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BONFERRONI-TYPE INEQUALITIES; CHEBYSHEV-TYPE INEQUALITIES FOR THE DISTRIBUTIONS ON [0, n]

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Abstract. An elementary "majorant-minorant method" to construct the most stringent Bonferroni-type inequalities is presented. These are essentially Chebyshev-type inequalities for discrete probability distributions on the set $\{0, 1, \dots, n\}$, where n is the number of concerned events, and polynomials with specific properties on the set lead to the inequalities. All the known results are proved easily by this method. Further, the inequalities in terms of all the lower moments are completely solved by the method. As examples, the most stringent new inequalities of degrees three and four are obtained. Simpler expressions of Mărgăritescu's inequality (1987, *Stud. Cerc. Mat.*, **39**, 246–251), improving Galambos' inequality, are given.

Key words and phrases: Binary random variable, Galambos' inequality, Kwerel's inequality, moment problem.

1. Introduction

Let $A = \{A_i\}_{i=1}^n$ be a set of events in a probability space, and let K denote the count of those A_i's which occur. Put $p_m = P\{K = m\}$, $q_m = P\{K \geq m\}$, $S_0 = 1$, and

$$(1.1) \qquad S_r = \sum_{0 \leq i_1 < i_2 < \cdots < i_r \leq n} P(A_{i_1} A_{i_2} \cdots A_{i_r}), \qquad r = 1, 2, \dots, n.$$

Inequalities which bound p_m or q_m by linear combinations $\sum_{r=0}^n b_r S_r$ are called Bonferroni-type inequalities. They are used to evaluate the distribution functions of the order statistics of dependent random variables, and are important in theories of extreme statistics, multiple comparisons, applied probability, and others. As introductions to Bonferroni-type inequalities, Alt (1982) and Galambos (1984, 1987) are recommended.

Lemmas 2.1 and 2.2 in Section 2 show the simple fact that a Bonferroni-type inequality is just a Chebyshev-type inequality of the probability distributions on the integer interval $[0, n] = \{0, 1, \dots, n\}$. Then Theorem 2.1 shows that there is a natural bijection between Bonferroni-type inequalities and "majorant-minorant polynomials".

In Section 3, the set of all possible values of $(S_1, S_2, \ldots, S_n)$ and that of some of its components are studied; in other words, the moment problem of K is discussed. Theorem 3.1 states that an inequality obtained in Theorem 2.1 is, in fact, the most stringent in a strict sense in the region that is easily specified. The relationship between the moment space and "the majorant-minorant of 0" helps us to understand the geometry of the former by that of the latter. Theorems 3.2 and 3.3 state that if a vector of all the lower moments is given, the most stringent inequality in the strict sense (defined in Section 2) is determined by the proposed method.

In Section 4, as examples, new inequalities of degree three (in terms of (S_1, S_2, S_3)) on p_m and q_m are obtained. In Section 5, new inequalities of degree four on p_0 and p_n are obtained. Propositions 4.1, 4.2 and 5.1 state new inequalities.

In Section 6, classical Bonferroni's and Galambos' inequalities on p_m and q_m based on $S_m, S_{m+1}, \ldots$ are simply proved. These inequalities, except for Galambos' ones on q_m, are shown to be the most stringent. Simpler expressions of Mărgăritescu's most stringent inequalities on q_m (Mărgăritescu (1987)), which improve Galambos' ones, are obtained. In Section 7, a general method to find exhaustively majorants and minorants is shown.

Kwerel (1975*a*, 1975*b*, 1975*c*) showed that Bonferroni-type inequalities can be obtained by solving a linear programming problem by the simplex method, and obtained all the inequalities of degree 2 on p_m (1975*a*) and the inequalities of degree 3 on p_0 and p_n (1975*b*). He showed also the general method for obtaining those of any degree on p_0 and p_n (1975*c*). The method of this paper is valid for any m, $0 \le m \le n$, for both p_m and q_m, and is simpler for obtaining explicit expressions than the simplex method. The point is, the use of geometry of the moment space and polynomials, as discussed in Section 3.

There are other type of inequalities, which strengthen the classical Bonferroni inequalities by using not S_r but a partial sum of the definition in (1.1). General results were given by Rényi (1961) (see also Galambos (1987)), Hailperin (1965), Kounias and Marin (1976), and others; a practical inequality was obtained by Hunter (1976) and rediscovered by Worsley (1982). There is another result using the ordered P-values (Simes (1986)). However, these types of inequalities are out of the scope of this paper.

2. Elementary facts

In a probability space $(\Omega, \mathcal{A}, P)$, let A denote a finite set of events $\{A_i \in \mathcal{A};\ i = 1, 2, \ldots, n\}$, and K the count of those A_i's which occur. That is,

$$K = K(\omega;\ A) = \sum_{i=1}^{n} I(\omega;\ A_i), \qquad \omega \in \Omega,$$

where I denotes an indicator function. If the probability space and the event set are arbitrary, then the random variable K can have any probability distribution on $[0, n] = \{0, 1, \ldots, n\}$.

LEMMA 2.1. *For any probability function* $(p_m)_{m=0}^n$ *on* $[0, n]$ $(p_m \geq 0, \sum_{m=0}^n p_m = 1)$, *there exists a probability space* $(\Omega, \mathcal{A}, P)$ *and a set of n events* $A = \{A_i\}_{i=1}^n$ *such that* $K = \sum_{i=1}^n I(\omega, A_i)$ *has the probability function,* $P\{K = m\} = p_m$, $m \in [0, n]$.

PROOF. Let $\Omega = [0, 1]$, $\mathcal{A}$ the family of Borel sets of Ω, and P the Lebesgue measure. Partition Ω into 2^n subintervals so that there are $\binom{n}{m}$ intervals of the length $p_m \Big/ \binom{n}{m}$, $m \in [0, n]$. There are possibly intervals which degenerate to a point. An interval of the length $p_m \Big/ \binom{n}{m}$ is regarded as an event $B_J = \bigcap_{i \in J} A_i \bigcap_{j \in J^c} A_j^c$ where J is a subset of $[0, n]$ with cardinality $|J| = m$ and c denotes the complement. The union $\bigcup_{|J|=m} B_J$ is the event that $K = m$, and the union $\bigcup_{i \in J} B_J$, where the union is with respect to J such as $i \in J \subset [0, n]$, defines A_i. Thus, the partition $\{B_J;\ J \subset [0, n]\}$ defines the event set $\{A_i\}_{i=1}^n$.

Lemma 2.1 tells that the sum of n dependent 0-1 random variables can have any distribution on $[0, 1]$.

In the above proof an exchangeable set A is chosen; that is, the probability of any Boolean function of A is invariant with respect to the permutation of the indices of $A_i \in A$. This fact was remarked by Galambos (1975), in a more specific context, and by Galambos (1987) and Takeuchi and Takemura (1987) in a general way. If A is independent, then K has a log-concave probability function, $p_m^2 \geq p_{m-1}p_{m+1}$, which is strongly unimodal (see Keilson and Gerber (1971)), and cannot be arbitrary.

It is known that S_r is the binomial moment, a version of the factorial moment, of K.

LEMMA 2.2.

$$S_r = \sum_{m=r}^{n} p_m \binom{m}{r} = E\left[\binom{K}{r}\right].$$

PROOF. The random variable K is related with the indicator functions by

$$\sum_{1 \leq i_1 < \cdots < i_r \leq n} \prod_{j=1}^{r} I(\omega, A_{i_j}) = \binom{K}{r}.$$

In fact, among $\binom{n}{r}$ terms of $\prod_{j=1}^r I(\omega, A_{i_j})$, $\binom{K}{r}$ are 1 and the others are zero. The expectation of the left-hand side is S_r by definition (1.1).

Thus, the inequalities to bound $p_m = P\{K = m\}$ or $q_m = P\{K \geq m\}$ in terms of the moments of K are Chebyshev-type inequalities. Except for $S_r = 0$, $r = n+1, n+2, \ldots$, it is not essential to use the binomial moments. In fact, some inequalities are better expressed in terms of the moments around the origin,

$M_r = E[K^r]$. At any rate, using Stirling numbers $\left[{r \atop m}\right]$ of the first kind (unsigned), and $\left\{{r \atop m}\right\}$ of the second kind (the notation by Knuth (1975), see also Jordan (1960) and Riordan (1968)),

$$r!S_r = \sum_{j=1}^{r} \left[{r \atop j}\right] (-1)^{r-j} M_j \quad \text{and} \quad M_r = \sum_{j=1}^{r} \left\{{r \atop j}\right\} j!S_j.$$

For example, $S_1 = M_1$, $2S_2 = M_2 - M_1$, $6S_3 = M_3 - 3M_2 + 2M_1$; $M_2 = 2S_2 + S_1$ and $M_3 = 6S_3 + 6S_2 + S_1$.

In the following, n is assumed to be known and fixed, and the dependence on n is sometimes implicit in the notations. The dependence of the probabilities p_m and q_m, and the moments M_j's and S_j's on an arbitrary random variable K on $[0, n]$ is also implicit.

A standard technique to prove a Chebyshev-type inequality is the "majorant-minorant method". Define

$$\xi_m(x) = \xi_m(x;\, n) = \begin{cases} 1, & \text{if} \quad x = m;\ m \in [0, n], \\ 0, & \text{if} \quad x \neq m \quad \text{and} \quad x \in [0, n], \end{cases}$$

and

$$\eta_m(x) = \eta_m(x;\, n) = \begin{cases} 0, & \text{if} \quad x \in [0, m-1];\ m \in [1, n-1], \\ 1, & \text{if} \quad x \in [m, n]. \end{cases}$$

Let θ denote a generic subset of $[0, n]$ with $r+1$ elements. A "majorant u_θ of $\xi_m(x)$ on θ" is a polynomial of degree r such that

$$\xi_m(x) \le u_\theta(x), \quad x \in [0, n], \qquad \text{and} \qquad \xi_m(x) = u_\theta(x), \quad x \in \theta.$$

A minorant of ξ_m and a majorant and a minorant of η_m are similarly defined.

