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Boolean Properties of Sets

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Summary. The text includes a number of theorems about Boolean operations on sets: union, intersection, difference, symmetric difference; and relations on sets: meets (having non-empty intersection), misses (being disjoint) and subset (inclusion).

The terminology and notation used here are introduced in the article [1]. For simplicity we adopt the following convention: x will have the type Any; X, Y, Z, V will have the type set. The scheme *Separation* concerns a constant $\mathcal{A}$ that has the type set and a unary predicate $\mathcal{P}$ and states that the following holds

$$\textbf{ex}\, X \textbf{ st for}\, x \textbf{ holds}\, x \in X \textbf{ iff}\, x \in \mathcal{A} \;\&\; \mathcal{P}[x]$$

for all values of the parameters.

We now define several new constructions. The constant $\emptyset$ has the type set, and is defined by

$$\textbf{not ex}\, x \textbf{ st}\, x \in \textbf{it}\,.$$

Let us consider X, Y. The functor

$$X \cup Y,$$

with values of the type set, is defined by

$$x \in \textbf{it iff}\, x \in X \textbf{ or}\, x \in Y.$$

The functor

$$X \cap Y,$$

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with values of the type set, is defined by

$$x \in \mathbf{it}\ \mathbf{iff}\ x \in X\ \&\ x \in Y.$$

The functor

$$X \setminus Y,$$

yields the type set and is defined by

$$x \in \mathbf{it}\ \mathbf{iff}\ x \in X\ \&\ \mathbf{not}\ x \in Y.$$

The predicate

$$X \text{ meets } Y \qquad \text{is defined by} \qquad \mathbf{ex}\ x\ \mathbf{st}\ x \in X\ \&\ x \in Y.$$

The predicate

$$X \text{ misses } Y \qquad \text{is defined by} \qquad \mathbf{for}\ x\ \mathbf{holds}\ x \in X\ \mathbf{implies}\ \mathbf{not}\ x \in Y.$$

Let us consider X, Y. The functor

$$X \dot{-} Y,$$

with values of the type set, is defined by

$$\mathbf{it} = (X \setminus Y) \cup (Y \setminus X).$$

We now state several propositions:

$$(1) \qquad Z = \emptyset\ \mathbf{iff}\ \mathbf{not}\ \mathbf{ex}\ x\ \mathbf{st}\ x \in Z,$$

$$(2) \qquad Z = X \cup Y\ \mathbf{iff}\ \mathbf{for}\ x\ \mathbf{holds}\ x \in Z\ \mathbf{iff}\ x \in X\ \mathbf{or}\ x \in Y,$$

$$(3) \qquad Z = X \cap Y\ \mathbf{iff}\ \mathbf{for}\ x\ \mathbf{holds}\ x \in Z\ \mathbf{iff}\ x \in X\ \&\ x \in Y,$$

$$(4) \qquad Z = X \setminus Y\ \mathbf{iff}\ \mathbf{for}\ x\ \mathbf{holds}\ x \in Z\ \mathbf{iff}\ x \in X\ \&\ \mathbf{not}\ x \in Y,$$

$$(5) \qquad X \subseteq Y\ \mathbf{iff}\ \mathbf{for}\ x\ \mathbf{holds}\ x \in X\ \mathbf{implies}\ x \in Y,$$

$$(6) \qquad X \text{ meets } Y\ \mathbf{iff}\ \mathbf{ex}\ x\ \mathbf{st}\ x \in X\ \&\ x \in Y,$$

$$(7) \qquad X \text{ misses } Y\ \mathbf{iff}\ \mathbf{for}\ x\ \mathbf{holds}\ x \in X\ \mathbf{implies}\ \mathbf{not}\ x \in Y.$$

Let us consider X, Y. Let us note that one can characterize the predicate

$$X = Y$$

by the following (equivalent) condition:

$$X \subseteq Y\ \&\ Y \subseteq X.$$

The following propositions are true:

$$(8) \qquad x \in X \cup Y\ \mathbf{iff}\ x \in X\ \mathbf{or}\ x \in Y,$$

(9) $$x \in X \cap Y \textbf{ iff } x \in X \ \& \ x \in Y,$$

(10) $$x \in X \setminus Y \textbf{ iff } x \in X \ \& \ \textbf{not } x \in Y,$$

(11) $$x \in X \ \& \ X \subseteq Y \textbf{ implies } x \in Y,$$

(12) $$x \in X \ \& \ X \text{ misses } Y \textbf{ implies not } x \in Y,$$

(13) $$x \in X \ \& \ x \in Y \textbf{ implies } X \text{ meets } Y,$$

(14) $$x \in X \textbf{ implies } X \neq \emptyset,$$

(15) $$X \text{ meets } Y \textbf{ implies ex } x \textbf{ st } x \in X \ \& \ x \in Y,$$

