# BOOK REVIEWS

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*Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*, by James E. Humphreys, Graduate Studies in Mathematics, vol. 94, American Mathematical Society, Providence, RI, 2008, xvi+289 pp., ISBN 978-0-8218-4678-0

1. Category $\mathcal{O}$ was introduced by J. Bernstein, I. Gelfand, and S. Gelfand in [BGG76]. I would bet that the authors of [BGG76] did not suspect at the time that their child would grow up in the following years to be a subject too large for a single book. The subject itself grew out of the study of the so-called Verma modules. Verma at first suspected those to have all Jordan–Hölder multiplicities at most one, but soon it was realized that things were much more complicated. Category $\mathcal{O}$ started out as a means to attack this question of Jordan–Hölder multiplicities of Verma modules, but it turned out to be useful far beyond that and is even used successfully in knot theory today. Let me now be a bit more precise.

2. Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be a semisimple complex Lie algebra along with a fixed Borel subalgebra and a fixed Cartan subalgebra. The reader not familiar with these notions will lose little by just considering $\mathfrak{g} = \mathfrak{sl}(n; \mathbb{C})$ with $\mathfrak{b}$ its upper triangular matrices and $\mathfrak{h}$ its diagonal matrices. By definition, category $\mathcal{O} = \mathcal{O}(\mathfrak{g}, \mathfrak{b}, \mathfrak{h})$ is the category of all representations of the Lie algebra $\mathfrak{g}$, which are finitely generated, locally finite under the action of the Borel subalgebra $\mathfrak{b}$, and semisimple under the action of the Cartan subalgebra $\mathfrak{h}$. The objects of category $\mathcal{O}$ are always of finite length, but in general of infinite dimension and far from semisimple. The authors of [BGG76] remarked that one could find in their category $\mathcal{O}$ a phenomenon similar to the Bauer–Nesbitt reciprocity known from the modular representation theory of finite groups. Namely, there are enough projective objects, every object has finite length, the so-called *Cartan matrix* (which expresses the characters of the indecomposable projective covers of simple objects by simple characters) is symmetric and it can even be written as the product of yet another matrix with its transpose. In the modular case this other matrix is the so-called *decomposition matrix*, encoding how irreducible representations over the complex numbers decompose when we descend integral lattices inside to a field of positive characteristic. In the category $\mathcal{O}$ case this other matrix can be taken to be the matrix of the Jordan–Hölder multiplicities of the Verma modules alluded to above, i.e., the representations of $\mathfrak{g}$ induced from the one-dimensional representations of the Borel subalgebra. In formulas the

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Verma modules are given as

$$\Delta(\lambda) = U(\mathfrak{g}) \bigotimes_{U(\mathfrak{b})} \mathbb{C}_\lambda,$$

where $\lambda \in \mathfrak{h}^*$ is a linear form on the Cartan subalgebra extended by zero to the Borel subalgebra and $\mathbb{C}_\lambda$ is the corresponding one-dimensional representation. If we let $L(\lambda)$ be the unique simple quotient of $\Delta(\lambda)$ and $P(\lambda)$ its unique up-to-isomorphism indecomposable projective cover in $\mathcal{O}$, then this analogue of the Bauer–Nesbitt reciprocity reads

$$[P(\lambda) : \Delta(\mu)] = [\Delta(\mu) : L(\lambda)] .$$

Here it has to be understood that the $[\Delta(\mu)]$ as well as the $[L(\lambda)]$ form a basis of the Grothendieck group $[\mathcal{O}]$ of our category $\mathcal{O}$, and on the left-hand side we mean the coefficients of $[P(\lambda)] \in [\mathcal{O}]$ in the basis $[\Delta(\mu)]$ given by the Verma modules, whereas on the right-hand side we mean the coefficients of $[\Delta(\mu)] \in [\mathcal{O}]$ in the basis $[L(\lambda)]$ given by the simple module alias the Jordan–Hölder multiplicities of the Verma module $\Delta(\mu)$. In fact, the category $\mathcal{O}$ case is even better than the modular case, since the projectives actually admit a filtration with Verma subquotients, a so-called *Verma flag*. The reciprocity explained above is generally called *BGG-reciprocity* after the authors of [BGG76]. So in a way, category $\mathcal{O}$ as defined above was tailored in such a way that the projective covers of simple objects are "just right", i.e., they satisfy this nice reciprocity law. This in turn opened a new line of attack on the fundamental question of Jordan–Hölder multiplicities of Verma modules.

