BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 46, Number 3, July 2009, Pages 489–497
S 0273-0979(09)01231-2
Article electronically published on January 15, 2009

Additive combinatorics, by Terence C. Tao and Van H. Vu, Cambridge Studies in Advanced Mathematics, 105, Cambridge University Press, Cambridge, 2006, xviii+512 pp., ISBN-13: 978-0-521-85386-6, ISBN-10: 0-521-85386-9

The term *additive combinatorics* was coined a few years ago by Terry Tao to describe a rapidly developing and rather exciting area of mathematics. My personal experience is that rather few people have heard the term, though they are often familiar with some of the landmark results. When asked to define the area, I often experience a little difficulty, and in this respect I perhaps have a little in common with Dr. M. Kirschner, head of the Harvard School of Systems Biology, who said,[1] "Systems biology is like the old definition of pornography: I don't know what it is, but I know it when I see it." He went on to say that "it's a marriage of the natural science [*sic*] and computer science with biology, to try and understand complex systems." Well one might say that additive combinatorics is a marriage of number theory, harmonic analysis, combinatorics, and ideas from ergodic theory, which aims to understand very *simple* systems: the operations of addition and multiplication and how they interact.

Even that definition is something of an oversimplification, as a glance at the choice of topics in the book under review shows. Let us begin by mentioning a selection of theorems whose pornographic appeal might be debated but which are undoubtedly additive combinatorics from the moment one sees them.

An extremely old result is the Cauchy-Davenport theorem.[2] Suppose that A and B are subsets of $\mathbb{Z}/p\mathbb{Z}$, where p is a prime. Define the sumset $A+B$ to be the set consisting of all elements of the form $a+b$ where $a \in A$ and $b \in B$. Then we have the inequality

$$|A+B| \geqslant \min(|A|+|B|-1, p).$$

This is proved in three different ways in the book: one combinatorial, one more algebraic, and the other somewhat Fourier-analytic. One starts to get some of the flavour of additive combinatorics when one asks under what conditions equality

2000 *Mathematics Subject Classification.* Primary 11–02; Secondary 05–02, 05D10, 11B13, 11P70, 11P82, 28D05, 37A45.

Part of this review was written, appropriately enough, while the author was attending the semester in Additive Combinatorics at the Institute for Advanced Study, Princeton. He thanks the Institute for their staging of this most enjoyable and productive programme.

[1]Quote taken from *Boston's Biotech Moment* by Charles P. Pierce, Boston Globe, December 14, 2003 (available online).

[2]Note that Cauchy died in 1857 and Davenport was born in 1907. See [1] for some remarks which explain how this theorem came to have this name.

can occur in the Cauchy-Davenport theorem. In this regard one has a theorem of Vosper: assuming that $|A|, |B| \geqslant 2$ and that $|A + B| \leqslant p - 2$ to avoid rather degenerate cases, equality occurs if and only if A and B are arithmetic progressions with the same common difference.

Vosper's theorem relates two somewhat different kinds of *structure*: the rather combinatorial notion of structure encoded in the size of the sumset $A + B$ and the extremely rigid, algebraic notion of structure that is an arithmetic progression.

A remarkable theorem of Freĭman-Ruzsa is in a similar spirit to Vosper's theorem but lies significantly deeper. Suppose that A is a finite subset of $\mathbb{Z}$. Then it is very easy to show that $|A + A| \geqslant 2|A| - 1$ and that equality can occur if and only if A is an arithmetic progression. But what if we assume only that $|A + A| \leqslant K|A|$, for some absolute constant K? Must A still have "rigid" structure?

The Freĭman-Ruzsa theorem says that the answer is *yes*: the set A must be contained within a *generalised arithmetic progression,* a set of the form

$$\{x_0 + l_1x_1 + \cdots + l_dx_d : 0 \leqslant l_i < L_i\},$$

where $d \leqslant f_1(K)$ and $L_1 \ldots L_d \leqslant f_2(K)|A|$. Such a set should be thought of as the projection of a box in $\mathbb{Z}^d$ down to $\mathbb{Z}$.

Freĭman discussed these issues in his 1966 book; an extremely elegant and quite short proof of the theorem was given by Imre Ruzsa some 25 years later. Ideas from both of these sources are a major part of the foundation of additive combinatorics, and they are discussed at length and with full background in the book under review.

