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BOREL STOCHASTIC GAMES WITH LIM SUP PAYOFF

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We consider two-person zero-sum stochastic games with limit superior payoff function and Borel measurable state and action spaces. The games are shown to have a value and the value function is calculated by transfinite iteration of an operator and proved to be upper analytic. The paper extends results of our earlier article [17] in which the same class of games was considered for countable state spaces and finite action sets.

1. Introduction. In [17], a class of two-person zero-sum stochastic games was formulated as follows. Let X be a countable, nonempty set of states, and let A and B be finite, nonempty sets of actions for players I and II, respectively. Let u be a bounded, real-valued utility function on X and let q be a function which assigns to each triple $(x, a, b) \in X \times A \times B$ a probability distribution on X. The game starts at some initial state x. Player I chooses an action $a_1 \in A$ and, simultaneously, player II chooses $b_1 \in B$. The next state x_1 has distribution $q(\cdot|x, a_1, b_1)$ and is announced to the players along with their chosen actions. The procedure is iterated so as to generate a random sequence $x_1, x_2, \dots$ and the payoff from player II to player I is

$$u^* = \limsup_n u(x_n). \tag{1.1}$$

It was proved in [17] that this game has a value.

The aim of the present article is to extend this result to a Borel measurable setting. The following assumptions will remain in effect throughout the paper:

(1.2)

(i) X, A, B will be nonempty Borel subsets of Polish spaces.
(ii) F, G will be Borel subsets of $X \times A$, $X \times B$, respectively, with nonempty vertical sections $F(x), G(x)$ for all $x \in X$. At the state x, $F(x)(G(x))$ is the set of actions that player I (II) is allowed to use.
(iii) $G(x)$ is compact for every $x \in X$.
(iv) q is a Borel measurable transition function on $J \times \mathscr{B}(X)$, where J is the Borel set $\{(x, a, b) \in X \times A \times B: a \in F(x) \text{ and } b \in G(x)\}$ and $\mathscr{B}(X)$ is the Borel σ-field of X.
(v) For every fixed set $E \in \mathscr{B}(X)$ and $(x, a) \in F$, the function $q(E|x, a, \cdot\,)$ is continuous on $G(x)$.
(vi) u is a bounded, upper analytic function on X, that is, for every real c, the set $\{u > c\}$ is analytic.

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Let $\mathscr{N}(u)(x)$ be the stochastic game with initial state x in which the payoff from player II to player I is u^*. (The play of the game is as described in the first paragraph above with measurability conditions which will be explained in Section 2.) Here is the main result of the paper.

THEOREM 1.1. *Assume that the conditions* (1.2) *hold. Then, for each* $x \in X$, *the game* $\mathscr{N}(u)(x)$ *has a value. The value function is upper analytic. Furthermore, for every* $\varepsilon > 0$, *both players have* ε*-optimal families of universally measurable strategies.*

The techniques which will be used to prove Theorem 1.1 are similar to the methods of [17]. However, there are problems of measurability which arise and which are solved by methods from the theory of inductive definability. Similar methods were used to resolve problems of measurability in the theory of gambling in [7] and [16].

Stochastic games were formulated by Shapley [23], with state and action spaces finite and payoff function equal to the total discounted reward. Shapley proved that his game had a value and that both players had optimal stationary strategies. Thereafter, a number of authors considered the problem when the payoff function is the average reward per day. Notable contributors to the average reward problem include Gillette [11], Hoffman and Karp [12], Blackwell and Ferguson [5] and Kohlberg [13], who solved different special cases of the problem. The definitive solution of the problem was provided by Mertens and Neyman [18], who based their proof on a difficult result of Bewley and Kohlberg [2] on the asymptotic behavior of the value of the discounted reward game as the discount factor tends to one. Stochastic games with general state and action spaces were considered by a number of authors. Nowak's article [21] provides an excellent bibliography. In the same article, Nowak formulated the conditions (1.2) and he considered both discounted and positive stochastic games under these assumptions. Indeed, his Theorem 5.1 will be the point of departure of the present article.

Blackwell [3] proposed a variant of Shapley's game in which the law of motion was eliminated but which allowed for payoff functions more general than either the total discounted reward or the average reward per day. He proved that a win–lose game, where the winning set for player I is a G_δ subset of the set of histories, has a value. In [4], he gave an operator solution of the same problem. This second paper of Blackwell is the inspiration for the present paper, as it was for our previous article [17].

