\{\sum_{i=1}^{2} \int_0^S \beta_i(v)dW_i(v) - (1/2) \sum_{i=1}^{2} \int_0^S \beta_i(v)^2 dv\}) =$

$\lim_{n\to\infty} E(M^n(S))$. But, $E(M^n(S)) = 1$. This completes the proof in one direction.

(STEP 2) Conversely, suppose

$$E(\exp\{\sum_{i=1}^{2} \int_0^S \beta_i(v)dW_i(v) - (1/2) \sum_{i=1}^{2} \int_0^S \beta_i(v)^2 dv\}) = 1.$$

We know $E(\exp\{\sum_{i=1}^{2} \int_0^S \beta_i(v)dW_i(v) - (1/2) \sum_{i=1}^{2} \int_0^S \beta_i(v)^2 dv | F_{T_n}) = M^n(S)$, hence

$M^n(S)$ is uniformly integrable.

Q.E.D.

Lemma 2.2: Assume conditions (C.1) - (C.3) hold for fixed $S,T \in [0,\tau]$ such that $0 < S < T \leq \tau$. Suppose conditions (i), (ii), (iii), and (iv) of Lemma 1.1 hold; then, $\gamma_i(t)$ for i = 1, 2 satisfying (i), (ii), (iii) and (iv) are unique (up to $\lambda \times Q$ equivalence)

if and only if

$$\begin{vmatrix} a_1(t,S) & a_2(t,S) \\ a_1(t,T) & a_2(t,T) \end{vmatrix} \text{ is singular with } (\lambda \times Q) \text{ measure zero.}$$

Proof:

Suppose $A(t) \equiv \begin{pmatrix} a_1(t,S) & a_2(t,S) \\ a_1(t,T) & a_2(t,T) \end{pmatrix}$ is singular with $(\lambda \times Q)$ measure zero. Then, by condition (i) of Lemma 1.1, $\gamma_i(t)$ for $i = 1, 2$ are unique (up to $\lambda \times Q$ equivalence).

Conversely, suppose $\Sigma \equiv \{t \times w: [0,S] \times \Omega: A(t) \text{ is singular}\}$ has $(\lambda \times Q)(\Sigma) > 0$. We want to show that the functions satisfying conditions (i), (ii), (iii), and (iv) are not unique. First, by hypothesis, we are given a pair of functions $(\gamma_1(t), \gamma_2(t))$ satisfying (i), (ii), (iii), and (iv).

(STEP 1) Show that there exists a bounded, adapted, measurable pair of functions $(\delta_1(t), \delta_2(t))$ non-zero on Σ such that

$$A(t) \begin{pmatrix} \delta_1(t) \\ \delta_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and}$$

$$g(t) \equiv \exp\{\sum_{i=1}^{2} \int_0^t \delta_i(v)dW_i(v) - \sum_{i=1}^{2} (\int_0^t \delta_i(v)\gamma_i(v)dv) - (1/2) \sum_{i=1}^{2} \int_0^t \delta_i^2(v)dv\}$$

is bounded a.e. Q.

Let $\Sigma_0 = \{(t,w): A(t) \text{ has rank } 0\}$ and

$\Sigma_1 = \{(t,w): A(t) \text{ has rank } 1\}$. Both Σ_0 and Σ_1 are measurable sets.

Then $\Sigma = \Sigma_0 \cup \Sigma_1$ and $\Sigma_0 \cap \Sigma_1 = \phi$.

Fix $\eta > 0$.

On the set Σ_0, set $\delta_1^{\eta}(t) = \min(\eta, 1/\gamma_1(t))$

$$\delta_2^{\eta}(t) = \min(\eta, 1/\gamma_2(t)).$$

This satisfies

$$A(t)\begin{pmatrix}\delta_1^\eta(t)\\ \\ \delta_1^\eta(t)\end{pmatrix} = \begin{pmatrix}0\\ \\ 0\end{pmatrix} \text{ on } \Sigma_0.$$

On the set Σ_1, define $\delta_1^\eta(t)$, $\delta_2^\eta(t)$ as the unique solution $(x_1(w),x_2(w))$ to

$$A(t,w)\begin{pmatrix}x_1(w)\\ x_2(w)\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}, \ \max\ (x_1(w),x_2(w)) = \min(\eta,\ 1/\gamma_1(t),\ 1/\gamma_2(t)),$$

$x_2(w) \geq 0$, and if $x_2(w) = 0$ then $x_1(w) = 1$.

