

# Boolean Information Sieves: A Local-to-Global Approach to Quantum Information

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## Abstract

We propose a sheaf-theoretic framework for the representation of a quantum observable structure in terms of Boolean information sieves. The algebraic representation of a quantum observable structure in the relational local terms of sheaf theory, effectuates a semantic transition from the axiomatic set-theoretic context of orthocomplemented partially ordered sets, *à la Birkhoff and Von Neumann*, to the categorical topos-theoretic context of Boolean information sieves, *à la Grothendieck*. The representation schema is based on the existence of a categorical adjunction, which is used as a theoretical platform for the development of a functorial formulation of information transfer, between quantum observables and Boolean localization devices in typical quantum measurement situations. We also establish precise criteria of integrability and invariance of quantum information transfer by cohomological means.

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**Keywords** : Quantum Observables; Quantum Information; Sheaves; Adjunction; Boolean Localization; Sieves; Cohomology; Curvature.

## 1 Introduction

The main objective of a sheaf-theoretic representation schema regarding quantum observable structures is the application of category-theoretic concepts and methods for the evaluation and interpretation of the information content of these structures. For this purpose we introduce the central notion of Boolean information sieves leading to a novel perspective regarding quantum information transfer into Boolean local contexts of quantum measurement. The basic guiding idea is of a topological origin, and concerns the representation of the information enfolded in a global quantum observable structure, in terms of localization systems of interlocking Boolean contexts of observation, satisfying certain well defined compatibility relations. The implementation of this idea, emphasizes the contextual character of quantum information retrieval in typical quantum measurement situations, via Boolean preparatory contexts, and furthermore, demonstrates that the former is not *ad hoc* but can be cast in a mathematical form that respects strictly the rules of topological transition from local to global. The language of category theory (Lawvere and Schanuel 1997, Mac Lane 1971) proves to be appropriate for the implementation of this idea in a universal way. The conceptual essence of this scheme is the development of a sheaf-theoretic topos perspective (Mac Lane and Moerdijk 1992, Bell 1988) on quantum observable structures, which will constitute the basis for a functorial formulation of information transfer between Boolean localization devices and quantum systems.

In quantum logical approaches the notion of event, associated with the measurement of an observable, is taken to be equivalent to a proposition describing the behavior of a physical system. This formulation of quantum theory is based on the identification of propositions with projection operators on a complex Hilbert space. In this sense, the Hilbert space formalism of quantum theory associates events with closed subspaces of a separable, complex Hilbert space corresponding to a quantum system. Then, the quantum events algebra is identified with the lattice of closed subspaces of the Hilbert space, ordered by inclusion and carrying an orthocomplementation operation which is given by the orthogonal complements of the closed subspaces (Varadarajan 1968, Birkhoff and von Neumann 1936). Equivalently, it is isomorphic to the partial Boolean algebra of closed subspaces of the Hilbert space of the system, or alternatively the partial Boolean algebra of projection operators of the system (Kochen and Specker 1967).

The starting point of our investigation is based on the observation that set-theoretic axiomatizations of quantum observable structures hide the fundamental significance of Boolean localization systems in the formation of these structures. This is not satisfactory, due to the fact that, all operational procedures employed in quantum measurement, are based in the preparation of appropriate Boolean environments. The construction of these contexts of observation are related with certain abstractions and can be metaphorically considered as Boolean pattern recognition systems. In the categorical language we adopt, we can explicitly associate them with appropriate Boolean covering systems of a structure of quantum observables. In this way, the real significance of a quantum structure proves to be, not at the level of events, but at the level of gluing together overlapping Boolean localization contexts. The development of the conceptual and technical machinery of lo-

calization systems for generating non-trivial global event structures, as it has been recently demonstrated in (Zafiris 2006a), effectuates a transition in the semantics of events and observables from a set-theoretic to a sheaf-theoretic one. This is a crucial semantic difference that characterizes the present approach in comparison to the vast literature on quantum measurement and quantum logic.

The formulation of information transfer proposed in this paper, is based on the sheaf-theoretic representation of a quantum observable structure in terms of Boolean information sieves, consisting of families of local Boolean reference frames, which can be pasted together using category-theoretic means. Contextual topos-theoretic approaches to quantum structures have been independently proposed, from the viewpoint of the theory of presheaves on partially ordered sets, in (Butterfield and Isham 1998 and 1999), and have been extensively discussed and critically analyzed in (Rawling and Selesnick 2000, Butterfield and Isham 2000). An interesting intuitionistic interpretation of quantum mechanics has been constructed in (Adelman and Corbett 1995), by using the real number continuum given by the sheaf of Dedekind reals in the topos of sheaves on the quantum state space. The idea of introduction of Boolean reference frames has been also appeared in the literature, from a non-category theoretic perspective, in (Takeuti 1978, Davis 1977). For a general mathematical discussion of sheaves, variable sets, and related structures, the interested reader should consult (Lawvere 1975). Finally, it is also worth mentioning that an alternative sheaf-theoretic approach to quantum structures has been recently initiated independently in (de Groote 2001). In a general setting, this approach proposes a theory of presheaves on the quantum lattice of closed subspaces of a complex Hilbert space, by transposing literally and generalizing the corresponding constructions from the lattice of

open sets of a topological space to the quantum lattice. In comparison, our approach emphasizes the crucial role of Boolean localization systems in the global formation of quantum structures, and thus, shifts the focus of relevant constructions to sheaves over suitable localization systems on a base category of Boolean subalgebras of global quantum algebras. Technical expositions of sheaf theory, being of particular interest in relation to the focus of the present work are provided by (Mac Lane and Moerdijk 1992, Bredon 1997, Mallios 1998, Mallios 2004). We mention that another local to global perspective on quantum information has been developed in the context of the System of Systems approach (Jamshidi 2009), in which post-measurement, the linear probabilistic quantum model may be viewed as giving rise to a system of systems each characterized by a linear probabilistic quantum system model. Graph models for dealing with quantum complexity have been developed in (Kitto 2008, Sahni, Srivastava and Satsangi 2009). Finally, various applications of sheaf-theoretic structures, based on the development of suitable localization schemes referring to the modeling and interpretation of quantum systems, have been communicated, both conceptually and technically by the author, in the literature (Zafiris 2000, 2006a, 2006b, 2006c, 2006d and 2009).

In Section 2, we define event and observable structures using category-theoretic means. In Section 3, we introduce the notion of a Boolean functor, we define the category of presheaves of Boolean observables, and also, develop the idea of fibered structures. In Section 4, we prove the existence of an adjunction between the topos of presheaves of Boolean observables and the category of quantum observables, and formulate a schema of functorial information transfer. In Section 5, we define Boolean information sieves as systems of Boolean localizations for quantum observable structures and analyze their operational role. In Section 6, we formulate an invariance property

of functorial information transfer using the adjunction established previously. In Section 7, we establish precise functorial criteria of intergrability and invariance of information transfer between quantum observable structures and Boolean localization systems. Finally, we conclude in Section 8.

## 2 Categories of Events and Observables

A **quantum event structure** is a category, denoted by  $\mathcal{L}$ , which is called the category of quantum event algebras.

Its objects, denoted by  $L$ , are quantum algebras of events, that is orthomodular  $\sigma$ -orthoposets. More concretely, each object  $L$  in  $\mathcal{L}$ , is considered as a partially ordered set of quantum events, endowed with a maximal element 1, and with an operation of orthocomplementation  $[-]^* : L \longrightarrow L$ , which satisfy, for all  $l \in L$ , the following conditions: [a]  $l \leq 1$ , [b]  $l^{**} = l$ , [c]  $l \vee l^* = 1$ , [d]  $l \leq \acute{l} \Rightarrow \acute{l}^* \leq l^*$ , [e]  $l \perp \acute{l} \Rightarrow l \vee \acute{l} \in L$ , [f] for  $l, \acute{l} \in L$ ,  $l \leq \acute{l}$  implies that  $l$  and  $\acute{l}$  are compatible, where  $0 := 1^*$ ,  $l \perp \acute{l} := l \leq \acute{l}^*$ , and the operations of meet  $\wedge$  and join  $\vee$  are defined as usually. We also recall that  $l, \acute{l} \in L$  are compatible if the sublattice generated by  $\{l, l^*, \acute{l}, \acute{l}^*\}$  is a Boolean algebra, namely if it is a Boolean sublattice. The  $\sigma$ -completeness condition, namely that the join of countable families of pairwise orthogonal events must exist, is also required in order to have a well defined theory of observables over  $L$ .

Its arrows are quantum algebraic homomorphisms, that is maps  $K \xrightarrow{H} L$ , which satisfy, for all  $k \in K$ , the following conditions: [a]  $H(1) = 1$ , [b]  $H(k^*) = [H(k)]^*$ , [c]  $k \leq \acute{k} \Rightarrow H(k) \leq H(\acute{k})$ , [d]  $k \perp \acute{k} \Rightarrow H(k \vee \acute{k}) \leq H(k) \vee H(\acute{k})$ , [e]  $H(\bigvee_n k_n) = \bigvee_n H(k_n)$ , where  $k_1, k_2, \dots$  countable family of mutually orthogonal events.

A **classical event structure** is a category, denoted by  $\mathcal{B}$ , which is called

the category of Boolean event algebras. Its objects are  $\sigma$ -Boolean algebras of events and its arrows are the corresponding Boolean algebraic homomorphisms.

The notion of observable corresponds to a physical quantity that can be measured in the context of an experimental arrangement. In any measurement situation the propositions that can be made concerning a physical quantity are of the following type: the value of the physical quantity lies in some Borel set of the real numbers. A proposition of this form corresponds to an event as it is apprehended by an observer using his measuring instrument. An observable  $\Xi$  is defined to be an algebraic homomorphism from the Borel algebra of the real line  $Bor(\mathbb{R})$  to the quantum event algebra  $L$ .

$$\Xi : Bor(\mathbb{R}) \rightarrow L \quad (2.1)$$

such that: [i]  $\Xi(\emptyset) = 0, \Xi(\mathbb{R}) = 1$ , [ii]  $E \cap F = \emptyset \Rightarrow \Xi(E) \perp \Xi(F)$ , for  $E, F \in Bor(\mathbb{R})$ , [iii]  $\Xi(\bigcup_n E_n) = \bigvee_n \Xi(E_n)$ , where  $E_1, E_2, \dots$  sequence of mutually disjoint Borel sets of the real line.