THEOREM 2.1. *Let n be any fixed positive integer. A polynomial $u(x) = \sum_{j=0}^{n} a_j x^j = \sum_{j=0}^{n} b_j \binom{x}{j}$ satisfies*

$$\xi_m(x) \le u(x), \quad x \in [0, n], \tag{2.1}$$

if and only if

$$p_m \le U(M_1, \ldots, M_n) = \sum_{j=0}^{n} a_j M_j = \sum_{j=0}^{n} b_j S_j, \tag{2.2}$$

for any possible values of p_m and M_j's.

Further, if u_θ is a majorant of ξ_m of degree r, the corresponding $U_\theta(M_1, \ldots, M_r)$ is the most stringent in the sense that there is no other linear combinations of $M_1, \ldots, M_r$ which is not greater than U_θ for "all" possible values of $(M_1, \ldots, M_r)$.

Conversely, the polynomial u corresponding to a most stringent upper bound U is a majorant. Such $U_\theta(M_1, \ldots, M_r)$ *is equal to* p_m *if and only if the distribution has probability one on* θ.

Similar statements on lower bounds on p_m, *or upper and lower bounds on* q_m *hold.*

PROOF. Sufficiency of the first part. Take the expectation of both sides of $\xi_m(K) \leq u(K)$. Necessity of the first part. A distribution on $[0, n]$ is a mixture of distributions which have probability one at a single point $x \in [0, n]$. Since (2.2) is valid for these degenerated distributions, for which $M_j = x^j$ and $S_j = \binom{x}{j}$, and $p_m = 1$ or 0 if $x = m$ or $\neq m$, respectively, (2.1) holds.

The second part. If the values of $u_\theta(x)$ are specified for $x \in \theta$, u_θ is uniquely determined and no polynomial, not smaller than ξ_m and of degree r, is uniformly smaller than u_θ. If a polynomial is not a majorant, a value $u(x_0) > \xi_m(x_0)$ can be decreased to $\xi_m(x_0)$ without changing the values $u(x) = \xi_m(x)$ at less than $r+1$ points, and the corresponding U cannot be the most stringent.

Remark. Theorems of this paper hold for discrete probability distributions on a finite set of any real values. We discuss distributions on $[0, n]$ to obtain simpler expressions.

In general there are several majorants, and the minimum of the corresponding bounds for given values of moments is "the most stringent in the strict sense". A most stringent upper bound (or lower bound) in the wide sense can be greater than one (or negative) for some values of moments (Schwager (1984)). Since each of the most stringent bounds in the strict sense is equal to the estimated probability for specific distributions of K, the bound is always between 0 and 1.

Let us go back to the event set A for a while. The above proof means that an inequality like (2.2) holds for any event set if it holds for any A such that the events $A_1, \ldots, A_n$ are mutually independent, $P(A_1) = \cdots = P(A_j) = 1$ and $P(A_{j+1}) = \cdots = P(A_n) = 0$, $j \in [1, n]$. Such an event set corresponds to K which is equal to j with probability one.

The method of indicators by Loève (1942, 1963) limits the check of (2.2) to such A that $P(A_i) = 0$ or 1, $i \in [1, n]$. Galambos (1975) limits to A of j independent events such that $P(A_1) = \cdots = P(A_j) = p$, $0 \leq p \leq 1$, $j \in [1, n]$. The new method in Theorem 2.1 is simpler than these methods: An inequality (2.2) on the multivariate function U reduces to (2.1) of the single-variable function u. Moreover the original events can be completely disregarded.

Kwerel (1975*a*, 1975*b*, 1975*c*) used Lemma 2.1 without mentioning it explicitly. He, however, did not use the conventional geometric approach to the Chebyshev-type inequalities but used the simplex method to solve linear programming problems. The equivalence of these two approaches has been well known (Isii (1964)).

Móri and Székely (1985) mentioned Lemma 2.1 and showed a geometric approach to (2.2) but did not go back to (2.1). Recently, Samuels and Studden (1989) remarked the relationship between Bonferroni-type inequalities and Chebyshev-type inequalities. They examined applications of the general theory of Chebyshev systems. Here, the sequence of all polynomials are mainly considered to get simply typical results.

3. Moment problems and the most stringent inequality in the strict sense

The most stringent inequality in the strict sense depends on the moment values. It is necessary, therefore, to find the possible range of the moment vector $(M_1, M_2, \ldots, M_n)$ or $(S_1, \ldots, S_n)$. The answer is simple and well-known (e.g. Theorem of Móri and Székely (1985)).

LEMMA 3.1. *The possible range of* $(M_1, \ldots, M_n)$ *is the closed convex hull* Γ *of* $n+1$ *points*

$$G_j = (j, j^2, \ldots, j^n), \qquad j \in [0, n], \tag{3.1}$$

in $\boldsymbol{R}^n$. *That is, by symbol,*

$$\Gamma = \text{C.H.}\{G_0, G_1, \ldots, G_n\}. \tag{3.2}$$

(It is rather an n-dimensional simplex.) The possible range of $(S_1, \ldots, S_n)$ *is*

$$\text{C.H.}\left\{\left(\binom{j}{1}, \binom{j}{2}, \ldots, \binom{j}{n}\right) : j \in [0, n]\right\}.$$

PROOF. The moments of a mixture (a convex combination) of distributions are the mixture of their moments. The moments $(M_r)_{r=1}^n$ of the distribution degenerated to j are $(j^r)_{r=1}^n$, and the moment of a general distribution is the corresponding convex combination of $(j^r)_{r=1}^m$.

In general, the possible range of $(E[f_1(K)], E[f_2(K)], \ldots, E[f_r(K)])$ is $\text{C.H.}\{(f_1(j), f_2(j), \ldots, f_r(j))\colon j \in [0, n]\}$. This can be applied to $(S_r)_{r=1}^n$.

Remark. The possible range of some moments, say (M_1, M_3), is the projection of Γ on the M_1M_3-subspace. The projection of Γ on the $(M_1, M_2, \ldots, M_r)$-subspace is denoted by Γ_r. In the following, the symbol G_j of (3.1) is intentionally abused to denote any projection of G_j on a context dependent subspace.

It can be shown that the most stringent bound obtained by the majorant-minorant method of Theorem 2.1 is the most stringent in the strict sense (defined in the previous section) in a specific region.

THEOREM 3.1. *Suppose* $u_\theta(x)$ *is a majorant of* ξ_m *of order* r *on* θ, $|\theta| = r+1$. *Then the corresponding bound* $p_m \le U_\theta(M_1, \ldots, M_r)$ *is the most stringent in the strict sense for* $(M_1, \ldots, M_r) \in \text{C.H.}\{G_j\colon j \in \theta\}$.

For the other bounds on p_m *or* q_m, *similar statements hold.*

PROOF. The first part. The equality $p_m = U_\theta(M_1, \ldots, M_r)$ holds if $(M_1, \ldots, M_r) = G_j$, $j \in \theta$, that is $P\{K = j\} = 1$, $j \in \theta$. Therefore the equality holds if $P\{K \in \theta\} = 1$, and under this condition $(E[K], E[K^2], \ldots, E[K^r]) \in$ C.H.$\{G_j\colon j \in \theta\}$. Conversely, a point of C.H.$\{G_j\colon j \in \theta\}$ corresponds to a distribution on θ. Therefore there is no other upper bound on p_m smaller than or equal to U_θ in C.H.$\{G_j\colon j \in \theta\}$.

Remark 1. As a minorant of ξ_m (or η_m) the constant 0 may be included. If $(M_1, \ldots, M_r) \in$ C.H.$\{G_j\colon j \neq m\}$ (or C.H.$\{G_j\colon j < m\}$) 0 is the most stringent lower bound of p_m (or q_m if $m \geq r+1$) in the strict sense. Similarly, the constant 1 may be included as a majorant of ξ_m if $n - m \geq r$.

Remark 2. Let Θ denote the index set of all the majorants of ξ_m of order r; $\{u_\theta(x)\colon \theta \in \Theta\}$. If the corresponding set of convex hulls is a partition of the possible range of $(M_1, \ldots, M_r)$; $\Gamma_r = \bigcup_{\theta \in \Theta}$ C.H.$\{G_j\colon j \in \theta\}$, then the best possible bound is determined for a given moment vector $(M_1, \ldots, M_r)$. The fact of the partition is shown in Theorem 3.2.

Remark 3. In applying Theorem 3.1 we have to find all the majorants or minorants of ξ_m or η_m. A method to find them is given in Theorem 7.1.

To examine Remark 2 of Theorem 3.1, the geometry of Γ_r is further studied. For this purpose a variant of the majorant-minorant method is again useful.

Let $u(x)$ be a monic polynomial of degree r (the coefficient of x^r is 1), such that $u(x) \geq 0$, $x \in [0, n]$, and $u(x) = 0$ for $x = k_1, \ldots, k_r \in [0, n]$. It will be called "a majorant of 0 of degree r". "A minorant $l(x)$ of 0 of degree r" is similarly defined. A method to find all the majorants and the minorants of 0 is discussed in Section 7. For this $u(x)$, $U(M_1, \ldots, M_r) = E[u(K)] \geq 0$ gives a lower bound of M_r in terms of $(M_1, \ldots, M_{r-1})$, and $U(M_1, \ldots, M_r) = 0$ if $P\{K \in \{k_1, \ldots, k_r\}\} = 1$. Similarly, $E[l(K)] \leq 0$ gives an upper bound of M_r.

