Tôhoku Math. Journ.
36 (1984), 233–251.

BOREL-Le POTIER DIAGRAMS—CALCULUS OF THEIR COHOMOLOGY BUNDLES

HANS R. FISCHER AND FLOYD L. WILLIAMS

(Received June 8, 1983)

Abstract. We compute the E_2-term of Borel's spectral sequence for certain holomorphic fibrations. Among some of the applications considered are the representation of automorphic cohomology of a flag domain, and the derivation of new cohomology vanishing theorems for certain compact projective varieties.

1. Introduction. In this paper we consider diagrams $\boldsymbol{E} \to X \to Y$ where $X \to Y$ is a holomorphic fibre bundle (with compact fibre F) and $\boldsymbol{E} \to X$ is a holomorphic vector bundle; the problem then is to relate the Dolbeault cohomology of X with coefficients in $\boldsymbol{E}$ to suitable cohomologies of Y and F. For general $\boldsymbol{E}$ there does not seem to be any way of achieving this for the space $H^{p,q}(X, \boldsymbol{E})$ with $p>0$ in a manner accessible to explicit computation. However if $\boldsymbol{E}$ is assumed to be locally trivial *over* Y the problem is more tractable: in this case there is the (generalized) *Borel spectral sequence* relating $H_{\bar\partial}(Y)$ and a suitable fibre cohomology to $H_{\bar\partial}(X, \boldsymbol{E})$ and a convenient form of the E_2-term of this spectral sequence (or, more accurately, family of spectral sequences) can be found by the techniques of [1], [3], [12], [13] and [14]. In all generality the E_2-terms are determined by holomorphic vector bundles $\boldsymbol{H}^{r,s}(\boldsymbol{E})$, associated with $\boldsymbol{E} \to X$, whose fibres are suitable (r, s)-cohomologies of the fibres of $X \to Y$; for $p = 0$ one concludes the bundles $\boldsymbol{H}^{r,s}(\boldsymbol{E})$ "represent" the direct images of the sheaf $\mathcal{O}(\boldsymbol{E})$ which thus are locally free.

The "cohomology bundles" $\boldsymbol{H}^{r,s}(\boldsymbol{E})$ thus are crucial for the description of the E_2-terms of the Borel spectral sequence and merit some attention; we present the calculation of such bundles in some important special cases and also indicate some applications. As an example if X and Y are homogeneous spaces of a Lie group G and if $\boldsymbol{E} \to X$ is a homogeneous vector bundle, it is locally trivial over Y and the cohomology boundles $\boldsymbol{H}^{r,s}(\boldsymbol{E})$ are homogeneous as well. This interesting fact has, among others, the following application: Let $\mathscr{Y}_D \to M_D$ be the linear deformation space of a maximal compact subvariety of a flag domain D. In [26] Wells and Wolf show that under suitable conditions M_D is a Stein manifold and establish a representation of the automorphic cohomology of D (with respect to a discrete subgroup Γ) in the space of Γ-invariant holomorphic sections of a certain bundle over M_D, cf. [26, Theorem 3.4.7]. We show below that if L is the stabilizer of the compact subvariety Y, then this bundle over M_D can be described as an associated vector bundle of a canonical principal L-bundle $A \to M_D$, induced by the action of L on the cohomology of Y. In a sense, this result is "best possible" since it is known that, in general, M_D is not a quotient of Lie groups.

In Section 4, we investigate the "transform" of $\boldsymbol{H}^{r,s}$ under a discontinuous action of a group Γ on a diagram $\boldsymbol{E} \to X \to Y$; when $\boldsymbol{E}$ is trivial over Y, this "transform" is determined by an automorphic factor which we compute explicitly in Theorem 4.4.

In Section 5, this result is combined with the Borel-Bott-Weil theorem and results of [27], [28] to derive new vanishing theorems for the cohomology of compact normal projective varieties $\Gamma \backslash G_0/T$; here G_0 is a connected, non-compact semi-simple Lie group, T is a Cartan subgroup of G_0 contained in some maximal compact subgroup $K \subset G_0$ such that G_0/K is Hermitian symmetric, and finally Γ is a discrete subgroup of G_0 acting freely on G_0/K. Theorems 5.24 and 5.27 are the main results of this paper which also subsumes a note announced in [3] as "Construction of cohomology bundles in the case of an open real orbit in a complex flag manifold".

2. Borel-Le Potier diagrams. Let X, Y be complex manifolds and assume that $\pi: X \to Y$ is a holomorphic fibre bundle with *compact* fibre F. Moreover let $\boldsymbol{E} \to X$ be a holomorphic vector bundle (with fibre E) with projection σ. One says that $\boldsymbol{E}$ is locally trivial over Y if there exists a holomorphic vector bundle $\boldsymbol{E}_0 \to F$ (with fibre E) such that $\pi \circ \sigma: \boldsymbol{E} \to Y$ is a holomorphic fibre bundle with fibre $\boldsymbol{E}_0$ and group $\mathrm{GL}(\boldsymbol{E}_0)$, the group of all holomorphic automorphisms of $\boldsymbol{E}_0$ (i.e., all fibrewise linear biholomorphisms $\boldsymbol{E}_0 \to \boldsymbol{E}_0$); it is known that $\mathrm{GL}(\boldsymbol{E}_0)$ is a complex Lie group, cf. e.g., [13]. In this case,

$$\boldsymbol{E} \to X \to Y \tag{2.1}$$

will be called a *Borel-Le Potier diagram* (BL-diagram). More explicitly this means that each $y \in Y$ has an open neighbourhood U over which there is a holomorphic trivialization $\phi_U: \pi^{-1}(U) \cong U \times F$ which is covered by a holomorphic isomorphism ψ_U of the vector bundle $\boldsymbol{E}|\pi^{-1}(U)$ onto $U \times \boldsymbol{E}_0$:

$$\begin{array}{ccc} \boldsymbol{E}|\pi^{-1}(U) & \xrightarrow{\psi_U} & U \times \boldsymbol{E}_0 \\ \downarrow & & \downarrow \\ \pi^{-1}(U) & \xrightarrow[\phi_U]{} & U \times F \\ & \searrow \quad U \quad \swarrow & \end{array} \tag{2.2}$$

the diagram commutes and ψ_U is fibrewise linear. The following are some examples:

(i) Given $\pi: X \to Y$, a holomorphic bundle with compact fibre, and the holomorphic vector bundle $\boldsymbol{W} \to Y$, $\pi^* \boldsymbol{W} \to X \to Y$ is a BL-diagram; this is the case originally considered in [1].

(ii) Let H be a complex Lie group, $L \subset H$ a closed complex subgroup and suppose that $\rho: Z \to Y$ is a holomorphic principal H-bundle. Set $X = Z/L$ and suppose that X has a complex structure such that the natural map $\sigma: Z \to X$ is a holomorphic principal L-bundle. Lastly, assume that H/L is compact. The natural map $\pi: X \to Y$ then yields a holomorphic bundle with fibre H/L such that $\pi \circ \sigma = \rho$. If $\lambda: H \to \mathrm{GL}(E)$ is a finite-dimensional holomorphic representation, then we can form the holomorphic vector bundles $\boldsymbol{E}_\lambda = Z \times {}_H E \to X$ and $\boldsymbol{E}_0 = H \times {}_L E \to H/L$. Under these conditions

$$\boldsymbol{E}_\lambda \to X \to Y \tag{2.3}$$

is a BL-diagram: If $U \subset Y$ is a sufficiently small open set, there is a holomorphic section $s: U \to Z$ of ρ and this section is used to construct both ϕ_U and ψ_U in the following manner: For $(z, e) \in Z \times E$ and $(h, e) \in H \times E$ let $[z, e]$, $[h, e]$ be their respective equivalence classes in $\boldsymbol{E}_\lambda$, $\boldsymbol{E}_0$. Then set $\phi_U^{-1}(y, hH) = \sigma(s(y)h)$ for $(y, h) \in U \times H$; this yields a trivialization of $Z|U = \rho^{-1}(U)$. A covering isomorphism ψ_U in the sense of (2.2) then is obtained by setting $\psi_U^{-1}(y, [h, e]) = (\phi_U^{-1}(y, hH), [s(y)h, e])$.

(iii) Let $P \subset \mathrm{SL}(4, C) = G$ be the parabolic subgroup defined by $a_{21} = a_{31} = a_{41} = a_{32} = a_{42} = 0$; let $V = \mathrm{SU}(2, 2) \cap P = S(U(1) \times U(1) \times U(2))$ and $K = S(U(2) \times U(2))$, so that K is a maximal compact subgroup of the real form $G_0 = \mathrm{SU}(2, 2)$ of G. Set $F_{12}^+ = G_0/V$, $M^+ = G_0/K$; thus there is the "double fibration"

$$P_3^+ \xleftarrow{\alpha} F_{12}^+ \xrightarrow{\beta} M^+$$

where $P_3^+ \subset P_3(\boldsymbol{C})$ is the "projective twister space", of importance in mathematical physics in connection with the so-called Penrose transform, cf. [24] for some details. Let $H \to P_3^+$ be the restriction of the hyperplane bundle of $P_3(\boldsymbol{C})$. Then it can be shown that for every integer m $\alpha^* H^m \to F_{12}^+ \to M^+$ is a BL-diagram.