If  $L$  is isomorphic with the orthocomplemented lattice of orthogonal projections on a Hilbert space, then it follows from von Neumann's spectral theorem (Varadarajan 1968) that the observables are in injective correspondence with the hypermaximal Hermitian operators on the Hilbert space.

A **quantum observable structure** is a category, denoted by  $\mathcal{O}_Q$ , which is called the category of quantum observables. Its objects are quantum observables  $\Xi : Bor(\mathbb{R}) \rightarrow L$  and its arrows  $\Xi \longrightarrow \Theta$  are commutative triangles, or equivalently the quantum algebraic homomorphisms  $L \xrightarrow{H} K$  in  $\mathcal{L}$ , preserving by definition the join of countable families of pairwise orthogonal events, such that  $\Theta = H \circ \Xi$  in Diagram 1 is again a quantum observable.

Correspondingly, a **Boolean observable structure** is a category, denoted by  $\mathcal{O}_B$ , which is called the category of Boolean observables. Its objects

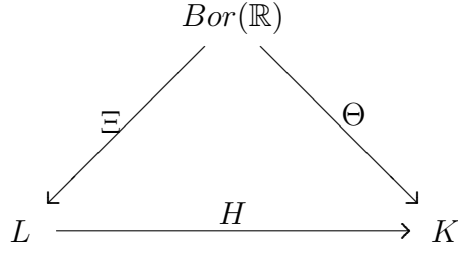


Diagram 1

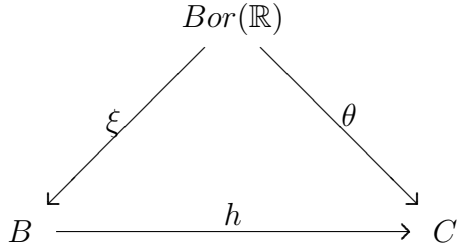


Diagram 2

are the Boolean observables  $\xi : Bor(\mathbb{R}) \rightarrow B$  and its arrows are the Boolean algebraic homomorphisms  $B \xrightarrow{h} C$  in  $\mathcal{B}$ , such that  $\theta = h \circ \xi$  in Diagram 2 is again a Boolean observable.

### 3 Functors Associated with Observables

#### 3.1 Functor Category of Boolean Observable Presheaves

If  $\mathcal{O}_B^{op}$  is the opposite category of  $\mathcal{O}_B$ , then  $\mathbf{Sets}^{\mathcal{O}_B^{op}}$  denotes the functor category of presheaves on Boolean observables. Its objects are all functors  $\mathbf{X} : \mathcal{O}_B^{op} \rightarrow \mathbf{Sets}$ , and its morphisms are all natural transformations between such functors. Each object  $\mathbf{X}$  in this category is a contravariant set-valued functor on  $\mathcal{O}_B$ , called a presheaf on  $\mathcal{O}_B$ .

For each Boolean observable  $\xi$  of  $\mathcal{O}_B$ ,  $\mathbf{X}(\xi)$  is a set, and for each arrow  $f : \theta \rightarrow \xi$ ,  $\mathbf{X}(f) : \mathbf{X}(\xi) \rightarrow \mathbf{X}(\theta)$  is a set function. If  $\mathbf{X}$  is a presheaf on  $\mathcal{O}_B$



and  $x \in \mathbf{X}(O)$ , the value  $\mathbf{X}(f)(x)$  for an arrow  $f : \theta \longrightarrow \xi$  in  $\mathcal{O}_B$  is called the restriction of  $x$  along  $f$  and is denoted by  $\mathbf{X}(f)(x) = x \cdot f$ .

Each object  $\xi$  of  $\mathcal{O}_B$  gives rise to a contravariant Hom-functor  $\mathbf{y}[\xi] := \text{Hom}_{\mathcal{O}_B}(-, \xi)$ . This functor defines a presheaf on  $\mathcal{O}_B$ . Its action on an object  $\theta$  of  $\mathcal{O}_B$  is given by

$$\mathbf{y}[\xi](\theta) := \text{Hom}_{\mathcal{O}_B}(\theta, \xi) \quad (3.1)$$

whereas its action on a morphism  $\eta \xrightarrow{w} \theta$ , for  $v : \theta \longrightarrow \xi$  is given by

$$\mathbf{y}[\xi](w) : \text{Hom}_{\mathcal{O}_B}(\theta, \xi) \longrightarrow \text{Hom}_{\mathcal{O}_B}(\eta, \xi) \quad (3.2)$$

$$\mathbf{y}[\xi](w)(v) = v \circ w \quad (3.3)$$

Furthermore  $\mathbf{y}$  can be made into a functor from  $\mathcal{O}_B$  to the contravariant functors on  $\mathcal{O}_B$

$$\mathbf{y} : \mathcal{O}_B \longrightarrow \mathbf{Sets}^{\mathcal{O}_B^{op}} \quad (3.4)$$

such that  $\xi \mapsto \text{Hom}_{\mathcal{O}_B}(-, \xi)$ . This is an embedding, called the Yoneda embedding (Mac Lane and Moerdijk 1992), and it is a full and faithful functor.

The functor category of presheaves on Boolean observables  $\mathbf{Sets}^{\mathcal{O}_B^{op}}$ , provides an exemplary case of a category known as topos (Mac Lane and Moerdijk 1992, Bell 1988, Lawvere 1975). A topos can be conceived as a well defined notion of a universe of variable sets. Furthermore, it provides a natural example of a many-valued truth structure, which remarkably is not ad hoc, but reflects genuine constraints of the surrounding universe.

### 3.2 Fibrations over Boolean Observables

Since  $\mathcal{O}_B$  is a small category, there is a set consisting of all the elements of all the sets  $\mathbf{X}(\xi)$ , and similarly there is a set consisting of all the functions  $\mathbf{X}(f)$ . This observation regarding  $\mathbf{X} : \mathcal{O}_B^{op} \longrightarrow \mathbf{Sets}$  permits us to take

$$\begin{array}{ccc}
& \int(\mathbf{X}, \mathcal{O}_B) & \\
& \downarrow \int_{\mathbf{X}} & \\
\mathcal{O}_B & \xrightarrow{\mathbf{X}} & \mathbf{Sets}
\end{array}$$

Diagram 3

the disjoint union of all the sets of the form  $\mathbf{X}(\xi)$  for all objects  $\xi$  of  $\mathcal{O}_B$ . The elements of this disjoint union can be represented as pairs  $(\xi, x)$  for all objects  $\xi$  of  $\mathcal{O}_B$  and elements  $x \in \mathbf{X}(\xi)$ . Thus the disjoint union of sets is made by labeling the elements. Now we can construct a category whose set of objects is the disjoint union just mentioned. This structure is called the category of elements of the presheaf  $\mathbf{X}$ , denoted by  $\int(\mathbf{X}, \mathcal{O}_B)$ . Its objects are all pairs  $(\xi, x)$ , and its morphisms  $(\acute{\xi}, \acute{x}) \rightarrow (\xi, x)$  are those morphisms  $u : \acute{\xi} \rightarrow \xi$  of  $\mathcal{O}_B$  for which  $x \cdot u = \acute{x}$ . Projection on the second coordinate of  $\int(\mathbf{X}, \mathcal{O}_B)$ , defines a functor  $\int_{\mathbf{X}} : \int(\mathbf{X}, \mathcal{O}_B) \rightarrow \mathcal{O}_B$ .  $\int(\mathbf{X}, \mathcal{O}_B)$  together with the projection functor  $\int_{\mathbf{X}}$  is called the split discrete fibration induced by  $\mathbf{X}$ , and  $\mathcal{O}_B$  is the base category of the fibration (Diagram 3). We note that the fibration is discrete because the fibers are categories in which the only arrows are identity arrows. If  $\xi$  is a Boolean observable object of  $\mathcal{O}_B$ , the inverse image under  $\int_{\mathbf{X}}$  of  $\xi$  is simply the set  $\mathbf{X}(\xi)$ , although its elements are written as pairs so as to form a disjoint union. The notion of discrete fibration induced by  $\mathbf{X}$ , is an application of the general Grothendieck construction in our context of enquiry.

It is instructive to remark that, that the construction of the split discrete fibration induced by  $\mathbf{X}$ , where  $\mathcal{O}_B$  is the base category of the fibration, incorporates the physically important requirement of uniformity (Zafiris (2006)). The notion of uniformity, requires that for any two events observed over

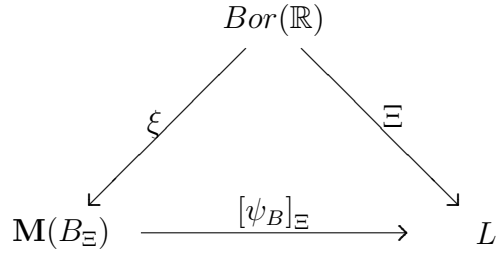


Diagram 4

the same domain of measurement, the structure of all Boolean contexts that relate to the first cannot be distinguished in any possible way from the structure of Boolean contexts relating to the second. In this sense, all the observed events within any particular Boolean context should be uniformly equivalent to each other. It is easy to notice that the composition law in the category of elements of the presheaf  $\mathbf{X}$ , expresses precisely the above uniformity condition.

### 3.3 Functor of Boolean coefficients

We define a Boolean coefficient or Boolean coordinatization functor,

$$\mathbf{M} : \mathcal{O}_B \longrightarrow \mathcal{O}_Q \quad (3.5)$$

which assigns to Boolean observables in  $\mathcal{O}_B$  (that plays the role of the category of coordinatization models) the underlying quantum observables from  $\mathcal{O}_Q$ , and to Boolean homomorphisms the underlying quantum algebraic homomorphisms.