(16) $$(\textbf{for } x \textbf{ st } x \in X \textbf{ holds } x \in Y) \textbf{ implies } X \subseteq Y,$$

(17) $$(\textbf{for } x \textbf{ st } x \in X \textbf{ holds not } x \in Y) \textbf{ implies } X \text{ misses } Y,$$

(18) $$(\textbf{for } x \textbf{ holds } x \in X \textbf{ iff } x \in Y \textbf{ or } x \in Z) \textbf{ implies } X = Y \cup Z,$$

(19) $$(\textbf{for } x \textbf{ holds } x \in X \textbf{ iff } x \in Y \ \& \ x \in Z) \textbf{ implies } X = Y \cap Z,$$

(20) $$(\textbf{for } x \textbf{ holds } x \in X \textbf{ iff } x \in Y \ \& \ \textbf{not } x \in Z) \textbf{ implies } X = Y \setminus Z,$$

(21) $$\textbf{not } (\textbf{ex } x \textbf{ st } x \in X) \textbf{ implies } X = \emptyset,$$

(22) $$(\textbf{for } x \textbf{ holds } x \in X \textbf{ iff } x \in Y) \textbf{ implies } X = Y,$$

(23) $$x \in X \dot{-} Y \textbf{ iff not } (x \in X \textbf{ iff } x \in Y),$$

(24) $$x \in X \ \& \ x \in Y \textbf{ implies } X \cap Y \neq \emptyset,$$

(25) $$(\textbf{for } x \textbf{ holds not } x \in X \textbf{ iff } (x \in Y \textbf{ iff } x \in Z)) \textbf{ implies } X = Y \dot{-} Z,$$

(26) $$X \subseteq X,$$

(27) $$\emptyset \subseteq X,$$

(28) $$X \subseteq Y \ \& \ Y \subseteq X \textbf{ implies } X = Y,$$

(29) $$X \subseteq Y \ \& \ Y \subseteq Z \textbf{ implies } X \subseteq Z,$$

(30) $$X \subseteq \emptyset \textbf{ implies } X = \emptyset,$$

(31) $$X \subseteq X \cup Y \ \& \ Y \subseteq X \cup Y,$$

(32) $$X \subseteq Z \ \& \ Y \subseteq Z \textbf{ implies } X \cup Y \subseteq Z,$$

(33) $$X \subseteq Y \textbf{ implies } X \cup Z \subseteq Y \cup Z \ \& \ Z \cup X \subseteq Z \cup Y,$$

(34) $$X \subseteq Y \;\&\; Z \subseteq V \textbf{ implies } X \cup Z \subseteq Y \cup V,$$

(35) $$X \subseteq Y \textbf{ implies } X \cup Y = Y \;\&\; Y \cup X = Y,$$

(36) $$X \cup Y = Y \textbf{ or } Y \cup X = Y \textbf{ implies } X \subseteq Y,$$

(37) $$X \cap Y \subseteq X \;\&\; X \cap Y \subseteq Y,$$

(38) $$X \cap Y \subseteq X \cup Z,$$

(39) $$Z \subseteq X \;\&\; Z \subseteq Y \textbf{ implies } Z \subseteq X \cap Y,$$

(40) $$X \subseteq Y \textbf{ implies } X \cap Z \subseteq Y \cap Z \;\&\; Z \cap X \subseteq Z \cap Y,$$

(41) $$X \subseteq Y \;\&\; Z \subseteq V \textbf{ implies } X \cap Z \subseteq Y \cap V,$$

(42) $$X \subseteq Y \textbf{ implies } X \cap Y = X \;\&\; Y \cap X = X,$$

(43) $$X \cap Y = X \textbf{ or } Y \cap X = X \textbf{ implies } X \subseteq Y,$$

(44) $$X \subseteq Z \textbf{ implies } X \cup Y \cap Z = (X \cup Y) \cap Z,$$

(45) $$X \setminus Y = \emptyset \textbf{ iff } X \subseteq Y,$$

(46) $$X \subseteq Y \textbf{ implies } X \setminus Z \subseteq Y \setminus Z,$$

(47) $$X \subseteq Y \textbf{ implies } Z \setminus Y \subseteq Z \setminus X,$$

(48) $$X \subseteq Y \;\&\; Z \subseteq V \textbf{ implies } X \setminus V \subseteq Y \setminus Z,$$