(iv) We shall give more examples later (Sections 3 and 4). Another interesting example is used by Fisher in the study of the cohomology of compact complex nilmanifolds, cf. [4].

In the situation of (2.1) we also write X_y for the fibre $\pi^{-1}(y)$ at y. With this set for each pair of natural numbers (r, s)

$$\boldsymbol{H}^{r,s}(\boldsymbol{E}) = \bigcup_y H^{r,s}(X_y, \boldsymbol{E}|X_y) . \tag{2.4}$$

Here $H^{r,s}$ denotes the (bundle-valued) Dolbeault cohomology of type (r, s). Since $X_y = F$ is compact these cohomologies all are finite-dimensional and one can prove the following:

THEOREM 2.5. *$\boldsymbol{H}^{r,s}(\boldsymbol{E})$ is a holomorphic vector bundle over Y with fibre $H^{r,s}(F, \boldsymbol{E}_0)$, associated with the bundle $\pi: X \to Y$.*

Explicit local trivializations will be indicated below; cf. also [1], [3], [13]. The importance of these "cohomology bundles" lies in their use in the computation of the E_2-terms of the *Borel spectral sequence* for the $\bar\partial$-cohomology of holomorphic fibre bundles with compact fibre; in brief the spectral sequence is obtained as follows: Let $A^{p,q}(X, \boldsymbol{E})$ be the space of smooth $\boldsymbol{E}$-valued forms of type (p, q) on X. This space has a natural decreasing filtration "in terms of base forms": one defines $F^rA^{p,q}(X, \boldsymbol{E})$ to be the space of those (p, q)-forms which may be written as finite sums of forms of the type $\pi^*\alpha \wedge \beta$ with $\alpha \in A^{a,b}(Y)$, $\beta \in A^{c,d}(X, \boldsymbol{E})$ such that $a + c = p$, $b + d = q$ and $a + b \geqq r$. Then $F^rA^{p,q} \supset F^{r+1}A^{p,q}$ and $\bar\partial(F^rA^{p,q}) \subset F^rA^{p,q+1}$. If one fixes p, one thus obtains a decreasing filtration of $A^{p,\cdot}(X, \boldsymbol{E}) = \bigoplus_q A^{p,q}(X, \boldsymbol{E})$ which is compatible with $\bar\partial$ and is regular, etc. Accordingly, one obtains a spectral sequence $({}^pE_r^{s,t})$ which converges to the $\bar\partial$-cohomology $H^{p,\cdot}(X, \boldsymbol{E})$. The main result, due to Borel in the case $\boldsymbol{E} = \pi^* \boldsymbol{W}$ and to Le Potier in the more general case, is the following:

THEOREM 2.6. *Let $\boldsymbol{E} \to X \to Y$ be a Borel-Le Potier diagram as in* (2.1). *For each $p \geqq 0$ the E_2-term of the Borel spectral sequence is given by*

$${}^pE_2^{s,t} = \bigoplus_i H^{i,s-i}(Y, \boldsymbol{H}^{p-i,t+i}(\boldsymbol{E})) . \tag{2.7}$$

For $p = 0$ in particular, one obtains ${}^0E_2^{s,t} = H^{0,s}(Y, \boldsymbol{H}^{0,t}(\boldsymbol{E})) = H^s(Y, \mathscr{O}(\boldsymbol{H}^{0,t}(\boldsymbol{E}))$ where $\mathscr{O}(\,..\,)$ denotes the sheaf of holomorphic sections. Now for $p = 0$ the Borel spectral sequence coincides with the Leray sequence and one can show that $\mathscr{O}(\boldsymbol{H}^{0,t}(\boldsymbol{E})) \cong R^t\pi_*(\mathscr{O}(\boldsymbol{E}))$, establishing that these direct image sheaves here are locally free; we omit all details and refer instead to [1], [12], [14] for more information—including the case $p > 0$ where the Borel sequence no longer is the Leray sequence of any "standard" locally free sheaf over X.

In the situation of Example (ii) above more can be said about the cohomology bundles: Again $\boldsymbol{E}_0$ is the homogeneous vector bundle $H \times_L E$ over $F = H/L$. In particular H acts on the cohomology $H^{r,s}(F, \boldsymbol{E}_0)$ "by left translations" and one now shows that $\boldsymbol{H}^{r,s}(\boldsymbol{E}_\lambda)$ is associated with the principal H-bundle $Z \to Y$ under this action of H:

$$\boldsymbol{H}^{r,s}(\boldsymbol{E}) = Z \times {}_H H^{r,s}(F, \boldsymbol{E}_0) . \tag{2.8}$$

This yields:

COROLLARY 2.9. *With the notations of Example* (ii), *for each* $p \geqq 0$ *there is a spectral sequence* $({}^pE_r^{s,t})$ *which converges to* $H^{p,\cdot}(Z/L, \boldsymbol{E}_\lambda)$ *and whose* E_2*-term is given by*

$${}^pE_2^{s,t} = \bigoplus_i H^{i,s-i}(Z/H, Z \times {}_H H^{p-i,t+i}(H/L, \boldsymbol{E}_0)) \tag{2.10}$$

where $\boldsymbol{E}_0 = H \times {}_L E$.

This corollary generalizes an earlier theorem of Bott [2] to the case $p \geqq 0$. In [4], Fisher obtains a result similar to (2.8) and uses it in conjunction with (2.7) to generalize the classical Mumford-Matsushima vanishing theorem for line bundle cohomologies on a torus (cf. also [15], [18]).

REMARK. Given a diagram (2.2), the restrictions $\phi_{U,y} = \phi_U | X_y \colon X_y \to F$ and $\psi_{U,y} \colon \boldsymbol{E}|X_y \to \boldsymbol{E}_0$ induce isomorphisms $H^{r,s}(F, \boldsymbol{E}_0) \cong H^{r,s}(X_y, \boldsymbol{E}|X_y)$ in an obvious way and these isomorphisms yield a holomorphic trivialization of the cohomology bundle $\boldsymbol{H}^{r,s}(\boldsymbol{E})$ over $U \subset Y$.

3. Remarks on a representation theorem of Wells and Wolf. In their paper [26], Wells and Wolf establish—among other things! — some conjectures of Griffiths ([6], [7]) on the geometric representation of certain *automorphic cohomologies*; cf. also [8], [22], [23], [25]. The framework is the following:

If D is a period domain or, more generally, a flag domain and $Y \subset D$ is a maximal compact subvariety of dimension s then there is a diagram

$$M_D \xleftarrow{\pi} \mathscr{Y}_D \xrightarrow{\tau} D \tag{3.1}$$

where τ is holomorphic, $\pi \colon \mathscr{Y}_D \to M_D$ is a holomorphic fibre bundle with fibre Y; M_D is the space of linearly deformed compact subvarieties of dimension s. Wells and Wolf prove the (difficult!) result that M_D is a *Stein manifold* provided that D has compact isotropy, D being a homogeneous space $D = G_0/V$, cf. below. They then establish their principal *representation theorem*: For non-degenerate homogeneous vector bundles $\boldsymbol{E}_\lambda = G_0 \times {}_V \lambda$ over $D = G_0/V$, there exists a Fréchet injection

$$H^s(D, \mathscr{O}(\boldsymbol{E}_\lambda)) \to H^0(M_D, R^s\pi_*(\mathscr{O}(\tau^*\boldsymbol{E}_\lambda))) . \tag{3.2}$$

In this assertion λ is an irreducible unitary representation of V; cf.

[26, Theorem 3.4.7]. The injection is G_0-equivariant and thus permits the representation of automorphic cohomology with respect to a discrete subgroup of G_0.

In this section we show that

$$\tau^* \boldsymbol{E}_\lambda \to \mathscr{Y}_D \to M_D \tag{3.3}$$

is, in fact, a BL-diagram; since the fibre Y is compact this amounts to showing that $\tau^*\boldsymbol{E}_\lambda$ is locally trivial over M_D. We then indicate how to compute the cohomology bundles $\boldsymbol{H}^{r,s}(\tau^*\boldsymbol{E}_\lambda)$. Furthermore, the direct image sheaf $R^s\pi_*(\tau^*\boldsymbol{E}_\lambda)$ is locally free and coincides with $\mathscr{O}(\boldsymbol{H}^{0,s}(\tau^*\boldsymbol{E}_\lambda))$; this yields an explicit description of the right-hand side of (3.2).

Some of the details are the the following: G is a connected complex semi-simple Lie group, $P \subset G$ a parabolic subgroup and G_0 a non-compact real form of G. We assume once and for all that $V = G_0 \cap P$ is *compact*.

If one chooses maximal compact subgroups $\tilde{M}$, K of G, G_0, respectively, such that $V \subset K \subset \tilde{M}$, then $V = K \cap P = \tilde{M} \cap P$, the real orbit $G_0 \cdot 0$ of the neutral coset $0 \in G/P$ is open in the complex flag manifold $X = G/P$ and thus $D = G_0/V = G_0 \cdot 0$ inherits a *complex* structure. $\tilde{M}/V$ and K/V also possess complex structures, being equal to G/P and $K^c/K^c \cap P$, respectively.