Equivalently the functor of Boolean coefficients can be characterized as,  $\mathbf{M} : \mathcal{B} \longrightarrow \mathcal{L}$ , which assigns to Boolean event algebras in  $\mathcal{B}$  the underlying quantum event algebras from  $\mathcal{L}$ , and to Boolean homomorphisms the underlying quantum algebraic homomorphisms, such that Diagram 4 commutes.

$$\begin{array}{ccc}
\mathbf{X}(\xi) & \xrightarrow{\tau_\xi} & \text{Hom}_{\mathcal{O}_Q}(\mathbf{M}(\xi), \Xi) \\
\mathbf{X}(u) \downarrow & & \downarrow \mathbf{M}(u) \\
\mathbf{X}(\xi') & \xrightarrow{\tau_{\xi'}} & \text{Hom}_{\mathcal{O}_Q}(\mathbf{M}(\xi'), \Xi)
\end{array}$$

Diagram 5

## 4 Functorial Information Transfer

### 4.1 Adjunctive correspondence between Presheaves of Boolean Observables and Quantum Observables

We consider the category of quantum observables  $\mathcal{O}_Q$  and the modeling functor  $\mathbf{M}$ , and we define the functor  $\mathbf{R}$  from  $\mathcal{O}_Q$  to the topos of presheaves of Boolean observables, given by

$$\mathbf{R}(\Xi) : \xi \mapsto \text{Hom}_{\mathcal{O}_Q}(\mathbf{M}(\xi), \Xi) \quad (4.1)$$

A natural transformation  $\tau$  between the topos of presheaves on the category of Boolean observables  $\mathbf{X}$  and  $\mathbf{R}(\Xi)$ ,  $\tau : \mathbf{X} \longrightarrow \mathbf{R}(\Xi)$  is a family  $\tau_\xi$  indexed by Boolean observables  $\xi$  of  $\mathcal{O}_B$  for which each  $\tau_\xi$  is a map

$$\tau_\xi : \mathbf{X}(\xi) \longrightarrow \text{Hom}_{\mathcal{O}_Q}(\mathbf{M}(\xi), \Xi) \quad (4.2)$$

of sets, such that Diagram 5 commutes for each Boolean homomorphism  $u : \xi' \rightarrow \xi$  of  $\mathcal{O}_B$ .

If we make use of the category of elements of the Boolean observable-variable set  $X$ , then the map  $\tau_\xi$ , defined above, can be characterized as:

$$\tau_\xi : (\xi, p) \longrightarrow \text{Hom}_{\mathcal{O}_Q}(\mathbf{M} \circ \int_{\mathbf{X}} (\xi, p), \Xi) \quad (4.3)$$

Equivalently such a  $\tau$  can be seen as a family of arrows of  $\mathcal{O}_Q$  which is being indexed by objects  $(\xi, p)$  of the category of elements of the presheaf of

$$\begin{array}{ccc}
\mathbf{M}(\xi) & \xlongequal{\quad} & \mathbf{M} \circ \int_{\mathbf{X}}(\xi, p) \\
\uparrow \mathbf{M}(u) & & \uparrow u_* \\
\mathbf{M}(\xi') & \xlongequal{\quad} & \mathbf{M} \circ \int_{\mathbf{X}}(\xi', p') \\
& & \searrow \tau_{\xi}(p) \\
& & \Xi \\
& & \nearrow \tau_{\xi'}(p')
\end{array}$$

Diagram 6

Boolean observables  $\mathbf{X}$ , namely

$$\{\tau_{\xi}(p) : \mathbf{M}(\xi) \rightarrow \Xi\}_{(\xi, p)} \quad (4.4)$$

From the perspective of the category of elements of  $\mathbf{X}$ , the condition of the commutativity of the preceding diagram is equivalent with the condition that for each Boolean homomorphism  $u : \xi' \rightarrow \xi$  of  $\mathcal{O}_B$ , Diagram 6 is commutative.

It is straightforward to see that the arrows  $\tau_{\xi}(p)$  form a cocone from the functor  $\mathbf{M} \circ \int_{\mathbf{X}}$  to the quantum observable  $\Xi$ . If we remind the categorical notion of colimit, being the universal construction of interconnection, we conclude that each such cocone emerges by the composition of the colimiting cocone with a unique arrow from the colimit  $\mathbf{LX}$  to the quantum observable  $\Xi$ . In other words, there is a bijection which is natural in  $\mathbf{X}$  and  $\Xi$

$$\text{Nat}(\mathbf{X}, \mathbf{R}(\Xi)) \cong \text{Hom}_{\mathcal{O}_Q}(\mathbf{LX}, \Xi) \quad (4.5)$$

From the above bijection we are driven to the conclusion that the functor  $\mathbf{R}$  from  $\mathcal{O}_Q$  to the topos of presheaves given by

$$\mathbf{R}(\Xi) : \xi \mapsto \text{Hom}_{\mathcal{O}_Q}(\mathbf{M}(\xi), \Xi) \quad (4.6)$$

has a left adjoint  $\mathbf{L} : \mathbf{Sets}^{\mathcal{O}_B^{op}} \rightarrow \mathcal{O}_Q$ , which is defined for each presheaf of Boolean observables  $\mathbf{X}$  in  $\mathbf{Sets}^{\mathcal{O}_B^{op}}$  as the colimit

$$\mathbf{L}(\mathbf{X}) = \mathit{Colim}\left\{ \int (\mathbf{X}, \mathcal{O}_B) \xrightarrow{\int \mathbf{X}} \mathcal{O}_B \xrightarrow{\mathbf{M}} \mathcal{O}_Q \right\} \quad (4.7)$$

For readers not feeling comfortable with the categorical notion of colimit we may construct it explicitly for the case of interest  $\mathbf{X} = \mathbf{R}(\Xi)$  in set-theoretical language as follows:

**Colimit construction:** We consider the set

$$\mathbf{L}(\mathbf{R}(\Xi)) = \{(\psi_\xi, q) / (\psi_\xi : \mathbf{M}(\xi) \longrightarrow \Xi) \in [\int (\mathbf{R}(\Xi), \mathcal{O}_B)]_0, q \in \mathbf{M}(\xi)\} \quad (4.8)$$

We notice that if there exists  $u : \psi_\xi \rightarrow \psi_\xi$  such that:  $u(q) = q$  and  $\psi_\xi \circ u = \psi_\xi$ , where  $[\mathbf{R}(\Xi)u](\psi_\xi) := \psi_\xi \circ u$  as usual, then we may define a transitive and reflexive relation  $\mathfrak{R}$  on the set  $\mathbf{L}(\mathbf{R}(\Xi))$ . Of course the inverse also holds true.

We notice then that

$$(\psi_\xi \circ u, q) \mathfrak{R} (\psi_\xi, u(q)) \quad (4.9)$$

for any  $u : \mathbf{M}(\xi) \rightarrow \mathbf{M}(\xi)$  in the category  $\mathcal{O}_B$ . The next step is to make this relation also symmetric by postulating that for  $\varphi, \chi$  in  $\mathbf{L}(\mathbf{R}(\Xi))$ , where  $\varphi, \chi$  denote pairs in the above set, we have:

$$\varphi \sim \chi \quad (4.10)$$

if and only if  $\varphi \mathfrak{R} \chi$  or  $\chi \mathfrak{R} \varphi$ . Finally by considering a sequence  $\varrho_1, \varrho_2, \dots, \varrho_k$  of elements of the set  $\mathbf{L}(\mathbf{R}(\Xi))$  and also  $\varphi, \chi$  such that:

$$\varphi \sim \varrho_1 \sim \varrho_2 \sim \dots \sim \varrho_{k-1} \sim \varrho_k \sim \chi \quad (4.11)$$

we may define an equivalence relation on the set  $\mathbf{L}(\mathbf{R}(L))$  as follows:

$$\varphi \boxtimes \chi := \varphi \sim \varrho_1 \sim \varrho_2 \sim \dots \sim \varrho_{k-1} \sim \varrho_k \sim \chi \quad (4.12)$$

Then for each  $\varphi \in \mathbf{L}(\mathbf{R}(\Xi))$  we define the quantum at  $\varphi$  as follows:

$$Q_\varphi = \{\iota \in \mathbf{L}(\mathbf{R}(\Xi)) : \varphi \bowtie \iota\} \quad (4.13)$$

We finally put

$$\mathbf{L}(\mathbf{R}(\Xi))/\bowtie := \{Q_\varphi : \varphi = (\psi_\xi, q) \in \mathbf{L}(\mathbf{R}(\Xi))\} \quad (4.14)$$

and use the notation  $Q_\varphi = \|(\psi_\xi, q)\|$ . If we remind that each quantum observable is defined as an algebraic homomorphism from the Borel algebra of the real line  $Bor(\mathbb{R})$  to a quantum event algebra  $L$ , we may finally write the quotient  $\mathbf{L}(\mathbf{R}(\Xi))/\bowtie$  in the form of a quantum observable as follows:

$$\mathbf{L}(\mathbf{R}(\Xi))/\bowtie: Bor(\mathbb{R}) \rightarrow \mathbf{L}(\mathbf{R}(L))/\bowtie \quad (4.15)$$

and verify that  $\mathbf{L}(\mathbf{R}(L))/\bowtie$  is actually a quantum event algebra, where in complete analogy with the definition of  $\mathbf{L}(\mathbf{R}(\Xi))/\bowtie$  we have:

$$\mathbf{L}(\mathbf{R}(L)) = \{(\psi_B, b)/(\psi_B : \mathbf{M}(B) \longrightarrow L), b \in \mathbf{M}(B)\} \quad (4.16)$$

The set  $\mathbf{L}(\mathbf{R}(L))/\bowtie$  is naturally endowed with a quantum algebra structure if we are careful to notice that:

- [1]. The orthocomplementation is defined as:  $Q_\zeta^* = \|(\psi_B, b)\|^* = \|(\psi_B, b^*)\|$ .
- [2]. The unit element is defined as:  $\mathbf{1} = \|(\psi_B, 1)\|$ .
- [3]. The partial order structure on the set  $\mathbf{L}(\mathbf{R}(L))/\bowtie$  is defined as:

