BOOK REVIEWS

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Chaotic billiards, by Nikolai Chernov and Roberto Markarian, Mathematical Surveys and Monographs, vol. 127, American Mathematical Society, Providence, RI, 2006, xii+316 pp., ISBN 0-8218-4096-7, US $85.00

Studies of the dynamical systems of billiard type (or simply billiards) form one of the most fascinating and notoriously difficult areas in the modern theory of dynamical systems. Billiards are visual and arise naturally in applications (primarily in classical mechanics, statistical mechanics, optics, acoustics, and quantum physics). For instance, the Boltzmann gas of elastically colliding balls in a box, which is the most fundamental and venerable model in statistical mechanics, is a billiard system. Billiards demonstrate a full variety of possible dynamics from the most regular (integrable) one to the strongest possible stochastic behavior, when typical (almost all) orbits cannot be distinguished from realizations of random processes with almost independent values.

Recall that a billiard is a dynamical system generated by the free motion of a point particle within a bounded domain Q with a piecewise smooth boundary ∂Q in Euclidean space or on a torus. When the particle reaches a regular point of the boundary (where there is a unique normal to ∂Q) it gets reflected according to the law of specular (or elastic) collisions; i.e., the angle of incidence equals the angle of reflection. These angles are measured between the velocity vector v of the particle (which can be always assumed to be a unit vector) and the inward unit normal vector $n(q)$, where q is the point of reflection, $q \in \partial Q$. Thus billiard orbits in the configuration space (a billiard table) Q are broken lines. Hence a billiard is a Hamiltonian system with a potential identically equal to 0 within Q and ∞ at ∂Q. Therefore, a billiard dynamics (billiard flow) Φ^t preserves a phase volume m (Liouville measure) in the phase space Ω. The set of all orbits which eventually hits singular points of the boundary has m-measure zero. Because the book under review deals exclusively with two-dimensional billiards, we assume from now on that the billiard table Q is a planar domain or belongs to a 2D torus. Then Ω becomes a 3D manifold.

Let $\partial Q = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k$, where Γ_i, $1 \leq i \leq k$, are smooth ($\mathcal{C}^\ell$, $\ell \geq 3$, to avoid pathologies) compact curves. A smooth component Γ of the boundary of a billiard table is called dispersing, focusing or neutral if it is convex outward, inward or has identically zero curvature, respectively. A billiard may have the most

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regular (integrable) dynamics such as billiards in circles and ellipses. The book under review deals with the billiards which have the most irregular (stochastic) behavior. Although it is called *Chaotic Billiards*, this title is, in some sense, misleading because the book is completely devoted to hyperbolic billiards rather than to chaotic ones. Hyperbolic dynamical systems demonstrate *strong* chaos but there are many classes of dynamical systems (including billiards) with different forms of *weakly* chaotic dynamics.

Recall that a dynamical system is called hyperbolic if its Lyapunov exponents do not vanish almost everywhere with respect to corresponding invariant measure. The fundamental Oseledec's theorem [23] establishes the existence of Lyapunov exponents for a very general class of measure preserving conservative dynamical systems. The celebrated geodesic flows on surfaces of negative curvature were the first class of hyperbolic systems, studied by Hadamard, Hedlund and Hopf. This class was dramatically generalized after the introduction of Anosov systems and Smale's axiom A-systems.