Finally, let $\delta_1^\eta(t)$, $\delta_2^\eta(t)$ be zero on Σ^c. Note that we shall always interpret superscripts on δ as the upper bound on the process, and not as an exponent.

By construction, $(\delta_1^\eta(t),\delta_2^\eta(t))$ are adapted, measurable, bounded by η,

$$A(t)\begin{pmatrix}\delta_1^\eta(t)\\ \\ \delta_2^\eta(t)\end{pmatrix} = \begin{pmatrix}0\\ \\ 0\end{pmatrix} \text{ a.e. } \lambda \times Q, \text{ and}$$

$$\left| \sum_{i=1}^{2} \int_0^t \delta_i^\eta(v)\gamma_i(v)dv + (1/2)\sum_{i=1}^{2}\int_0^t \delta_i^\eta(v)^2 dv \right| \leq [2+\eta^2]\tau \text{ a.e. } Q.$$

Let $\alpha = \inf\{j \in \{1, 2, 3, \ldots\}: (1/2)^{2j}\, S < 1\}$.

Define inductively the stopping times.

$$\tau_1 = \inf\{t \in [0,S]:\ \sum_{i=1}^{2}\int_0^t \delta_i^{(1/2)^\alpha}(v)dW_i(v) \geq (1/2)\}$$

$$\tau_j = \inf\{t \in [0,S]: \sum_{i=1}^{2} \int_{\tau_{j-1}}^{t} \delta_i^{(1/2)^{2j+\alpha}}(v)dW_i(v) \geq (1/2)^j\}$$

for $j = 2, 3, 4, \ldots$

We claim that $Q(\lim_{j\to\infty} \tau_j = S) = 1$.

Indeed,

$$Q(\tau_j < S \mid F_{\tau_{j-1}}) \leq Q(|\sum_{i=1}^{2} \int_{\tau_{j-1}}^{S} \delta_i^{(1/2)^{2j+\alpha}}(v)dW_i(v)| \geq (1/2)^j \mid F_{\tau_{j-1}})$$

$$\leq \frac{1}{[1/2]^{2j}} \int_{\tau_{j-1}}^{S} (\delta_i^{(1/2)^{2j+\alpha}}(v))^2 dv$$ by Chebyshev's inequality [Durrett [1984, p. 296]]

$$\leq \frac{1}{[1/2]^{2j}} ([1/2]^{2j+\alpha})^2 S < (1/2)^{2j}$$ by choice of α.

Hence $E[Q(\tau_j < S \mid F_{\tau_{j-1}})] = Q(\tau_j < S) < E((1/2)^{2j}) = (1/2)^{2j}$. Since,

$Q(\lim_{j\to\infty} \tau_j = S) = 1 - Q(\lim_{j\to\infty} \tau_j < S)$ and

$$Q(\lim_{j\to\infty} \tau_j < S) \leq Q(\bigcap_{j=1}^{\infty} (\tau_j < S))$$

$\leq \inf\{Q(\tau_j < S): j = 1,2,3,\ldots\} = 0$. This proves the claim.

Set $\delta_i(t) = \sum_{j=0}^{\infty} 1_{[\tau_j,\tau_{j+1}]}(t) \; \delta_i^{(1/2)^{2j+\alpha}}(t)$ for $i = 1, 2$.

$\delta_i(t)$ is bounded, adapted, and measurable and satisfies $A(t) \begin{pmatrix} \delta_1(t) \\ \delta_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ a.e. $\lambda \times Q$.

Note that for all $t \in [0,S]$, $|\sum_{i=1}^{2} \int_0^t \delta_i(t)dW_i(t)| \leq \sum_{j=0}^{\infty} (1/2)^j = 2$, so

$$\exp\{\sum_{i=1}^{2} \int_0^t \delta_i(t)dW_i(t) - \sum_{i=1}^{2} \int_0^t \delta_i(v)\gamma_i(v)dv - (1/2) \sum_{i=1}^{2} \int_0^t \delta_i^2(v)dv\}$$

is bounded a.e. $\lambda \times Q$. This completes (STEP 1).

(STEP 2) Show that $(\gamma_1(t) + \delta_1(t), (\gamma_2(t) + \delta_2(t))$ satisfies conditions (i), (ii), (iii), and (iv) of Lemma 1.1. This step will complete the proof.