$\|(\psi_B, b)\| \preceq \|(\psi_C, r)\|$  if and only if  $d_1 \preceq d_2$  where we have made the following identifications:  $\|(\psi_B, b)\| = \|(\psi_D, d_1)\|$  and  $\|(\psi_C, r)\| = \|(\psi_D, d_2)\|$ , with  $d_1, d_2 \in \mathbf{M}(D)$  according to the fibered product Diagram 7 of event algebras, such that  $\beta(d_1) = b, \gamma(d_2) = r$ . The rest of the requirements such that  $\mathbf{L}(\mathbf{R}(L))/\bowtie$  actually carries the structure of a quantum event algebra are obvious.

$$\begin{array}{ccc}
\mathbf{M}(D) & \xrightarrow{\beta} & \mathbf{M}(B) \\
\downarrow \gamma & & \downarrow \\
\mathbf{M}(C) & \longrightarrow & L
\end{array}$$

Diagram 7

$$\begin{array}{ccc}
\text{Nat}(\mathbf{X}, \mathbf{R}(\Xi)) & \xrightarrow{\mathbf{r}} & \text{Hom}_{\mathcal{O}_Q}(\mathbf{LX}, \Xi) \\
\parallel & & \parallel \\
\text{Nat}(\mathbf{X}, \mathbf{R}(\Xi)) & \xleftarrow{\mathbf{l}} & \text{Hom}_{\mathcal{O}_Q}(\mathbf{LX}, \Xi)
\end{array}$$

Diagram 8

The conclusion being drawn from the analysis presented in this Section can be summarized as follows: There exists a **pair of adjoint functors**  $\mathbf{L} \dashv \mathbf{R}$  according to the bidirectional correspondence;

$$\mathbf{L} : \mathbf{Sets}^{\mathcal{O}_B^{op}} \xrightleftharpoons{\quad} \mathcal{O}_Q : \mathbf{R} \quad (4.17)$$

This pair of functors forms a categorical adjunction consisting of the functors  $\mathbf{L}$  and  $\mathbf{R}$ , called left and right adjoints with respect to each other respectively, as well as the natural bijection:

$$\text{Nat}(\mathbf{X}, \mathbf{R}(\Xi)) \cong \text{Hom}_{\mathcal{O}_Q}(\mathbf{LX}, \Xi) \quad (4.18)$$

The existence of the categorical adjunctive correspondence explained above, provides a theoretical platform for the formulation of a functorial schema of interpretation, concerning the information transfer that takes place in quantum measurement situations. If we consider that  $\mathbf{Sets}^{\mathcal{B}^{op}}$  is the universe of Boolean observable event structures modeled in  $\mathbf{Sets}$ , and  $\mathcal{L}$  that of quantum



event structures, then the topos theoretical specification of the first category represents the varying world of Boolean localization filters of information associated with abstraction mechanisms of observation. In this perspective the functor  $\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \longrightarrow \mathcal{L}$  can be comprehended as a translational code from Boolean information filters to the quantum species of structure, whereas the functor  $\mathbf{R} : \mathcal{L} \longrightarrow \mathbf{Sets}^{\mathcal{B}^{op}}$  as a translational code in the inverse direction. In general, the content of the information is not possible to remain completely invariant translating from one language to another and back, in any information transfer mechanism. However, there remain two ways for a Boolean-event algebra variable set  $\mathbf{P}$ , or else Boolean filter of information, to communicate a message to a quantum event algebra  $L$ . Either the information is transferred in quantum terms with  $\mathbf{P}$  translating, which we can be represented as the quantum homomorphism  $\mathbf{LP} \longrightarrow L$ , or the information is transferred in Boolean terms with  $L$  translating, that, in turn, can be represented as the natural transformation  $\mathbf{P} \longrightarrow \mathbf{R}(L)$ . In the first case, from the perspective of  $L$  information is being received in quantum terms, while in the second, from the perspective of  $\mathbf{P}$  information is being sent in Boolean terms. The natural bijection then corresponds to the assertion that these two distinct ways of communicating are equivalent. Thus, the physical meaning of the adjunctive correspondence, signifies a co-dependency of the involved languages in communication. This is realized operationally in the process of extraction of the information content enfolded in a quantum observable structure through the pattern recognition characteristics of specified Boolean domain preparatory contexts. In turn, this process gives rise to a variation of the information collected in Boolean filtering systems for an observed quantum system, which is not always compatible. In the next section, we will specify the necessary and sufficient conditions for a full and

faithfull representation of the informational content included in a quantum observable structure in terms of Boolean information sieves, or equivalently Boolean localization systems. At the present stage we may observe that the representation of a quantum observable as a categorical colimit, resulting from the same adjunctive correspondence, reveals a theoretical concept that can admit a multitude of Boolean coordinatizations, specified mathematically by different Boolean coefficients in Boolean information filtering systems.

## 5 Boolean Information Sieves

### 5.1 Functor of Boolean Points of Quantum Observables

The development of the ideas contained in the proposed scheme are based on the notion of the functor of Boolean points of quantum observables, so it is worthwhile to explain its meaning in some detail. The conceptual background behind this notion has its roots in the work of Grothendieck in algebraic geometry (Mac Lane and Moerdijk 1992). If we consider the opposite of the category of quantum observables, that is the category with the same objects but with arrows reversed  $\mathcal{O}_Q^{op}$ , each object in the context of this category can be thought of as the locus of a quantum observable, or else it carries the connotation of space. The crucial observation is that any such space is determined up to canonical isomorphism if we know all morphisms into this locus from any other locus in the category. For instance, the set of morphisms from the one-point locus to  $\Xi$  in  $\mathcal{O}_Q^{op}$  determines the set of points of the locus  $\Xi$ . The philosophy behind this approach amounts to considering any morphism in  $\mathcal{O}_Q^{op}$  with target the locus  $\Xi$  as a generalized point of  $\Xi$ .

It is obvious that the description of a locus  $\Xi$  in terms of all possible morphisms from all other objects of  $\mathcal{O}_Q^{op}$  is redundant. For this reason we may restrict the generalized points of  $\Xi$  to all those morphisms in  $\mathcal{O}_Q^{op}$  having as domains measurement loci corresponding to Boolean observables. Evidently such measurement loci correspond, if we take into account Stone's representation theorem for Boolean algebras, to a replacement of each Boolean algebra  $B$  in  $\mathcal{B}$  by its set-theoretical representation  $[\Sigma, B_\Sigma]$ , consisting of a local measurement space  $\Sigma$  and its local field of subsets  $B_\Sigma$ .

Variation of generalized points over all domain-objects of the subcategory of  $\mathcal{O}_Q^{op}$  consisting of Boolean observables produces the functor of points of  $\Xi$  restricted to the subcategory of Boolean objects, identified with  $\mathcal{O}_B^{op}$ . This functor of Boolean points of  $\Xi$  is made then an object in the category of presheaves  $\mathbf{Sets}^{\mathcal{O}_B^{op}}$ , representing a quantum observable -(in the sequel for simplicity we talk of an observable and its associated locus tautologically)- in the environment of the topos of presheaves over the category of Boolean observables. This methodology will prove to be successful if it could be possible to establish an isomorphic representation of  $\Xi$  in terms of the information being carried by its Boolean points  $\xi \rightarrow \Xi$  collated together by appropriate means.

## 5.2 Boolean Information Sieves of Prelocalization

We coordinatize the information contained in a quantum observable  $\Xi$  in  $\mathcal{O}_Q$  by means of Boolean points, namely morphisms  $\xi \rightarrow \Xi$  having as their domains, locally defined Boolean observables  $\xi$  in  $\mathcal{O}_B$ . Any single map from a Boolean coordinate domain to a quantum observable is not enough for a complete determination of its information content, and hence, it contains only a limited amount of information about it. More concretely, it includes only

the amount of information related to a prepared Boolean local context, and thus, it is inevitably constrained to represent the abstractions associated with its preparation. In order to cope with this problem we consider a sufficient number of localizing morphisms from the domains of Boolean preparatory contexts simultaneously, such that the information content of a quantum observable can be eventually covered completely. In turn, the information available about each morphism of the specified covering may be used to determine the quantum observable itself. In this case, we say that, the family of such morphisms generate a Boolean information sieve of prelocalizations for a quantum observable, induced by measurement. We may formalize these intuitive ideas as follows:

A **Boolean information sieve of prelocalizations** for a quantum observable  $\Xi$  in  $\mathcal{O}_Q$  is a subfunctor of the Hom-functor  $\mathbf{R}(\Xi)$  of the form  $\mathbf{S} : \mathcal{O}_B^{op} \rightarrow \mathbf{Sets}$ , namely for all  $\xi$  in  $\mathcal{O}_B$  it satisfies  $\mathbf{S}(\xi) \subseteq [\mathbf{R}(\Xi)](\xi)$ . According to this definition, a Boolean information sieve of prelocalizations for a quantum observable  $\Xi$  in  $\mathcal{O}_Q$ , can be understood as a right ideal  $\mathbf{S}(\xi)$  of quantum algebraic homomorphisms of the form

$$\psi_\xi : \mathbf{M}(\xi) \longrightarrow \Xi, \quad \xi \in \mathcal{O}_B \quad (5.1)$$

such that  $\langle \psi_\xi : \mathbf{M}(\xi) \longrightarrow \Xi \text{ in } \mathbf{S}(\xi), \text{ and } \mathbf{M}(v) : \mathbf{M}(\xi') \rightarrow \mathbf{M}(\xi) \text{ in } \mathcal{O}_Q \text{ for } v : \xi' \rightarrow \xi \text{ in } \mathcal{O}_B, \text{ implies } \psi_\xi \circ \mathbf{M}(v) : \mathbf{M}(\xi') \longrightarrow \mathcal{O}_Q \text{ in } \mathbf{S}(\xi) \rangle$ .

We observe that the operational role of a Boolean information sieve, viz. of a subfunctor of the Hom-functor  $\mathbf{R}(\Xi)$  is tantamount to the depiction of an ideal of localizing morphisms acting as local coverings of a quantum observable by coordinatizing Boolean information points. We may characterize the morphisms  $\psi_\xi : \mathbf{M}(\xi) \longrightarrow \Xi, \quad \xi \in \mathcal{O}_B$  in a sieve of prelocalizations, as Boolean covers for the filtration of information associated with a quantum observable structure. Their domains  $B_\Xi$  provide Boolean coefficients, asso-

$$\begin{array}{ccc}
\text{Bor}(\mathbb{R}) & \xrightarrow{\xi} & \mathbf{M}(\acute{B}_{\Xi}) \\
\downarrow \xi & \searrow \Xi & \downarrow [\psi_{\acute{B}}]_{\Xi} \\
\mathbf{M}(B_{\Xi}) & \xrightarrow{[\psi_B]_{\Xi}} & L
\end{array}$$

Diagram 9

ciated with measurement situations according to Diagram 9.

