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BORDISM ALGEBRAS OF PERIODIC TRANSFORMATIONS

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For the equivariant bordism groups of C^∞-manifolds with differentiable actions of $S^1=U(1)$ and its subgroups Z_n, the cases of free actions have been studied by Conner-Floyd [3], Conner [2], Su [11], Uchida [13], Kamata [5, 6] and others.

The purpose of this note is to study the ring structure of bordism for the cases of semi-free actions (cf. Alexander [1], Miščenko [8]).

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1. The ring structure of $\mathcal{M}_*(S^i)$ ($i=1, 3$).

It was shown by Conner-Floyd [3] and Uchida [12] that the following sequences are exact (and also split):

$$(1.1)\qquad 0 \to \mathcal{I}_*(Z_2) \xrightarrow{\nu} \mathcal{M}_*(Z_2) \xrightarrow{\partial} \mathfrak{N}_*(Z_2) \to 0\,,$$

$$(1.2)\qquad 0 \to \mathcal{O}_*(S^1) \xrightarrow{\nu} \mathcal{M}_*(S^1) \xrightarrow{\partial} \Omega_*(S^1) \to 0\,,$$

$$(1.3)\qquad 0 \to \mathcal{O}_*(S^3) \xrightarrow{\nu} \mathcal{M}_*(S^3) \xrightarrow{\partial} \Omega_*(S^3) \to 0\,,$$

where $\mathcal{I}_*(Z_2)$ is the bordism group of unoriented manifolds with involution and $\mathcal{O}_*(S^i)$ ($i=1, 3$) are the bordism groups of oriented manifolds with semi-free S^i-action. Corresponding to these bordsim groups, the cases of free involution and free S^i-action are denoted by $\mathfrak{N}_*(Z_2)$ and $\Omega_*(S^i)$ respectively. And $\mathcal{M}_*(Z_2)=\Sigma_{k\geq 0}\mathfrak{N}_*(BO(k))$, $\mathcal{M}_*(S^1)=\Sigma_{k\geq 0}\Omega_*(BU(k))$ and $\mathcal{M}_*(S^3)=\Sigma_{k\geq 0}\Omega_*(BSp(k))$.

The above three exact sequences are apparently analogous, and in fact we can study them under a uniform argument.

Let F denote either one of the fields of real numbers R, complex numbers C, or quaternions H. Let $d=\dim_R F$, and let $FP(n)$ denote the n-dimensional projective space.

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Proposition (1.4) (cf. Ossa [9]) (1) *The bordsim groups* $\mathfrak{N}_*(Z_2)$ *and* $\Omega_*(S^{d-1})$ $(d-1=1, 3)$ *are free modules over* $\mathfrak{N}_*$ *and* Ω_* *respectively with generating set* $\{\alpha_{dn-1}=[S^{dn-1}, T_0]; n\geq 1\}$ *where* $T_0: S^{d-1}\times S^{dn-1}\rightarrow S^{dn-1}$, $(S^{dn-1}\subset F^n$, *the unit sphere), is the usual scalar multiplication.*

(2) *The bordism groups* $\mathcal{M}_*(Z_2)\approx\mathfrak{N}_*[\theta_0, \theta_1, \cdots]$ *and* $\mathcal{M}_*(S^1)\approx\Omega_*[\theta_0, \theta_1, \cdots]$ $(i=1, 3)$ *are the polynomial algebras in* $\theta_0, \theta_1, \cdots$ *over the Thom bordism rings* $\mathfrak{N}_*$ *and* Ω_* *respectively, where* $\theta_n=[\eta_n\rightarrow FP(n)]$, *and* η_n *is the canonical line bundle.*

Proof. (1) is well known (cf. [3]). For (2), we shall only show the case of $\mathcal{M}_*(S^1)$. The other cases are analogous. The weak direct sum $\Sigma_{k\geq 0}H_*(BU(k); Z)$ and $\mathcal{M}_*(S^1)=\Sigma_{k\geq 0}\,\Omega_*(BU(k))$ can be given the structure of graded rings. The multiplications are given by

$$H_i(BU(k); Z)\otimes H_j(BU(l); Z) \rightarrow H_{i+j}(BU(k+l); Z)$$

and

$$\Omega_i(BU(k))\otimes\Omega_j(BU(l)) \rightarrow \Omega_{i+j}(BU(k+l))$$

which are induced by the Whitney sum map: $BU(k)\times BU(l)\rightarrow BU(k+l)$.