Another major theme in the subject was initiated by Klaus Roth in his 1953 paper *On certain sets of integers,* the title being somewhat of a masterpiece of understatement. In this paper Roth addressed a question of Erdős and Turán, proving that every "large" subset $A \subseteq \{1, \ldots, N\}$ contains three distinct integers in arithmetic progression. He showed that a suitable notion of large in this context is that $|A| \geqslant cN/\log\log N$; the important feature of this bound is that the denominator tends to infinity with N, so that one may assert in a certain sense that sets of positive density contain three term progressions.

It is natural to ask what happens for progressions of length $k \geqslant 4$. This issue was not resolved until the landmark work of Szemerédi, who proved in 1969 that sets of positive density contain 4-term progressions and then generalized this to k-term progressions in 1975. His proof of the latter assertion, now known as *Szemerédi's theorem*, is legendarily difficult,[3] but aside from its intrinsic importance the paper led to one of the most important ideas in graph theory, the Szemerédi regularity lemma.

Remarkably there have been several subsequent proofs of Szemerédi's theorem, and it would scarcely be an exaggeration to say that each of them has opened up an entirely new field of study. In 1977 Furstenberg proved the result by an ergodic-theoretic approach. In 1998 Gowers obtained the first sensible bounds, similar in strength to Roth's bound mentioned above, using a kind of "higher Fourier analysis". Intruigingly, this used Freĭman's theorem as an essential tool. Around 2003 Nagle, Rödl, Skokan, and Schacht and independently Gowers gave a fourth proof by developing a *hypergraph regularity lemma*.

Tao has remarked that the many proofs of Szemerédi's theorem act as a kind of "Rosetta Stone". There is much to be gained by studying the relations between

[3] Tim Gowers remarked that "it takes a few seconds even to check that the diagram near the beginning of the dependences between the various lemmas really does indicate a valid proof."

the different arguments, and indeed in proving that the primes contain arbitrarily long arithmetic progressions, Tao and the reviewer studied aspects of all four of the proofs mentioned above.

The theorems stated so far have only involved *addition*. Once multiplication is introduced as well, an incredibly rich variety of questions may be asked. Some of the most classical results in additive number theory, such as Lagrange's theorem that every integer is the sum of 4 squares or Vinogradov's result that every large odd number is the sum of three primes, can be regarded as additive-combinatorial questions about multiplicatively defined sets. However those sets, referred to in the books of Nathanson as the classical bases, are very particular. Though I am shying away from a definition of additive combinatorics, the subject often concerns more general situations. A wonderfully general setting in which one can say something nontrivial about the interaction of addition and multiplcation is provided by the *sum-product phenomenon*: in a wide variety of settings, a set cannot have both additive and multiplicative structure. The first such result, proved by Erdős and Szemerédi, showed that if $A \subseteq \mathbb{Z}$ is a set of n integers, then at least one of $A + A$ and $A \cdot A = \{aa' : a, a' \in A\}$ has size greater than an absolute constant times n^{1+c}, for some $c > 0$. Extremely recently Solymosi showed that one can take c to be anything less than $1/3$, the best result currently known; it is a fascinating open question to decide whether or not an arbitrary $c < 1$ is permissible.

Study of the sum-product phenomenon in finite fields has proved very fruitful. A result of Bourgain, Katz, and Tao, refined by Bourgain, Glibichuk, and Konyagin, tells us that a similar conclusion to the above holds if A is taken to be a subset of $\mathbb{Z}/p\mathbb{Z}$, p a prime, unless A is nearly all of $\mathbb{Z}/p\mathbb{Z}$. This is a rather different result, since multiplication does not "blow things up" in $\mathbb{Z}/p\mathbb{Z}$ the same way it does in $\mathbb{Z}$ or, more precisely, there is no order relation on $\mathbb{Z}/p\mathbb{Z}$ which interacts nicely with multiplication.

One application of this result is Bourgain's estimate for exponential sums over subgroups of the multiplicative subgroup $(\mathbb{Z}/p\mathbb{Z})^{\times}$. If H is such a subgroup and if H is not incredibly small ($|H| > p^{\epsilon}$ will do), then we have

$$\frac{1}{|H|}\Big|\sum_{x \in H} e(xr/p)\Big| = o(1)$$

for all nonzero $r \in \mathbb{Z}/p\mathbb{Z}$. Although not obvious at first sight, this is a rather strong statement to the effect that multiplicative subgroups of $\mathbb{Z}/p\mathbb{Z}$ have rather little additive structure. Before the sum-product technology was imported, this statement was known only under the much weaker condition that $|H| > p^{1/4+o(1)}$; this result, due to Heath-Brown, required deep number-theoretical arguments.