The boundary $\partial\Gamma$ of the closed simplex Γ (3.2) in $\boldsymbol{R}^n$ is the set of n pieces of $(n-1)$-dimensional simplexes

$$\gamma_{i,n} = \text{C.H.}\{G_j\colon j \neq i, j \in [0, n]\}, \qquad i \in [0, n].$$

A simplex $\gamma_{i,n}$ is on a hyperplane determined by the equation

$$T(i, n;\ M_1, \ldots, M_n) = E\left[\prod_{j \neq i}(K - j)\right] = 0, \tag{3.3}$$

and the boundary of $\gamma_{i,n}$ is the set of $n-1$ pieces of $(n-2)$-dimensional simplexes

$$\text{C.H.}\{G_j\colon j \neq i, l\}, \qquad l \neq i, \quad l \in [0, n]. \tag{3.4}$$

A polynomial $\prod_{j \neq i}(x - j)$ in (3.3) of degree n is a majorant of 0 if $i = n, n-2, \ldots$, or a minorant of 0 if $i = n-1, n-3, \ldots$. That is,

$$T(i, n;\ M_1, \ldots, M_n) \begin{cases} \geq 0, & \text{if} \quad i = n, n-2, \ldots, \\ \leq 0, & \text{if} \quad i = n-1, n-3, \ldots, \text{ for } (M_1, \ldots, M_n) \in \Gamma. \end{cases}$$

This means that

$$\text{(3.5)}\qquad \bigcup_{i=0}^{[n/2]} \gamma_{n-2i,n} \quad \text{and} \quad \bigcup_{i=1}^{[n/2]} \gamma_{n-2i+1,n}$$

are the upper and lower part of $\partial\Gamma$, respectively. The border of the upper and lower parts is the set of segments $\bigcup_{j=0}^{n-1} \overline{G_j G_{j+1}} \cup \overline{G_n G_0}$. The unions of (3.5) form a partition of the upper and lower surfaces, and being projected on the $(M_1, \dots, M_{n-1})$-subspace each subset is a partition of Γ_{n-1}. The partition boundary is the projection of $(n-2)$-dimensional simplexes, (3.4), ($i = n, n-2, \dots$ or $i = n-1, n-3, \dots$), which are still $(n-2)$-dimensional simplexes. Some of them are inside Γ_{n-1} and others form the boundary $\partial\Gamma_{n-1}$ of Γ_{n-1}. The upper (or lower) part of $\partial\Gamma_{n-1}$ is formed by the $(n-2)$-dimensional simplexes, whose corresponding polynomial $\prod_{j\neq i,l}(x-j)$ is a majorant (or minorant) of zero of degree $n-1$.

The projection can be continued, and the corresponding fact holds:

LEMMA 3.2. *The boundary $\partial\Gamma_r$ of the closed convex hull Γ_r in the $(M_1, \dots, M_r)$-space consists of the set of all $(r-1)$-dimensional simplexes*

$$\text{C.H.}\{G_j \in \Gamma_r\colon\ j \in k = \{k_1, \dots, k_r\}\}$$

such that $\prod_{j\in k}(x - x_j)$ is a majorant or a minorant of zero. Those corresponding to majorants (or minorants) form the lower (or upper) part of $\partial\Gamma_r$.

We are ready to state a theorem confirming that the most stringent upper bound of p_m in the strict sense is determined by using the majorant-minorant method for any given value of $(M_1, \dots, M_r)$.

THEOREM 3.2. *Let Θ denote the index set of all the majorants of ξ_m of order r: $\{u_\theta(x) : \theta \in \Theta\}$. Then,*

$$\bigcup_{\theta\in\Theta} \text{C.H.}\{G_j \in \Gamma_r\colon\ j \in \theta\}$$

is a partition of Γ_r, the possible range of $(M_1, \dots, M_r)$. That is, for any given value of $(M_1, \dots, M_r) \in \Gamma_r$, there is a most stringent upper bound $U_\theta(M_1, \dots, M_r)$ of p_m in the strict sense.

PROOF. Suppose that $\theta = \{k_1, \dots, k_r, m\}$. Since $0 \le \xi_m(x) \le u_\theta(x)$, $x \in [0, n]$, the unary polynomial $\prod_{j=1}^{r}(x - k_j)$, which is equal to $u_\theta(x)$ divided by its coefficient of x^r, is a majorant or a minorant of 0. Lemma 3.2 shows that

$$\text{C.H.}\{G_j \in \Gamma_r\colon\ j \in \theta - \{m\}\}, \qquad \theta \in \Theta,$$

are the set of all "$(r-1)$-dimensional" simplexes on $\partial\Gamma_r$ but not those with G_m at a vertex. Therefore, the set of "r-dimensional" simplexes with G_m at a vertex

$$\text{C.H.}\{G_j \in \Gamma_r\colon\ j \in \theta\}, \qquad \theta \in \Theta,$$

form a partition of Γ_r.

Example. The case $r = 2$. All the majorants are

$$(x-j)(x-j-1)/(m-j)(m-j-1), \quad j \in [0,\, m-2] \cup [m+1,\, n-1]; \quad \text{and}$$
$$x(n-x)/m(n-m).$$

The monic polynomials $(x-j)(x-j-1)$ are majorants and $x(x-n)$ is a minorant of 0. Segments $\overline{G_jG_{j+1}}$, $j \in [0,\, m-2] \cup [m+1,\, n-1]$, and $\overline{G_nG_0}$ form $\partial\Gamma_2$ if $\overline{G_{m-1}G_m}$ and $\overline{G_mG_{m+1}}$ are added. The set of triangles $\triangle G_jG_{j+1}G_m$, $j \in [0,\, m-2] \cup [m+1,\, n-1]$ and $\triangle G_0G_mG_n$ is a partition of Γ_2. The case $r = 3$ is explained in Section 4.

In Theorem 3.2 only the upper bound of p_m was discussed. The cases of the lower bounds of p_m and the upper and lower bounds of q_m are examined.

First, the upper bound of q_m is examined. Theorem 7.1 shows that if $u_\theta(x)$ is a majorant of η_m there exists a majorant of ξ_m with the same θ. The opposite is not always true since a majorant of ξ_m can be equal to ξ_m only on $[m,\, n]$, provided that $n - m \geq r$. The corresponding majorant of η_m is the constant 1. That is, for the given vector of moments in C.H.$\{G_j\colon\, j \in [m,\, n]\}$, $n - m \geq r$, 1 is the most stringent inequality in the strict sense. Except for this special feature, Theorem 3.2 holds for the upper bound of q_m.

Secondly, the lower bound of q_m is examined. Theorem 7.1 again shows that if $l_\theta(x)$ is a minorant of η_m there exists a majorant of ξ_{m-1} (not ξ_m) with the same θ. The opposite is not always true since a majorant of ξ_{m-1} can be equal to ξ_{m-1} only on $[0,\, m-1]$, provided that $r \leq m$. For the given vector of moments in C.H.$\{G_j\colon\, j \in [0,\, m-1]\}$, $r \leq m$, 0 is the most stringent inequality in the strict sense. Except for this feature, Theorem 3.2 holds for the lower bound of q_m.

Lastly, the lower bound of p_m is examined. Put

$$\Gamma^-_{m,r} = \text{C.H.}\{G_j \in \Gamma_r\colon\, j \neq m,\, j \in [0,\, n]\}, \qquad 0 \leq m \leq n.$$

In $\Gamma^-_{m,r}$ the most stringent lower bound in the strict sense is 0. The discussion before Lemma 3.2 can be applied on the distributions on $[0,\, m-1] \cup [m+1,\, n]$. If $\prod_{j=1}^r (x - k_j)$ is a majorant (or minorant) of 0 on $[0,\, m-1] \cup [m+1,\, n]$, the $(r-1)$-dimensional simplex

$$\text{C.H.}\{G_j \in \Gamma_r\colon\, j \in k = \{k_1, \ldots,\, k_r\}\}$$

is a part of the lower (or upper) boundary of $\Gamma^-_{m,r}$. There is a majorant (or minorant) such that $\{m-1,\, m+1\} \in k$ (if $m = 0$ or n, $1 \in k$ or $n-1 \in k$, respectively). Theorem 7.1 shows that if $\theta = k \cup \{m\}$ for such a set k, then

$$L_\theta(M_1, \ldots,\, M_r) = E\left[-\prod_{j\in\theta}(K-j)/(m-j)\right] \quad \left(\text{or } E\left[\prod_{j\in\theta}(K-j)/(m-j)\right]\right)$$

is the most stringent lower bound of p_m in the strict sense, for the moment vector in the r-dimensional simplex C.H.$\{G_j\colon\ j \in \theta\}$. This simplex has a surface C.H.$\{G_j\colon\ j \in k\} \in \partial\Gamma^-_{m,r}$, and the total of simplexes of this type is a partition of $\Gamma_r - \Gamma^-_{m,r}$. Thus, for any moment vector in Γ_r, L_θ or 0 is the most stringent lower bound of p_m in the strict sense.

The discussions are summarized as Theorem 3.3.

THEOREM 3.3. *For any value of* $(M_1, \ldots, M_r) \in \Gamma_r$, $1 \le r \le n$, *the most stringent lower and upper bounds of* p_m, $0 \le m \le n$, *and of* q_m, $1 \le m \le n-1$, *in the strict sense, are determined by the majorant-minorant method.*

Remark. Theorems 3.2 and 3.3 can be proved as a special case of the linear programming theory. The proofs of this paper help us to understand the geometric meaning of the results.