The natural map μ: $\Sigma\Omega_*(BU(k))\rightarrow\Sigma H_*(BU(k)\colon Z)$ (Conner Floyd [3]) is then a ring homomorphism.

Let $a_n=\mu(\theta_n)=\{CP(n)\}\in H_{2n}(CP(\infty); Z)=H_{2n}(BU(1); Z)$ for $\theta_n=[\eta_n\rightarrow CP(n)]\in\Omega_{2n}(BU(1))$. Then $\{a_n; n\geq 0\}$ is an additive base of $H_*(BU(1); Z)$.

To show $\mathcal{M}_*(S^1)\approx\Omega_*[\theta_0, \theta_1\cdots]$, it suffices to show that $\{\theta_{i_1}\cdots\theta_{i_k}; 0\leq i_1\leq\cdots\leq i_k\}$ is an Ω_*-base of $\Omega_*(BU(k))$. To see this, it is only necessary to show that $\{\mu(\theta_{i_1}\cdots\theta_{i_k})=a_{i_1}\cdots a_{i_k}\}$ is an additive base of $H_*(BU(k); Z)$ (Conner-Floyd [3], Theorem 18.1). On the other hand, the map f: $(CP(\infty))^k\rightarrow BU(k)$ induces a monomorphism f^*: $H^*(BU(k); Z)\rightarrow H^*(CP(\infty))^k; Z)$ whose image is the ring of symmetric polynomials in $x_1, \cdots, x_k$ where $H^*(CP(\infty))^k; Z)=Z[x_1, \cdots, x_k]$ with deg $x_i=2$. Let $s_\omega=\Sigma x_1^{i_1}\cdots x_k^{i_k}$ (the symmetric sum without repetition) where $\omega=(i_1, i_2, \cdots, i_k)$, $0\leq i_1\leq\cdots\leq i_k$. Then $\{s_\omega; \omega=(i_1, \cdots, i_k), 0\leq i_1\leq\cdots\leq i_k\}$ is an additive base of $f^*(H^*(BU(k); Z)\approx H^*(BU(k); Z)$. Since $a_\omega=a_{i_1}\cdots a_{i_k}=f_*\{CP(i_1)\times\cdots\times CP(i_k)\}$, we can easily obtain the following (Milnor [7]):

$$\begin{aligned}\langle s_{\omega'}, a_\omega\rangle &= \langle x_1^{i_1'}, \{CP(i_1)\}\rangle\cdots\langle x_k^{i_k'}, \{CP(i_k)\}\rangle\\ &= \begin{cases} 0\ (\omega'\neq\omega)\,, \\ 1\ (\omega'=\omega)\,. \end{cases}\end{aligned}$$

The assertion thus follows.

2. The ring structure of $\mathcal{O}_*(S^i)$ $(i=1, 3)$

Alexander [1] studied the ring structure of $\mathcal{I}_*(Z_2)$ by using the exact sequence

(1.1). If we make further use of Proposition (1.4), the ring structures of $\mathcal{I}_*(Z_2)$ and $\mathcal{O}_*(S^i)$ may be determined in a definite form. We shall treat the case of $\mathcal{O}_*(S^1)$ in the following.

In the exact sequence (1.2)

$$0 \to \mathcal{O}_*(S^1) \xrightarrow{\nu} \mathcal{M}_*(S^1) \xrightarrow{\partial} \Omega_{*-1}(S^1) \to 0\,,$$

ν is a ring homomorphism defined by $\nu[M^n, T]=\Sigma_i[\nu_{F_i}\to F_i]$ where F_i is a connected component of the fixed point set F_T of T and itself an oriented closed submanifold of M^n, and ν_{F_i} is the normal bundle to F_i in M^n. Also $\partial[\xi\to M^n]=[S(\xi), T]$ where $S(\xi)=\partial D(\xi)$ is the boundary of the disk bundle $D(\xi)$, and T is the standard fibre-preserving $U(1)$-action. We then have

$$\partial\theta_n = [S^{2n+1}, \mathrm{T}_0] = \alpha_{2n+1},\ \partial\theta_0^n = \alpha_{2n-1}\,. \tag{2.1}$$

Now, let

$\sigma_n = [CP(n+1), T]$, $n\geq 1$, with $T(s, (z_0, \cdots, z_{n+1})) = (sz_0, \cdots, sz_n, z_{n+1})$.
Then

$$\nu(\sigma_n) = \theta_n - \theta_0^{n+1} \qquad (n\geq 1) \tag{2.2}$$

holds.