The studies of hyperbolic billiards were initiated by Sinai's remarkable groundbreaking 1970 paper [27] which is fundamental to the entire theory of hyperbolic systems with singularities. In hyperbolic billiards there are singularities of two types. The first one is related to the existence of collisions tangent to the boundary, and the second with the existence of singular ("corner") points on the boundary of a billiard table. Therefore, a standard proof of ergodicity of smooth hyperbolic dynamical systems via a Hopf chain which connects almost any pair of points in a phase space and consists of smooth local manifolds alternating between the stable and unstable ones is not applicable. Sinai developed a very elegant although technically extremely sophisticated and challenging theory to account for that. Obviously a billiard dynamical system is completely defined by the geometry of the boundary ∂Q of a billiard table. This connection becomes especially transparent if one studies the so called billiard map. Consider a 2D cross-section $M \subset \Omega$ consisting of all unit vectors with footpoints on the boundary ∂Q and pointing to the interior of the billiard table. Then the billiard map $\mathcal{F}$ sends a point from one collision to the point of the next collision with the boundary. $\mathcal{F}$ preserves an absolutely continuous probability measure μ which is the projection of m onto M. It is easy to see that topologically M is a union of not more than k cylinders. In fact, a cylinder corresponds to every connected component of ∂Q. There exist natural coordinates (r, φ) on M, where r is a length coordinate on the boundary ∂Q and φ, $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$, is the angle between the unit normal vector $n(q)$ at the point of reflection and the velocity vector v of the particle.

Dynamics of hyperbolic billiards can be conveniently described by the so-called wave fronts, which are (C^1-smooth) orthogonal cross-sections of narrow families of directed lines (beams of rays) representing billiard orbits. The curvature of such beams changes in the course of the dynamics. It is easy to compute that if the beam did not experience reflections from ∂Q on the interval $[0, t]$, then

$$\kappa_t = \frac{\kappa_0}{1 + t\kappa_0}, \tag{1}$$

where κ_0 and κ_t are the initial curvature of the wave front and its curvature at the moment t. Thus (1) describes the evolution of wave fronts during a free path between the reflections from ∂Q. At the moments of reflections at the boundary, the curvature of a wave front changes according to the fundamental mirror formula of the geometric optics

$$\kappa_+ = \kappa_- + \frac{2k(q)}{\cos\varphi}, \tag{2}$$

where κ_- and κ_+ are the curvatures of the wave front at the moments just before and just after reflection off the boundary at the point $q \in \partial Q$, respectively, $k(q)$ is the curvature of the boundary at this point and $\cos\varphi$ is the corresponding incidence angle.

All the properties of the hyperbolic billiards could be deduced from the formulas (1) and (2). However, in attempting to do that, one encounters "enormous technical difficulties", as the authors correctly write, and this book does a very good job in presenting a clear and detailed exposition of fundamentals of these techniques [28], [29], [25]. On the other hand, for billiards in polygons the fundamental mirror formula (2) gives nothing because the curvature of wave fronts does not change at the reflections. The billiards in polygons (and polyhedra) are nonhyperbolic. Therefore, studies of such billiards require absolutely different techniques [19], [29], [22]. It is a beautiful area of mathematics and I respectfully disagree with the statement of the authors that billiards studied in their book are the "least elementary" ones. The theory of billiards in polygons and polyhedra is in my view mathematically not easier, but of course it deals with a very tiny and specific subclass of billiards. Thus, the book under review deals with billiards where the curvature of the boundary is not identically zero because billiards in polygons and polyhedra are nonhyperbolic.

Sinai [27] introduced a very important class of dispersing billiards which plays a fundamental role in the theory of nonuniformly hyperbolic dynamical systems as geodesic flows on manifolds of negative curvature play in the theory of smooth hyperbolic systems. A billiard is called dispersing if all regular components of the boundary are dispersing; i.e., they have a positive curvature. It is easy to see from (1) and (2) that any wave front with positive curvature at $t = 0$ in dispersing billiards will have a positive curvature at any $t > 0$; i.e., the corresponding beam of rays will be divergent and the distance in phase space between the billiard or bits 2in such a beam will (locally) increase. This mechanism of hyperbolicity is naturally called the mechanism of dispersing. It is analogous to the mechanism of hyperbolicity in geodesic flows on manifolds with negative curvature. The most celebrated example of dispersing billiards is a 2D plane torus with removed convex subset (scatterer) with a piecewise smooth boundary. This very billiard table, besides being an instructive example introduced by Sinai, is also of great importance in nonequilibrium statistical mechanics. By letting a billiard particle move on the plane between a periodic array of convex disjoint scatterers, one gets the celebrated periodic Lorentz gas. Lorentz introduced this classical model (with arbitrary configuration of scatterers) of statistical mechanics in 1905. Although the natural invariant measure for the periodic Lorentz gas is infinite, this dynamical system can be reduced to Sinai's billiards by making use of the fact that the dynamics commutes with $\mathbb{Z}^2$ action. Dispersing billiards enjoy the strongest statistical and stochastic properties. They are ergodic, mixing, Kolmogorov systems (K-systems), B-systems (metrically isomorphic to Bernoulli shifts), satisfy the Central Limit Theorem of the probability theory and enjoy the exponential decay of correlations for "good" functions on the phase space. (A more or less standard choice is the class of Hölder continuous functions.)