Having set the scene, let us say something about the book under review. It is organised into 12 chapters. Starting with a discussion of basic tools from probability theory in Chapter 1, the early chapters develop basic tools common to many different papers in the subject, whilst the later chapters discuss particular topics. The material in Chapter 1 covers some of the same ground as the wonderful book *The Probabilistic Method* by Alon and Spencer, but it is much more condensed. This is a nice discussion of the material, and it also serves to introduce some of the notation, particularly the expectation notation $\mathbb{E}$, which is important to the subject.

Students of mine have found Chapter 2 to be extremely useful. It gives a rather systematic and comprehensive treatment of inequalities for sumsets. Prior to this account one generally had to refer to the somewhat scattered literature and in particular to a number of brilliant but not always easy-to-access papers of Imre Ruzsa. Let us give some examples to illustrate the flavour. Suppose that A is a subset of some abelian group G satisfying the doubling condition $|A+A| \leqslant K|A|$. Then the size of the iterated sumsets $sA - tA = A + \cdots + A - A - \cdots - A$ (here there are $s+t$ copies of A) can be controlled in the sense that one has an inequality

$$|sA - tA| \ll K^{C(s,t)}|A|.$$

This is extremely useful in practice, as it allows us to think of the theory of sets with small doubling as a kind of approximate abelian group theory, a concept which is elaborated upon in the book. Analogues of these inequalities in the nonabelian setting are also given; these have proved very useful indeed and appeared for the first time in this book, though they have now been covered in a separate paper of Tao.

Much is also made of the relation between the small doubling condition and another property that a set A may enjoy, that of having large *additive energy*: many solutions to $a_1 + a_2 = a_3 + a_4$. A simple application of the Cauchy-Schwarz inequality shows that a set with small doubling has large additive energy, but the converse fails (A could be the union of an arithmetic progression of length $n/2$ and a further $n/2$ random points). A remarkable result known as the Balog-Szemerédi-Gowers theorem provides a partial converse: if A has large additive energy, then there is a big subset $A' \subseteq A$ with small doubling. This result has proved invaluable because sets with large additive energy often arise in nature, for example as a byproduct of Fourier-analytic arguments, whereas the small doubling condition allows one to prove strong structural results such as the Freĭman-Ruzsa theorem discussed earlier.

The Balog-Szemerédi-Gowers theorem is discussed at some length. A nonabelian version is stated, as well as an asymmetric version which the authors extracted from Bourgain's work on exponential sums over subgroups. This latter result does not, to my knowledge, appear anywhere else in the literature. The actual proof of the Balog-Szemerédi-Gowers theorem is deferred to Chapter 6.

A nice feature of Chapter 2 is the introduction of the *Ruzsa distance* between two sets A and B in some ambient group G. This provides a unified discussion of a number of the results just mentioned.

Chapter 3 discusses a number of results with the flavour of the geometry of numbers, a subject which, according to Peter Swinnerton-Dyer[4] "went out of fashion in England in the 1950s and elsewhere considerably earlier." These results about intersections of lattices with convex bodies have found a new lease of life in additive combinatorics, particularly in association with the Freĭman-Ruzsa theorem and in more recent work of the authors on random matrices. This chapter is very useful on account of its brevity and also its perspective, which has the additive-combinatorial applications in mind. Many of the results here, though in some sense classical, would be rather hard to extract from the literature. There is also a brief discussion of the Brunn-Minkowski inequality and related issues.

Chapter 4 is, with Chapter 2, found to be the most useful by students. The Fourier transform is introduced, and its role in additive combinatorics discussed at length. The use of exponential sums in additive number theory goes back a long

[4]In a Cambridge lecture course in 1999.

way—at least to the work of Hardy and Littlewood. Since then it has appeared systematically in papers on combinatorial number theory, for example in the work of Roth, Freĭman-Ruzsa, and Gowers discussed in the introduction. This book represents the first attempt that I know of to give a systematic discussion of the Fourier transform as applied in this area; before this, one had to go to a disparate collection of original papers. The topics are rather different from those that would be featured in a course on Euclidean harmonic analysis, and much of the discussion centres on the subtle relationship between the additive structure of a set or function and properties of its Fourier transform. Particularly useful in this regard is the introduction and subsequent development of the notation $\mathrm{Spec}_\alpha(A)$ for the α-large spectrum of a set A, defined to be the set of points where the Fourier transform of A is at least α of its maximum value.