The introduction of these systems is justified by the consequences of the Kochen-Specker theorem, according to which, it is not possible to understand completely a quantum mechanical system with the use of a single Boolean experimental arrangement. Equivalently, there are no two-valued homomorphisms on the algebra of quantum events, and thus, it cannot be embedded into a Boolean one. On the other side, in every concrete experimental context, the set of events that have been actualized in this context forms a Boolean algebra. Consequently, any Boolean domain object  $(B_{\Xi}, [\psi_B]_{\Xi} : \mathbf{M}(B_{\Xi}) \longrightarrow L)$  in a sieve of prelocalizations for a quantum event algebra, such that the diagram above commutes, corresponds to a set of Boolean events that become actualized in the experimental context of  $B$ . These Boolean objects play the role of Boolean information localizing devices in a quantum event structure, which are induced by local preparatory contexts of quantum measurement situations. The above observation is equivalent to the statement that a measurement-induced Boolean algebra serves as a reference frame, relative to which a measurement result is being coordinatized, in accordance to the informational specification of the corresponding localization context.

A family of Boolean covers  $\psi_{\xi} : \mathbf{M}(\xi) \longrightarrow \Xi$ ,  $\xi \in \mathcal{O}_B$  is the generator of a Boolean information sieve of prelocalization  $\mathbf{S}$ , if and only if, this sieve is

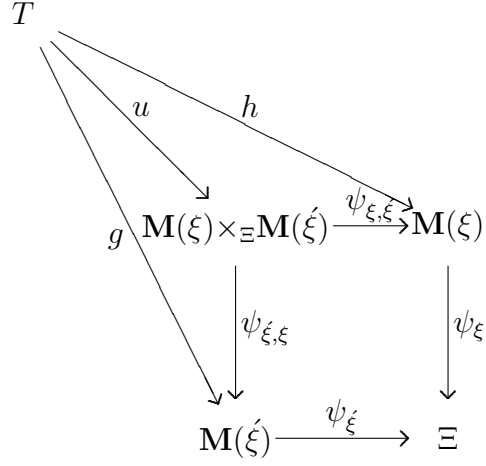
the smallest among all that contains that family. It is evident that a quantum observable, and correspondingly the quantum event algebra over which it is defined, may be covered by a multitude of Boolean information sieves of prelocalizations, that, significantly, form an ordered structure. More specifically, sieves of prelocalization constitute a partially ordered set under inclusion. The minimal sieve is the empty one, namely  $\mathbf{S}(\xi) = \emptyset$  for all  $\xi \in \mathcal{O}_B$ , whereas the maximal sieve is the Hom-functor  $\mathbf{R}(\Xi)$  itself, or equivalently, the set of all quantum algebraic homomorphisms  $\psi_\xi : \mathbf{M}(\xi) \longrightarrow \Xi$ .

### 5.3 Boolean Information Sieves of Localization

The transition from a sieve of prelocalizations to a Boolean information sieve of localizations for a quantum observable, is necessary for the compatibility of the information content gathered in different Boolean filtering mechanisms. A Boolean information sieve of localizations contains all the necessary and sufficient conditions for the representation of the information content of a quantum observable structure as a sheaf of Boolean coefficients associated with Boolean localization contexts. The notion of sheaf expresses exactly the pasting conditions that the local filtering devices have to satisfy, or else, the specification by which local data, providing Boolean coefficients obtained in measurement situations, can be collated.

In order to define a Boolean information sieve of localizations, it is necessary to explain the notion of pullback in the category  $\mathcal{O}_Q$ .

The pullback of the Boolean information filtering covers  $\psi_\xi : \mathbf{M}(\xi) \longrightarrow \Xi$ , where  $\xi \in \mathcal{O}_B$ , and  $\psi_{\xi'} : \mathbf{M}(\xi') \longrightarrow \Xi$ , where  $\xi' \in \mathcal{O}_B$ , with common codomain the quantum observable  $\Xi$ , consists of the object  $\mathbf{M}(\xi) \times_{\Xi} \mathbf{M}(\xi')$  and two arrows  $\psi_{\xi\xi'}$  and  $\psi_{\xi'\xi}$ , called projections, as shown in Diagram 10. The square commutes and for any object  $T$  and arrows  $h$  and  $g$  that make the outer



square commute, there is a unique  $u : T \longrightarrow \mathbf{M}(\xi) \times_{\Xi} \mathbf{M}(\xi')$  that makes the whole diagram commutative. Hence we obtain the condition:

$$\psi_{\xi'} \circ g = \psi_{\xi} \circ h \quad (5.2)$$

We emphasize that if  $\psi_{\xi}$  and  $\psi_{\xi'}$  are injective maps, then their pullback is isomorphic with the intersection  $\mathbf{M}(\xi) \cap \mathbf{M}(\xi')$ . Then we can define the pasting map, which is an isomorphism, as follows:

$$\Omega_{\xi, \xi'} : \psi_{\xi \xi'}(\mathbf{M}(\xi) \times_{\Xi} \mathbf{M}(\xi')) \longrightarrow \psi_{\xi \xi'}(\mathbf{M}(\xi) \times_{\Xi} \mathbf{M}(\xi')) \quad (5.3)$$

by putting

$$\Omega_{\xi, \xi'} = \psi_{\xi \xi'} \circ \psi_{\xi \xi'}^{-1} \quad (5.4)$$

Then we have the following cocycle conditions:

$$\Omega_{\xi, \xi} = 1_{\xi} \quad 1_{\xi} := id_{\xi} \quad (5.5)$$

$$\Omega_{\xi, \xi'} \circ \Omega_{\xi', \xi''} = \Omega_{\xi, \xi''} \quad \text{if } \mathbf{M}(\xi) \cap \mathbf{M}(\xi') \cap \mathbf{M}(\xi'') \neq \emptyset \quad (5.6)$$

$$\Omega_{\xi, \xi'} = \Omega_{\xi', \xi}^{-1} \quad \text{if } \mathbf{M}(\xi) \cap \mathbf{M}(\xi') \neq \emptyset \quad (5.7)$$

The pasting map assures that  $\psi_{\xi\xi}(\mathbf{M}(\xi) \times_{\Xi} \mathbf{M}(\xi'))$  and  $\psi_{\xi\xi'}(\mathbf{M}(\xi) \times_{\Xi} \mathbf{M}(\xi'))$  cover the same part of the informational content of a quantum observable in a compatible way.

Given a sieve of prelocalizations for quantum observable  $\Xi \in \mathcal{O}_Q$ , and correspondingly for the quantum event algebra over which it is defined, it is called a **Boolean information sieve of localizations**, if and only if, the above compatibility conditions are satisfied.

We assert that the above compatibility conditions provide the necessary relations for understanding a Boolean information sieve of localizations for a quantum observable, as a sheaf of Boolean coefficients representing the information encoded in local Boolean observables. In essence, the pullback compatibility conditions express gluing relations on overlaps of Boolean domain information covers. The concept of sheaf (Mac Lane and Moerdijk 1992, Bredon 1997, Mallios 1998) expresses exactly the amalgamation conditions that local coordinatizing Boolean points have to satisfy, namely, the way by which local data, providing Boolean coefficients obtained in measurement situations, can be collated globally.

In this sense, the specification of a Boolean information sieve of localization, as a sheaf of Boolean coefficients associated with the variation of the information obtained in multiple Boolean localization contexts, permits the conception of a quantum observable (or of its associated quantum event algebra) as a global manifestation of local Boolean observable information collation, obtained by pasting the  $\psi_{\xi\xi}(\mathbf{M}(\xi) \times_{\Xi} \mathbf{M}(\xi'))$  and  $\psi_{\xi\xi'}(\mathbf{M}(\xi) \times_{\Xi} \mathbf{M}(\xi'))$  covers together by the transition functions  $\Omega_{\xi, \xi'}$ .



## 6 Conditions for Invariant Functorial Information Transfer

The interpretational framework for the comprehension of functorial information transfer, as established by the adjunctive correspondence between preheaves of Boolean localization coefficients, associated with information filtering contexts of observation, and, quantum observable structures, can be completed by the formulation of a property characterizing the conditions for invariance of the information transferred in the totality of Boolean localization environments.

The existence of this invariance property is equivalent to the representation of quantum observables and their associated quantum event algebras, in terms of Boolean information sieves, capable of encoding the whole informational content included in a quantum structure. The intended representation can be obtained from the established adjunction itself as follows:

Every categorical adjunction is completely characterized by the unit and counit natural transformations (Mac Lane and Moerdijk 1992). For the adjunctive correspondence between presheaves of Boolean observables and quantum observables the unit and counit morphisms are defined as follows:

For any presheaf  $\mathbf{P} \in \mathbf{Sets}^{\mathcal{O}_B^{op}}$ , the unit is defined as

$$\delta_{\mathbf{P}} : \mathbf{P} \longrightarrow \mathbf{RLP} \quad (6.1)$$

On the other side, for each quantum observable  $\Xi$  of  $\mathcal{O}_Q$  the counit is

$$\epsilon_{\Xi} : \mathbf{LR}(\Xi) \longrightarrow \Xi \quad (6.2)$$

The counit corresponds to the vertical morphism in Diagram 11.