Next, we define an Ω_*-map

$$\Gamma\colon \mathcal{O}_*(S^1) \to \mathcal{O}_{*+2}(S^1) \tag{2.3}$$

as follows (Conner-Floyd [3], p.119). If T_0 is the standard S^1-action on D^2, then for a manifold (M^n, T) with a semi-free S^1-action T, we form a manifold $(\tilde{M}^{n+2}, \tilde{T})$ from $(-D^2\times M^n, T_0\times 1)$ and $(D^2\times M^n, T_0\times T)$ by identifying the boundaries via the equivariant diffeomorphism φ: $(S^1\times M^n, T_0\times 1)\to(S^1\times M^n, T_0\times T)$ which is defined by $\varphi(s, x)=(s, sx)$. We then define Γ by

$$\Gamma(M^n, T) = (\tilde{M}^{n+2}, \tilde{T}) = (-D^2\times M^n, T_0\times 1)_1 \underset{\varphi}{\cup} (D^2\times M^n, T_0\times T)_2\,. \tag{2.4}$$

Since the fixed point set of $\tilde{T}$ is $F_{\tilde{T}}=((0)\times M^n)_1\cup((0)\times F_T)_2$, we have

$$\nu[\tilde{M}^{n+2}, \tilde{T}] = \nu[M^n, T]\cdot\theta_0 - [M^n]\cdot\theta_0\,. \tag{2.5}$$

By using the following notations

$$\begin{aligned} &\iota\colon \mathcal{M}_n(S^1) \to \mathcal{M}_{n+2}(S^1),\ \iota(x) = x\cdot\theta_0\,,\\ &\varepsilon\colon \mathcal{O}_n(S^1) \to \Omega_n,\ \varepsilon[M^n, T] = [M^n]\,,\\ &\tau\colon \Omega_n \to \mathcal{M}_{n+2}(S^1),\ \tau[M^n] = [M^n]\cdot\theta_0\,. \end{aligned}$$

we can express (2.5) in the form

$$\nu\Gamma = \iota\nu - \tau\varepsilon\,. \tag{2.6}$$

Lemma (2.7). (cf. Alexander [1])

$$\Gamma(ab) = \Gamma(a)b+\varepsilon(a)\Gamma(b)\,,$$
$$= a\Gamma(b)+\varepsilon(b)\Gamma(a) \quad \textit{for} \quad a, b\in\mathcal{O}_*(S^1)\,.$$

Proof. It follows easily from (1.2) and (2.6).

Theorem (2.8). *The bordism group $\mathcal{O}_*(S^1)$ is a free Ω_*-module with generating set $\{\Gamma^l(\sigma_{j_1}\cdots\sigma_{j_k});\ l\geq 0,\ 1\leq j_1\leq \cdots \leq j_k\}\cup\{1\}$. Its ring structure is then given as the quotient of the polynomial algebra $\Omega_*[\Gamma^l(\sigma_j);\ l\geq 0, j\geq 1]$ by the ideal generated by*

$$\Gamma^l(\sigma_j)\cdot\Gamma^{m+1}(\sigma_k)-\Gamma^m(\sigma_k)\cdot\Gamma^{l+1}(\sigma_j)-\varepsilon\Gamma^l(\sigma_j)\cdot\Gamma^{m+1}(\sigma_k)$$
$$+\varepsilon\Gamma^m(\sigma_k)\cdot\Gamma^{l+1}(\sigma_j)\ (l,\ m\geq 0;\ j,\ k\geq 1)\,.$$

Similarly for $\mathcal{O}_*(S^3)$.

Proof. Turning our attention to (2.6), we see that if the monomials of $\mathcal{M}_*(S^1)$ are given a suitable order, then

$$\nu(\Gamma^l(\sigma_{j_1}\cdots\sigma_{j_k}))=\theta_0^l\theta_{j_1}\cdots\theta_{j_k}+\text{lower terms}$$

holds. Since $\{\theta_0^l\theta_{j_1}\cdots\theta_{j_k}\}$ is an Ω_*-base of $\mathcal{M}_*(S^1)$, the first half of the theorem follows.

For the second half, we first observe that $\{\Gamma^l(\sigma_j);\ l\geq 0,\ j\geq 1\}$ forms a generating set of the Ω_*-algebra $\mathcal{O}_*(S^1)$ in virtue of (2.7). Then the assertion of the theorem can be verified easily by comparing the ranks of the associated graded modules of the suitably filtered modules in question.