3. From my point of view, a truly convincing argument for the study of category $\mathcal{O}$ was only given later by J. Bernstein and S. Gelfand in [BG80]. They established fully faithful exact embeddings of this category into the category of Harish-Chandra modules for the simply connected complex algebraic group $G$ with Lie algebra $\mathfrak{g}$ considered as a real Lie group, mapping simple modules to simple modules and Verma modules to principal series representations. In our standard example from above, we would land within Harish-Chandra modules for the real Lie group $G = \mathrm{SL}(n;\mathbb{C})$. In fact Bernstein and Gelfand succeeded in this way to give an algebraic and very transparent proof of the Langlands classification in the case of complex groups. Admittedly the images of category $\mathcal{O}$ under these embeddings look rather artificial from the point of view of Harish-Chandra modules, and in fact the category of all Harish-Chandra modules of a semisimple complex algebraic group as above may seem a more natural object to study as compared to category $\mathcal{O}$. Category $\mathcal{O}$ however has the great advantage of being better accessible and simpler in many ways, and as the two are intimately linked anyway, it may not be a bad idea to concentrate on $\mathcal{O}$ in the first place. Lost in this simplification is an underlying monoidal structure that we discuss later in more detail.

4. A more direct line of attack on the question of the composition factor multiplicities of Verma modules was developed by Jantzen in [Jan79]. He introduced the so-called Jantzen filtration on a Verma module and was able to give an interesting relation between the characters of the proper submodules in this filtration and the characters of "smaller" Verma modules. This leads to the determination of some additional Jordan–Hölder multiplicities for Verma modules. But to get them all by this approach, it would have been necessary to solve the so-called Jantzen conjectures, which up to now can only be established by using even more refined

methodes of algebraic geometry, as compared to what is already needed to get the multiplicities themselves.

5. Let me discuss next the Kazhdan–Lusztig conjectures, which integrate the inductive procedure to determine the Jordan–Hölder multiplicities of Verma modules $[\Delta(\lambda) : L(\mu)]$ implied by the Jantzen conjectures. Our algebraic group $G$ from above can in fact be defined over the integers as can its closed connected subgroup $B$ with $\operatorname{Lie} B = \mathfrak{b}$. This gives rise to finite groups $G(\mathbb{F}_q) \supset B(\mathbb{F}_q)$, which in our standard example will be the finite group $\mathrm{SL}(n; \mathbb{F}_q)$ with its subgroup of upper triangular matrices. To this pair of finite groups in turn corresponds what is known as the *Hecke algebra* $\mathcal{H}(G(\mathbb{F}_q), B(\mathbb{F}_q))$, by which we understand the subring of the group ring $\mathbb{Z}G(\mathbb{F}_q)$ consisting of $B(\mathbb{F}_q)$-biinvariant functions, with its multiplication renormalized by a factor $|B(\mathbb{F}_q)|^{-1}$ to get again a ring with unit. By the Bruhat decomposition $G(\mathbb{F}_q) = \coprod_{x \in W} B(\mathbb{F}_q)xB(\mathbb{F}_q)$, the constant functions $T_x$ on the double cosets form a $\mathbb{Z}$-basis of our Hecke algebra. Here $x$ runs over the Weyl group $W$, which in our standard example can be identified with the symmetric group $\mathcal{S}_n$. Now the structure constants for this basis turn out to be polynomial in $q$, hence we can use them to form what is called the universal Hecke algebra

$$\mathcal{H} = \bigoplus_{x \in W} \mathbb{Z}[q]T_x.$$

When $q = 1$, the Hecke algebra is known to specialize to the group ring of the Weyl group $\mathbb{Z}W$ with $T_x$ specializing to $x$. Under the *functions-faisceaux* correspondence of Grothendieck, the intersection cohomology complexes of the closures of the double cosets $\overline{BxB} \subset G$ now correspond to certain elements $C_x \in \mathcal{H}$, which constitute up to a suitable power of $q$ the *canonical basis* of Kazhdan–Lusztig. Further, the conjecture of Kazhdan–Lusztig on the Jordan–Hölder multiplicities of a Verma module can, in light of BGG-reciprocity, be rewritten in the group ring $\mathbb{Z}W$ as the identity

$$C_x(1) = \sum_{y \in W} [P(x \cdot 0) : \Delta(y \cdot 0)]\, y.$$

Here the dot means the *translated action* of $W$ on $\mathfrak{h}$ with a fixed point in $-\rho$ for $\rho$ the half-sum of roots from our Borel subalgebra, given by the formula $x \cdot \lambda = x(\lambda + \rho) - \rho$, so that we have $x \cdot 0 = x\rho - \rho$. This conjecture of Kazhdan–Lusztig was established by Beilinson–Bernstein and Brylinski–Kashiwara using the theory of $\mathcal{D}$-modules, but it would take me too far afield to go into these matters in any detail.