Conditions (i) and (ii) are obvious. To obtain condition (iii),

Define $T_n = \inf\{t \in [0,T]: E(\exp\{(1/2) \sum_{i=1}^{2} \int_0^t (\gamma_i(t) + \delta_i(t))^2 dv) \geq n)\}$

$$M^n(t) = \exp\{\sum_{i=1}^{2} \int_0^{\min(T_n,t)} [\gamma_i(v) + \delta_i(v)]dW_i(v) - (1/2) \sum_{i=1}^{2} \int_0^{\min(T_n,t)} [\gamma_i(v) + \delta_i(v)]^2 dv\}.$$

By Lemma 2.1, we need to show that $M^n(S)$ is uniformly integrable.

But $M^n(S) = \exp\{\sum_{i=1}^{2} \int_0^{\min(S,T_n)} \gamma_i(v)dW_i(v) - (1/2) \sum_{i=1}^{2} \int_0^{\min(S,T_n)} \gamma_i(v)^2 dv\}\exp\{\sum_{i=1}^{2} \int_0^{\min(S,T_n)} \delta_i(v)dW_i(v)$

$$- (1/2) \sum_{i=1}^{2} \int_0^{\min(S,T_n)} [2\gamma_i(v)\delta_i(v) + \delta_i(v)^2]dv\}.$$

Since $\exp\{\sum_{i=1}^{2} \int_0^{\min(S,T_n)} \delta_i(v)dW_i(v) - (1/2) \sum_{i=1}^{2} \int_0^{\min(S,T_n)} [2\gamma_i(v)\delta_i(v) + \delta_i(v)^2]dv\}$

is bounded,

$$0 \leq M^n(S) \leq K \exp\{\sum_{i=1}^{2} \int_0^{\min(S,T_n)} \gamma_i(v)dW_i(v) - (1/2) \sum_{i=1}^{2} \int_0^{\min(S,T_n)} \gamma_i(v)^2 dv\} \text{ for some } K > 0.$$

By Lemma 2.1 since $\gamma_i(t)$ satisfies (iii), the right hand side is uniformly integrable. By Kopp [1984; p. 29], it can be shown that $M^n(S)$ is uniformly integrable. Finally, an analogous argument used to prove (iii) shows (iv) holds as well.

Q.E.D.

Proof of Proposition 4:

(STEP 1) Assume (C.6) holds. Let $\{N_0(t), N_{T_1}(t), \ldots, N_{T_n}(t): t \in [0,\tau]\}$ be an admissible s.f.t.s. which is an arbitrage opportunity. Then its value process, $V(t)$, satisfies $V(0) = 0$, $Q(V(\tau) \geq 0) = 1$, $Q(V(\tau) > 0) > 0$. But, $\tilde{Q} \simeq Q$ implies $\tilde{E}(V(\tau)) > 0$, i.e., $\tilde{E}(V(\tau)) > V(0)$. This contradicts the condition that $V(t)/B(t)$ is a $\tilde{Q}$-martingale.

(STEP 2) Suppose there are no arbitrage opportunities, yet there exists $\{S_1, T_1, S_2, T_2\}$ with $S_1 < S_2$, $0 < S_1 < T_1 \leq \tau$, $0 < S_2 < T_2 \leq \tau$ and $A \in F_{S_1}$ such that $\tilde{Q}_{S_1T_1}(A) \neq \tilde{Q}_{S_2T_2}(A)$. Without loss of generality, let $\tilde{Q}_{S_1T_1}(A) < \tilde{Q}_{S_2T_2}(A)$. Consider the contingent claim

$$1_A = \begin{cases} 1 \text{ if } \omega \in A \\ 0 \text{ otherwise.} \end{cases}$$

By the analysis after expression (23) in the text, for each pair $\{S_i, T_i\}$ for $i = 1, 2$ there exists an admissible s.f.t.s. $\{N_0^i(t), N_{S_i}(t), N_{T_i}(t): t \in [0, S_1]\}$ for $i = 1, 2$ such that for $i = 1,2$,

$$1_A B(S) = N_0^i(s) + N_{S_i}(S)P(S,S_i) + N_{T_i}(S)P(S,T_i) \text{ a.e. } Q.$$