Incidentally, Lemma 3.2 gives a method to check whether a given r-vector $\boldsymbol{v}$ belongs to Γ_r or not:

LEMMA 3.3. *Put*

$$T(k,\, n;\ M_1, \ldots, M_r) = E\left[\prod_{j\in k}(K-j)\right], \qquad k = \{k_1, \ldots, k_r\} \subset [0,\, n],$$

for such k *that the polynomial* $\prod_{j\in k}(x-j)$ *is a majorant or a minorant of zero. Then* $\boldsymbol{v} = (v_1, \ldots, v_r) \in \Gamma_r$ *if and only if*

$$T(k,\, n;\ v_1, \ldots, v_r)\begin{cases} \ge 0 & \textit{for} \quad k \textit{ of a majorant}, \\ \le 0 & \textit{for} \quad k \textit{ of a minorant}, \end{cases}$$
$$\boldsymbol{v} \in \text{C.H.}\{G_j \in \Gamma_r\colon\ j \in k\}.$$

To apply Lemma 3.3 we have to know $\boldsymbol{v}^* = (v_1, \ldots, v_{r-1}) \in$ C.H.$\{G_j \in \Gamma_{r-1}\colon\ j \in k\}$. Similarly, to apply Theorem 3.1, we have to know $\boldsymbol{v} \in$ C.H.$\{G_j \in \Gamma_r\colon\ j \in \theta\}$.

Locating algorithm. Put

$$T(i,\, k,\, n;\ M_1, \ldots, M_{r-1}) = E\left[\prod_{j\in k-\{i\}}((x-j)/(i-j))\right], \qquad i \in k.$$

Then, $\boldsymbol{v}^* \in$ C.H.$\{G_j \in \Gamma_{r-1}\colon\ j \in k\}$ if and only if

$$T(i,\, k,\, n;\ v_1, \ldots, v_{r-1}) \ge 0, \qquad i \in k.$$

This condition means that $\boldsymbol{v}^*$ is the same side of G_i with respect to the $(r-2)$-dimensional hyperplane determined by $\{G_j \in \Gamma_{r-1};\ j \in k - \{i\}\}$.

4. Most stringent inequalities based on (M_1, M_2, M_3)

Theorems 3.2 and 3.3 in the case $r = 2$ leads to the results by Kwerel (1975*a*), Sathe *et al.* (1980) and Platz (1985). The derivation is simple and is omitted. For the case $r = 3$ Kwerel (1975*b*) obtained the inequalities on p_0 and p_n. Here, new inequalities on p_m and q_m, for a general m, based on (M_1, M_2, M_3) are shown. Before stating the propositions the shape of Γ_3 is studied again. The polyhedron

$$\Gamma_3 = \text{C.H.}\{G_0, G_1, \ldots, G_n\}, \qquad G_j = (j, j^2, j^3),$$

is separated into two parts by the triangle $\triangle G_0 G_m G_n$. (The following discussions are valid for $m \in [0, n]$, although some shapes degenerate unless $m \in [3, n-3]$.) Assume m fixed. The points $G_1, G_2, \ldots, G_{m-1}$ are above the plane determined by G_0, G_m and G_n, while the points $G_{m+1}, G_{m+2}, \ldots, G_{n-1}$, are below it. Thus, the upper part is C.H.$\{G_0, G_1, \ldots, G_m, G_n\}$ and the lower part is C.H.$\{G_0, G_m, G_{m+1}, \ldots, G_n\}$. Each is partitioned into two parts A and B, and C and D, respectively, and finally all are partitioned into tetrahedrons as follows:

$$A = \text{C.H.}\{G_0, G_1, \ldots, G_m\} = \bigcup_{j=1}^{m-2} \text{C.H.}\{G_0, G_j, G_{j+1}, G_m\},$$

$$B = \text{C.H.}\{G_0, G_1, \ldots, G_m, G_n\} - A = \bigcup_{j=0}^{m-2} \text{C.H.}\{G_j, G_{j+1}, G_m, G_n\},$$

$$C = \text{C.H.}\{G_m, G_{m+1}, \ldots, G_n\} = \bigcup_{j=m+1}^{n-2} \text{C.H.}\{G_m, G_j, G_{j+1}, G_n\}, \qquad \text{and}$$

$$D = \text{C.H.}\{G_0, G_m, G_{m+1}, \ldots, G_n\} - C = \bigcup_{j=m+1}^{n-1} \text{C.H.}\{G_0, G_m, G_j, G_{j+1}\}.$$

In fact, the shape of the part A (or C) is the same as Γ_3. The upper (or lower) boundary of A (or C) is given by triangles $\triangle G_j G_{j+1} G_m$, $j \in [0, m-2]$, (or $G_m G_j G_{j+1}$, $j \in [m+1, n-1]$), and the tetrahedrons C.H.$\{G_j, G_{j+1}, G_m, G_n\}$ (or C.H.$\{G_0, G_m, G_j, G_{j+1}\}$) form the remaining part B (or D).

To identify the tetrahedron to which a given point (M_1, M_2, M_3) belongs, the following procedure can be used. A point on the boundary can be on either side. Put

$$T(i, j, l) = T((i, j, l), n;\, M_1, M_2, M_3) = M_3 - (i+j+l)M_2 + (ij+jl+li)M_1 - ijl.$$

$T(i, j, l) = 0$ on the plane determined by G_i, G_j and G_l.

Locating algorithm.

1) Put

$$\mu = [(M_3 - mM_2)/(M_2 - mM_1)]. \tag{4.1}$$

If $T(0, \mu, \mu+1) \geq 0$ and $T(\mu, \mu+1, m) \leq 0$, then

$$(M_1, M_2, M_3) \in \text{C.H.}\{G_0, G_\mu, G_{\mu+1}, G_m\} \subset \begin{cases} A, & \text{if} \quad m \in [3, n] \text{ and } \mu \in [1, m-2], \\ D, & \text{if} \quad m \in [1, n-2] \text{ and } \mu \in [m+1, n-1]. \end{cases}$$

2) Otherwise, put

$$\nu = [(M_3 - (m+n)M_2 + mnM_1)/(M_2 - (m+n)M_1 + mn)]. \tag{4.2}$$

If $T(\nu, \nu+1, n) \leq 0$ and $T(\nu, \nu+1, m) \geq 0$, then

$$(M_1, M_2, M_3) \in \text{C.H.}\{G_m, G_\nu, G_{\nu+1}, G_n\} \subset \begin{cases} B, & \text{if} \quad m \in [2, n-1] \text{ and } \nu \in [0, m-2], \\ C, & \text{if} \quad m \in [0, n-3] \text{ and } \nu \in [m+1, n-2]. \end{cases}$$

If none of these conditions are satisfied, then $(M_1, M_2, M_3) \notin \Gamma_3$.

PROPOSITION 4.1. *The most stringent inequalities on $p_m = P\{K = m\}$ in the strict sense in terms of (M_1, M_2, M_3) are as follows:*

(1) *The upper bound.*

(1-1) *In* C.H.$\{G_0, G_m, G_\mu, G_{\mu+1}\}$ *in A or D which is characterized by μ of* (4.1),

$$p_m \leq \frac{1}{m(m-\mu)(m-\mu-1)}(M_3 - (2\mu+1)M_2 + \mu(\mu+1)M_1). \tag{4.3}$$

(1-2) *In* C.H.$\{G_m, G_\nu, G_{\nu+1}, G_n\}$ *in B and C which is characterized by ν of* (4.2)

$$p_m \leq \frac{1}{(m-\nu)(m-\nu-1)(m-n)} \cdot (M_3 - (n+2\nu+1)M_2 + ((2\nu+1)n + \nu(\nu+1))M_1 - \nu(\nu+1)n). \tag{4.4}$$

To locate a tetrahedron to which the given (M_1, M_2, M_3) belongs, the above mentioned procedure can be used.

(2) *The lower bound.*

(2-1) *For $m \in [2, n-1]$, in* C.H.$\{G_0, G_{m-1}, G_m, G_{m+1}\}$,

$$p_m \geq \frac{1}{m}(-M_3 + 2mM_2 - (m^2-1)M_1). \tag{4.5}$$

For $m \in [1, n-2]$, in C.H.$\{G_{m-1}, G_m, G_{m+1}, G_n\}$,

$$p_m \geq \frac{1}{n-m}(M_3 - (2m+n)M_2 + (2mn + m^2 - 1)M_1 - n(m^2-1)). \tag{4.6}$$

The triangle $\triangle G_{m-1}G_mG_{m+1}$ *is the boundary between* C.H.$\{G_0, G_{m-1}, G_m, G_{m+1}\}$ *and* C.H.$\{G_{m-1}, G_m, G_{m+1}, G_n\}$ *and characterized by* $T(m-1, m, m+1) = 0$. *Outside these tetrahedrons there is no available lower bound better than* 0.

(2-2) *For* $m = 0$ *or* n. *In* C.H.$\{G_0, G_1, G_\mu, G_{\mu+1}\}$ *which is characterized by*

$$\begin{aligned} &\mu = [(M_3 - M_2)/(M_2 - M_1)], \qquad 2 \le \mu \le n-1, \\ &p_0 \ge 1 - \frac{1}{\mu(\mu+1)}(M_3 - 2(\mu+1)M_2 + (2\mu+1+\mu(\mu+1))M_1). \end{aligned} \tag{4.7}$$

In C.H.$\{G_\nu, G_{\nu+1}, G_{n-1}, G_n\}$ *which is characterized by*

$$\begin{aligned} \nu = {} & [(M_3 - (2n-1)M_2 + (n-1)nM_1)/(M_2 - (2n-1)M_1 + (n-1)n)], \\ p_n \ge {} & \frac{1}{(n-\nu)(n-\nu-1)} \\ & \cdot (M_3 - (2\nu+n)M_2 \\ & \quad + ((2\nu+1)(n-1) + \nu(\nu+1))M_1 - \nu(\nu+1)(n-1)). \end{aligned} \tag{4.8}$$

In the other parts of Γ_3 *there is no available lower bound better than* 0. *For example, in* C.H.$\{G_1, G_j, G_{j+1}, G_{n-1}\}$, $j \in [2, n-3]$, *the possible bound of* p_0 *or* p_n *is* 0.

PROOF.

(1) The upper bounds.

(1-1) The cubic polynomial

$$\xi_m(x) \le x(x-\mu)(x-\mu-1)/m(m-\mu)(m-\mu-1),$$

where $\mu \in [1, m-2]$ (Region A) or $\mu \in [m+1, n-1]$ (Region D), gives (4.3).