If at least one component of the boundary ∂Q is focusing, then the situation changes drastically and the corresponding billiards demonstrate all variety of behavior from the most regular (integrable in circles and ellipses) to the completely hyperbolic ones with strong statistical and stochastic properties. In 1973 Lazutkin established that if the boundary ∂Q of a billiard table is a sufficiently smooth convex curve, then there exists an uncountable family of caustics converging to ∂Q [21]. Recall that a curve γ is a caustic for a billiard if from the existence of one link of a billiard orbit tangent to γ it follows that any other link of this orbit is also tangent to γ. If there is just one caustic, then the corresponding billiard is nonergodic. The mirror formula (2) shows that any reflection from the focusing boundary pushes to the focusing (i.e., convergence) of wave fronts rather than to their divergence, required for hyperbolicity. However, there are billiards with some focusing components [4] and even without dispersing components [5] which are hyperbolic. The mechanism of hyperbolicity in such billiards is different from the dispersing one and is called the mechanism of defocusing. In dealing with billiards with at least one focusing component, one must consider the evolution of wave fronts with negative curvature, i.e., the evolution of convergent (focusing) beams of rays. It follows immediately from (1) that dispersing wave fronts continue to be dispersing during the entire free path between two consecutive reflections from the boundary. To the contrary, a focusing wave front can get transformed into a dispersing one if $t > |\kappa_0|^{-1}$, i.e., if a free path of the particle is long enough. At the moment $t_d = |\kappa_0|^{-1}$, the event of defocusing of a focusing beam occurs and this beam becomes a dispersing one. Thus, a free path $\tau = \tau_c + \tau_d$, where τ_d (τ_c) is the time interval in which this beam of rays was divergent (convergent). The mechanism of defocusing generates hyperbolicity if divergence (dispersing) dominates convergence (focusing) and thus such billiards enjoy most of the ergodic and statistical properties that dispersing billiards have. The simplest example of a hyperbolic focusing billiard arises by cutting a circle by a chord, which is not a diameter, and then taking the bigger piece as a billiard table. After the discovery of the mechanism of defocusing, the theory of billiards in some sense assumed a leading role to the theory of geodesic flows. In fact, by using the same geometric ideas, Osserman and then Donnay constructed hyperbolic geodesic flows on surfaces that have pieces with positive curvature [17].

Because the billiards with focusing components demonstrate all possible dynamical behaviors ranging from the integrable to the hyperbolic, a natural question is which focusing components can serve as boundary components for hyperbolic billiards [6]. To a large extent, this problem is now completely resolved [18], [7], [10]. The claim is that such focusing components must be absolutely focusing [6]. The notion of absolute focusing seems to be a new one in a geometric optics. A smooth component Γ of a billiard table's boundary is called absolutely focusing [6], [7] if any narrow parallel beam of rays which falls on Γ becomes focused (convergent) after the last reflection in a series of consecutive reflections from Γ. Observe that the standard notion of a focusing component takes into account just the first reflection from the boundary by a parallel beam of rays which according to (2) immediately becomes convergent. Absolute focusing can be characterized in local terms as well by requiring that any initially parallel beam of rays that falls on Γ becomes focused between any two consecutive reflections in the series of reflections from Γ (as well as after the last reflection in this series) [18], [7]. Although the last definition seems to be more restrictive, these two are in fact equivalent [7]. By choosing all focusing components of the boundary to be absolutely focusing and putting each of them sufficiently far from any other nonneutral component of the boundary ∂Q, one can ensure hyperbolicity of a billiard [16], [32], [4], [5]. On another hand this strategy does not work for nonabsolutely focusing curves [10].