Finally, if $\lambda: V \to \mathrm{GL}(E)$ is an irreducible unitary representation, it extends uniquely to an irreducible *holomorphic* representation of P and it follows that the homogeneous vector bundles $G_0 \times_V E \to D$, $K \times_V E \to K/V$ inherit holomorphic structures as holomorphic pull-backs from $G_0 \cdot 0$ and $K^c K^c \cap P$.

We put $Y = K \cdot 0 \subset D$, $A = \{a \in G \mid aY \subset D\} (= G_C\{D\}$ in the notations of [26]) $L = \{a \in G \mid aY = Y\} \subset A$, a closed complex Lie subgroup of G, and we let $\sigma: G \to G/L$, $\beta: G \to G/L \cap P$ be the natural maps (which are holomorphic principal bundles). Now A is open in G, $AL = A$; furthermore setting

$$\begin{aligned} M = M_D &= \sigma A \subset G/L \quad \text{(open)} \\ \mathscr{Y} = \mathscr{Y}_D &= \beta A \subset G/L \cap P \quad \text{(open)}\,; \end{aligned} \tag{3.4}$$

it is clear that e.g., $\sigma^{-1}(M) = A$ and we conclude that $\sigma | A: A \to M$ is a holomorphic principal L-bundle. Similarly, $\beta^{-1}(\mathscr{Y}) = A$ and $\beta | A: A \to Y$ is a holomorphic principal $(L \cap P)$-bundle. If $\varepsilon: G/L \cap P \to G/L$ is the natural fibration, $\varepsilon^{-1}(M) = \mathscr{Y}$ and the fibration $\varepsilon | \mathscr{Y}: \mathscr{Y} \to M$ is the *linear deformation space* of Y.

Setting $\tilde{A} = A/L \cap P$, let $\pi_2: A \to \tilde{A}$ be the quotient map. It then is clear that the map $\beta a \to \pi_2 a$, $a \in A$, identifies $\tilde{A}$ and $\mathscr{Y}$ and also that $\pi_2 \colon A \to \tilde{A}$ is a holomorphic principal $(L \cap P)$-bundle. We are thus in the situation of Example (ii) of Section 2 (with $H = L$, $L = L \cap P$, $Z = A$, etc.) and any holomorphic representation λ of $L \cap P$ on a finite-dimensional vector space E yields a BL-diagram

$$\tilde{\boldsymbol{E}}_\lambda \to \tilde{A} \to M \tag{3.5}$$

where $\pi \colon \tilde{A} \to M = A/L$ again is the natural map. If we set $\boldsymbol{E}_0 = L \times {}_{L \cap P} E$, then the cohomology bundles of (3.5) are given by

$$\boldsymbol{H}^{r,s}(\tilde{\boldsymbol{E}}_\lambda) = A \times_L H^{r,s}(L/L \cap P, \boldsymbol{E}_0) \,.$$

In the applications λ will be the restriction to $L \cap P$ of a holomorphic representation of P.

By the very definition of A the natural map $\tau \colon G/L \cap P \to X = G/P$ restricts to a map $\tau \colon \mathscr{Y} \to D (\tau\beta a = a \cdot 0$ for $a \in A)$. Let also $i \colon D \to X$ be the inclusion. A direct, albeit somewhat lengthy computation then yields the following:

THEOREM 3.6. *Let $\tilde{\lambda}$ be a holomorphic representation of P on the finite dimensional vector space E and $\boldsymbol{E}_{\tilde{\lambda}} = G \times {}_P E$ the corresponding homogeneous vector bundle over $X = G/P$. Set $\lambda = \tilde{\lambda} | L \cap P$ and let $\tilde{\boldsymbol{E}}_\lambda \to A$ be the induced bundle. Then, under the bundle isomorphism of $\varepsilon \colon \mathscr{Y} \to M$ onto $\pi \colon \tilde{A} \to M$ mentioned above, the diagram*

$$\tau^* \boldsymbol{E}_{\tilde{\lambda}} \to \mathscr{Y} \to M \tag{3.7}$$

is isomorphic to

$$\tilde{\boldsymbol{E}}_\lambda \to A \to M \,.$$

In particular (3.7) is a BL-diagram (as claimed in (3.3)) and its cohomology bundle of type (r, s) is given by

$$\boldsymbol{H}^{r,s}(\tau^* \boldsymbol{E}_{\tilde{\lambda}}) = A \times {}_L H^{r,s}(L/L \cap P, \boldsymbol{E}_0) \tag{3.8}$$

where $\boldsymbol{E}_0 = L \times {}_{L \cap P} E$.

One concludes that the E_2-term of the Leray spectral sequence of (3.7) is given by ${}^0E_2^{s,t} = H^{0,s}(M, A \times {}_L H^{0,t}(L/L \cap P, \boldsymbol{E}_0))$. Since we assume V to be compact, the main result of [26, Section 2.5] asserts that M is a Stein manifold; accordingly, the spectral sequence degenerates: ${}^0E_2^{s,t} = 0$ for $s > 0$ and we see that

$$H^{0,q}(\mathscr{Y}, \tau^* \boldsymbol{E}_{\tilde{\lambda}}) \cong H^{0,0}(M, A \times {}_L H^{0,q}(L/L \cap P, \boldsymbol{E}_0)) \tag{3.9}$$

for $q \geqq 0$.

Suppose, in particular, that $\tilde{\lambda}$ is the holomorphic extension to P of an irreducible unitary representation of V in E and let $\boldsymbol{E}_\lambda = G_0 \times {}_V E$ be the corresponding homogeneous bundle over D with the holomorphic structure described earlier. Then if $\boldsymbol{E}_\lambda$ is non-degenerate in the sense of [26], the results of Schmid [21] imply that $H^q(D, \boldsymbol{E}_\lambda) = 0$ for $q \neq s = \dim Y$ and that the induced map

$$H^s(D, \boldsymbol{E}_\lambda) \to H^s(\mathscr{Y}, \tau^* \boldsymbol{E}_\lambda) \tag{3.10}$$

is a Fréchet injection. Lastly, one has to argue that (3.9) is an isomorphism of Fréchet spaces (using the open mapping theorem as in [26]). (3.10) and (3.9) then imply the representation theorem (3.2).

As a by-product one obtains the following:

COROLLARY 3.11. *Let $\pi^{0,s}$ be the representation of L on $H^{0,s}(L/L \cap P, \boldsymbol{E}_0)$ induced by left multiplication. Then the space $H^0(M_D, R^s\pi_*(\mathscr{O}(\tau^*\boldsymbol{E}_\lambda)))$ of* (3.2) *coincides with the space of all maps*

$$f: A \to H^{0,s}(L/L \cap P, \boldsymbol{E}_0)$$

satisfying the conditions:

(i) *f is holomorphic*

(ii) *$f(al) = \pi^{0,s}(l^{-1})f(a)$ for $(a, l) \in A \times L$.*

4. **Discontinuous group actions and automorphic factors.** Let $\boldsymbol{E} \to X \to Y$ be a BL-diagram and suppose that the group Γ acts freely and properly discontinuously on $\boldsymbol{E}$, X and Y such that $\pi: X \to Y$ and $\sigma: \boldsymbol{E} \to X$ are equivariant and that the action on $\boldsymbol{E}$ is fibrewise linear. We then show that $\Gamma \backslash \boldsymbol{E} \to \Gamma \backslash X \to \Gamma \backslash Y$ again is a BL-diagram and we relate the cohomologies of the two diagrams. In the special case where $\boldsymbol{E}$ is globally trivial over Y (i.e., $\boldsymbol{E} = X \times \boldsymbol{E}_0$, $X = Y \times F$ in the earlier notations), the cohomology bundles of the quotient diagram are determined by an automorphic factor which we compute below; applications will follow in Section 5.

First of all, we recall some well-known results (which, in any case, are easily verified): Let X be a complex manifold and Γ a group acting on X, say on the left, by holomorphic maps: $\Gamma \times X \to X$ maps (γ, x) to γx and $x \to \gamma x$ is holomorphic; the group Γ is considered to be discrete. The action is properly discontinuous (p.d., for short) if for each compact set $K \subset X$, the set of $\gamma \in \Gamma$ with $\gamma K \cap K \neq \emptyset$ is finite. If Γ acts freely and properly discontinuously, then the quotient $\Gamma \backslash X$ is a complex manifold in a natural way such that the quotient map $q: X \to \Gamma \backslash X$ is a holomorphic submersion (and is, in fact, locally biholomorphic).

Let $\boldsymbol{E} \to \Gamma \backslash X$ be a holomorphic vector bundle with fibre E such that

$X \times E \cong q^*\boldsymbol{E}$ and let ϕ be a fixed such trivialization. Since $(q^*\boldsymbol{E})_x = \boldsymbol{E}_{q(x)} = \boldsymbol{E}_{q(\gamma x)} = (q^*\boldsymbol{E})_{\gamma x}$, the trivialization induces the linear maps $\phi_{\gamma x}^{-1} \circ \phi_x$ of E, denoted by $j(\gamma, x)$. Clearly $j(\gamma, x) \in \mathrm{GL}(E)$ and $x \to j(\gamma, x)$ is holomorphic. Moreover $j(\gamma\delta, x) = j(\gamma, \delta x) \cdot j(\delta, x)$ for $\gamma, \delta \in \Gamma$ and $x \in X$: j is an *automorphic factor* $\Gamma \times X \to \mathrm{GL}(E)$. In turn j defines a left operation of Γ on $X \times E$ by: $\gamma \cdot (x, e) = (\gamma x, j(\gamma, x)e)$ and one shows that $\boldsymbol{E} \cong \Gamma \backslash (X \times E)$ as a vector bundle over $\Gamma \backslash X$. The action of Γ on $X \times E$ is automatically free and p.d. and we also denote $\Gamma \backslash (X \times E)$ by $\boldsymbol{E}(j)$.