Chapter 5 deals with *inverse problems* in additive combinatorics. In particular the theorems of Vosper and Freĭman-Ruzsa mentioned earlier are proved here, the latter by the interesting new method of universal ambient groups which does not appear elsewhere in the literature. A number of refinements of these theorems due to Freĭman and later authors are presented, in particular an argument of Chang giving reasonably good bounds for the Freĭman-Ruzsa theorem. In addition there is an extended discussion on the important notions of Freĭman homomorphism and Freĭman isomorphism. Freĭman isomorphisms preserve structures that result from adding at most some fixed number s of elements of a set A; because of the restriction on s, they are rather weaker as a concept than group homomorphisms, but consequently more flexible. It is often very advantageous to work with a particular Freĭman-isomorphic copy of a set A, as, for example, in Ruzsa's proof of the Freĭman-Ruzsa theorem.

Chapter 6 is entitled "Graph-theoretic methods", and it is a rather eclectic mix of topics, all of which have at least something to do with a graph. The chapter begins with a pleasant collection of topics from extremal graph theory and Ramsey theory. These are presented with an eye towards applications in additive combinatorics: for example, Ramsey's theorem is used to prove Schur's beautiful theorem that in any finite colouring of the integers there is a monochromatic solution to $x+y=z$. There is then a brief digression away from graph theory, as a proof is given of the Hales-Jewitt theorem and (as a corollary) van der Waerden's theorem on the existence of monochromatic progressions of arbitrary length in finite colourings of the integers. The rest of the chapter is devoted to two important topics. Firstly, the proof of the Balog-Szemerédi-Gowers theorem, deferred from Chapter 2, is at last furnished. Secondly, there is a discussion of Ruzsa's proof of Plünnecke's inequality which involves, among other ingredients, Menger's theorem from elementary graph theory. This inequality gives a much cleaner version of the iterated sumset inequalities of Chapter 2; for example, if $|A+A| \leqslant K|A|$, then $|sA - tA| \leqslant K^{s+t}|A|$ for every $s,t \geqslant 0$. For many applications the rougher results of Chapter 2 are sufficient, but the elegance of this proof makes this section highly recommended reading.

The first six chapters of the book are, in large part, essential reading for any student of the subject. From Chapter 7 onwards the material is more topics-based.

Chapter 7 begins with a discussion of the Littlewood-Offord problem: take elements $v_1, \dots, v_d$ in some abelian group G, and make a random choice of d signs. What is the probability $\mathbb{P}_{\mathbf{v}}$ that $\pm v_1 \pm v_2 \pm \cdots \pm v_d$ is zero? When $G = \mathbb{Z}$ and none of the v_i is zero, this probability is bounded by $C/\sqrt{d}$, a result that is clearly sharp (take all the v_i to equal one). The *inverse* Littlewood-Offord problem asks what can be said about $v_1, \ldots, v_d$ if many of these sums do vanish. One of the highlights of this chapter is a result of the authors, stating that if $G = \mathbb{Q}$ and $\mathbb{P}_{\mathbf{v}} \geqslant d^{-A}$, then the elements v_i must be efficiently contained inside a reasonably small generalised arithmetic progression of dimension bounded in terms of A. Results such as this are then applied to random matrices and, in particular, to the beautiful problem of estimating the probability p_n that a random $n \times n$ matrix with ± 1 entries is singular. A sketch is given of the authors' proof that $p_n \ll (\frac{3}{4} + o(1))^n$. This gets about half way to the expected truth, which is that $p_n \sim 2n^2(\frac{1}{2})^n$. The thinking behind this guess is that by far the most likely way for a ± 1 matrix to be singular is for two rows to be multiples of one another. There is also a discussion of random *symmetric* matrices with ± 1 entries; remarkably, all that is known about the corresponding singularity probabilty $\tilde{p}_n$ here is the bound $\tilde{p}_n \ll n^{-1/8+o(1)}$. The elegant proof of this result of Costello, Tao, and Vu finishes the chapter.

It is certainly worth noting that the authors' work in this area has led to an impressive string of papers, most recently a preprint [3] establishing a very general universality principle for the eigenvalue distribution of random matrices.