3. On the ring structure of $\mathcal{O}_*(Z_3)$

Let p be an odd prime. We treat here the case of Z_p-action, particularly the case p=3, in this section. We already have the following exact sequence (Conner [2], Wu [14]):

$$0 \to \Omega_* \xrightarrow{i_*} \mathcal{O}_*(Z_p) \xrightarrow{\nu} \mathcal{M}_*(Z_p) \xrightarrow{\partial} \tilde{\Omega}_*(Z_p) \to 0 \tag{3.1}$$

where $\mathcal{O}_*(Z_p)$, similarly in the previous section, denotes the bordism group of semi-free Z_p-action, and $\mathcal{M}_*(Z_p)=\mathcal{M}^*(S^1)\underset{\Omega_*}{\otimes}\cdots\underset{\Omega_*}{\otimes}\mathcal{M}_*(S^1)$ ($(p-1)/2$-fold tensor product over Ω_*) The reduced group $\tilde{\Omega}^*(Z_p)=\text{Ker }\bar{\varepsilon}$, where $\Omega_*(Z_p)$ is the bordism group of free Z_p-action, and $\bar{\varepsilon}:\Omega_*(Z_p)\to\Omega_*$ is defined by $\bar{\varepsilon}[M^n, \tau]=[M^n/\tau]$. The homomorphism i_*: $\Omega_*\to\mathcal{O}_*(Z_p)$ is defined by $i_*[M^n]=[M^n]\cdot\mu_0=[M^n\times Z_p, 1\times\sigma]$, where $\mu_0=[Z_p, \sigma]\in\mathcal{O}_0(Z_p)$. And the homomorphisms ν and ∂ are to be analogously defined as in the previous section.

For the sake of simplicity, we consider only the case of p=3 in the following. We have the following commutative diagram (Wu [14]):

$$
\begin{array}{ccccccccc}
 & & 0 \to & \mathcal{O}_*(S^1) & \xrightarrow{\nu} & \mathcal{M}_*(S^1) & \xrightarrow{\partial} & \Omega_*(S^1) & \to 0 \\
(3.2) & & & \downarrow \lambda & & \approx \downarrow \bar{\lambda} & & \downarrow \bar{\bar{\lambda}} & \\
0 \to & \Omega_* & \xrightarrow{i_*} & \mathcal{O}_*(Z_3) & \xrightarrow{\nu} & \mathcal{M}_*(Z_3) & \xrightarrow{\partial} & \tilde{\Omega}_*(Z_3) & \to 0
\end{array}
$$

where each vertical map λ is the natural map obtained by restricting the corresponding S^1-action to its subgroup Z_3. We then have the following results (Conner-Floyd [3], §36 and §46):

1) The map $\bar{\bar{\lambda}}$: $\Omega_*(S^1) \to \tilde{\Omega}_*(Z_3)$ is an epimorphism.

2) $\Omega_*(S^1) \approx \Omega_{*-1}(CP(\infty))$ is a free Ω_*-module generated by $\{\alpha_{2n-1} = [S^{2n-1}, T]; n \geq 1\}$.

3) There is a sequence of oriented closed manifolds, M^4, M^8, $\cdots$, M^{4k}, $\cdots$ such that if we put

$$\beta_{2n-1} = 3\alpha_{2n-1} + [M^4]\alpha_{2n-5} + [M^8]\alpha_{2n-9} + \cdots, \ (n \geq 1), \tag{3.3}$$

then $\{\beta_{2n-1};\ n \geq 1\}$ constitutes a generating set for $K = \operatorname{Ker} \bar{\bar{\lambda}}$.

Now, put

$$\bar{\beta}_n = 3\theta_0^n + [M^4]\theta_0^{n-2} + [M^8]\theta_0^{n-4} + \cdots, \ (n \geq 1), \tag{3.4}$$

and identify $\mathcal{M}_*(Z_3)$ with $\mathcal{M}_*(S^1)$ by the isomorphism $\bar{\lambda}$. Then $\bar{\beta}_n$ is in the kernel of ∂: $\mathcal{M}_*(Z_3) \to \tilde{\Omega}_*(Z_3)$ for each $n \geq 1$:

$$\partial(\bar{\beta}_n) = 0. \tag{3.5}$$

Therefore, from the exact sequence (3.1), there exists $\mu_n \in \mathcal{O}_*(Z_3)$ such that $\nu(\mu_n) = \bar{\beta}_n$ for each $n \geq 1$. We thus obtain the following theorem (Wu [14]).