REMARKS. Given the automorphic factor $j: \Gamma \times X \to \mathrm{GL}(E)$ and a holomorphic map $h: X \to \mathrm{GL}(E)$, $j_h(\gamma, x) = h(\gamma x) \circ j(\gamma, x) \circ h(x)^{-1}$ defines another automorphic factor and we see that $\boldsymbol{E}(j_h) \cong \boldsymbol{E}(j)$ — and conversely.

The holomorphic sections of $\boldsymbol{E}(j)$ coincide with those holomorphic functions $f: X \to E$ which satisfy $f(\gamma x) = j(\gamma, x) f(x)$ for $(\gamma, x) \in \Gamma \times X$ (=holomorphic automorphic forms).

One now obtains the following basic result:

THEOREM 4.1. *Let $\boldsymbol{E} \to X \to Y$ be a* BL-*diagram, $\sigma: \boldsymbol{E} \to X$ and $\pi: X \to Y$ the projections. Suppose that the group Γ acts on the left on $\boldsymbol{E}$, X and Y by holomorphic maps such that*

(a) *the actions are free and properly discontinuous;*

(b) *the maps σ, π are equivariant;*

(c) *the action on $\boldsymbol{E}$ is fibrewise linear.*

Then there are induced maps $\tilde{\sigma}: \Gamma \backslash \boldsymbol{E} \to \Gamma \backslash X$ and $\tilde{\pi}: \Gamma \backslash X \to \Gamma \backslash Y$ such that

$$\Gamma \backslash \boldsymbol{E} \to \Gamma \backslash X \to \Gamma \backslash Y$$

is a BL-*diagram. Moreover the cohomology bundles of the two diagrams are related by*

$$q^* \boldsymbol{H}^{r,s}(\Gamma \backslash \boldsymbol{E}) \cong \boldsymbol{H}^{r,s}(\boldsymbol{E}) \tag{4.2}$$

with $q: Y \to \Gamma \backslash Y$ the natural map.

In the proof one uses the following fact: each $y \in Y$ has an open neighbourhood U such that $\gamma U \cap U = \varnothing$ for $\gamma \neq 1$ and then $U \to q(U)$ is biholomorphic. This shows, e.g., that $\tilde{\pi}: \Gamma \backslash X \to \Gamma \backslash Y$ is a holomorphic fibre bundle with fibre F(=fibre of $X \to Y$). Similar arguments then imply that $\Gamma \backslash \boldsymbol{E}$ is a holomorphic vector bundle over $\Gamma \backslash X$ with fibre E, the fibre of $\boldsymbol{E}$ and that it is also locally trivial over $\Gamma \backslash Y$ with fibre $\boldsymbol{E}_0$. The verifications are straightforward and are omitted here.

Let $p: X \to \Gamma \backslash X$ be the natural projection. Then $p_y = p | X_y$ maps the fibre $X_y = \pi^{-1}(y)$ biholomorphically onto $\tilde{\pi}^{-1}(q(y)) \subset \Gamma\setminus X$ and is covered by a bundle isomorphism $\boldsymbol{E}|X_y \to \Gamma\setminus\boldsymbol{E}|\tilde{\pi}^{-1}(q(y))$; thus it induces an isomorphism p_y^* of $\boldsymbol{H}^{r,s}(\Gamma\setminus\boldsymbol{E})_{q(y)}$ onto $\boldsymbol{H}^{r,s}(\boldsymbol{E})_y$ since these simply are fibre cohomologies. The maps p_y^* yield the isomorphism (4.2).

Next we consider the case where the basic diagram (2.1) simply is $\boldsymbol{E} = X\times\boldsymbol{E}_0 \to Y\times F \to Y$ where $\boldsymbol{E}_0$ is a holomorphic vector bundle; in other words $\boldsymbol{E}$ is globally trivial over Y with fibre $\boldsymbol{E}_0$. In this case $\boldsymbol{H}^{r,s}(\boldsymbol{E})$ is the trivial bundle $Y\times H^{r,s}(F, \boldsymbol{E}_0)$. (4.2) therefore yields an isomorphism

$$\phi: Y \times H^{r,s}(F, \boldsymbol{E}_0) \cong q^*\boldsymbol{H}^{r,s}(\Gamma\setminus\boldsymbol{E}) . \tag{4.3}$$

Accordingly, there is an automorphic factor $j_\phi: \Gamma \times Y \to \mathrm{GL}(H^{r,s}(F, \boldsymbol{E}_0))$ such that $\boldsymbol{H}^{r,s}(\Gamma\setminus\boldsymbol{E}) \cong \boldsymbol{E}(j_\phi)$ and the following theorem determines j_ϕ:

Observe, firstly, that the action of Γ on $X = Y \times F$ necessarily is of the form $\gamma\cdot(y, f) = (\gamma y, I(\gamma, yf), f \to I(\gamma, y)f)$ holomorphic in f (and also in y). By assumption Γ acts on $\boldsymbol{E}$ by bundle automorphisms covering this action on $Y \times F$ and this implies that the holomorphic automorphism $I(\gamma, y)$ of F is covered by an automorphism $\tilde{I}(\gamma, y)$ of $\boldsymbol{E}_0$; $y \to \tilde{I}(\gamma, y)$ still is holomorphic. Accordingly, there are induced automorphisms of the vector spaces $H^{r,s}(F, \boldsymbol{E}_0)$, denoted by $I(\gamma, y)^*$. With these notations:

THEOREM 4.4. *The automorphic factor j_ϕ derived from* (4.3) *is given by*

$$j_\phi(\gamma, y) = (I(\gamma, y)^{-1})^* = I(\gamma^{-1}, y)^* \tag{4.5}$$

for $(\gamma, y) \in \Gamma \times Y$. The cohomology bundles $\boldsymbol{H}^{r,s}(\boldsymbol{E})$ are the trivial bundles $Y \times H^{r,s}(F, \boldsymbol{E}_0)$ and

$$\boldsymbol{H}^{r,s}(\Gamma\setminus\boldsymbol{E}) = \Gamma\setminus(Y \times H^{r,s}(F, \boldsymbol{E}_0)) \tag{4.6}$$

where Γ acts on the product by $\gamma\cdot(y, h) = (\gamma y, I(\gamma^{-1}, \gamma y)^* h)$.

Once again the proof is straightforward and will not be reproduced here.

5. Vanishing theorem for projective varieties $\Gamma\setminus G_0\backslash T$. Let G_0 be a connected non-compact semi-simple Lie group admitting a faithful finite-dimensional representation; G_0 is a real form of a connected semi-simple complex Lie group G. We assume here that G is *simply connected.*

Let $K \subset G_0$ be a maximal compact subgroup such that G_0/K has a G_0-invariant complex structure (thus is a Hermitian symmetric space). Since G_0 and K now have the same rank, we can choose a Cartan subgroup T of G_0 such that $T \subset K$; G, G_0 and K satisfy the assumptions of Section 3.

Let $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{t}$ be the complexifications of the Lie algebras $\mathfrak{g}_0$, $\mathfrak{k}_0$ and $\mathfrak{t}_0$ of G_0, K and T, respectively, and for a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, set $\mathfrak{p} = \mathfrak{p}_0^C$; here $\mathfrak{p}_0 = \mathfrak{k}_0^\perp$ with respect to the Killing form $(,)$ of $\mathfrak{g}_0$. Let Δ be the set of non-zero roots of $(\mathfrak{g}, \mathfrak{t})$ and let Δ_n, Δ_k be the sets of those roots $\alpha \in \Delta$ whose root spaces $\mathfrak{g}_\alpha$ satisfy $\mathfrak{g}_\alpha \subset \mathfrak{p}$ respectively $\mathfrak{g}_\alpha \subset \mathfrak{k}$ (compact, non-compact roots). Choose a system of positive roots compatible with the complex structure of G_0/K, i.e., such that the following holds: If $\Delta_n^+ = \Delta^+ \cap \Delta_n$ and if $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ is the splitting of the complexified tangent space at $0 \in G_0/K$ induced by the complex structure, then

$$\mathfrak{p}^\pm = \Sigma\{\mathfrak{g}_{\pm\alpha} \mid \alpha \in \Delta_n^+\} . \tag{5.1}$$

The compatibility condition on Δ^+ may be rephrased as follows: Every non-compact root $\alpha \in \Delta^+$ is *totally positive*: this means that if $\beta \in \Delta_k$ is such that $\alpha + \beta \in \Delta$, then in fact $\alpha + \beta \in \Delta_n^+$. Equivalently one can say that $\mathfrak{p}^\pm$ are K-stable abelian subalgebras.