Diagram 11 has been obtained by the categorical representation of the colimit in the category of elements of the functor  $\mathbf{R}(\Xi)$  as a coequalizer of

$$\begin{array}{ccc}
\coprod_{v:\acute{\xi}\rightarrow\xi}\mathbf{M}(\acute{\xi}) & \xrightarrow[\eta]{\zeta} & \coprod_{(\xi,p)}\mathbf{M}(\xi) \longrightarrow \mathbf{LR}(\Xi) \\
& & \searrow \downarrow \epsilon_{\Xi} \\
& & \Xi
\end{array}$$

Diagram 11

coproduct (Mac Lane and Moerdijk 1992). More specifically, in the coequalizer representation of the colimit, the second coproduct is over all the objects  $(\xi, p)$  with  $p \in \mathbf{R}(\Xi)(\xi)$  of the category of elements, while the first coproduct is over all the maps  $v : (\acute{\xi}, \acute{p}) \longrightarrow (\xi, p)$  of that category, so that  $v : \acute{\xi} \longrightarrow \xi$  and the condition  $p \cdot v = \acute{p}$  is satisfied.

In general, by means of that representation, we can show that the left adjoint functor of the adjunction is like the tensor product  $-\otimes_{\mathcal{O}_B}\mathbf{M}$ . More specifically, using the coequalizer representation of the colimit  $\mathbf{LX}$  we can easily show that the elements of  $\mathbf{X}\otimes_{\mathcal{O}_B}\mathbf{M}$ , considered as a set endowed with a quantum algebraic structure, are all of the form  $\chi(p, q)$ , or in a suggestive notation,

$$\chi(p, q) = p \otimes q, \quad p \in \mathbf{X}(\xi), q \in \mathbf{M}(\xi) \quad (6.3)$$

satisfying the coequalizer condition  $pv \otimes \acute{q} = p \otimes v\acute{q}$ .

From Diagram 11, it is clear that the representation of a quantum observable  $\Xi$  in  $\mathcal{O}_Q$ , and thus, of a quantum event algebra  $L$  in  $\mathcal{L}$ , in terms of a Boolean information sieve of localizations, is full and faithful, if and only if the counit of the established adjunction, restricted to this sieve, is an isomorphism, that is, structure-preserving, injective and surjective.

The physical significance of this representation lies on the following proposition:

**Quantum Information Preservation Principle :** The whole information content enfolded in a quantum observable structure, is preserved by some covering Boolean system, if and only if, that system forms a Boolean information sieve of localizations.

The preservation principle is established by the counit isomorphism. It is remarkable, that the categorical notion of adjunction provides the appropriate formal tool for the formulation of invariant properties, giving rise to preservation principles of a physical character.

Concerning the representation above, we realize that the surjective property of the counit guarantees that the Boolean information filtering mechanisms, being themselves objects in the category of elements,  $\int(\mathbf{R}(L), B)$ , cover entirely the quantum event algebra  $L$ , whereas its injective property, guarantees that any two information filters are compatible in a sieve of localizations. Moreover, since the counit is also a homomorphism, the algebraic structure is preserved.

We observe that each Boolean filtering device gives rise to a set of Boolean events actualized locally in a measurement situation. The equivalence classes of Boolean events represent quantum events in  $L$ , through compatible coordinatizations by Boolean coefficients. Consequently, the structure of a quantum event algebra is being generated by the information carried from its structure preserving morphisms, encoded as Boolean information filters in localization sieves, together with their compatibility relations.

We may clarify that the underlying invariance property specified by the adjunction is associated with the informational content of all these, different or overlapping information filtering mechanisms in various Boolean localization contexts, and can be explicitly formulated as follows:

**Invariance Property:** The information content of a quantum observable structure remains invariant, with respect to measurement contexts of Boolean coordinatizations, if and only if, the counit of the adjunction, restricted to covering systems, qualified as Boolean information sieves of localizations, is an isomorphism.

In turn, the counit of the adjunction, restricted to a Boolean information sieve of localizations is an isomorphism, if and only if the right adjoint functor is full and faithful, or equivalently, if and only if the cocone from the functor  $\mathbf{M} \circ \int_{\mathbf{R}(\Xi)}$  to the quantum observable  $\Xi$  is universal for each object  $\Xi$  in  $\mathcal{O}_Q$  (Mac Lane and Moerdijk 1992). In the latter case we characterize the functor  $\mathbf{M} : \mathcal{O}_B \longrightarrow \mathcal{O}_Q$ , a proper functor of Boolean coefficients.

From a physical perspective, we conclude that the counit isomorphism, provides a categorical equivalence, signifying an invariance in the translational code of communication between Boolean information filtering contexts, acting as localization devices for measurement, and quantum systems.

## 7 Cohomological Criterion of Functoriality

We have reached the conclusion that if the right adjoint functor of the adjunction is a full and faithful functor, then the counit is an isomorphism and conversely. In this case, we have established a functoriality property, referring to invariant information transfer between quantum observables algebras and Boolean information sieves of localizations. In this Section, we are going to establish a cohomological criterion elucidating that functoriality property. For this purpose, we consider the counit of the adjunction, expressed in terms of the quantum event algebra over which observables are defined;

$$\epsilon_L : \mathbf{G}L := \mathbf{L}\mathbf{R}(L) = \mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M} \longrightarrow L \quad (7.1)$$

$$\begin{array}{ccc}
& \mathbf{R}(L) \otimes_B \mathbf{M} & \\
& \uparrow \psi_B \otimes [-] & \searrow \epsilon_L \\
\mathbf{M}(B) & \xrightarrow{\psi_B} & L \\
& \uparrow \xi & \nearrow \Xi \\
& \text{Bor}(\mathbb{R}) & 
\end{array}$$

Diagram 12

$$L \xleftarrow{\epsilon_0 := \epsilon} \mathbf{G}L \xleftarrow[\epsilon_1]{\epsilon_0} \mathbf{G}^2L \xleftarrow{\epsilon_{0,1,2}} \dots \xleftarrow{\epsilon_{0,1,\dots,n-1}} \mathbf{G}^nL \xleftarrow{\epsilon_{0,1,\dots,n}} \mathbf{G}^{n+1}L \dots$$

Diagram 13

such that Diagram 12 commutes. The counit  $\epsilon_L : \mathbf{G}L \rightarrow L$  is the first step of a functorial free resolution of an object  $L$  in  $\mathcal{L}$ . Thus, by iteration of  $\mathbf{G}$ , we may extend  $\epsilon_L$  to a free simplicial resolution of  $L$  in  $\mathcal{L}$ , denoted by  $\mathbf{G}_\star L \rightarrow L$ , according to Diagram 13. In the simplicial resolution represented by Diagram 13,  $\epsilon_{0,1,2}$  denotes a triplet of arrows etc. Notice that,  $\mathbf{G}^{n+1}$  is the term of degree  $n$ , whereas the face operator  $\epsilon_i : \mathbf{G}^{n+1} \rightarrow \mathbf{G}^n$  is  $\mathbf{G}^i \circ \epsilon \circ \mathbf{G}^{n-i}$ , where  $0 \leq i \leq n$ . We can verify the following simplicial identities;

$$\epsilon_i \circ \epsilon_j = \epsilon_{j+1} \circ \epsilon_i \tag{7.2}$$

where  $i \leq j$ . The resolution  $\mathbf{G}_\star L \rightarrow L$  induces obviously a resolution in the comma category  $[\mathcal{L}/L]$ , which we still denote by  $\mathbf{G}_\star L \rightarrow L$ .

Now, having at our disposal the resolution  $\mathbf{G}_\star L \rightarrow L$ , it is possible to define the cohomology groups  $\tilde{\mathbf{H}}^n(L, X_L)$ ,  $n \geq 0$ , of a quantum event algebra

$L$  in  $\mathcal{L}$  with coefficients in an  $L$ -module  $X_L$ , relative to the given underlying functor of points  $\mathbf{R} : \mathcal{L} \rightarrow \mathbf{Sets}^{\mathcal{B}^{op}}$ , defined by  $\mathbf{R}(L) : B \mapsto \text{Hom}_{\mathcal{L}}(\mathbf{M}(B), L)$ , having a left adjoint  $\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \rightarrow \mathcal{L}$ .

First of all we define the notion of an  $L$ -module  $X_L$  by the requirement that it is equivalent to an abelian group object in the comma category  $[\mathcal{L}/L]$ . This follows from the general definition of categorical modules introduced in (Barr and Beck 1966, Beck 1956) according to which: Let  $\mathcal{Y}$  be a category and let  $Y$  be an object in  $\mathcal{Y}$ . Then the category of  $\mathbf{Mod}(Y)$  is the category of abelian group objects in the comma category  $[\mathcal{Y}/Y]$ . This definition is appropriate for the kind of module that is a coefficient module for cohomology. For the interested reader we have included an appendix which contains an exposition of the relevant details for the case of commutative rings, as well as its functionality for setting up the derivations functor, reproducing well-known algebraic results.

Since  $X_L$  can be characterized as an abelian group object in  $[\mathcal{L}/L]$ , the set  $\text{Hom}_{\mathcal{L}}(L, X_L)$  has an abelian group structure for every object  $L$  in  $\mathcal{L}$ , and moreover, for every arrow  $\acute{L} \rightarrow L$  in  $\mathcal{L}$ , the induced map of sets  $\text{Hom}_{\mathcal{L}}(L, X_L) \rightarrow \text{Hom}_{\mathcal{L}}(\acute{L}, X_L)$  is a map of abelian groups.

Under the above specifications, an  $n$ -cochain of a quantum event algebra  $L$  with coefficients in an  $L$ -module  $X_L$ , where, by definition,  $X_L$  is an object in  $[\mathcal{L}/L]_{\mathbf{Ab}}$ , is defined as a map  $\mathbf{G}^{n+1}L \rightarrow \Upsilon_L(X_L)$  in the comma category  $[\mathcal{L}/L]$ , where;

$$\Upsilon_L : [\mathcal{L}/L]_{\mathbf{Ab}} \hookrightarrow [\mathcal{L}/L] \quad (7.3)$$

denotes the corresponding inclusion functor of abelian group objects. Furthermore, we define the derivations functor from the comma category  $\mathcal{L}/L$  to the category of abelian groups  $\mathbf{Ab}$ :

$$\mathbf{Der}(-, X_L) : \mathcal{L}/L \rightarrow \mathbf{Ab} \quad (7.4)$$

$$\begin{array}{ccccccc}
0 & \xrightarrow{d^0} & \mathbf{Der}(\mathbf{G}L, X_L) & \xrightarrow{d^1} & \mathbf{Der}(\mathbf{G}^2L, X_L) & & \\
& & & & & & \\
& & \xrightarrow{d^2} & \dots & \xrightarrow{d^n} & \mathbf{Der}(\mathbf{G}^{n+1}L, X_L) & \xrightarrow{d^{n+1}} \dots
\end{array}$$

Diagram 14

where  $X_L$  is an  $L$ -module, or equivalently, an abelian group object in  $\mathcal{L}/L$ , by the following requirement: If  $K : E \rightarrow L$  is an object of  $\mathcal{L}/L$ , then we have the isomorphism:

$$\mathbf{Der}(E, X_L) \cong \mathit{Hom}_{\mathcal{L}/L}(E, \Upsilon_L(X_L)) \quad (7.5)$$

Thus, we may finally identify the set of  $n$ -cochains with the abelian group of derivations of  $\mathbf{G}^{n+1}L$  into the abelian group object  $X_L$  in  $[\mathcal{L}/L]_{\mathbf{Ab}}$ . Hence, we consider an  $n$ -cochain as a derivation map  $\mathbf{G}^{n+1}L \rightarrow X_L$ .