6. Let me discuss now in more detail the monoidal structure alluded to above. Let $Z \subset U(\mathfrak{g})$ be the center of the enveloping algebra and let $\mathcal{M} \subset \mathfrak{g}$-mod be the full subcategory of all locally $Z$-finite $\mathfrak{g}$-modules. As shown in [BG80], for every finite-dimensional representation $E$ of $\mathfrak{g}$, the functor of tensoring with $E$ stabilizes $\mathcal{M}$ and thus leads to an exact functor $(E\otimes) : \mathcal{M} \to \mathcal{M}$. The direct summands of such functors are what J. Bernstein and S. Gelfand call *projective functors*. These functors clearly form a monoidal category under composition. Let us concentrate now for simplicity on the *principal block* $\mathcal{M}_{\mathrm{triv}}$ of $\mathcal{M}$, consisting of all representations, in which every vector is annihilated by some power of $Z^+ = \operatorname{Ann}_Z \mathbb{C}$, and let $(\mathcal{P}_{\mathrm{triv}}, \circ)$ be the monoidal category of all functors $\mathcal{M}_{\mathrm{triv}} \to \mathcal{M}_{\mathrm{triv}}$ which can be obtained as restrictions of projective functors. Furthermore, let us concentrate on the principal block $\mathcal{O}_{\mathrm{triv}} = \mathcal{O} \cap \mathcal{M}_{\mathrm{triv}}$ of $\mathcal{O}$. Its simple objects are

the simple highest weight modules $L(x \cdot 0)$ with $x \in W$ running over the Weyl group. Bernstein and Gelfand then show that

(1) The functor $\mathcal{P}_{\mathrm{triv}} \to \mathcal{O}_{\mathrm{triv}}$, $F \mapsto F\Delta(0)$ from our functor category to category $\mathcal{O}$ induces a bijection between isomorphism classes of the indecomposable objects from $\mathcal{P}_{\mathrm{triv}}$ and the indecomposable projective objects from $\mathcal{O}_{\mathrm{triv}}$, that is the $P(x \cdot 0)$ for $x \in W$;
(2) The map $J : \mathcal{P}_{\mathrm{triv}} \to \mathbb{Z}W$ given by $F \mapsto \sum [F\Delta(0) : \Delta(y \cdot 0)]\, y$ is compatible with multiplication up to interchanging the factors; i.e., we have $J(F \circ H) = J(H)J(F)$ for all $F, H \in \mathcal{P}_{\mathrm{triv}}$.

In this way, we see that $\mathcal{O}_{\mathrm{triv}}$, or rather its subcategory of projective objects $p\mathcal{O}_{\mathrm{triv}}$, might be considered a "reasonably faithful right module for the monoidal category $\mathcal{P}_{\mathrm{triv}}$", which in turn seems to me a most natural object to study. In fact, it seems even more natural to consider the sheaf-theoretic categorification of our Hecke algebra to the $B$-biequivariant constructible derived category with complex coefficients

$$\mathrm{Der}^c_{B \times B}(G)$$

of our complex group $G$ with its monoidal structure given by some sheaf-theoretic convolution. Its subcategory $\mathrm{Der}^{c,ss}_{B \times B}(G)$ of all perversely semisimple objects alias direct sums of shifted intersection cohomology complexes is stable under convolution. If we *degrade* it by letting

$$\overline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}) = \prod_{n \in \mathbb{Z}} \mathrm{Hom}(\mathcal{F}, \mathcal{G}[n]),$$

we just get back the monoidal category $\mathcal{P}_{\mathrm{triv}}$ of projective functors discussed before, to be taken for the Langlands dual group if you want more canonicity. Thus hidden behind the study of category $\mathcal{O}$ is the study of the sheaf-theoretic categorification of the Hecke algebra. It is even hidden in two ways, since the $\mathcal{D}$-module techniques alluded to above lead to an equivalence between category $\mathcal{O}_{\mathrm{triv}}$ and $N$-$B$-equivariant perverse sheaves on the group $G$ for $N \subset B$ the unipotent radical, no Langlands dual this time. This makes it look less of a surprise that category $\mathcal{O}$ turned out to be a rich subject and was even used lately with success in knot theory [BFK99, Str05].

7. The book under review is situated clearly on what one might call the "enveloping algebra side" of this story, as opposed to its geometric side. It starts with recollecting what the reader should know about representation theory of semisimple Lie algebras and proving the character formulas of Weyl and Kostant in parallel to developing the basic properties of category $\mathcal{O}$. It goes on to discuss more refined properties of $\mathcal{O}$ like BGG-reciprocity, homomorphisms between Verma modules, block decomposition, Jantzen sum formula, and BGG-resolution, and quite carefully develops the theory up to the introduction of translation functors, which are special cases of the projective functors discussed above. We are now about half-way through the book. From there on, beginning with Chapter 8 on Kazhdan–Lusztig theory, the author switches, as he nicely puts it himself, from "textbook mode" to "survey mode". I found it somewhat difficult to survey this survey part. Let me just add that there is a big and still rather textbook-style chapter on the parabolic version of category $\mathcal{O}$, a chapter on its relation to Harish-Chandra modules and projective functors, one on tilting modules, one on the twisting, shuffling and completion functors, which all are shadows of the underlying monoidal structure under various illuminations, and finally a chapter entitled "Complements", which

really is very much in survey mode. The textbook part puts together what you had to gather from, say, [MP95, Jan83, Jan79, Dix74] and the literature before, into a nicely readable source. The survey part, on the other hand, also includes material for which textbook-style references are in fact available, and which are then cited for the reader interested in more details. I know several students of mine and of colleagues who read the book or at least parts of it with pleasure and profit. It certainly is a valuable addition to the rather rare textbook literature on the subject.

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