(1-2) The cubic polynomial

$$\xi_m(x) \le (x-\nu)(x-\nu-1)(n-x)/(m-\nu)(m-\nu-1)(n-m),$$

where $\nu \in [0, m-2]$ (Region B) or $\nu \in [m+1, n-2]$ (Region C), gives (4.4).

(2) The lower bounds.

(2-1) The cubic polynomials

$$\begin{aligned} &\xi_m(x) \ge x(x-m+1)(m+1-x)/m, \qquad \text{and} \\ &\xi_m(x) \ge (x-m+1)(x-m-1)(x-n)/(n-m), \end{aligned}$$

give the inequalities (4.5) and (4.6), respectively.

(2-2) The cubic polynomials.

$$\begin{aligned} &\xi_0(x) \ge -(x-1)(x-\mu)(x-\mu-1)/\mu(\mu+1), \qquad \mu \in [2, n-1], \qquad \text{and} \\ &\xi_n(x) \ge (x-\nu)(x-\nu-1)(x-n+1)/(n-\nu)(n-\nu-1), \qquad \nu \in [0, n-3], \end{aligned}$$

give (4.7) and (4.8), respectively.
There is no other cubic minorant of ξ_m positive somewhere on $[0,\, n]$.

Next, new results on q_m in terms of $(M_1,\, M_2,\, M_3)$ are given.

PROPOSITION 4.2. *The most stringent inequalities on* $q_m = P\{K \geq m\}$ *in the strict sense in terms of* $(M_1,\, M_2,\, M_3)$ *are as follows:*

(1) *The upper bounds.*

(1-1) *In* C.H.$\{G_0,\, G_m,\, G_\mu,\, G_{\mu+1}\}$ *with* μ *of* (4.1), *if* $\mu \in [1,\, m-2]$ (*Region* A), *then*

$$q_m \leq \frac{1}{m(m-\mu)(m-\mu-1)}(M_3 - (2\mu+1)M_2 + \mu(\mu+1)M_1), \tag{4.9}$$

else if, $\mu \in [m+1,\, n-1]$ (*Region* D), *then*

$$q_m \leq \frac{1}{m\mu(\mu+1)}(M_3 - (m+2\mu+1)M_2 + (m(2\mu+1)+\mu(\mu+1))M_1). \tag{4.10}$$

(1-2) *In* C.H.$\{G_\nu,\, G_{\nu+1},\, G_m,\, G_n\}$ *in* B *with* $\nu \in [0,\, m-2]$ *of* (4.2),

$$\begin{aligned} q_m \leq\ & \frac{-1}{(m-\nu)(m-\nu-1)(n-m)} \\ & \cdot (M_3 - (n+2\nu+1)M_2 + (n(2\nu+1)+\nu(\nu+1))M_1 - n\nu(\nu+1)) \\ & + \frac{1}{(n-\nu)(n-\nu-1)(n-m)} \\ & \cdot (M_3 - (m+2\nu+1)M_2 + (m(2\nu+1)+\nu(\nu+1))M_1 - m\nu(\nu+1)). \end{aligned} \tag{4.11}$$

In Region C *there is no available upper bound better than* 1. *To locate a tetrahedron to which the given* $(M_1,\, M_2,\, M_3)$ *belongs, the above mentioned procedure can be used.*

(2) *The lower bounds.*

In the following lower bounds, Regions B, C *and* D *are defined as in the beginning of this section with* $m-1$ *replacing* m.

(2-1) *In* C.H.$\{G_\nu,\, G_{\nu+1},\, G_{m-1},\, G_n\}$ *with* ν *of* (4.2) (*with* $m-1$ *replacing* m), *if* $\nu \in [0,\, m-3]$ (*Region* B), *then*

$$\begin{aligned} q_m \geq\ & \frac{1}{(n-\nu)(n-\nu-1)(n-m+1)} \\ & \cdot (M_3 - (m+2\nu)M_2 \\ & \quad + ((m-1)(2\nu+1)+\nu(\nu+1))M_1 - (m-1)\nu(\nu+1)), \end{aligned} \tag{4.12}$$

else if, $\nu \in [m,\, n-2]$ (*Region* C) *then*

$$\begin{aligned} q_m \geq\ & 1 + \frac{1}{(\nu-m+1)(\nu-m+2)(n-m+1)} \\ & \cdot (M_3 - (2\nu+n+1)M_2 + ((2\nu+1)n+\nu(\nu+1))M_1 - n\nu(\nu+1)). \end{aligned} \tag{4.13}$$

(2-2) *In* C.H.$\{G_0, G_{m-1}, G_\mu, G_{\mu+1}\}$ *in Region D with* $\mu \in [m, n-1]$ *of* (4.1) (*with* $m-1$ *replacing* m),

$$(4.14)\quad q_m \geq -\frac{1}{\mu(\mu-m+1)}(M_3-(m+\mu)M_2+(m-1)(\mu+1)M_1) + \frac{1}{(\mu+1)(\mu-m+2)}(M_3-(m+\mu-1)M_2+(m-1)\mu M_1).$$

Remark. The bound (4.9) on q_m is the same as (4.3) on p_m, and the bound (4.12) on q_m, $m = n$, is the same as (4.8) on p_n. The bound (4.13) on q_m is equivalent with (4.4) on p_{m-1} (not p_m), and (4.10) on q_m, $m = 1$, to (4.7) on p_0.

PROOF.

(1) The upper bounds.

(1-1) The cubic polynomial

$$\eta_m \leq x(x-\mu)(x-\mu-1)/m(m-\mu)(m-\mu-1), \qquad \mu \in [1, m-2],$$

gives (4.9); and the cubic polynomial

$$\eta_m \leq 1 + (x-m)(x-\mu)(x-\mu-1)/m\mu(\mu+1), \qquad \mu \in [m+1, n-1],$$

gives (4.10).

(1-2) The cubic polynomial

$$\eta_m \leq (x-\nu)(x-\nu-1) \cdot \left\{\frac{x-n}{(m-\nu)(m-\nu-1)(m-n)} + \frac{x-m}{(n-\nu)(n-\nu-1)(n-m)}\right\},$$

for $\nu \in [0, m-2]$, gives (4.11).

(2) The lower bounds.

(2-1) The cubic polynomial

$$\eta_m \geq (x-\nu)(x-\nu-1)(x-m+1)/(n-\nu)(n-\nu-1)(n-m+1),$$

for $\nu \in [0, m-3]$, gives (4.12); and the cubic polynomial

$$\eta_m \geq 1 - \frac{(x-\nu)(x-\nu-1)(x-n)}{(m-1-\nu)(m-2-\nu)(m-1-n)}, \qquad \nu \in [m, n-2],$$

gives (4.13).

(2-2) The cubic polynomial

$$\eta_m \geq x(x-m+1)\left\{\frac{x-\mu}{(\mu+1)(\mu-m+2)} - \frac{x-\mu-1}{\mu(\mu-m+1)}\right\}, \qquad \mu \in [m, n-1],$$

gives (4.14).

5. Most stringent inequalities on p_0 and p_n based on (M_1, M_2, M_3, M_4)

In this section new inequalities of degree 4 on p_0 and p_n are shown. Because, the algorithm to determine the most stringent inequalities in the strict sense is simpler for $m = 0$ and n. Inequalities on p_m and q_m in general are very close to these. For example, the upper bound on p_m is obtained by replacing the denominator $\lambda(\lambda+1)\mu(\mu+1)$ of (5.2) by $(\lambda-m)(\lambda-m+1)(\mu-m)(\mu-m+1)$ and adjusting the possible values of λ and μ.

In this section $G_j = (j, j^2, j^3, j^4)$, $j \in [0, n]$, and the possible range of (M_1, M_2, M_3, M_4) is $\Gamma_4 = \text{C.H.}\{G_0, G_1, \ldots, G_n\}$, which is partitioned into the four-dimensional simplexes, of which the boundary consists of five hyperplanes. A hyperplane determined by four points G_{j_1}, G_{j_2}, G_{j_3} and G_{j_4} has the equation

$$\begin{aligned}(5.1)\qquad T(j_1, j_2, j_3, j_4) &= T((j_1, j_2, j_3, j_4), n;\, M_1, M_2, M_3, M_4)\\ &= M_4 - \left(\sum_{i=1}^{4} j_i\right) M_3 + \left(\sum_{1\le i_1<i_2\le 4} j_{i_1}j_{i_2}\right) M_2\\ &\quad - \left(\sum_{1\le i_1<i_2<i_3\le 4} j_{i_1}j_{i_2}j_{i_3}\right) M_1 + \prod_{i=1}^{4} j_i = 0.\end{aligned}$$

PROPOSITION 5.1. *The most stringent inequalities in the strict sense on p_0 and p_n in terms of (M_1, M_2, M_3, M_4).*