There are two techniques that are used to establish hyperbolicity. The first one deals with special continued fractions which are intimately related to billiard dynamics. Consider a (nonsingular) billiard orbit of a point $x \in M$. Let τ_n, $n = 1, 2, \ldots$, be a time (free path) between the $(n-1)$th and the nth reflections of this trajectory from the boundary ∂Q, and $x_n = \mathcal{F}^n x$, $R_n = \frac{2kn}{\cos\varphi(x_n)}$, where k_n is the curvature of ∂Q at the point of the nth reflection and $\varphi(x_n)$ is the corresponding angle of incidence. A remarkable infinite continued fraction discovered by Sinai,

$$\kappa^s(x) = \cfrac{1}{\tau_1 + \cfrac{1}{R_1 + \cfrac{1}{\tau_2 + \cfrac{1}{R_2 + \ddots}}}}, \tag{3}$$

gives a (formal) expression for a tangent to a (local) stable manifold passing through x [29]. By reversing the time, one gets the corresponding expression for an unstable manifold at x. Therefore, the first step in proving hyperbolicity is to establish that the infinite continued fraction (3) exists for almost all points $x \in M$. For dispersing billiards, the signs of all elements of the continued fraction (3) are the same and therefore there exists a general Seidel-Sterne criterion of convergence. Thus, for dispersing billiards the problem of convergence of (3) becomes essentially trivial. If a billiard table has at least one focusing boundary component Γ, then the blocks of the continued fraction (3) that correspond to the series of consecutive reflections off Γ have elements with alternating signs. There are no general criteria of convergence of such continued fractions. Therefore, the discovery of new classes of hyperbolic billiards becomes a challenge.

The second approach to proving hyperbolicity goes back to Alekseev [1] and is called a cone method. This method was extended to any dimension by Alekseev's former student Wojtkowski [31]. Now the cone method has become almost universal and has been applied to a large variety of dynamical systems, including billiards. From a general point of view, the continued fraction technique deals directly with the local stable and unstable manifolds, whereas the cone approach deals with the action of the differential $\mathcal{DF}$ of the billiards map and the invariant under $\mathcal{DF}$ families of cones in the tangent space TM.

Although hyperbolicity is the fundamental ingredient to ensure stochasticity of the dynamics, many questions remain on how "strong" is this "chaos" in the dynamics. Because billiards have a natural invariant measure, the corresponding questions immediately lead to the studies of ergodic and statistical properties of hyperbolic billiards. The first fundamental problem is ergodicity. In fact, the entire area of hyperbolic billiards (and the ergodic theory itself) developed from the attempts to prove ergodicity of the Boltzmann gas (Boltzmann hypothesis, BH). Sinai proved BH for two discs on the torus [27]. This system is equivalent to dispersing billiards. When the number of particles is greater than two, then the corresponding billiard becomes only semi-dispersing; i.e., its boundary consists of pieces of cylinders. (Each cylinder corresponds to collisions of a certain pair of particles.) This enormously complicates the analysis of this system. However, the developments in recent years in the sharpening of these techniques gives hope that the full proof of BH is within reach [24], [25]. For nonuniformly hyperbolic dynamical systems, a theory has been developed which allows one to deduce from ergodicity stronger ergodic properties, such as mixing, K-property and B-property [29], [28], [20], [15]. Therefore, ergodicity is a key property for such systems that include most hyperbolic billiards.