(49) $$X \setminus Y \subseteq X,$$

(50) $$X \subseteq Y \setminus X \textbf{ implies } X = \emptyset,$$

(51) $$X \subseteq Y \;\&\; X \subseteq Z \;\&\; Y \cap Z = \emptyset \textbf{ implies } X = \emptyset,$$

(52) $$X \subseteq Y \cup Z \textbf{ implies } X \setminus Y \subseteq Z \;\&\; X \setminus Z \subseteq Y,$$

(53) $$(X \cap Y) \cup (X \cap Z) = X \textbf{ implies } X \subseteq Y \cup Z,$$

(54) $$X \subseteq Y \textbf{ implies } Y = X \cup (Y \setminus X) \;\&\; Y = (Y \setminus X) \cup X,$$

(55) $$X \subseteq Y \;\&\; Y \cap Z = \emptyset \textbf{ implies } X \cap Z = \emptyset,$$

(56) $$X = Y \cup Z \textbf{ iff } Y \subseteq X \;\&\; Z \subseteq X \;\&\; \textbf{for } V \textbf{ st } Y \subseteq V \;\&\; Z \subseteq V \textbf{ holds } X \subseteq V,$$

(57) $$X = Y \cap Z \textbf{ iff } X \subseteq Y \;\&\; X \subseteq Z \;\&\; \textbf{for } V \textbf{ st } V \subseteq Y \;\&\; V \subseteq Z \textbf{ holds } V \subseteq X,$$

(58) $$X \setminus Y \subseteq X \dot{-} Y,$$

(59) $$X \cup Y = \emptyset \textbf{ iff } X = \emptyset \;\&\; Y = \emptyset,$$

(60) $$X \cup \emptyset = X \;\&\; \emptyset \cup X = X,$$

(61) $$X \cap \emptyset = \emptyset \;\&\; \emptyset \cap X = \emptyset,$$

(62) $$X \cup X = X,$$

(63) $$X \cup Y = Y \cup X,$$

(64) $$(X \cup Y) \cup Z = X \cup (Y \cup Z),$$

(65) $$X \cap X = X,$$

(66) $$X \cap Y = Y \cap X,$$

(67) $$(X \cap Y) \cap Z = X \cap (Y \cap Z),$$

(68) $$X \cap (X \cup Y) = X$$
$$\& \; (X \cup Y) \cap X = X \;\&\; X \cap (Y \cup X) = X \;\&\; (Y \cup X) \cap X = X,$$

(69) $$X \cup (X \cap Y) = X$$
$$\& \; (X \cap Y) \cup X = X \;\&\; X \cup (Y \cap X) = X \;\&\; (Y \cap X) \cup X = X,$$

(70) $$X \cap (Y \cup Z) = X \cap Y \cup X \cap Z \;\&\; (Y \cup Z) \cap X = Y \cap X \cup Z \cap X,$$

(71) $$X \cup Y \cap Z = (X \cup Y) \cap (X \cup Z) \;\&\; Y \cap Z \cup X = (Y \cup X) \cap (Z \cup X),$$

(72) $$(X \cap Y) \cup (Y \cap Z) \cup (Z \cap X) = (X \cup Y) \cap (Y \cup Z) \cap (Z \cup X),$$

(73) $$X \setminus X = \emptyset,$$

(74) $$X \setminus \emptyset = X,$$

(75) $$\emptyset \setminus X = \emptyset,$$

(76) $$X \setminus (X \cup Y) = \emptyset \;\&\; X \setminus (Y \cup X) = \emptyset,$$

(77) $$X \setminus X \cap Y = X \setminus Y \;\&\; X \setminus Y \cap X = X \setminus Y,$$

(78) $$(X \setminus Y) \cap Y = \emptyset \;\&\; Y \cap (X \setminus Y) = \emptyset,$$

(79) $$X \cup (Y \setminus X) = X \cup Y \;\&\; (Y \setminus X) \cup X = Y \cup X,$$

(80) $$X \cap Y \cup (X \setminus Y) = X \;\&\; (X \setminus Y) \cup X \cap Y = X,$$