Chapter 8 discusses point-line arrangements in Euclidean space and Erdős problems such as the distinct distances problem: what is the minimum number of distinct distances $g_d(n)$ defined by a set of n points in $\mathbb{R}^d$? A key result here is the Szemerédi-Trotter theorem, which gives an upper bound for the number of incidences between l lines and p points. This is applied to give sum-product type results in $\mathbb{R}$. For example, we find a proof of Solymosi's result that if $A \subseteq \mathbb{R}$ has size n, then either $|A + A|$ or $|A \cdot A|$ has size $\gg n^{14/11-o(1)}$ (very recently, as mentioned earlier, he has improved the 14/11 here to 4/3). These sum-product phenomena in Euclidean space have a very different flavour to those in $\mathbb{Z}/p\mathbb{Z}$, as geometry—and, in particular, order relations and convexity—come into play. The chapter concludes with a discussion of the sum-product phenomenon in $\mathbb{C}$. The arguments in this chapter are uniformly very elegant, and the chapter could stand alone as a 3- or 4-lecture course.

Chapter 9 has a rather different flavour in that it concerns algebraic methods which are more rigid than the $O(\ ), o(\ )$-style material of much of the rest of the book. The first topic is the *Combinatorial Nullstellensatz* and its applications, discussed in a 2000 paper of Noga Alon (the authors have overlooked the fact that the actual result is of somewhat earlier vintage, dating back at least as far as a 1992 paper of Alon and Tarsi). Among the pleasant applications of this result is a short proof of the Erdős-Heilbronn conjecture,[5] an analogue of the Cauchy-Davenport theorem in which only *distinct* sums are allowed. If $A, B \subseteq \mathbb{Z}/p\mathbb{Z}$ are two sets, then we write $A \hat{+} B$ for the set of all sums $a + b$ with $a \in A, b \in B$, and $a \neq b$. The assertion is then that $|A \hat{+} B| \geqslant \min(|A| + |B| - 3, p)$. After this, the Chevalley-Warning theorem is discussed and used to prove the lovely Erdős-Ginzberg-Ziv theorem: given any $2n - 1$ integers $a_1, \ldots, a_{2n-1}$, there is some subset of exactly n of them whose sum is divisible by n. There is an interesting discussion of Stepanov's method and its use in obtaining sum-product estimates in $\mathbb{Z}/p\mathbb{Z}$ (in this specific context, however, more elementary methods are now available). Finally, the first author's elegant *discrete uncertainty principle* is proved: the size of the support of a function $f : \mathbb{Z}/p\mathbb{Z} \to \mathbb{C}$ and that of its Fourier transform must be at least $p + 1$.

[5]This conjecture was first established by Dias de Silva and Hamidoune using different methods.

This is then used to supply the third proof of the Cauchy-Davenport theorem in the book.

In Chapter 10 the first case of Szemerédi's theorem is discussed, that of arithmetic progressions of length three. Roth's 1953 argument is discussed, as is Bourgain's proof of the best bound for this problem that was known in 2006, namely that any subset of $\{1,\dots,N\}$ of cardinality at least $C\sqrt{\frac{\log\log N}{\log N}}N$ contains three elements in progression (recently, Bourgain has improved the power of log here to 2/3). The exposition here incorporates a significant technical refinement in the part of the argument that has to do with understanding the large spectrum of a Bohr set—this first appeared in UCLA lecture notes of Tao. A variant of my result on versions of Roth's theorem relative to the primes is discussed, and there is another proof of Roth's theorem using a kind of "finitary ergodic" argument close to a technique of Bourgain from 1989. The reason for this, and indeed the motivation for the rather unusual presentation of some of the results here, is to get the reader thinking along the lines that eventually lead to the theorem of Tao and Green (the reviewer) on long arithmetic progressions of primes. Finally, two arguments of Szemerédi are discussed. Firstly, there is material on his extremely important regularity lemma for graphs and the observation of Ruzsa and Szemerédi that this implies Roth's theorem. Lastly, there is a combinatorial proof of Roth's theorem, which was a precursor to his 1975 *tour de force* on progressions of arbitrary length.

Chapter 11 discusses Szemerédi's theorem for longer progressions. It begins with a discussion of the Gowers uniformity norms U^k. Let G be a finite abelian group and let $f: G \to \mathbb{C}$ be a function. Then the U^2-norm of f is the 4th root of

$$\frac{1}{|G|^3}\sum_{x,h_1,h_2} f(x)\overline{f(x+h_1)f(x+h_2)}f(x+h_1+h_2),$$

a kind of average of f over 2-dimensional parallelograms. The U^3-norm of f is the 8th root of

$$\frac{1}{|G|^4}\sum_{x,h_1,h_2,h_3} f(x)\overline{f(x+h_1)}\dots\overline{f(x+h_1+h_2+h_3)},$$

an average of f over 3-dimensional parallelepipeds, and the higher norms are defined similarly. The Gowers U^k-norm can perhaps best be understood when $f(x) = e^{2\pi i\phi(x)}$ is a pure phase function, in which case it detects biases in the kth iterated difference (discrete derivative) of ϕ. Since Gowers's results and work by Host-Kra in ergodic theory these norms have assumed a central role in additive combinatorics.