By expression (23),

$$\begin{aligned} \tilde{Q}_{S_iT_i}(A) &= \tilde{E}_{S_iT_i}(1_A) \\ &= N_0^i(0) + N_{S_i}(0)P(0,S_i) + N_{T_i}(0)P(0,T_i) \text{ for } i = 1, 2. \end{aligned}$$

Consider the following portfolio consisting of five securities:

$$\bar{N}_0(t) = \tilde{Q}_{S_2T_2}(A) - \tilde{Q}_{S_1T_1}(A) + N_0^1(t) - N_0^2(t)$$

$$\bar{N}_{S_1}(t) = N_{S_1}(t)$$

$$\bar{N}_{T_1}(t) = N_{T_1}(t)$$

$$\bar{N}_{S_2}(t) = - N_{S_2}(t)$$

$$\bar{N}_{T_2}(t) = - N_{T_2}(t) \text{ for all } t \in [0,S_1].$$

This is an arbitrage opportunity. Indeed, its value process at time 0 is

$$V(0) = \tilde{Q}_{S_2T_2}(A) - \tilde{Q}_{S_1T_1}(A) + N_0^1(0) + N_{S_1}(0)P(0,S_1) + N_{T_1}(0)P(0,T_1)$$

$$- N_0^2(0) - N_{S_2}(0)P(0,S_2) - N_{T_2}(0)P(0,T_2) = 0.$$

It is a s.f.t.s. since $\{N^i, N_{S_i}, N_{T_i}\}$ are s.f.t.s. for $i = 1, 2$. Its time S value is

$$V(S) = [\tilde{Q}_{S_2T_2}(A) - \tilde{Q}_{S_1T_1}(A)]B(S) > 0 \text{ a.e. } Q.$$

This contradiction completes the proof.

Q.E.D.

Proof of Expression (40):

$$C(0) = \tilde{E}[\max(P(t^*,T)-K,0)/B(t^*)]$$

$$= \tilde{E}[\max(P(t^*,T)-K,0)/\exp(\int_0^{t^*} r(y)dy)]$$

By substitution of expression (36) this gives

$$C(0) = \exp(-\int_0^{t^*} f(0,y)dy-\sigma^2(t^*)^3/6)\tilde{E}[\exp(-\sigma\int_0^{t^*} \tilde{W}(y)dy)\max[\exp(-\int_{t^*}^{T} f(0,y)dy-(\sigma^2/2)Tt^*(T-t^*)-\sigma(T-t^*)\tilde{W}(t^*))-K,0]]$$

$$= \exp(-\int_0^{t^*} f(0,y)dy-\sigma^2(t^*)^3/6)\int_{-\infty}^{g} \tilde{E}[\exp(-\sigma\int_0^{t^*} \tilde{W}(y)dy)\,|\tilde{W}(t^*)=u].$$

$$(\exp(-\int_{t^*}^{T} f(0,y)dy-(\sigma^2/2)Tt^*(T-t^*)-\sigma(T-t^*)u)-K)e^{-(u^2/2t^*)}/\sqrt{2\pi t^*}du$$

where $\quad g \equiv [-\log K - \int_{t^*}^{T} f(0,y)dy-(\sigma^2/2)Tt^*(T-t^*)]/\sigma(T-t^*)$

Consider $\tilde{E}[\exp(-\sigma\int_0^{t^*} \tilde{W}(y)dy)\,|\tilde{W}(t^*)=u]$.

Let $\bar{W}(t)$ be the Brownian bridge conditioned to hit u at time t^* defined by

$$\bar{W}(t) = \tilde{W}(t) - \frac{t}{t^*}(\tilde{W}(t^*)-u).$$

Then $\quad \tilde{E}(\exp(-\sigma\int_0^{t^*} \tilde{W}(y)dy)\,|\tilde{W}(t^*)=u) = \tilde{E}(\exp(-\sigma\int_0^{t^*} \bar{W}(y)dy)).$