Theorem (3.6). *$\mathcal{O}_*(Z_3)$ is isomorphic as a free Ω_*-module to the direct sum of $\Omega_*\{\mu_0, \mu_1, \cdots\}$ and $\lambda(\mathcal{O}_*(S^1))$.*

We go on to study the multiplicative structure of $\mathcal{O}_*(Z_3)$. It is evident from the previous arguments that $\{\mu_k (k \geq 0), \Gamma^l(\sigma_j)\ (l \geq 0, j \geq 1)\}$ can be taken as a generating set of Ω_*-algebra $\mathcal{O}_*(Z_3)$ where $\lambda(\Gamma^l(\sigma_j))$ is simply denoted by $\Gamma^l(\sigma_j)$. The map $\lambda: \mathcal{O}_*(S^1) \to \mathcal{O}_*(Z_3)$ is a ring isomorphism of $\mathcal{O}_*(S^1)$ into $\mathcal{O}_*(Z_3)$, and $\operatorname{Im} \lambda \approx \mathcal{O}_*(S^1)$ is a subalgebra of $\mathcal{O}_*(Z_3)$. Also, $\operatorname{Im} i_* = \operatorname{Ker} \nu = \Omega_* \cdot \mu_0 (\mu_0 = [Z_3, \sigma])$ is an ideal of $\mathcal{O}_*(Z_3)$. In fact, if we let ε: $\mathcal{O}_*(Z_3) \to \Omega_*$ be defined by $\varepsilon[M^n, \tau] = [M^n]$, we have

$$\mu_0 \cdot a = \varepsilon(a)\mu_0 \quad \text{for} \quad a \in \mathcal{O}_*(Z_3). \tag{3.7}$$

We next have to appropriately choose and fix $\mu_n (n \geq 1)$ so as to study the relations among them. First, let

$$\mu_1 = [M^2, \tau_1], \tag{3.8}$$

where M^2 is the algebraic curve $z_0^3 + z_1^3 + z_2^3 = 0$ in $CP(2)$ which is non-singular and of genus 1. The action τ_1 is defined by $\tau_1(z_0, z_1, z_2) = (z_0, z_1, \rho z_2)$ with $\rho = \exp(2\pi i/3)$. The fixed points of τ_1 are $(1, -1, 0)$, $(\rho, -1, 0)$ and $(\rho^2, -1, 0)$, (Conner [2]), so we have

$$\nu(\mu_1) = 3\theta_0. \tag{3.9}$$

Next, let

$$\mu_2 = [CP(2), \tau_2] \tag{3.10}$$

where τ_2 is defined by $\tau_2(z_0, z_1, z_2) = (z_0, \rho z_1, \rho^2 z_2)$.
Since the fixed points of τ_2 are $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, we have

$$\nu(\mu_2) = 3\theta_0^2. \tag{3.11}$$

Before determining appropriate $\mu_n (n \geq 3)$, we need some preparation. First note that for an oriented closed maifold (M^n, T) with any S^1-action T, not necessarily semi-free, we can define also the Γ-operation just as in §2;

$$\Gamma[M^n, T] = [\tilde{M}^{n+2}, \tilde{T}] \tag{3.12}$$

which is an operation on the bordism group of all S^1-actions.

We may also define the manifold with Z_3-action $\lambda(T)$ (the restriction of an S^1-action T) as follows:

$$\lambda(M^n, T) = (M^n, \lambda(T)). \tag{3.13}$$

Since the fixed point set of $\lambda(\tilde{T})$ for $(\tilde{M}^{n+2}, \lambda(\tilde{T}))$ is $F_{\lambda(\tilde{T})} = ((0) \times M^n)_1 \cup ((0) \times F_{\lambda(T)})_2$, we have

$$\nu[\tilde{M}^{n+2}, \lambda(\tilde{T})] = \nu[M^n, \lambda(T)] \cdot \theta_0 - [M^n] \cdot \theta_0, \tag{3.14}$$

or equivalently,

$$\nu\lambda\Gamma = \iota\nu\lambda - \tau\varepsilon\lambda. \tag{3.15}$$

We thus have, by induction, the formula

$$\nu\lambda\Gamma^n = \iota^n \nu\lambda - \sum_{i=0}^{n-1} \iota^{n-1-i} \tau\varepsilon\lambda\Gamma^i. \tag{3.16}$$