(1) *The upper bound on p_0.*

$$\begin{aligned}(5.2)\qquad p_0 \le 1 + &\frac{1}{\lambda(\lambda+1)\mu(\mu+1)}\\ &\cdot (M_4 - 2(\lambda+\mu+1)M_3\\ &\quad + (\lambda(\lambda+1) + \mu(\mu+1) + (2\lambda+1)(2\mu+1))M_2\\ &\quad - (\lambda(\lambda+1)(2\mu+1) + (2\lambda+1)\mu(\mu+1))M_1),\end{aligned}$$

for $(M_1, M_2, M_3, M_4) \in \text{C.H.}\{G_0, G_\lambda, G_{\lambda+1}, G_\mu, G_{\mu+1}\}$, $1 \le \lambda$, $\lambda+2 \le \mu \le n-1$. *If μ is given, the simplex is located by*

$$(5.3)\qquad \lambda = \left[\frac{M_4 - (2\mu+1)M_3 + \mu(\mu+1)M_2}{M_3 - (2\mu+1)M_2 + \mu(\mu+1)M_1}\right],$$

and if λ is given, the simplex is located by (5.3) *with λ and μ exchanged. An approximate value of λ and μ is the solution of the quadratic equation*

$$(5.4)\qquad (M_3M_1 - M_2^2)z^2 - (M_4M_1 - M_3M_2)z + (M_4M_2 - M_3^2) = 0.$$

(2) *The upper bound on p_n.*

$$\begin{aligned}(5.5)\qquad p_n \le &\frac{1}{(n-\lambda)(n-\lambda-1)(n-\mu)(n-\mu-1)}\\ &\cdot (M_4 - 2(\lambda+\mu+1)M_3\\ &\quad + (\lambda(\lambda+1) + \mu(\mu+1) + (2\lambda+1)(2\mu+1))M_2\\ &\quad - (\lambda(\lambda+1)(2\mu+1) + (2\lambda+1)\mu(\mu+1))M_1) + \lambda(\lambda+1)\mu(\mu+1),\end{aligned}$$

for $(M_1, M_2, M_3, M_4) \in \text{C.H.}\{G_\lambda, G_{\lambda+1}, G_\mu, G_{\mu+1}, G_n\}$, $0 \le \lambda$, $\lambda + 2 \le \mu \le n-2$. *If* μ *is given, the simplex is located by*

$$(5.6)\quad \lambda = \left[\frac{M_4 - (2\mu+1+n)M_3 + (n(2\mu+1)+\mu(\mu+1))M_2 - n\mu(\mu+1)M_1}{M_3 - (2\mu+1+n)M_2 + (n(2\mu+1)+\mu(\mu+1))M_1 - n\mu(\mu+1)}\right],$$

and if λ *is given, the simplex is located by* (5.6) *with* λ *and* μ *exchanged. An approximate value of* λ *and* μ *is the solution of*

$$(5.7)\quad \begin{aligned} &((M_3 - nM_2)(M_1 - n) - (M_2 - nM_1)^2)z^2 \\ &\quad - ((M_4 - nM_3)(M_1 - n) - (M_3 - nM_2)(M_2 - nM_1))z \\ &\quad + ((M_4 - nM_3)(M_2 - nM_1) - (M_3 - nM_2)^2) = 0. \end{aligned}$$

(3) *The lower bound on* p_0.

$$(5.8)\quad \begin{aligned} p_0 \ge 1 + &\frac{1}{\lambda(\lambda+1)n} \\ &\cdot(M_4 - (2\lambda+n+2)M_3 + (\lambda(\lambda+1) + (2\lambda+1)(n+1) + n)M_2 \\ &\qquad - (\lambda(\lambda+1)(n+1) + (2\lambda+1)n)M_1), \end{aligned}$$

for $(M_1, M_2, M_3, M_4) \in \text{C.H.}\{G_0, G_1, G_\lambda, G_{\lambda+1}, G_n\}$, $\lambda \in [2, n-2]$. *The simplex is located by*

$$(5.9)\quad \lambda = \left[\frac{M_4 - (n+1)M_3 + nM_2}{M_3 - (n+1)M_2 + nM_1}\right].$$

(4) *The lower bound on* p_n.

$$(5.10)\quad \begin{aligned} p_n \ge &\frac{1}{n(n-\lambda)(n-\lambda-1)} \\ &\cdot (M_4 - 2(\lambda+n)M_3 \\ &\quad + ((2\lambda+1)(n-1) + \lambda(\lambda+1))M_2 - \lambda(\lambda+1)(n-1)M_1), \end{aligned}$$

for $(M_1, M_2, M_3, M_4) \in \text{C.H.}\{G_0, G_\lambda, G_{\lambda+1}, G_{n-1}, G_n\}$, $\lambda \in [1, n-3]$. *The simplex is located by*

$$(5.11)\quad \lambda = \left[\frac{M_4 - (2n-1)M_3 + (n-1)nM_2}{M_3 - (2n-1)M_2 + (n-1)nM_1}\right].$$

In cases (2)–(4), *outside the specified simplexes, the available bounds are* 1 *or* 0. *The consistency of the moments is checked in each simplex comparing it with the boundary. For example, in* $\text{C.H.}\{G_0, G_1, G_\lambda, G_{\lambda+1}, G_n\}$ *for the case* (3), *if the value of* λ *is on* $[2, n-2]$, *using the function* (5.1)

$$T(0, 1, \lambda, \lambda+1) \ge 0, \quad T(0, \lambda, \lambda+1, n) \le 0 \quad \text{and} \quad T(1, \lambda, \lambda+1, n) \ge 0$$

should be checked.

PROOF.

(1) The quartic polynomial

$$\xi_0(x) \le (x-\lambda)(x-\lambda-1)(x-\mu)(x-\mu-1)/\lambda(\lambda+1)\mu(\mu+1)$$

gives the inequality (5.2). Given μ, the value of λ is determined by $T(0,\ \lambda,\ \mu,\ \mu+1) = 0$. The quadratic equation to obtain an approximate value of λ and μ is obtained from the system of this equation and $T(0,\ \lambda,\ \lambda+1,\ \mu) = 0$.

(2) The quartic polynomial

$$\xi_n(x) \le (x-\lambda)(x-\lambda-1)(x-\mu)(x-\mu-1)/(n-\lambda)(n-\lambda-1)(n-\mu)(n-\mu-1)$$

gives the inequality (5.5). The discussions are similar to those in case (1).

(3) The quartic polynomial

$$\xi_0(x) \ge (x-1)(x-\lambda)(x-\lambda-1)(x-n)/\lambda(\lambda+1)n$$

gives the inequality (5.8). The value (5.9) of λ is determined by $T(0,\ 1,\ \lambda,\ n) = 0$.

(4) The quartic polynomial

$$\xi_n(x) \ge x(x-\lambda)(x-\lambda-1)(x-n+1)/n(n-\lambda)(n-\lambda-1)$$

gives the inequality (5.10). The value (5.11) of λ is determined by $T(0,\ \lambda,\ n-1,\ n) = 0$.

6. The classical Bonferroni inequalities and the Galambos inequalities

Unless all the lower moments are used, Theorems 3.2 and 3.3 are not applicable. Still Theorems 2.1 and 3.1 are useful for obtaining Bonferroni-type inequalities. In this section a new proof is given to the five groups of inequalities; the classical Bonferroni, Galambos' and Mărgăritescu's (1987) on p_m and q_m. The proof is shorter than those of Walker (1981), Galambos (1987) and Recsei and Seneta (1987). Here, only elementary properties of the binomial coefficients are used. Moreover, all the inequalities are shown to be stringent. Galambos' inequalities on q_m (1977) are not stringent, and were improved by Mărgăritescu (1987). Proposition 6.2 expresses the improved ones in a shorter form, and proves them simply.

PROPOSITION 6.1. *The following inequalities* (6.1)–(6.4) *hold. Except for* (6.4), *they are the most stringent.*

The classical Bonferroni inequalities,

$$\sum_{r=m}^{m+2u-1} (-1)^{r-m}\binom{r}{m} S_r \le p_m \le \sum_{r=m}^{m+2u} (-1)^{r-m}\binom{r}{m} S_r, \tag{6.1}$$

($0 \le m \le n$; $2 \le 2u \le n-m+1$ *for the l.h.s. and* $0 \le 2u \le n-m$ *for the r.h.s.*),

$$\sum_{r=m}^{m+2u-1} (-1)^{r-m}\binom{r-1}{m-1} S_r \le q_m \le \sum_{r=m}^{m+2u} (-1)^{r-m}\binom{r-1}{m-1} S_r, \tag{6.2}$$

($1 \le m \le n$; u *is the same as* (6.1)); *and Galambos' inequalities,*

$$\sum_{r=m}^{m+2u-1}(-1)^{r-m}\binom{r}{m}S_r + \frac{2u}{n-m}\binom{m+2u}{m}S_{m+2u} \le p_m \le \sum_{r=m}^{m+2u}(-1)^{r-m}\binom{r}{m}S_r - \frac{2u+1}{n-m}\binom{m+2u+1}{m}S_{m+2u+1}, \tag{6.3}$$

($0 \le m \le n$; $2 \le 2u \le n-m$ *for the l.h.s. and* $0 \le 2u \le n-m-1$ *for the r.h.s.*),

$$\sum_{r=m}^{m+2u-1}(-1)^{r-m}\binom{r-1}{m-1}S_r + \frac{2u}{n-m}\binom{m+2u-1}{m-1}S_{m+2u} \le q_m \le \sum_{r=m}^{m+2u}(-1)^{r-m}\binom{r-1}{m-1}S_r - \frac{2u+1}{n-m}\binom{m+2u}{m-1}S_{m+2u+1}, \tag{6.4}$$

($1 \le m \le n$; u *is the same as* (6.3)).

PROOF. Firstly, let

$$\begin{aligned} t_1(x) = t_1(x;\ m,\ m+k) &= \binom{x}{m}\binom{x-m-1}{k}\Big/\binom{-1}{k} \\ &= (-1)^k\binom{x}{m}\sum_{i=0}^{k}\binom{x-m}{i}\binom{-1}{k-i} \\ &= \binom{x}{m}\sum_{i=0}^{k}(-1)^i\binom{x-m}{i} = \sum_{r=m}^{m+k}(-1)^{r-m}\binom{r}{m}\binom{x}{r}. \end{aligned}$$

$\left(\text{Notice that } \binom{-1}{k} = (-1)^k.\right)$ The first expression shows that

$$t_1(x) = \begin{cases} 0, & \text{if} \quad x \in [0,\ m-1] \cup [m+1,\ m+k], \\ 1, & \text{if} \quad x = m, \end{cases}$$

and that, for $x \ge m+k+1$,

$$t_1(x) \begin{cases} \ge \binom{m+k+1}{m} \ge 1, & \text{if} \quad k \text{ is even}, \\ \le -\binom{m+k+1}{m}, & \text{if} \quad k \text{ is odd}. \end{cases}$$

Thus, $t_1(x)$ is an upper or lower bound of $\xi_m(x)$ if k is even or odd, respectively, and $t_1(x) = \xi_m(x)$ at $m+k+1$ points if the degree of $t_1(x)$ is $m+k$. Taking the expectation of $t_1(K)$ in the last expression, (6.1) and its stringency is proved.