With $\Delta_k^+ = \Delta^+ \cap \Delta_k$, set $\mathfrak{b}_k = \mathfrak{t} \oplus \Sigma\{g_{-\alpha} \mid \alpha \in \Delta_k^+\}$, $\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{p}^-$, $\mathfrak{b} = \mathfrak{k} \oplus \Sigma\{g_{-\alpha} \mid \alpha \in \Delta^+\}$, and let now K^C, $P^\pm$, U, B_k and B be the closed complex subgroups of G corresponding to these Lie algebras. $B_k \subset K^C$ is a Borel subgroup such that $K \cap B_k = T$, $B_k = K^C \cap B$ and we set

$$F = K/T = K^C/B_k ; \tag{5.2}$$

in the notations of Section 3, $V = T$ for the choice $P = B$. The following is fundamental:

THEOREM 5.3. (Harish-Chandra [9], [19], [31]). *The subgroups K^C, $P^\pm$ and U are closed in G and $P^\pm$ are simply connected. The exponential maps $\mathfrak{p}^\pm \to P^\pm$ are diffeomorphisms, K^C normalizes $P^\pm$ and $U = K^C P^-$, a semi-direct product, is a parabolic subgroup of G such that $G_0 \cap U = K$. The map $(x, k, y) \to (\exp x)k(\exp y)$ of $\mathfrak{p}^+ \times K^C \times \mathfrak{p}^-$ into G is a biholomorphism onto a dense open subset $\Omega = P^+K^CP^-$ in G containing G_0. Given $a \in \Omega$ let*

$$a = a^+k(a)a^- \tag{5.4}$$

be the corresponding decomposition, $k(a) \in K^C$. In particular, $(ak)^+ = a^+$, $k(ak) = k(a)k$ for $a \in \Omega$, $k \in K^C$. Then the map $\zeta: \Omega \to \mathfrak{p}^+$ given by

$$\zeta(a) = \log(a^+) \tag{5.5}$$

induces a biholomorphism of G_0/K onto $\zeta(G_0)$; $\zeta(G_0)$ is a bounded domain in $\mathfrak{p}^+$.

Now set $Y = G_0/K$ and define $J: G_0 \times Y \to K^C$, following Satake [16], [20], by

$$J(a, y) = k(a \exp \zeta(y)) ; \tag{5.6}$$

one has $J(ab, y) = J(a, by)J(b, y)$ for $a, b \in G_0$ and letting $0 = 1K$ be the neutral coset, $J(a, 0) = k(a)$, in particular: $J(k, 0) = k$. $J(a, y)$ is C^∞ in (a, y) and holomorphic in y and is called the *canonical automorphic factor* of Y. If moreover $\tau: K^C \to \mathrm{GL}(E)$ is a holomorphic representation, we set $j_\tau = \tau \circ J$ and obtain what is called the canonical automorphic factor "of type τ" ([16]).

With the notations introduced above, $B \subset G$ is a Borel subgroup such that $G_0 \cap B = T$; hence G_0/T inherits a complex structure as the open (real) orbit $G_0 \cdot 0 \subset G/B$. Similarly, the complex structure of $Y = G_0/K$ is the one of the orbit $G_0 \cdot 0 \subset G/U$.

From [10; Lemma 2], one obtains the following:

PROPOSITION 5.7. *The map* $\phi(aT) = (aK, J(a, 0)B_k) = (aK, k(a)B_k)$ *of* G_0/T *onto* $Y \times F$ *is biholomorphic and the action of* G_0 *on* G_0/T *transforms into the following action on* $Y \times F$:

$$a(y, f) = (ay, J(a, y)f) . \tag{5.8}$$

Since the argument in [10] appears to be somewhat incomplete we include a proof of the assertion: ϕ is injective since $K \cap B_k = T$ and $k(ak) = k(a)k$ for $a \in G_0$, $k \in K$. Next, $k(a)^{-1}kB_k \in F$ for $a \in G_0$, $k \in K^C$ and so we can write $k(a)^{-1}k = k_0 b_0$ with $k_0 \in K$, $b_0 \in B_k$. With this $\phi(ak_0T) = (ak, kB_k)$ and ϕ is surjective. Using once more that $J(a, 0)k_0 = k(a)k_0 = kb_0^{-1}$, one derives (5.8) by a direct computation. Note also that ϕ certainly is C^∞.

Next, by the definition of the holomorphic structure of G_0/T, ϕ^{-1} will be holomorphic if and only if the composite map $(aK, kB_k) \to ak_0B \in G_0 \cdot 0 \subset G/B$ is holomorphic. Since $B_k \subset B$ and K^C normalizes $P^- \subset B$, we have $ak_0B = ak(a)^{-1}kB = a^+kB$ (cf. Theorem 5.3) and by (5.5), $aK \to a^+$ is holomorphic and, of course, so is $kB_k \to kB$. Accordingly, $(aK, kB_k) \to a^+kB$ is holomorphic and maps $Y \times F$ to $G_0 \cdot 0 \subset G/B$; hence ϕ^{-1} is holomorphic. Thus ϕ is a diffeomorphism such that ϕ^{-1} is holomorphic and, therefore, ϕ itself is holomorphic. This completes the argument.

Now we fix a C^∞ character λ of T and form the line bundle $\boldsymbol{L}_\lambda = G_0 \times {}_T\boldsymbol{C} \to G_0/T$; since λ extends uniquely to a holomorphic character of B, $\boldsymbol{L}_\lambda$ has the structure of a holomorphic line bundle over G_0/T("$\subset G/B$"). Also define $\boldsymbol{E}_0 = K^C \times {}_{B_k}\boldsymbol{C} \to F = K^C/B_k$. Then:

PROPOSITION 5.9. *Let again* $Y = G_0/K$. *Then* $\boldsymbol{L}_\lambda \to G_0/T \to Y$ *is a* BL-*diagram with cohomology bundles* $\boldsymbol{H}^{r,s}(\boldsymbol{L}_\lambda) = Y \times H^{r,s}(F, \boldsymbol{E}_0)$.

For the proof, observe first of all that the map ϕ of Proposition 5.7

is a global trivialization of the holomorphic bundle $G_0/T \to Y$. We define a map ψ from $\boldsymbol{L}_\lambda$ to $F \times \boldsymbol{E}_0$ covering ϕ by

$$\psi([a, z]) = (aK, [k(a), z]) \tag{5.10}$$

for $(a, z) \in G_0 \times \boldsymbol{C}$. Since λ extends to B_k and $k(at) = k(a)t$ for $a \in G_0$, $t \in T$, ψ is well-defined. A simple verification shows that ψ is a fibrewise linear bijection and it is obvious that ψ covers ϕ. There still remains to be shown that ψ is holomorphic, in which case it will be a biholomorphic bundle isomorphism.

The point here is to show that $[a, z] \to [k(a), z]$ is holomorphic from $\boldsymbol{L}_\lambda$ to $\boldsymbol{E}_0$ since $[a, z] \to aT \to aK$ clearly is holomorphic. Now the representation λ extends up to B and therefore $\boldsymbol{E}_0$ is the bundle induced on F by the bundle $G \times {}_B\boldsymbol{C} \to G/B$ under the natural map $F = K^C/B_k \to G/B$. On the other hand, if $j: G_0/T \to G/B$ again is the natural map, the definition of $\boldsymbol{L}_\lambda$ shows that this bundle is holomorphically isomorphic to $j^*(G \times {}_B\boldsymbol{C})$; explicitly, these bundle isomorphisms are given by

$$i([k, z]) = (kB_k, [k, z]) , \quad (k, z) \in K^C \times \boldsymbol{C} ,$$

for $\boldsymbol{E}_0$, and

$$j([a, z]) = (aT, [a, z]) , \quad (a, z) \in G_0 \times \boldsymbol{C} ,$$

for $\boldsymbol{L}_\lambda$.

Now the map $[a, z] \to k(a)B_k$ is the composition $[a, z] \to aT \to \phi(aT) = (aK, k(a)B_k) \to k(a)B_k$ and thus is holomorphic. There remains the map $[a, z] \to [k(a), z]$: $P^- \subset [B, B]$ implies $\lambda(P^-) = 1$ and so in $G \times {}_B\boldsymbol{C}$, one has $[k(a), z] = [(a^+)^{-1}a, z]$ where a^+ again is defined as in Theorem 5.3; by the same theorem, this is holomorphic in $[a, z]$ since it is holomorphic in aK. With this, the proposition is established.

COROLLARY 5.11. *Under the isomorphism* $\psi: \boldsymbol{L}_\lambda \cong Y \times \boldsymbol{E}_0$ *the action of* G_0 *on* $\boldsymbol{L}_\lambda$ *transforms into the action* $a \cdot (y, [k, z]) = (ay, [J(a, y)k, z]) := (ay, J(a, y)[k, z])$ *for* $a \in G_0$, $y \in Y$, $(k, z) \in K^C \times \boldsymbol{C}$.

In order to mention explicitly the representations involved in their construction, it will again be convenient to denote homogeneous bundles such as L_λ, $\boldsymbol{E}_0$, etc., by $K^C \times {}_{B_k}\lambda$, $G_0 \times {}_T\lambda$, etc.