The next circle of questions deal with statistical properties which are essentially based on the rate of mixing, i.e., on the rate of correlations decay. These properties are the Central Limit Theorem as well as other limit theorems of probability theory such as the local limit theorem, the law of iterated logarithm, the almost-sure invariance principle, etc.

The studies of statistical properties of billiards are closely related to the problems of statistical mechanics. For instance, the fundamental problem of irreversibility refers to the relationship between the microdynamics of particles, described by the time-invertible Newton's law, and time-noninvertible macro-dynamics described by the partial differential equations of hydrodynamics [29]. The theory of dispersing billiards allows us to derive the diffusion equation for the periodic Lorentz gas with bounded free path from its completely deterministic and time-invertible dynamics [11].

Recent impressive progress in this area is essentially based on the remarkable advances due to Young [33], [34] of previously existing techniques which allowed researchers to obtain, in particular, exponential estimates for the rate of decay of correlations in dispersing billiards as well as power-like estimates for billiards with slow decay of correlations, e.g., for some focusing billiards, and to prove numerous limit theorems for various important billiard systems (see [14] and references therein, [29], [12], [13], [3]). This area is in full bloom now.

However, some fundamental (and intrinsic for billiards) questions about the dynamics of hyperbolic billiards remain. They are mostly related to the fundamental problem of the mechanisms of hyperbolicity. Even in two dimensions it is not known whether there is any other general strategy to design hyperbolic billiards besides choosing all focusing components to be absolutely focusing and placing other components of the boundary sufficiently "far" from them. In higher dimensions the situation becomes much more complex even for dispersing billiards [2]. Needless to say, the complications for billiards with focusing components has become extremely severe. The reason for all these complications is provided by a fundamental optical phenomenon called astigmatism. In fact, the mechanism of defocusing requires that wave fronts must experience a strong focusing when they collide with the boundary ∂Q. However, in view of astigmatism, the strength of focusing varies in different planar sections of the wave front and becomes quite weak in some sections. Therefore, there were doubts that the mechanism of defocusing works in higher dimensions. It does work though [9]. However, the classes of focusing components used so far are restricted to the pieces of spheres. To extend this class and construct open sets of admissible focusing components is a challenging problem.

Recall also that a generic Hamiltonian system is neither hyperbolic nor integrable. Such a system has instead a so-called divided phase space where the sets with regular dynamics (KAM-tori) coexist with the sets with chaotic dynamics, and each of these sets has positive measure. To understand the dynamics of such systems is the most fundamental problem in the theory of Hamiltonian systems. Although billiards do have some specific features, their studies may advance this area at least by providing exact and visual examples of coexistence of any number of KAM-islands with any number of chaotic components [8].

The book under review will play an outstanding role in all future developments of the theory of hyperbolic billiards and its applications. The fundamentals of the theory of two-dimensional hyperbolic billiards, which have existed for almost 40 years, were dispersed so far in numerous lengthy papers (mostly written in Russian). The authors provide all the technical details for proving hyperbolicity, ergodicity and statistical properties of dispersing billiards. Recent powerful techniques for studying statistical properties, such as Young towers and coupling, are also very clearly presented. In addition, the book gives a fairly complete analysis of hyperbolic billiards with constant curvature focusing components (arcs of circles). The last short chapter dealing with focusing billiards of general type is much more sketchy. However, the different approaches are carefully discussed as well as their comparison. The authors also provide a good number of historic remarks which are with a very high probability correct. (An anecdotal example of a "small probability event" is the reference to R. L. Dobrushin as a physicist.)

Overall, this book is an invaluable source for students and individual researchers to learn the fascinating and flourishing area of hyperbolic billiards and to contribute to it. It contains many carefully chosen exercises and of course many figures. The well-written appendices on measure theory, probability theory and ergodic theory make the exposition essentially self-contained. I highly recommend this book as the only source for graduate and undergraduate courses on the theory of 2D hyperbolic billiards as well as for individual studies.

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