Since $\int_0^{t^*} \bar{W}(y)dy$ is Gaussian with $\tilde{E}[\int_0^{t^*} \bar{W}(y)dy] = ut^*/2$ and $\tilde{E}[(\int_0^{t^*} \bar{W}(y)dy)^2] = (t^*)^3/12 + (t^*)^2u^2/4$, using the moment generating function gives $\tilde{E}(\exp(-\sigma\int_0^{t^*} \bar{W}(y)dy)) = \exp((t^*)^3\sigma^2/24 - ut^*\sigma/2)$. Substitution of this expression into the previous equation for C(0) generates:

$$C(0) = \exp((t^*)^3\sigma^2/24)\exp(-\int_0^{t^*} f(0,y)dy-\sigma^2(t^*)^3/6)\int_{-\infty}^{g} (\exp(-\int_{t^*}^{T} f(0,y)dy-(\sigma^2/2)Tt^*(T-t^*)).$$

$$\exp(-\sigma(T-t^*)u-\sigma ut^*/2-u^2/2t^*)/\sqrt{2\pi t^*})du$$

$$-\exp((t^*)^3\sigma^2/24)\exp(-\int_0^{t^*} f(0,y)dy-\sigma^2(t^*)^3/6)K\int_{-\infty}^{g} (\exp(-\sigma ut^*/2-u^2/2t^*)/\sqrt{2\pi t^*})du$$

Completing the square in the integrals, using expression (2), and simplification generates

$$= P(0,T)\Phi([g + 1/2\sigma t^*(2T-t^*)]/\sqrt{t^*})$$

$$- KP(0,t^*)\Phi([g + 1/2\sigma t^{*2}]/\sqrt{t^*})$$

Define $h = [g + (1/2)\sigma t^*(2T-t^*)]/\sqrt{t^*}$

$$= [-\log(KP(0,t^*)/P(0,T)) + \sigma^2t^*(T-t^*)^2/2]/\sigma\sqrt{t^*(T-t^*)}$$

to get the final result. Q.E.D.

<u>Proof (of Lemma 1 in Section 8)</u>:

Suppose that the condition involving (57) is satisfied.

Define $\alpha(t,T) = -\sum_{i=1}^{2} \sigma_i(t,T,f(t,T))\phi_i(t) + \tilde{\alpha}(t,T)$. Substitution into (57) yields $df(t,T) = \alpha(t,T)dt + \sum_{i=1}^{2} \sigma_i(t,T,f(t,T))dW_i(t)$, which satisfies (55) since ϕ_i is bounded.

Conversely, suppose that the condition involving (56) is satisfied.

Define $\tilde{W}_i(t) = W_i(t) - \int_0^t \phi_i(y)dy$. Since ϕ_i is bounded, by Girsanov's theorem, $\tilde{W}_i(t)$ is a Brownian motion as stated in the lemma.

Define $\tilde{\alpha}(t,T) = \alpha(t,T) + \sum_{i=1}^{2} \sigma_i(t,T,f(t,T))\phi_i(t)$.

Now $df(t,T) = \alpha(t,T)dt + \sum_{i=1}^{2} \sigma_i(t,T,f(t,T))dW_i(t)$ satisfies (55).

$$\text{So}\quad df(t,T) = \tilde{\alpha}(t,T)dt - \sum_{i=1}^{2} \sigma_i(t,T,f(t,T))\phi_i(t)dt + \sum_{i=1}^{2} \sigma_i(t,T,f(t,T))dW_i(t)$$

$$= \tilde{\alpha}(t,T)dt + \sum_{i=1}^{2} \sigma_i(t,T,f(t,T))d\tilde{W}_i(t) \text{ satisfies (55) with}$$

$\tilde{W}_i(t)$ replacing $W_i(t)$, $\tilde{\alpha}$ replacing α, and $\tilde{Q}$ replacing Q.

QED.

Proof of Proposition 5:

Under the hypotheses on ϕ_i for i = 1, 2, by Elliott [1982, Theorem 13.36, p. 178] and Girsanov's theorem, $\tilde{W}_i(t) = W_i(t) - \int_0^t \phi_i(v)dv$ for i = 1, 2 are independent Brownian motions on $\{(\Omega,F,\tilde{Q}), \{F_t: t \in [0,T]\}\}$ where $d\tilde{Q}/dQ = \exp\{\sum_{i=1}^{2} \int_0^T \phi_i(t)dW_i(t) - (1/2)\sum_{i=1}^{2} \int_0^T \phi_i(t)^2 dt\}$. Applying Lemma 1 and Lemma 2 in sequence, guarantees there exists a forward rate process with drifts

$$\alpha(t,T) = -\sum_{i=1}^{2} \sigma_i(t,T,f(t,T))[\phi_i(t) - \int_t^T \sigma_i(t,v,f(t,v))dv]$$

satisfying (55).

Q.E.D.

Proof of Proposition 6:

Fix a T_0.