Given $k \in K^C$, let l_k denote left translation by k in $F = K^C/B_k$ as well as, e.g., in $\boldsymbol{E}_0$. With this, we set

$$I(a, y) = l_{J(a,y)}: F \to F; \ \tilde{I}(a, y) = l_{J(a,y)}: \boldsymbol{E}_0 \to \boldsymbol{E}_0 \tag{5.12}$$

for $(a, y) \in G_0 \times Y$. It is clear that $\tilde{I}(a, y)$ is a bundle map over $I(a, y)$. If $l_k^*: H^{r,s}(F, \boldsymbol{E}_0) \to H^{r,s}(F, \boldsymbol{E}_0)$ is the induced action, then the representation

$\pi^{r,s}$ of K^C in $H^{r,s}(F, \boldsymbol{E}_0)$ is given by $\pi^{r,s}(k) = l^*_{k^{-1}}$.

Recall that G_0 acts on $Y \times F$ by $a(y, f) = (ay, I(a, y)f)$. Let now Γ be a discrete subgroup of G_0 which acts freely on $G_0/K = Y$. Then the action of G_0 restricts to a free and p.d. action of Γ on $Y \times F$ and the same holds for the action on $Y \times \boldsymbol{E}_0$; the projections $Y \times \boldsymbol{E}_0 \to Y \times F$ and $Y \times F \to Y$ are Γ-equivariant. Thus, all the assumption of Theorem 4.4 are satisfied and, since $(I(\gamma, y)^{-1})^* = \pi^{r,s}(I(\gamma, y)) = j_{\pi^{r,s}}(\gamma, y)$, one has:

THEOREM 5.13. *$\Gamma\backslash \boldsymbol{L}_\lambda \to \Gamma\backslash G_0/T \to \Gamma\backslash Y$ is a* BL*-diagram and its cohomology bundle of type (r, s) is $\boldsymbol{H}^{r,s}(\Gamma\backslash \boldsymbol{L}_\lambda) = \Gamma\backslash(Y \times H^{r,s}(F, \boldsymbol{E}_0))$ where Γ acts by $\gamma\cdot(y, h) = (\gamma y, j_{\pi^{r,s}}(\gamma, y)h)$.*

An equivalent description of $\boldsymbol{H}^{r,s}(\Gamma\backslash \boldsymbol{L}_\lambda)$ may be obtained as follows: Suppose that $\tau: K \to \mathrm{GL}(E)$ is a finite dimensional representation of K in the complex vector space E; τ extends holomorphically to K^C and then to $U = K^C P^-$ by requiring that $\tau | P^- = 1$. Using [16], [17] and [19], one concludes that the bundles $\boldsymbol{E}(j_\tau | \Gamma \times Y)$ and $\Gamma\backslash(G \times_U \tau) | Y$ are holomorphically equivalent where the restriction to Y of $G \times_U \tau$ is taken with respect to the Borel embedding $Y = G_0/K \to G_0 \cdot 1U \subset G/U$. With this we have:

COROLLARY 5.14. $\boldsymbol{H}^{r,s}(\Gamma\backslash \boldsymbol{L}_\lambda) = \Gamma\backslash(G \times_U \pi^{r,s}) | Y$.

Applying Theorem 2.6, we obtain:

COROLLARY 5.15. *Under the assumptions of Theorem* 5.13 *there is for each $p \geqq 0$ a spectral sequence $({}^pE_r^{s,t})$ which converges to $H^{p,\cdot}(\Gamma\backslash G_0/T, \Gamma\backslash \boldsymbol{L}_\lambda)$ and whose E_2-term is*

$$(5.16) \qquad {}^pE_2^{s,t} = \bigoplus_i H^{i,s-i}(\Gamma\backslash Y, \Gamma\backslash(G \times_U \pi^{p-i,t+i}) | Y) .$$

In particular ${}^0E_2^{s,t} = H^{0,s}(\Gamma\backslash Y, \Gamma\backslash(G \times_U \pi^{0,t}) | Y)$.

Next, the Borel-Weil theorem [11] implies that the representations $\pi^{0,t}$ vanish for all t except $t = q_0$, an integer determined by λ and described in detail below. Thus we conclude:

COROLLARY 5.16.

$$H^{0,q}(\Gamma\backslash G_0/T, \Gamma\backslash \boldsymbol{L}_\lambda) = H^{0,q-q_0}(\Gamma\backslash Y, \Gamma\backslash(G \times_U \pi^{0,q_0}) | Y)$$

for every q; cf. also (5.18) below.

This result was first established by Ise [10, Proposition 8] under the additional assumption that $\Gamma\backslash Y$ is compact; we do not require this restriction here.

Next we investigate when the spaces in Corollary 5.16 vanish: Identify λ with an integral element λ in the dual $\mathfrak{t}^*$ of $\mathfrak{t}$, i.e., a linear form λ such that

$$2\frac{(\lambda, \alpha)}{(\alpha, \alpha)} \in \boldsymbol{Z}, \quad \alpha \in \Delta ;$$

recall that (,) denotes the Killing form of $\mathfrak{g}$. Also set $2\delta = \Sigma_{\Delta^+}\alpha$, $2\delta_k = \Sigma_{\Delta_k^+}\alpha$, $2\delta_n = \Sigma_{\Delta_n^+}\alpha = 2\delta - 2\delta_k$ and let W_k be the subgroup of the Weyl group W of $(\mathfrak{g}, \mathfrak{t})$ generated by the compact root reflections. Since Δ^+ is compatible with the complex structure of G_0/K, one knows that $w\Delta_n^+ = \Delta_n^+$ for $w \in W_k$ and $(\delta_n, \alpha) = 0$ for $\alpha \in \Delta_k^+$. A linear form $\eta \in \mathfrak{t}^*$ is said to be Δ-regular (Δ_k-regular) if $(\eta, \alpha) \neq 0$ for $\alpha \in \Delta(\alpha \in \Delta_k)$. With this, we define $F_0' \subset t^*$ and $P^{(\Lambda)} \subset \Delta$ as follows: $\Lambda \in F_0'$ if and only if Λ is integral, $\Lambda + \delta$ is Δ-regular and

(5.17) $(\Lambda + \delta, \alpha) > 0$ for $\alpha \in \Delta_k^+$; $\alpha \in P^{(\Lambda)}$ for $\alpha \in \Delta$ if and only if $(\Lambda + \delta, \alpha) > 0$ whenever $\Lambda + \delta$ is Δ-regular.

Thus $P^{(\Lambda)}$ is a system of positive roots corresponding to the Δ-regular element $\Lambda + \delta$. The Borel-Weil theorem states that $\pi^{0,t}$ vanishes for all t if there is $\alpha \in \Delta_k^+$ such that $(\lambda + \delta_k, \alpha) = 0$; if this is not the case, then $\lambda + \delta_k$ is Δ_k^+-regular and the value of q_0 in Corollary 5.16 is

(5.18) $$q_0 = |\{\alpha \in \Delta_k^+ \,|\, (\lambda + \delta_k, \alpha) < 0\}| = |w(-\Delta_k^+) \cap \Delta_k^+|$$

where $w \in W_k$ is the unique element such that $(w(\lambda + \delta_k), \alpha) > 0$ for every $\alpha \in \Delta_k^+$ and where $|s|$ denotes the cardinality of the set s.

Moreover π^{0,q_0} is an irreducible representation of K with Δ_k^+-highest weight

(5.19) $$\tau(\lambda, w) := w(\lambda + \delta_k) - \delta_k .$$

With the above choice of w it is a straightforward computation to prove:

PROPOSITION 5.20. *$\tau(\lambda, w) \in F_0'$ if and only if $\lambda + \delta$ is Δ-regular. In this case $P^{(\tau(\lambda, w))} = wP^{(\lambda)}$.*

At this point we nearly are in a position to apply some results of [27] and [28] to obtain vanishing theorems for the spaces $H^{0,q}(\Gamma \backslash G_0/T, \Gamma \backslash \boldsymbol{L}_\lambda)$; however some additional notation will be needed:

Let Λ be integral and such that $\Lambda + \delta$ is Δ-regular. Given (w_1, τ) in $W \times W_k$, we define:

(5.21)
$$Q_\Lambda = \{\alpha \in \Delta_n^+ \,|\, (\Lambda + \delta, \alpha) > 0\}, \quad P_n^{(\Lambda)} = P^{(\Lambda)} \cap \Delta_n ,$$
$$2\delta^{(\Lambda)} = \Sigma\{\alpha \,|\, \alpha \in P^{(\Lambda)}\} ,$$
$$\Phi_{w_1}^{(\Lambda)} = w_1(-P^{(\Lambda)}) \cap P^{(\Lambda)} ,$$

$$\Phi^k_\tau = \tau(-\Delta^+_k) \cap \Delta^+_k ,$$
$$A_{\Lambda,\tau,w_1} = \{\alpha \in P^{(\Lambda)}_n \mid w_1^{-1}\tau\alpha \in -P^{(\Lambda)}\} .$$