Consequently, the face operators  $\epsilon_i$ , induce abelian group morphisms;

$$\mathbf{Der}(\epsilon_i L, X_L) : \mathbf{Der}(\mathbf{G}^n L, X_L) \rightarrow \mathbf{Der}(\mathbf{G}^{n+1} L, X_L) \quad (7.6)$$

Thus, the cohomology can be established by application of the contravariant functor  $\mathbf{Der}(-, X_L)$  on the free simplicial resolution of a quantum event algebra  $L$  in  $\mathcal{L}$ , obtaining the cochain complex of abelian groups represented by Diagram 14, where because of the aforementioned simplicial identities we have:

$$d^{n+1} = \sum_i (-1)^i \mathbf{Der}(\epsilon_i L, X_L) \quad (7.7)$$

where  $0 \leq i \leq n+1$ , and also;

$$d^{n+1} \circ d^n = 0 \quad (7.8)$$

written symbolically as;

$$d^2 = 0 \quad (7.9)$$

Now, we define the cohomology groups  $\tilde{\mathbf{H}}^n(L, X_L)$ ,  $n \geq 0$ , of a quantum event algebra  $L$  in  $\mathcal{L}$  with coefficients in an  $L$ -module  $X_L$  as follows:

$$\tilde{\mathbf{H}}^n(L, X_L) := \mathbf{H}^n[\mathbf{Der}(\mathbf{G}_*L, X_L)] = \frac{Ker(d^{n+1})}{Im(d^n)} \quad (7.10)$$

Notice that we may construct the  $L$ -module  $X_L$  by considering the abelian group object in the comma category  $[\mathcal{L}/L]$ , that corresponds to a quantum observable  $\Xi = \psi_B \circ \xi : Bor(\mathbf{R}) \rightarrow L$ , where,  $\xi = Bor(\mathbf{R}) \rightarrow \mathbf{M}(B)$ , and  $\psi_B : \mathbf{M}(B) \rightarrow L$ . The cohomology groups  $\tilde{\mathbf{H}}^n(L, X_L)$ , express obstructions to the preservation of the information content of a quantum event algebra with respect to measurement contexts of Boolean coordinatizations. It is clear that if the counit of the adjunction  $\epsilon_L : \mathbf{G}L \rightarrow L$  is an isomorphism, then the cohomology groups vanish at all orders and conversely.

Finally, it is instructive to connect the cohomological analysis presented above, with the exactness properties of the right adjoint functor of the adjunction. We remind that, if the right adjoint functor of the adjunction is a full and faithful functor, then the counit is an isomorphism and conversely. For this purpose we define an  $L$ -module  $\Omega_L$ , called suggestively a module of quantum 1-forms, by means of the following split short exact sequence:

$$0 \rightarrow J \rightarrow \mathbf{R}(L) \otimes_B \mathbf{M} \rightarrow L \quad (7.11)$$

where  $J = \mathbf{Ker}(\epsilon_L)$  denotes the kernel of the counit of the adjunction, in case that, the right adjoint is not a full and faithful functor. According to the above, we define the  $L$ -module  $\Omega_L$  as follows :

$$\Omega_L = \frac{J}{J^2} \quad (7.12)$$

In this setting, we notice that the functor of points of a quantum event algebra restricted to Boolean points, viz.,  $\mathbf{R}(L)$ , is a left exact functor, because it is the right adjoint functor of the established adjunction (MacLane 1971).



Thus, it preserves the short exact sequence defining the object of quantum 1-forms, in the following form:

$$0 \rightarrow \mathbf{R}(J) \rightarrow \mathbf{R}(\mathbf{G}(L)) \rightarrow \mathbf{R}(L) \quad (7.13)$$

Hence, we immediately obtain that:  $\mathbf{R}(\Omega_L) = \frac{Z}{Z^2}$ , where  $Z = \mathbf{Ker}(\mathbf{R}(\epsilon_L))$ .

Then, we introduce the notion of a functorial quantum connection, denoted by  $\nabla_{\mathbf{R}(L)}$ , in terms of the following natural transformation:

$$\nabla_{\mathbf{R}(L)} : \mathbf{R}(L) \rightarrow \mathbf{R}(\Omega_L) \quad (7.14)$$

Thus, the quantum connection  $\nabla_{\mathbf{R}(L)}$  induces a sequence of natural transformations as follows:

$$\mathbf{R}(L) \rightarrow \mathbf{R}(\Omega_L) \rightarrow \dots \rightarrow \mathbf{R}(\Omega^n_L) \rightarrow \dots \quad (7.15)$$

Let us denote by;

$$\mathbf{R}_{\nabla} : \mathbf{R}(L) \rightarrow \mathbf{R}(\Omega^2_L) \quad (7.16)$$

the composition  $\nabla^1 \circ \nabla^0$  in the obvious notation, where  $\nabla^0 := \nabla_{\mathbf{R}(L)}$ . The natural transformation  $\mathbf{R}_{\nabla}$  is called the curvature of the functorial quantum connection  $\nabla_{\mathbf{R}(L)}$ . Furthermore, the latter sequence of natural transformations, is actually a complex if and only if  $\mathbf{R}_{\nabla} = 0$ . We say that the quantum connection  $\nabla_{\mathbf{R}(L)}$  is integrable or flat if  $\mathbf{R}_{\nabla} = 0$ , referring to the above complex as the functorial de Rham complex of the integrable connection  $\nabla_{\mathbf{R}(L)}$  in that case. In this setting, a non-vanishing curvature  $\nabla_{\mathbf{R}(L)}$  is understood as the geometric effect being caused by cohomological obstructions, that prevent the above sequence of natural transformations from being a complex. Thus we arrive at the conclusion that a non-vanishing curvature  $\nabla_{\mathbf{R}(L)}$ , in case that, the right adjoint is not a full and faithful functor, prevents integrability of information transfer from quantum event algebras to Boolean information sieves of localizations.

## 8 Conclusions

We have proposed a sheaf-theoretic representation of quantum event algebras and quantum observables, by means of Boolean information sieves of localization. According to this schema of interpretation, quantum information structures are being understood by means of overlapping Boolean reference frames for the measurement of observables, being pasted together by sheaf-theoretic means. The proposed schema has been formalized categorically, as an instance of the adjunction concept. Moreover, the latter has been also used for the formulation of the physically important notions of integrability and invariance pertaining to information transfer from quantum events algebras to Boolean coordinatization systems. These notions have been technically implemented using the counit of the established adjunction, as well as, its iterations forcing a free simplicial resolutions of a quantum event algebra, by cohomological means. Conclusively, it has been demonstrated that:

[i]. The information transfer from quantum events algebras to Boolean coordinatization systems is integrable if the curvature of the functorial quantum connection  $\nabla_{\mathbf{R}(L)}$  vanishes, viz.,  $\mathbf{R}_{\nabla} = 0$ .

[ii]. The information content of a quantum observable structure remains invariant, with respect to measurement contexts of Boolean coordinatizations, if and only if, the counit of the adjunction, restricted to covering systems, qualified as Boolean information sieves of localizations, is an isomorphism. The latter property is equivalent to the triviality of the cohomology groups  $\tilde{\mathbf{H}}^n(L, X_L)$ , meaning the absence of obstructions to gluing information globally among Boolean measurement contexts.

The physical significance of the sheaf-theoretic representation boils down to the proof that, the totality of the content of information included in a quantum observable structure, is functorially preserved by some covering

Boolean system, if and only if, that system forms a Boolean information sieve of localizations, such that, the counit of the adjunction is an isomorphism. In this perspective, efficient handling of quantum information becomes precisely the area of application of the core relationship between quantum observables and interconnected localized Boolean information resources, bypassing in this manner, the global classical information encoding limits.