Consider $\eta(t) \equiv -\bar{\sigma}\min(f(t,T_0),\lambda)\int_t^{T_0}\bar{\sigma}\min(f(t,s),\lambda)ds/\bar{\sigma}\min(f(t,T_0),\lambda)$

$$= -\int_t^{T_0}\bar{\sigma}\min(f(t,s),\lambda)ds$$

Since $\bar{\sigma}\min(f(t,s),\lambda)$ is bounded, $\eta(t)$ is bounded. Hence,

$$E(\exp\{(1/2)\int_0^{T_0}\eta(t)^2dt\}) < +\infty.$$

By Girsanov's theorem, there exists an equivalent probability measure $\bar{Q}$ and a Brownian motion $\bar{W}(t)$ such that

$$df(t,T_0) = \bar{\sigma}\min(f(t,T_0),\lambda)d\bar{W}(t).$$

Define $t_0 = \inf\{t \in [0,T_0]: f(t,T_0) = 0\}$. By Karlin and Taylor [1981; lemma 15. 6.2], zero is an unattainable boundary, i.e.,

$$\bar{Q}\{t_0 \leq T_0\} = Q\{t_0 \leq T_0\} = 0.$$

Since $f(t,T_0)$ has continuous sample paths, $f(t,T_0) > 0$ a.e.

Let $\{T_i: i=1,2,3,\ldots\}$ be the rationals in $[0,\tau]$.

$Q\{f(t,T_i) = 0$ for some T_i and some $t \in [0,T_i]\}$ =

$$Q\{\bigcup_{i=1}^{\infty}\{f(t,T_i) = 0 \text{ for some } t \in [0,T_i]\}\}$$

$$\leq \sum_{i=1}^{\infty} Q\{f(t,T_i) = 0 \text{ for some } t \in [0,T_i]\} = 0.$$

By the joint continuity of $f(t,T)$, $Q\{f(t,T) \geq 0$ for all $T \in [0,\tau]$ and all $t \in [0,T]\} = 1$.

Q.E.D.

Proof that a unique continuous solution $\theta(s)$ to expression (67) exists.

Let $\{f(0,T): T \in [0,\tau]\}$ be twice continuously differentiable.

Note that $\bar{B}_T(t,T) \equiv \partial\bar{B}(t,T)/\partial T$ and $\bar{B}_{TT}(t,T) \equiv \partial^2\bar{B}_T(t,T)/\partial T$ are continuous on $0 \leq t \leq T \leq \tau$ with $\bar{B}(t,t) = 0$ and $\bar{B}_T(t,t) = 1$.

Expression (67) is

$$f(0,T) = r(0)\bar{B}_T(0,T) + K \int_0^T \theta(s)\bar{B}_T(s,0)ds.$$

Differentiating with respect to T yields

$$[\partial f(0,T)/\partial T - r(0)\bar{B}_{TT}(0,T)]/K = \theta(T) + \int_0^T \theta(s)(K\bar{B}_{TT}(s,T))ds.$$

This is a Volterra integral equation of the second kind with a unique continuous solution $\theta(\cdot)$ on $[0,\tau]$, see Taylor an Lay [1980; p. 200].

Q.E.D.

Proof that condition (69) is satisfied.

Condition (69) is (using notation from the previous proof)

$$r(t)(\bar{B}_{Tt}(t,T) - \bar{B}_T(t,T)) = \lambda r(t)\bar{B}_T(t,T) + \sigma^2\bar{B}_T(t,T)r(t) \int_t^T \bar{B}_T(t,s)ds$$

$$= \lambda r(t)\bar{B}_T(t,T) + \sigma^2\bar{B}_T(t,T)\bar{B}(t,T)$$

since $\bar{B}(t,t) = 0$.

Using

$$\bar{B}_T(t,T) = 4\gamma^2 e^{\gamma(T-t)}/[(\gamma+K+\lambda)(e^{\gamma(T-t)}-1) + 2\gamma]^2 \text{ and}$$

$$\bar{B}_{Tt}(t,T) = -\gamma + \frac{2\gamma(\gamma+K+\lambda)e^{\gamma(T-t)}}{[(\gamma+K+\lambda)(e^{\gamma(T-t)}-1) + 2\gamma]}$$

and $\gamma^2 = (K+\lambda)^2 + 2\sigma^2$ shows that condition (69) is valid.

Q.E.D.

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