Assume now that $\Gamma \setminus Y$ is *compact*. In this case, the main theorem [28, Theorem 4.3] applies to the right-hand side of Corollary 5.16. Among other things this theorem states that if π_Λ is an irreducible K-module with Δ^+_k-highest weight $\Lambda \in F'_0$ and if $H^{0,q}(\Gamma \setminus Y, \Gamma \setminus (G \times_U \pi_\Lambda) | Y) \neq 0$, then there is a pair $(w_1, \tau) \in W \times W_k$ such that

$$q = |A_{\Lambda,\tau,w_1}| - 2|Q_\Lambda \cap A_{\Lambda,\tau,w_1}| + |Q_\Lambda| . \tag{5.22}$$

One has $\Delta^+_k \subset w_1 P^{(\Lambda)}$, $A_{\Lambda,\tau,w_1} = \Phi^{(\Lambda)}_{\tau^{-1}w_1} - \Phi^k_\tau$ and $\tau(\delta + \delta - \delta^{(\Lambda)}) = w_1(\Lambda + \delta - \delta^{(\Lambda)}) = \Lambda + \delta - \delta^{(\Lambda)}$; also, A_{Λ,τ,w_1}, $\Phi^{(\Lambda)}_{w_1}$ and $\{\alpha \in P^{(\Lambda)}_n \mid \tau\alpha \in - P^{(\Lambda)}_n\}$ are contained in $\{\alpha \in P^{(\Lambda)}_n \mid (\Lambda + \delta - \delta^{(\Lambda)}, \alpha) = 0\}$, with $\Phi^k_{\tau^{-1}} \subset \{\alpha \in \Delta^+_k \mid (\Lambda + \delta - \delta^{(\Lambda)}, \alpha) = 0\}$.

We now assume that $\lambda \in \mathfrak{t}^*$ is integral and such that $\lambda + \delta$ is Δ-regular; one notes that $\lambda + \delta_k$ is Δ^+_k-regular, so that the Borel-Weil theorem gives the highest weight $\tau(\lambda, w)$ of (5.19) and Proposition 5.20 yields $\tau(\lambda, w) \in F'_0$ as well as $P^{(\tau(\lambda,w))} = wP^{(\lambda)}$. One concludes that $P^{(\tau(\lambda,w))}_n = wP^{(\lambda)}_n$ and hence that

$$A_{\tau(\lambda,w),\tau,w_1} = wA_{\lambda,\tau w,w_1 w} . \tag{5.23}$$

Similar arguments show that $Q_{\tau(\lambda,w)} = wQ_\lambda$, $\tau(\lambda, w) + \delta - \delta^{(\tau(\lambda,w))} = w(\lambda + \delta - \delta^{(\lambda)})$, $\Phi^{(\tau(\lambda,w))}_{w_1} = w_1 w(-P^{(\lambda)}) \cap wP^{(\lambda)}$. Hence Corollary 5.16 and the equation (5.22) yield:

THEOREM 5.24. *Let $\lambda \in \mathfrak{t}^*$ be integral, $\boldsymbol{L}_\lambda \to G_0/T$ the corresponding holomorphic line bundle. Suppose that the discrete subgroup $\Gamma \subset G_0$ acts freely on $Y = G_0/K$ such that $\Gamma \setminus Y$ is compact. If $\lambda + \delta_k$ is not Δ^+_k-regular, then $H^{0,q}(\Gamma \setminus G_0/T, \Gamma \setminus \boldsymbol{L}_\lambda) = 0$ for every q. On the other hand if $\lambda + \delta$ is Δ-regular then $\lambda + \delta_k$ is Δ^+_k-regular and there is a unique element $w \in W_k$ such that $(w(\lambda + \delta_k), \alpha) > 0$ for every $\alpha \in \Delta^+_k$. Then for every q*

$$H^{0,q}(\Gamma \setminus G_0/T, \Gamma \setminus \boldsymbol{L}_\lambda) = H^{0,q-q_0}(\Gamma \setminus Y, \Gamma \setminus (G \times_U \pi^{0,q_0}) | Y)$$

where π^{0,q_0} is the representation of K with Δ^+_k-highest weight $w(\lambda + \delta_k) - \delta_k$ and q_0 is given by (5.18). If $H^{0,q}(\Gamma \setminus G_0/T, \Gamma \setminus \boldsymbol{L}_\lambda) \neq 0$ there is a pair $(w_1, \tau) \in W \times W_k$ such that the following hold:

(i) $q = q_0 + |A_{\lambda,\tau w,w_1 w}| - 2|Q_\lambda \cap A_{\lambda,\tau w,w_1 w}| + |Q_\lambda|$;

(ii) $\Delta^+_k \subset w_1 w P^{(\lambda)}$, $wA_{\lambda,\tau w,w_1 w} = \tau^{-1} w_1(-P^{(\lambda)}) \cap (wP^{(\lambda)} - \Phi^k_{\tau^{-1}})$, $\tau w(\lambda + \delta - \delta^{(\lambda)}) = w_1 w(\lambda + \delta - \delta^{(\lambda)}) = w(\lambda + \delta - \delta^{(\lambda)})$, $\Phi^k_{\tau^{-1}} \subset \{\alpha \in \Delta^+_k \mid (w(\lambda + \delta - \delta^{(\lambda)}), \alpha) = 0\}$;

(iii) $wA_{\lambda,\tau w,w_1 w}$, $w_1 w(-P^{(\lambda)} \cap wP^{(\lambda)})$ *and* $\{\alpha \in wP^{(\lambda)}_n \mid \tau\alpha \in - wP^{(\lambda)}_n\}$ *are contained in* $\{\alpha \in P^{(\lambda)}_n \mid (w(\lambda + \delta - \delta^{(\lambda)}), \alpha) = 0\}$.

Because of its generality this theorem—like [28, Theorem 4.3] — has several corollaries of which we mention the following:

Firstly, assume that $(\lambda + \delta - \delta^{(\lambda)}, \alpha) \neq 0$ for every $\alpha \in P_n^{(\lambda)}$. By (iii), one has $A_{\lambda, \tau w, w_1 w} = \varnothing$ and so (i) gives $q = q_0 + |Q_\lambda|$:

COROLLARY 5.25. *If λ is integral, $\lambda + \delta$ is Δ-regular and $(\lambda + \delta - \delta^{(\lambda)}, \alpha)$ is $\neq 0$ for every $\alpha \in P_n^{(\lambda)}$, then $H^{0,q}(\Gamma \backslash G_0/T, \Gamma \backslash \boldsymbol{L}_\lambda) = 0$ for $q \neq q_0 + |Q_\lambda|$.*

Next suppose that λ is Δ_k^+-dominant. Then we must have $w = 1$ and so $q_0 = 0$. If moreover $(\lambda + 2\delta, \alpha) < 0$ for $\alpha \in \Delta_n^+$, one finds that $Q_\lambda = \varnothing$ and $(\lambda + \delta - \delta^{(\lambda)}, \alpha) < 0$ for $\alpha \in \Delta_n^+ = -P_n^{(\lambda)}$; this yields the following known result:

COROLLARY 5.26. *If λ is Δ_k^+-dominant integral and $(\lambda + 2\delta, \alpha) < 0$ for $\alpha \in \Delta_n^+$, then $H^{0,q}(\Gamma \backslash G_0/T, \Gamma \backslash \boldsymbol{L}_\lambda) = 0$ for $q \neq 0$.*

This result can also be obtained directly from the Kodaira vanishing theorem. Another specialization of λ leads to the following result:

THEOREM 5.27. *Let λ be integral such that $\lambda + \delta$ is Δ-regular and suppose that $P^{(\lambda)}$ is compatible with an invariant complex structure on $Y = G_0/K$ (cf. the beginning of this section). If $H^{0,q}(\Gamma \backslash G_0/T, \Gamma \backslash \boldsymbol{L}_\lambda) \neq 0$, there exists a parabolic subalgebra $\theta = \mathfrak{r} \oplus \mathfrak{u}$ of $\mathfrak{g}$, $\mathfrak{r}$ the reductive and $\mathfrak{u}$ the unipotent part of θ, such that if $\theta_{\mathfrak{u},n}$ denotes the set of non-compact roots in $\mathfrak{u}$ and $\Delta(\mathfrak{r})$ the set of all roots in $\mathfrak{r}$, then*

(i) *$q = q_0 + 2|\theta_{\mathfrak{u},n} \cap wQ| + |\Delta_n^+ - wQ| - |\theta_{\mathfrak{u},n}|$ with q_0, w as in Theorem* 5.24;

(ii) *θ contains the Borel subalgebra $\mathfrak{t} + \Sigma\{\mathfrak{g}_\alpha | \alpha \in wP^{(\lambda)}\}$;*

(iii) *$(w(\lambda + \delta - \delta^{(\lambda)}), \alpha) = 0$ for $\alpha \in \Delta(\mathfrak{r})$.*

The result follows from Proposition 5.20, the calculation in (5.23), and [27, Theorems 5.24 and 2.3], once one observes that since $P^{(\tau(\lambda,w))} = wP_n^{(\lambda)}$ and $w \in W_k$, every non-compact root in $P^{(\tau(\lambda,w))}$ actually is totally positive.