Secondly, let

$$t_2(x) = t_2(x;\ m,\ m+k) = \sum_{j=m}^{m+k} t_1(x;\ j,\ m+k) = \sum_{j=m}^{m+k}\sum_{r=j}^{m+k}(-1)^{r-j}\binom{r}{j}\binom{x}{r}$$
$$= \sum_{r=m}^{m+k}(-1)^r\binom{x}{r}\sum_{j=m}^{r}(-1)^{r-j}\binom{r}{j} = \sum_{r=m}^{m+k}(-1)^{r-m}\binom{r-1}{m-1}\binom{x}{r}.$$

From the definition,

$$t_2(x) = \begin{cases} 0, & \text{if} \quad x \in [0,\ m-1], \\ 1, & \text{if} \quad x \in [m,\ m+k], \\ 1 + (-1)^k \displaystyle\sum_{j=1}^{h}\binom{m+k+j-1}{m-1}\binom{k+j-1}{k} & \\ & \text{if} \quad x = m+k+h,\ h = 1,\ 2, \dots . \end{cases}$$

The same argument as $t_1(x)$ proves (6.2) and its stringency.

Thirdly, let

$$\begin{aligned} t_3(x) &= t_3(x;\ m,\ m+k) \\ &= t_1(x;\ m,\ m+k) + (-1)^{k+1}\frac{k+1}{n-m}\binom{m+k}{m}\binom{x}{m+k+1} \\ &= (-1)^k\binom{x}{m}\binom{x-m-1}{k} + (-1)^{k+1}\binom{x}{m}\frac{x-m}{n-m}\binom{x-m-1}{k} \\ &= t_1(x;\ m,\ m+k)\frac{n-x}{n-m}. \end{aligned}$$

Therefore

$$t_3(x) = \begin{cases} 1, & \text{if} \quad x = m, \\ 0, & \text{if} \quad x \in [0,\ m-1] \cup [m+1,\ m+k] \cup \{n\}, \end{cases}$$

and t_3 has the sign $(-1)^k$ for $x \in [m+k+1,\ n-1]$, and (6.3) and its stringency are proved.

Finally, let

$$t_4(x) = t_4(x;\ m,\ m+k) = t_2(x;\ m,\ m+k) + (-1)^{k+1}\frac{k+1}{n-m}\binom{m+k}{m-1}\binom{x}{m+k+1}.$$

The last term vanishes if $x \in [0,\ m+k]$, and since

$$\binom{m+k+h}{m+k+1} = \sum_{j=1}^{h}\binom{m+k+j-1}{m+k},$$

$$t_4(m+k+h) = 1 + (-1)^k\sum_{j=1}^{h}\binom{m+k+j-1}{m+k}\frac{(m+k)!}{(m-1)!k!}\left(\frac{1}{k+j} - \frac{1}{n-m}\right).$$

The summand is nonnegative if $n \geq m+k+j$, thus t_4 has the same property as t_2, and (6.4) is proved. The inequality cannot be stringent. Since $t_4(m+k+h) \neq 1$ for $h = 1, 2, \ldots$ unless $m+k+1 = n$. Using up to S_n, however, (6.2) and (6.4) become exact.

Galambos' inequalities imply the classical ones. If one uses just $\{S_m, S_{m+1}, \ldots, S_{m+k}\}$ with even k, however, the bounds appear as an upper bound in (6.1) and (6.2), and as a lower bound in (6.3) and (6.4). With odd k, the bounds appear in the opposite sides. In this sense, Galambos' inequalities are complementary to the classical ones and not improvements. It happens, therefore, that (6.2) is the most stringent but (6.4) is not.

Proposition 4.1 suggests the possibility to improve (6.4) by obtaining the most stringent inequality like (6.3).

PROPOSITION 6.2. *The following inequality* (6.5) *is an improvement of Galambos' inequality* (6.4) *on* q_m*, and it is the most stringent.*

$$\text{(6.5)} \quad \sum_{r=m}^{m+2u-1} (-1)^{r-m}\binom{r-1}{m-1} S_r + A(m+2u;\ m,\ n) S_{m+2u} \leq q_m$$
$$\leq \sum_{r=m}^{m+2u} (-1)^{r-m}\binom{r-1}{m-1} S_r - A(m+2u+1;\ m,\ n) S_{m+2u+1},$$

where

$$\text{(6.6)} \quad A(m+k+1;\ m,\ n)$$
$$= \sum_{j=1}^{n-m-k} \binom{m+k+j-1}{m-1}\binom{k+j-1}{k} \Big/ \binom{n}{m+k+1}$$
$$= \left(\sum_{l=0}^{k} (-1)^{k-l}\binom{m+l-1}{m-1}\binom{n}{m+l} + (-1)^{k+1}\right) \Big/ \binom{n}{m+k+1}.$$

For example,

$$\text{(6.6a)} \qquad A(k+2;\ 1,\ n) = \frac{k+2}{n},$$

$$\text{(6.6b)} \qquad A(k+3;\ 2,\ n) = \frac{k+3}{n(n-1)}((k+1)n+1),$$

and

$$\text{(6.6c)} \qquad A(m+1;\ m,\ n) = \left(\binom{n}{m} - 1\right) \Big/ \binom{n}{m+1}.$$

Remark. Mărgăritescu (1987) solved the same problem and obtained a more complex expression

$$A(m+k+1,\ m,\ n) = \frac{k+1}{n-m}\binom{m+k}{m-1} \sum_{i=0}^{n-m-k-1} \frac{i!(n-m-k-1)^{(i)}}{(n-m-1)^{(i)}(m+k+1+i)^{(i)}}$$

by using an integral expression of $t_2(x)$. The equality of this expression with (6.6) is discussed elsewhere (Sibuya (1991)).

PROOF. Using the notation in the proof of Proposition 6.1, define

$$t_5(x;\ m,\ m+k) = t_2(x;\ m,\ m+k) + (-1)^{k+1} A(m+k+1;\ m,\ n)\binom{x}{m+k+1}.$$

Then, as $t_4(x;\ m,\ m+k)$,

$$t_5(x) = \begin{cases} 0, & \text{if} \quad x \in [0,\ m-1], \\ 1, & \text{if} \quad x \in [m,\ m+k], \end{cases}$$

and by similar computations as t_4,

$$t_5(m+k+h) = 1 + (-1)^k \left(\sum_{j=1}^{h} \binom{m+k+j-1}{m-1}\binom{k+j-1}{k} \right.$$
$$\left. -A(m+k+1;\ m,\ n)\binom{m+k+h}{m+k+1} \right),$$

and $t_5(n) = 1$ from the definition (6.6) of $A(m+k+1;\ m,\ n)$. From the definitions $t_4(x) - t_5(x) = 0$, $x \in [0,\ m+k]$, and the difference is of degree $m+k+1$ and monotone outside $[0,\ m+k]$. Moreover,

$$t_4(n) - t_5(n) \begin{cases} > 0, & \text{if} \quad k \text{ is even}, \\ < 0, & \text{if} \quad k \text{ is odd}, \end{cases}$$

and the inequalities hold for $x \in [m+k+1,\ n]$. Thus, (6.5) is an improvement of (6.4).

The improvement on (6.4) is evident for $m = 1$ and 2, (6.6a) and (6.6b), respectively. The inequality (6.5) with $m = 1$ is equivalent to the inequality (6.3) with $m = 0$, but the inequality (6.4) with $m = 1$ is not so.

7. Finding majorants and minorants

The following fact is basic to find exhaustively majorants and minorants of ξ_m and η_m.

LEMMA 7.1.

(1) *Let h: $R \to R$ be a polynomial of degree $r = s + t$ such that*

$$h(a_i) = 0, \quad i = 1, \ldots, s; \quad h(m) = 1; \qquad \text{and} \qquad h(b_j) = 0, \quad j = 1, \ldots, t;$$

where $a_s < \cdots < a_1 < m < b_1 < \cdots < b_t$, then

$$h(x) \begin{cases} < 0 & \text{if} \quad a_{2i} < x < a_{2i-1} \text{ or } b_{2j-1} < x < b_{2j}, \\ > 0 & \text{if} \quad a_{2i+1} < x < a_{2i},\ b_{2j} < x < b_{2j+1} \text{ or } a_1 < x < b_1, \end{cases}$$
$$i = 1,\ 2, \ldots\ , \text{ and } j = 1,\ 2, \ldots$$

(2) *Let* h: $R \to R$ *be a polynomial of order* $r = s + t - 1$ *such that*

$$h(a_i) = 0, \quad i = 1, \ldots, s; \qquad \textit{and} \qquad h(b_j) = 1, \quad j = 1, \ldots, t;$$

where $a_s < \cdots < a_1 < b_1 < \cdots < b_t$, *then*

(i) $h'(x) > 0 \quad a_1 < x < b_1$

(ii) $h(x) \begin{cases} < 0 & \textit{if} \quad a_{2i} < x < a_{2i-1}, \\ > 0 & \textit{if} \quad a_{2i+1} < x < a_{2i}, \quad i = 1, 2, \ldots \end{cases}$

(iii) $h(x) \begin{cases} > 1 & \textit{if} \quad b_{2j-1} < x < b_{2j}, \\ < 1 & \textit{if} \quad b_{2j} < x < b_{2j+1}, \quad j = 1, 2, \ldots . \end{cases}$

PROOF. The first part. Because of the mean-value theorem $h'(x) = 0$ somewhere in (a_i, a_{i+1}), $i \in [1, s-1]$, (a_1, b_1) and (b_j, b_{j+1}), $j \in [1, t-1]$. That is, there are $r - 1$ zeroes of $h'(x)$, and there cannot be zero outside (a_s, b_t). These facts mean that

$$h'(x) > 0, \quad x = a_{2i-1} \text{ or } b_{2j}; \qquad \text{and} \qquad h'(x) < 0, \quad x = a_{2i} \text{ or } b_{2j-1}.$$

The last part is similarly proved.