## A Categorical Modules and Derivations

The first basic objective of the categorical perspective on abstract differential calculus is to express the notions of modules and derivations of a commutative unital ring  $\mathcal{B}$  in  $\mathfrak{B}$ , where  $\mathfrak{B}$  denotes the category of commutative unital rings of scalars, intrinsically with respect to the information contained in the category  $\mathfrak{B}$ . This can be accomplished by using the method of categorical relativization, which is based on the passage to the comma category  $\mathfrak{B}/\mathcal{B}$ . More concretely, the basic problem has to do with the possibility of representing the information contained in an  $\mathcal{B}$ -module, where  $\mathcal{B}$  is a commutative unital ring in  $\mathfrak{B}$ , with a suitable object of the relativization of  $\mathfrak{B}$  with respect to  $\mathcal{B}$ , viz., with an object of the comma category  $\mathfrak{B}/\mathcal{B}$ . For this purpose, we define the split extension of the commutative ring  $\mathcal{B}$  by an  $\mathcal{B}$ -module  $M$ , denoted by  $\mathcal{B} \oplus M$ , as follows: The underlying set of  $\mathcal{B} \oplus M$  is the cartesian product  $\mathcal{B} \times M$ , where the group and ring theoretic operations are defined respectively as;

$$(a, m) + (b, n) := (a + b, m + n)$$

$$(a, m) \bullet (b, n) := (ab, a \cdot n + b \cdot m)$$

Notice that the identity element of  $\mathcal{B} \oplus M$  is  $(1_{\mathcal{B}}, 0_M)$ , and also that, the split extension  $\mathcal{B} \oplus M$  contains an ideal  $0_{\mathcal{B}} \times M := \langle M \rangle$ , that corresponds naturally to the  $\mathcal{B}$ -module  $M$ . Thus, given a commutative ring  $\mathcal{B}$  in  $\mathfrak{B}$ , the information of an  $\mathcal{B}$ -module  $M$ , consists of an object  $\langle M \rangle$  (ideal in  $\mathcal{B} \oplus M$ ), together with a split short exact sequence in  $\mathfrak{B}$ ;

$$\langle M \rangle \hookrightarrow \mathcal{B} \oplus M \rightarrow \mathcal{B}$$

We infer that the ideal  $\langle M \rangle$  is identified with the kernel of the epimorphism  $\mathcal{B} \oplus M \rightarrow \mathcal{B}$ , viz.,

$$\langle M \rangle = \text{Ker}(\mathcal{B} \oplus M \rightarrow \mathcal{B})$$

From now on we focus our attention to the comma category  $\mathfrak{B}/\mathcal{B}$ , noticing that  $id_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{B}$  is the terminal object in this category. If we consider the split extension of the commutative ring  $\mathcal{B}$ , by an  $\mathcal{B}$ -module  $M$ , that is  $\mathcal{B} \oplus M$ , then the morphism:

$$\lambda : \mathcal{B} \oplus M \rightarrow \mathcal{B}$$

$$(a, m) \mapsto a$$

is obviously an object of  $\mathfrak{B}/\mathcal{B}$ . Moreover, it is easy to show that it is actually an abelian group object in the comma category  $\mathfrak{B}/\mathcal{B}$ . This equivalently means that for every object  $\xi$  in  $\mathfrak{B}/\mathcal{B}$  the set of morphisms  $Hom_{\mathfrak{B}/\mathcal{B}}(\xi, \lambda)$  is an abelian group in **Sets**. Moreover, the arrow  $\gamma : \kappa \rightarrow \lambda$  is a morphism of abelian groups in  $\mathfrak{B}/\mathcal{B}$  if and only if for every  $\xi$  in  $\mathfrak{B}/\mathcal{B}$  the morphism;

$$\hat{\gamma}_{\xi} : Hom_{\mathfrak{B}/\mathcal{B}}(\xi, \kappa) \rightarrow Hom_{\mathfrak{B}/\mathcal{B}}(\xi, \lambda)$$

is a morphism of abelian groups in **Sets**. We denote the category of abelian group objects in  $\mathfrak{B}/\mathcal{B}$  by the suggestive symbol  $[\mathfrak{B}/\mathcal{B}]_{\mathbf{Ab}}$ . Based on our previous remarks, it is straightforward to show that the category of abelian group objects in  $\mathfrak{B}/\mathcal{B}$  is equivalent with the category of  $\mathcal{B}$ -modules, viz.:

$$[\mathfrak{B}/\mathcal{B}]_{\mathbf{Ab}} \cong \mathcal{M}^{(\mathcal{B})}$$

Thus, we have managed to characterize intrinsically  $\mathcal{B}$ -modules as abelian group objects in the relativization of the category of commutative unital rings  $\mathcal{B}$  with respect to  $\mathcal{B}$ , and moreover, we have concretely identified them as kernels of split extensions of  $\mathcal{B}$ .

The characterization of  $\mathcal{B}$ -modules as abelian group objects in the comma category  $\mathfrak{B}/\mathcal{B}$  is particularly useful if we consider an  $\mathcal{B}$ -module  $M$  as a codomain for derivations of objects of  $\mathfrak{B}/\mathcal{B}$ . For this purpose, let us initially

notice that if  $k : \mathcal{A} \rightarrow \mathcal{B}$  is an arbitrary object in  $\mathfrak{B}/\mathcal{B}$ , then any  $\mathcal{B}$ -module  $M$  is also an  $\mathcal{A}$ -module via the morphism  $k$ . We define a derivations functor from the comma category  $\mathfrak{B}/\mathcal{B}$  to the category of abelian groups  $\mathbf{Ab}$ :

$$\mathbf{Der}(-, M) : \mathfrak{B}/\mathcal{B} \rightarrow \mathbf{Ab}$$

Then, if we evaluate the derivations functor at the commutative arithmetic  $\mathcal{A}$  we obtain:

$$\mathbf{Der}(\mathcal{A}, M) \cong \mathit{Hom}_{\mathfrak{B}/\mathcal{B}}(\mathcal{A}, \mathcal{B} \oplus M)$$

This means that, given an object  $k : \mathcal{A} \rightarrow \mathcal{B}$  in  $\mathfrak{B}/\mathcal{B}$ , then a derivation  $d : \mathcal{A} \rightarrow M$  is the same as the following morphism in  $\mathfrak{B}/\mathcal{B}$ :

$$\begin{array}{ccc} & \mathcal{B} & \\ & \nearrow & \nwarrow \\ \mathcal{A} & & \mathcal{B} \oplus M \\ & \xrightarrow{\tilde{d}} & \end{array}$$

Now we notice that the morphism:  $\lambda : \mathcal{B} \oplus M \rightarrow \mathcal{B}$  is actually an object in  $[\mathfrak{B}/\mathcal{B}]_{\mathbf{Ab}}$ . Hence, we consider it as an object of  $[\mathfrak{B}/\mathcal{B}]$  via the action of an inclusion functor:

$$\begin{aligned} \Upsilon_{\mathcal{B}} : [\mathfrak{B}/\mathcal{B}]_{\mathbf{Ab}} &\hookrightarrow [\mathfrak{B}/\mathcal{B}] \\ [\lambda : \mathcal{B} \oplus M \rightarrow \mathcal{B}] &\mapsto [\Upsilon_{\mathcal{B}}(\lambda) : \Upsilon_{\mathcal{B}}(M) \rightarrow \mathcal{B}] \end{aligned}$$

Thus we obtain the isomorphism:

$$\mathbf{Der}(\mathcal{A}, M) \cong \mathit{Hom}_{\mathfrak{B}/\mathcal{B}}(\mathcal{A}, \Upsilon_{\mathcal{B}}(M))$$

The inclusion functor  $\Upsilon_{\mathcal{B}}$  has a left adjoint functor;

$$\Omega_{\mathcal{B}} : [\mathfrak{B}/\mathcal{B}] \rightarrow [\mathfrak{B}/\mathcal{B}]_{\mathbf{Ab}}$$

Consequently, if we further take into account the equivalence of categories

$$[\mathfrak{B}/\mathcal{B}]_{\mathbf{Ab}} \cong \mathcal{M}^{(\mathcal{B})}$$

the above isomorphism takes the following final form:

$$\mathbf{Der}(\mathcal{A}, M) \cong \text{Hom}_{\mathcal{M}^{(\mathcal{B})}}(\Omega_{\mathcal{B}}(\mathcal{A}), M)$$

We conclude that the derivations functor  $\mathbf{Der}(-, M) : \mathfrak{B}/\mathcal{B} \rightarrow \mathbf{Ab}$  is being represented by the abelianization functor  $\Omega_{\mathcal{B}} : [\mathfrak{B}/\mathcal{B}] \rightarrow [\mathfrak{B}/\mathcal{B}]_{\mathbf{Ab}}$ . Furthermore, the evaluation of the abelianization functor  $\Omega_{\mathcal{B}}$  at an object  $k : \mathcal{A} \rightarrow \mathcal{B}$  of  $\mathfrak{B}/\mathcal{B}$ , viz.  $\Omega_{\mathcal{B}}(\mathcal{A})$ , is interpreted as the  $\mathcal{B}$ -module of differentials on  $\mathcal{A}$ . Finally, it is straightforward to see that, evaluating at the terminal object of  $\mathfrak{B}/\mathcal{B}$  we obtain:

$$\mathbf{Der}(\mathcal{B}, M) \cong \text{Hom}_{\mathcal{M}^{(\mathcal{B})}}(\Omega_{\mathcal{B}}(\mathcal{B}), M)$$

This means that the covariant functor of  $\mathcal{B}$ -modules valued derivations of  $\mathcal{B}$ , denoted by  $\widehat{\mathbf{Der}}(\mathcal{B}, -)$ , is being representable by the free  $\mathcal{B}$ -module of differential 1-forms of  $\mathcal{B}$ , denoted by  $\Omega_{\mathcal{B}} := \Omega^1_{\mathcal{B}}$  in the category of  $\mathcal{B}$ -modules, according to the isomorphism:

$$\widehat{\mathbf{Der}}(\mathcal{B}, M) \cong \text{Hom}_{\mathcal{M}^{(\mathcal{B})}}(\Omega_{\mathcal{B}}, M)$$

Furthermore, if  $\mathcal{B}$  is a  $\mathcal{C}$ -algebra then, the covariant functor of  $\mathcal{B}$ -modules  $\mathcal{C}$ -valued derivations of  $\mathcal{B}$ , denoted by  $\widehat{\mathbf{Der}}_{\mathcal{C}}(\mathcal{B}, -)$ , is being representable by the free  $\mathcal{B}$ -module of differential 1-forms of  $\mathcal{B}$  over  $\mathcal{C}$ , denoted by  $\Omega_{\mathcal{B}/\mathcal{C}} := \Omega^1_{\mathcal{B}/\mathcal{C}}$  in the category of  $\mathcal{B}$ -modules, according to the isomorphism:

$$\widehat{\mathbf{Der}}_{\mathcal{C}}(\mathcal{B}, M) \cong \text{Hom}_{\mathcal{M}^{(\mathcal{B})}}(\Omega_{\mathcal{B}/\mathcal{C}}, M)$$

Hence, in general if  $\mathcal{B}$  is a  $\mathcal{C}$ -algebra the object  $\Omega^1_{\mathcal{B}/\mathcal{C}}$  is characterized categorically as the universal object of relative differential 1-forms in  $\mathcal{M}^{(\mathcal{B})}$  and the derivation  $d_{\mathcal{B}/\mathcal{C}} : \mathcal{B} \rightarrow \Omega^1_{\mathcal{B}/\mathcal{C}}$  as the universal derivation.

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