Consider now the manifold

$$(CP(2), T_2) \tag{3.17}$$

where T_2 is an S^1-action defined by $T_2(e^{i\theta}, (z_0, z_1, z_2))=(z_0, e^{i\theta}z_1, e^{2i\theta}z_2)$. It is seen that

$$\lambda[CP(2), T_2] = [CP(2), \tau_2] = \mu_2 . \tag{3.18}$$

We then define

$$\mu_n = \lambda\Gamma^{n-2}[CP(2), T_2], (n\geq 2) . \tag{3.19}$$

It now follows from (3.16) that

$$\begin{aligned} \nu(\mu_n) &= 3\theta_0^n - \sum_{i=0}^{n-3} \mathcal{E}(\mu_{i+2})\theta_0^{n-2-i} \\ &= 3\theta_0^n - \sum_{i=2}^{n-1} \mathcal{E}(\mu_i)\theta_0^{n-i} . \end{aligned} \tag{3.20}$$

This may a little differ from the condition $\nu(\mu_n)=\bar{\beta}_n$ of (3.6). However, it does not matter, because $\{\bar{\beta}_n;\ n\geq 1\}$ is still a free base of the kernel of ∂: $\Omega_*[\theta_0] \to\tilde{\Omega}_*(Z_3)$ when we take for $\bar{\beta}_n$ the right-hand side of (3.20) instead of (3.4).

From the above definition of μ_n, we first obtain the following relations,

$$\begin{aligned} &\mu_1^2 = 3\mu_2 - \mathcal{E}(\mu_2)\mu_0 , \quad \mu_1\mu_2 = 3\mu_3 + \mathcal{E}(\mu_2)\mu_1 , \\ &\mu_1\mu_n = 3\mu_{n+1} + \mathcal{E}(\mu_n)\mu_1 - \mathcal{E}(\mu_{n+1})\mu_0 , \quad (n\geq 1) , \\ &\mu_2\mu_n = 3\mu_{n+2} + \mathcal{E}(\mu_n)\mu_2 + \mathcal{E}(\mu_{n+1})\mu_1 - \mathcal{E}(\mu_{n+2})\mu_0 , \quad (n\geq 2) \end{aligned} \tag{3.21}$$

which can be proved by operating ν and $\mathcal{E}$ on both sides of the equations. Here notice that $\mathcal{E}|\mathrm{Ker}\ \nu$ is a monomorphism, $\mathcal{E}(\mu_1)=0$ and $\mathcal{E}(\mu_3)=0$.

Moreover, we have

$$\begin{aligned} &\mu_1\Gamma^l(\sigma_i) = 3\Gamma^{l+1}(\sigma_i) + \mathcal{E}(\Gamma^l(\sigma_i))\mu_1 - \mathcal{E}(\Gamma^{l+1}(\sigma_i))\mu_0 , \\ &\mu_2\Gamma^l(\sigma_i) = 3\Gamma^{l+2}(\sigma_i) + \mathcal{E}(\Gamma^l(\sigma_i))\mu_2 + \mathcal{E}(\Gamma^{l+2}(\sigma_i))\mu_1 - \mathcal{E}(\Gamma^{l+2}(\sigma_i))\mu_0 \end{aligned} \tag{3.22}$$

We have finally the relations among μ_n for $n\geq 3$ and $\Gamma^l(\sigma_i)$ as follows:

$$\begin{aligned} &\mu_n\mu_m = \mu_{n-1}\mu_{m+1} + \mathcal{E}(\mu_m)\mu_n - \mathcal{E}(\mu_{n-1})\mu_{m+1} , \\ &\mu_n\Gamma^l(\sigma_i) = \mu_{n-1}\Gamma^{l+1}(\sigma_i) + \mathcal{E}(\Gamma^l(\sigma_i))\mu_n - \mathcal{E}(\mu_{n-1})\Gamma^{l+1}(\sigma_i), \ (n\geq 3) . \end{aligned} \tag{3.23}$$

The relations (3.23) could be also derived from (2.7), if we formally put $\mu_n= \Gamma(\mu_{n-1})$. Hence the multiplicative structure of $\mathcal{O}_*(Z_3)$ is essentially ruled by (2.7), except for (3.7), (3.21) and (3.22).

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