The *inverse question* for the Gowers norms, which asks under what circumstances the Gowers U^k-norm of bounded function can be large, is introduced. The inverse question for the U^3-norm in the case that G is a vector space over a finite field is resolved (f must correlate with a quadratic phase), and this is used to establish Szemerédi's theorem for progressions of length 4. This whole argument corresponds very closely to Gowers's original argument.

The rest of the chapter is quite sketchy and merely offers a taste of what lies beyond. The first topic is a discussion of the first author's very nice "quantitative ergodic theory" proof of Szemerédi's theorem, which arguably represents the most accessible proof of the theorem currently known. Some of the ideas here are also of relevance to the theorem of Tao and Green on progressions of primes, which is sketched in the last section of the chapter. Before that, there are short discussions of Furstenberg's famous ergodic-theoretic proof of Szemerédi's theorem and also of the hypergraph regularity approach of Gowers and Rödl et al.

Chapter 12 feels rather like a savoury course, following the dessert as something of an afterthought. Nonetheless it contains some very elegant results of Szemerédi and the second author, chiefly concerning their resolution of a conjecture of Erdős and Folkman: if C is sufficiently large and if $A = \{a_1 < a_2 < a_3 < \dots\}$ is a set of natural numbers such that $|A \cap \{1, \dots, N\}| \geqslant C\sqrt{N}$ for all N, then the set of sums $\{\sum_{i \in I} a_i : |I| < \infty\}$ contains an infinite arithmetic progression.

Last, but by no means least, there is an extremely extensive bibliography of some 388 papers.

The choice of material in this book is for the most part in very good taste, and the proofs are efficient and frequently elegant. As regards the choice of topics, it is important to acknowledge (as the authors do) the debt the subject owes to the two books of M. Nathanson, who first brought many of these topics (the Freĭman-Ruzsa theorem, Plünnecke's inequalities, ...) to a wider audience. Perhaps the most obvious omission from this book is any serious discussion of nilsequences, which seem set to play a major role in the future development of the subject. This is certainly forgivable, however, since at the time the book was written there had been little exploration of nilsequences outside of the ergodic literature.

Some parts of this book are better exposited elsewhere. In my opinion this is particularly true of Chapters 10 and 11, where the first author and others have subsequently written more accessible accounts. Of particular note in this regard are Tao's Montréal notes [2], which should probably be consulted long before Chapter 11 of this book. These comments notwithstanding, there are some parts of Chapter 10 in particular where a novel perspective is taken which is certainly welcome in the literature.

This book is more suitable as a reference text than for a course, though it would certainly be a useful accompaniment to a graduate course designed around a carefully selected subset of the material such as that given by Gowers in Cambridge in 1999 (which is where I learnt much of this material). I plan to use parts of it myself when lecturing a similar such course in 2009.

There are a few parts of the book that are rather heavy going, although this often reflects the difficulty of the underlying material. This is particularly true of parts of Chapter 10. Occasionally the use of the $O(\ )$ notation is taken too far, for example in Theorem 2.35 and Theorem 4.42 (the former argument is particularly difficult to scan, and here it might have been beneficial to write in explicit constants).

In my opinion the typesetting of this book leaves something to be desired. I am not fond of the font used in this series by Cambridge University Press, and some of the bracketing in, for example, Section 7.2 is rather heavy and unattractive. It surprises me that the notation and typesetting on pages 394 and 395, where the argument is already difficult enough, made it past the proof stage. The setting of citations in italics (*[195],[374]*) in the statement of theorems but not elsewhere seems very ugly to my admittedly untrained eye.

In summary, the book under review is a vital contribution to the literature, and it has already become required reading for a new generation of students as well as for experts in adjacent areas looking to learn about additive combinatorics (Chapter 4, for example, might be found very interesting to some theoretical computer scientists). This was very much a book that needed to be written at the time it was, and the authors are to be highly commended for having done so in such an effective way. I have three copies myself: one at home, one in the office, and a spare in case either of those should become damaged.

References

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[3] T.C. Tao and V. H. Vu, *Random matrices: Universality of ESDs and the circular law,* preprint available at `http://arxiv.org/abs/0807.4898`.

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