A very simple application of Theorem 5.27 is the following: Assume that λ actually is Δ^+-dominant. Then $P^{(\lambda)} = \Delta^+$ (so that every non-compact root in $P^{(\lambda)}$ is totally positive), $\delta^{(\lambda)} = \delta$, $w = 1$, $q_0 = 0$, $Q_\lambda = \Delta_n^+$, $\theta_{\mathfrak{u},n} = wP^{(\lambda)} - \Delta(\mathfrak{r}) = \Delta_n^+ - \Delta(\mathfrak{r}) \subset \Delta_n^+$; hence by (i) of Theorem 5.27, $q = 2|\theta_{\mathfrak{u},n}| - |\theta_{\mathfrak{u},n}| = |\theta_{\mathfrak{u},n}|$.

COROLLARY 5.28. *If λ is Δ^+-dominant integral and if $H^{0,q}(\Gamma \backslash G_0/T, \Gamma \backslash \boldsymbol{L}_\lambda) \neq 0$, then $q = |\theta_{\mathfrak{u},n}|$ for some parabolic subalgebra $\theta = \mathfrak{r} \oplus \mathfrak{u} \subset \mathfrak{g}$ containing $\mathfrak{t} + \Sigma_{\Delta^+}\mathfrak{g}_\alpha$.*

Moreover $(\lambda, \Delta(\mathfrak{r})) = 0$. If G_0 is simple then the set of numbers $|\theta_{\mathfrak{u},n}|$

for θ such that $\theta \supset \mathfrak{t} + \Sigma_{\Delta_T} \mathfrak{g}_\alpha$ is determined completely in [27, *Table* 3.4]. *In particular $H^{0,q}(\Gamma\backslash G_0/T, \Gamma\backslash \boldsymbol{L}_\lambda) = 0$ for $q < |\{\alpha \in \Delta_n^+ | (\lambda, \alpha) > 0\}|$.*

We conclude with some (more or less known) remarks about the cohomology of G_0/T:

By Proposition 5.9 and the fact that the spectral sequence ${}^0E_r^{s,t}$ degenerates since $Y = G_0/K$ is Stein, we obtain:

THEOREM 5.29. *With the above notations, for any integral λ and all $q \geqq 0$*

(i) $H^{0,q}(G_0/T, \boldsymbol{L}_\lambda) \cong H^{0,0}(Y, Y \times H^{0,q}(F, \boldsymbol{E}_0)) \cong H^{0,0}(Y) \otimes H^{0,q}(F, \boldsymbol{E}_0)$. *Hence if there is $\alpha \in \Delta_k^+$ such that $(\lambda + \delta_k, \alpha) = 0$ then by the Borel-Weil theorem $H^{0,q}(G_0/T, \boldsymbol{L}_\lambda) = 0$ for all q. If $\lambda + \delta_k$ is Δ_k-regular let w, q_0 be as in Theorem* 5.24. *Then $H^{0,q}(G_0/T, \boldsymbol{L}_\lambda) = 0$ for $q \neq q_0$ and $H^{0,q_0}(G_0/T, \boldsymbol{L}_\lambda) \cong H^{0,0}(Y) \otimes H^{0,q_0}(F, \boldsymbol{E}_0)$ where $H^{0,q_0}(F, \boldsymbol{E}_0)$ is an irreducible K-module with Δ_k^+-highest weight $w(\lambda + \delta_k) - \delta_k$.*

COROLLARY 5.30. *In particular suppose that $(\lambda + \delta_k, \alpha) < 0$ for $\alpha \in \Delta_k^+$. Then $q_0 = |\Delta_k^+| = s = \dim_C K/T$ and hence $H^{0,q}(G_0/T, \boldsymbol{L}_\lambda) = 0$ for $q \neq s$.*

Equation (i) of Theorem 5.29 may be regarded as a Hermitian version of the representation formula (3.2): in the present situation the fibration $\mathscr{Y}_D \to M_D$ of (3.1) collapses to $G_0/T \to G_0/K$ by [26, Proposition 2.4.7].

REFERENCES

[1] A. BOREL, A spectral sequence for complex analytic bundles, in: Appendix II to F. Hirzebruch, Topological Methods in Algebraic Geometry, Grundlehren der Math. Wissenschaften, Band 131, Springer-Verlag, Berlin, Heidelberg, New York, 1966, 202–217.

[2] R. BOTT, Homogeneous vector bundles, Ann. Math. (2) 66 (1957), 203–248.

[3] H. R. FISCHER AND F. L. WILLIAMS, The Borel spectral sequence-some remarks and applications, in: Topology-Calculus of Variations and Their Applications, volume dedicated to L. Euler, Marcel Dekker Publishing Company, New York, 1983.

[4] R. FISHER, JR., On the Dolbeault cohomology of a compact complex nilmanifold with values in a line bundle, preprint (1981), Dept. of Math., University of Oklahoma, Norman, OK.

[5] R. GODEMENT, Topologie Algébrique et Théorie des Faisceaux, Hermann, Paris, 1964.

[6] P. GRIFFITHS, Periods of integrals on algebraic manifolds, I, Amer. J. of Math. 90 (1968), 568–626.

[7] P. GRIFFITHS, Periods of integrals on algebraic manifolds: Summary of main results and discussion of open problems, Bull. Amer. Math. Soc. 76 (1970), 228-296.

[8] P. GRIFFITHS AND W. SCHMID, Locally homogeneous complex manifolds, Acta Math. 123 (1969), 253-302.

[9] HARISH-CHANDRA, Representations of semisimple Lie groups, VI, Amer. J. of Math. 78 (1956), 564–628.

[10] M. ISE, Generalized automorphic forms and certain holomorphic vector bundles, Amer. J. of Math. 86 (1964), 70–108.

[11] B. KOSTANT, Lie algebra cohomology and the generalized Borel-Weil theorem, Ann. of Math. (2) 74 (1961), 329-387.
[12] J. LE POTIER, Sur la suite spectrale de A. Borel, C. R. Acad. Sc. Paris, 276 (1973), série A, 463-466.
[13] J. LE POTIER, "Vanishing theorem" pour un fibré vectoriel holomorphe positif de rang quelconque, Journées Géom. analytique de Poitiers 1972, Bull. Soc. Math. France 38 (1974), 107-119.
[14] J. LE POTIER, Annulation de la cohomologie à valeurs dans un fibré vectoriel positif de rang quelconque, Math. Ann. 218 (1975), 35-53.
[15] Y. MATSUSHIMA, On the intermediate cohomology group of a holomorphic line bundle over a complex torus, Osaka J. Math. 16 (1979), 617-631.
[16] Y. MATSUSHIMA and S. Murakami, On vector bundle valued harmonic forms and automorphic forms on symmetric Riemannian manifolds, Ann. of Math. 78 (1963), 365-416.
[17] Y. MATSUSHIMA and S. Murakami, On certain cohomology groups attached to Hermitian symmetric spaces, Osaka J. Math. 2 (1965), 1-35.
[18] D. MUMFORD, Abelian varieties, Tata Inst. Studies in Math., Oxford Univ. Press, 1970.
[19] S. MURAKAMI, Cohomology groups of vector-valued forms on symmetric spaces, Lecture Notes, Univ. of Chicago, 1966.
[20] I. SATAKE, Factors of automorphy and Fock representation, Advances in Math. 7 (1971), 83-110.
[21] W. SCHMID, Homogeneous complex manifolds and representations of semisimple Lie groups, Thesis, Univ. of California, Berkeley, 1967, announced in Proc. Nat. Acad. Sc. U. S. A. 59 (1968), 56-59.
[22] R. WELLS, JR., Parametrizing the compact submanifolds of a period matrix domain by a Stein manifold, Symposium on Several Complex Variables, Park City, Utah, 1970, Lecture Notes in Math. 184, Springer-Verlag, Berlin, Heidelberg and New York.
[23] R. WELLS, JR., Automorphic cohomology on homogeneous complex manifolds, Rice Univ. Studies 59 (1973), 147-155.
[24] R. WELLS, JR., Complex manifolds and mathematical physics, Bull. Amer. Math. Soc. 1, (1979), 296-336.
[25] R. WELLS. JR. AND J. WOLF, Poincaré theta series and L_1-cohomology, Proc. of Symposia in Pure Math. 30 (several complex variables, part II), (1977), 59-66.
[26] R. WELLS, JR. AND J. WOLF, Poincaré series and automorphic cohomology of flag domains, Ann. of Math. 105 (1977), 397-448.
[27] F. WILLIAMS, Vanishing theorems for type (o, q) cohomology of locally symmetric spaces, Osaka J. Math. 18 (1981), 147-160.
[28] F. WILLIAMS, Vanishing theorems for type (o, q) cohomology of locally symmetric spaces II, Osaka J. Math. 20 (1983), 95-108.
[29] F. WILLIAMS, Frobenius reciprocity and Lie group representations on $\bar{\partial}$-cohomology spaces, L'Enseignement Math. 28 (1982), 3-30.
[30] J. WOLF, The action of a real semi-simple group on a complex flag manifold I, Bull. Amer. Math. Soc. 75 (1969), 1121-1237.
[31] J. WOLF, Fine structure of Hermitian symmetric spaces, in: Symmetric Spaces (short courses, Wash. Univ.), (W. Boothby and G. Weiss, eds.), Marcel Dekker, New York, 1972.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MASSACHUSETTS
AMHERST, MA 01003
U.S.A.