All the majorants and minorants of $\xi_m(x)$ and $\eta_m(x)$ are determined from Lemma 7.1. A majorant u_θ of ξ_m of degree r satisfies

$$u_\theta(x) \geq \xi_m(x), \qquad x \in [0, n] \tag{7.1}$$

and

$$u_\theta(x) = \xi_m(x), \quad x \in \theta, \quad \text{for a set } \theta \subset [0, n] \text{ and } |\theta| = r + 1, \tag{7.2}$$

and is determined by specifying θ. The point $x = m$ must always be selected, otherwise $u_\theta(x) \equiv 0$. If a point of $[1, m-1] \cup [m+1, n-1]$ is selected, it must be in a pair of adjacent points $[j, j+1]$ which are included within $[0, m-1]$ or within $[m+1, n]$, otherwise $u_\theta(x) < 0$ at one of the adjacent points. Both ends are exceptional.

THEOREM 7.1.

(1) *A majorant of* ξ_m. *A polynomial* u_θ, $|\theta| = r + 1$, *is a majorant of* ξ_m *if and only if* $\theta \subset [0, n]$ *is selected as follows*:

(i) *Always* $m \in \theta$ $(m \in [0, n])$.

(ii) *If the degree* r *of* u_θ *is even, the end points* 0 *and* n *are not included or included as a pair* $\{0, n\}$. *If* $m = 0$ *or* n, *the other cannot be included. The other even number of points are pairs of adjacent points within* $[0, m-1]$ *or within* $[m+1, n]$ (*if* $\{0, n\}$ *is selected, within* $[1, m-1]$ *or within* $[m+1, n]$).

(iii) *If* r *is odd, one of* $\{0, n\}$ *must be included. If* $m = 0$ *or* n, *the other must be included. The other even number of points are pairs of adjacent* *points within* $[0, m-1]$ *or within* $[m+1, n]$ *excluding the selected end point* 0 *or* n.

(2) *A minorant of* ξ_m.

(i) *If* $1 < m < n$, *three points of* $[m-1, m+1]$ *must be selected. If* $m = 0$ *or* $m = n$, $[0, 1]$ *or* $[n-1, n]$, *respectively, must be selected.*

(ii) *If the number of remaining points are even, pairs* $\{0, n\}$ *(if both are not selected yet) or pairs of adjacent points within* $[0, m-2]$ *or within* $[m+2, n]$, *excluding the selected end points, are selected.*

(iii) *If the number of remaining points are odd, one of* $\{0, n\}$ *must be selected. The other even number of points must be pairs of adjacent points as in the above case* (ii).

(3) *A majorant of* η_m. *The selection of* θ *is the same as a majorant of* ξ_m, *except that at least one point of* $[0, m-1]$, *where* $\eta_m = 0$, *must be selected.*

(4) *A minorant of* η_m.

(i) *Always* $m-1 \in \theta$ $(m \in [1, n])$. *In the following selection, at least one point of* $[m, n]$ *must be included.*

(ii) *If* r *is even, the pair* $\{0, n\}$ *or pairs of adjacent points within* $[0, m-2]$ *or within* $[m, n]$, *excluding the selected end points, must be selected.*

(iii) *If* r *is odd, one of* $\{0, n\}$ *must be selected. If* $m = 1$ *or* n, *the pair is selected by* (i). *The remaining even number of points must be pairs of adjacent points within* $[0, m-2]$ *or within* $[m, n]$ *excluding the selected end points.*

In the discussions for Theorem 3.2, majorants and minorants of 0 were compared with those of ξ_m. For that purpose, and just for checking the consistency of a given vector of all the lower moments, the following method to find a majorant and a minorant of 0 is necessary.

THEOREM 7.2. *A monic polynomial of degree* r *is a majorant or a minorant of* 0 *if and only if it is of the form* $h(x;\ k) = \prod_{j=1}^{r}(x - k_j)$, *and* $k = \{k_1, \dots, k_r\}$, $k_j \in [0, n]$, *satisfies the following condition.*

(1) *The case* r *is even. If the set* k *consists of pairs of adjacent points within* $[0, n]$, h *is a majorant. If* k *consists of the pair* $\{0, n\}$ *and* $(r/2) - 1$ *pairs of adjacent points within* $[1, n-1]$, h *is a minorant.*

(2) *The case* r *is odd. If* k *consists of* n *and* $(r-1)/2$ *pairs of adjacent points within* $[0, n-1]$, h *is a majorant. If* k *consists of* 0 *and* $(r-1)/2$ *pairs of adjacent points within* $[1, n]$, h *is a minorant.*

REFERENCES

Alt, F. B. (1982). Bonferroni inequalities and intervals, *Encyclopedia of Statistical Sciences* (eds. S. Kotz and N. L. Johnson), Vol. 1, 294–300, Wiley, New York.

Galambos, J. (1975). Methods for proving Bonferroni type inequalities, *J. London Math. Soc. (2)*, **9**, 561–564.

Galambos, J. (1977). Bonferroni inequalities, *Ann. Probab.*, **5**, 577–581.

Galambos, J. (1984). Order statistics, *Handbook of Statistics, Vol. 4, Nonparametric Methods* (eds. P. R. Krishnaiah and P. K. Sen), 359–382, North Holland, Amsterdam.

Galambos, J. (1987). *The Asymptotic Theory of Extreme Order Statics*, 2nd ed., Robert E. Krieger Publ., Malabar, Florida.

Hailperin, T. (1965). Best possible inequalities for the probability of a logical function of events, *Amer. Math. Monthly*, **72**, 343–359.

Hunter, D. (1976). An upper bound for the probability of a union, *J. Appl. Probab.*, **13**, 597–603.

Isii, K. (1964). Inequalities of the types of Chebyshev and Cramér-Rao and mathematical programming, *Ann. Inst. Statist. Math.*, **16**, 277–293.

Jordan, C. (1960). *Calculus of Finite Difference*, Chelsea, New York.

Keilson, J. and Gerber, H. (1971). Some results for discrete unimodality, *J. Amer. Statist. Assoc.*, **66**, 386–389.

Knuth, D. (1975). *Fundamental Algorithms, The Art of Computer Programming*, Vol. 1, 2nd ed., Addison-Wesley, Reading, Massachusetts.

Kounias, S. and Marin, J. (1976). Best linear Bonferroni bounds, *SIAM J. Appl. Math.*, **30**, 307–323.

Kwerel, S. M. (1975*a*). Most stringent bounds on aggregated probabilities of partially specified dependent probability systems, *J. Amer. Statist. Assoc.*, **70**, 472–479.

Kwerel, S. M. (1975*b*). Bounds on the probability of the union and intersection of m events, *Adv. in Appl. Probab.*, **7**, 431–448.

Kwerel, S. M. (1975*c*). Most stringent bounds on the probability of the union and intersection of m events for systems partially specified by $S_1, S_2, \dots, S_k$, $2 \le k \le m$, *J. Appl. Probab.*, **12**, 612–619.

Loève, M. (1942). Sur les systèmes d'événements, *Ann. Univ. Lyon, A*, **5**, 55–74.

Loève, M. (1963). *Probability Theory*, 2nd ed., Van Nostrand, Princeton, New Jersey.

Mărgăritescu, E. (1987). On some Bonferroni inequalities, *Stud. Cerc. Mat.*, **39**, 246–251.

Móri, T. F. and Székely, G. J. (1985). A note on the background of several Bonferroni-Galambos-type inequalities, *J. Appl. Probab.*, **22**, 836–843.

Platz, O. (1985). A sharp upper probability bound for the occurrence of at least m out of n events, *J. Appl. Probab.*, **12**, 978–981.

Recsei, E. and Seneta, E. (1987). Bonferroni-type inequalities, *Adv. in Appl. Probab.*, **19**, 508–511.

Rényi, A. (1961). A general method for proving theorems in probability theory and some applications, *MTA* III. *Oszt. Közl*, **11**, 79–105. (English translation (1976). *Selected Papers of A. Rényi*, Vol. 2, 581–602, Akadémiai Kiadó, Budapest.)

Riordan, J. (1968). *Combinatorial Identities*, Wiley, New York.

Samuels, S. M. and Studden, W. J. (1989). Bonferroni-type probability bounds as an application of the theory of Tchebycheff systems, *Probability, Statistics, and Mathematics, Papers in Honor of Samuel Karlin* (eds. T. W. Anderson, K. B. Athreya and D. L. Iglehart), Academic Press, Boston, Massachussets.

Sathe, Y. S., Pradhan, M. and Shah, S. P. (1980). Inequalities for the probability of the occurrence of at least m out of n events, *J. Appl. Probab.*, **17**, 1127–1132.

Schwager, S. J. (1984). Bonferroni sometimes loses, *Amer. Statist.*, **38**, 192–197.

Sibuya, M. (1991). An identity for sums of binomial coefficients, *SIAM Rev.* (in print).

Simes, R. J. (1986). An improved Bonferroni procedure for multiple tests of significance, *Biometrika*, **73**, 751–754.

Takeuchi, K. and Takemura, A. (1987). On sum of 0-1 random variables I. Univariate case, *Ann. Inst. Statist. Math.*, **39**, 85–102.

Walker, A. M. (1981). On the classical Bonferroni inequalities and the corresponding Galambos inequalities, *J. Appl. Probab.*, **18**, 757–763.

Worsley, K. J. (1982). An improved Bonferroni inequality and applications, *Biometrika*, **69**, 297–302.