(81) $$X \setminus (Y \setminus Z) = (X \setminus Y) \cup X \cap Z,$$

(82) $$X \setminus (X \setminus Y) = X \cap Y,$$

$$(X \cup Y) \setminus Y = X \setminus Y, \tag{83}$$

$$X \cap Y = \emptyset \textbf{ iff } X \setminus Y = X, \tag{84}$$

$$X \setminus (Y \cup Z) = (X \setminus Y) \cap (X \setminus Z), \tag{85}$$

$$X \setminus (Y \cap Z) = (X \setminus Y) \cup (X \setminus Z), \tag{86}$$

$$(X \cup Y) \setminus (X \cap Y) = (X \setminus Y) \cup (Y \setminus X), \tag{87}$$

$$(X \setminus Y) \setminus Z = X \setminus (Y \cup Z), \tag{88}$$

$$(X \cup Y) \setminus Z = (X \setminus Z) \cup (Y \setminus Z), \tag{89}$$

$$X \setminus Y = Y \setminus X \textbf{ implies } X = Y, \tag{90}$$

$$X \dot{-} Y = (X \setminus Y) \cup (Y \setminus X), \tag{91}$$

$$X \dot{-} \emptyset = X \ \& \ \emptyset \dot{-} X = X, \tag{92}$$

$$X \dot{-} X = \emptyset, \tag{93}$$

$$X \dot{-} Y = Y \dot{-} X, \tag{94}$$

$$X \cup Y = (X \dot{-} Y) \cup X \cap Y, \tag{95}$$

$$X \dot{-} Y = (X \cup Y) \setminus X \cap Y, \tag{96}$$

$$(X \dot{-} Y) \setminus Z = (X \setminus (Y \cup Z)) \cup (Y \setminus (X \cup Z)), \tag{97}$$

$$X \setminus (Y \dot{-} Z) = X \setminus (Y \cup Z) \cup X \cap Y \cap Z, \tag{98}$$

$$(X \dot{-} Y) \dot{-} Z = X \dot{-} (Y \dot{-} Z), \tag{99}$$

$$X \text{ meets } Y \cup Z \textbf{ iff } X \text{ meets } Y \textbf{ or } X \text{ meets } Z, \tag{100}$$

$$X \text{ meets } Y \ \& \ Y \subseteq Z \textbf{ implies } X \text{ meets } Z, \tag{101}$$

$$X \text{ meets } Y \cap Z \textbf{ implies } X \text{ meets } Y \ \& \ X \text{ meets } Z, \tag{102}$$

$$X \text{ meets } Y \textbf{ implies } Y \text{ meets } X, \tag{103}$$

$$\textbf{not } (X \text{ meets } \emptyset \textbf{ or } \emptyset \text{ meets } X), \tag{104}$$

$$X \text{ misses } Y \textbf{ iff not } X \text{ meets } Y, \tag{105}$$

$$X \text{ misses } Y \cup Z \textbf{ iff } X \text{ misses } Y \ \& \ X \text{ misses } Z, \tag{106}$$

$$X \text{ misses } Z \ \& \ Y \subseteq Z \textbf{ implies } X \text{ misses } Y, \tag{107}$$

(108) $$X \text{ misses } Y \textbf{ or } X \text{ misses } Z \textbf{ implies } X \text{ misses } Y \cap Z,$$

(109) $$X \text{ misses } \emptyset \;\&\; \emptyset \text{ misses } X,$$

(110) $$X \text{ meets } X \textbf{ iff } X \neq \emptyset,$$

(111) $$X \cap Y \text{ misses } X \setminus Y,$$

(112) $$X \cap Y \text{ misses } X \dot{-} Y,$$

(113) $$X \text{ meets } Y \setminus Z \textbf{ implies } X \text{ meets } Y,$$

(114) $$X \subseteq Y \;\&\; X \subseteq Z \;\&\; Y \text{ misses } Z \textbf{ implies } X = \emptyset,$$

(115) $$X \setminus Y \subseteq Z \;\&\; Y \setminus X \subseteq Z \textbf{ implies } X \dot{-} Y \subseteq Z,$$

(116) $$X \cap (Y \setminus Z) = (X \cap Y) \setminus Z,$$

(117) $$X \cap (Y \setminus Z) = X \cap Y \setminus X \cap Z \;\&\; (Y \setminus Z) \cap X = Y \cap X \setminus Z \cap X,$$

(118) $$X \text{ misses } Y \textbf{ iff } X \cap Y = \emptyset,$$

(119) $$X \text{ meets } Y \textbf{ iff } X \cap Y \neq \emptyset,$$

(120) $$X \subseteq (Y \cup Z) \;\&\; X \cap Z = \emptyset \textbf{ implies } X \subseteq Y,$$

(121) $$Y \subseteq X \;\&\; X \cap Y = \emptyset \textbf{ implies } Y = \emptyset,$$

(122) $$X \text{ misses } Y \textbf{ implies } Y \text{ misses } X.$$

References

[1] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1, 1990.

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