Our paper is organized as follows. The next section sets down definitions, notation and some preliminary results. Auxiliary games are treated in Sections 3 and 4. Section 5 handles the measurability problems involved in iterating the auxiliary games a transfinite number of times. Theorem 1.1 is proved in Section 6.

2. Preliminaries. Let $Z = A \times B \times X$ and define the *space of histories* to be $H = Z \times Z \times \cdots$. Elements of H will be denoted by $h = (z_1, z_2, \dots)$. We use $p_n(h)$, or more briefly, p_n to denote the *partial history* $(z_1, z_2, \dots, z_n)$.

Let $P(A)$ and $P(B)$ be the sets of probability measures on the Borel subsets of A and B, respectively. Equip $P(A)$ and $P(B)$ with the weak topology, so that $P(A)$ and $P(B)$ are both Borel subsets of Polish spaces. Moreover, the sets $\{(x,\mu) \in X \times P(A): \mu(F(x)) = 1\}$ and $\{(x,\nu) \in X \times P(B): \nu(G(x)) = 1\}$ are Borel in $X \times P(A)$ and $X \times P(B)$, respectively, and for every $x \in X$, the set $\{\nu \in P(B): \nu(G(x)) = 1\}$ is compact. See Parthasarathy [22, Chapter 2] for details.

Let $K = \{(x,\mu,\nu) \in X \times P(A) \times P(B): \mu(F(x)) = 1 = \nu(G(x))\}$. Then K is a Borel subset of $X \times P(A) \times P(B)$. We now define a function m on $K \times \mathscr{B}(Z)$ by the formula

$$m(A_1 \times B_1 \times X_1 | x, \mu, \nu) = \int_{B_1} \int_{A_1} q(X_1 | x, a, b)\, d\mu(a)\, d\nu(b),$$

where A_1, B_1, X_1 are Borel subsets of A, B, X, respectively. It is easy to verify that m is a Borel measurable transition function.

A *universally measurable* (or, more briefly, *measurable*) *strategy* σ for player I available at x is a sequence $\sigma_0, \sigma_1, \ldots$ where $\sigma_0 \in P(A)$, such that $\sigma_0(F(x)) = 1$, and for $n \geq 1$, σ_n is a universally measurable mapping from Z^n to $P(A)$ (i.e., measurable when Z^n is endowed with its σ-field of universally measurable subsets and $P(A)$ is equipped with its Borel σ-field) such that, for every $(z_1, z_2, \ldots, z_n) \in Z^n$, $\sigma_n(z_1, z_2, \ldots, z_n)(F(x_n)) = 1$, where $z_n = (a_n, b_n, x_n)$. A *universally measurable* (or more briefly, *measurable*) *strategy* τ for II available at x is defined analogously with $P(B)$ and $G(x)$ in place of $P(A)$ and $F(x)$, respectively. Measurable strategies σ and τ available at x determine a probability measure $P_{\sigma,\tau} = P_{x,\sigma,\tau}$ on the Borel subsets of H. (The initial state x will usually be clear from the context and we will suppress it.) Namely, the $P_{\sigma,\tau}$-distribution of the first coordinate $z_1 = (a_1, b_1, x_1)$ is $P_{\sigma_0,\tau_0} = m(\cdot | x, \sigma_0, \tau_0)$ and the $P_{\sigma,\tau}$ conditional distribution of $z_{n+1} = (a_{n+1}, b_{n+1}, x_{n+1})$ given $z_1, z_2, \ldots, z_n$ is $P_{\sigma,\tau}(\cdot | z_1, z_2, \ldots, z_n) = m(\cdot | x_n, \sigma_n(z_1, z_2, \ldots, z_n), \tau_n(z_1, z_2, \ldots, z_n))$. The existence of $P_{\sigma,\tau}$ is proved in [1, Proposition 7.45]. If g is a bounded, universally measurable function on H, we write its expectation under $P_{\sigma,\tau}$ as $\int g\, dP_{\sigma,\tau}$ or $E_{\sigma,\tau}(g)$.

If σ is a measurable strategy for player I available at some x and $p = (z_1, z_2, \ldots, z_n)$ a partial history, the *conditional strategy* $\sigma[p]$ is defined by

$$\sigma[p]_0 = \sigma_n(p),$$

$$\sigma[p]_m(z_1', z_2', \ldots, z_m') = \sigma_{n+m}(z_1, z_2, \ldots, z_n, z_1', z_2', \ldots, z_m')$$

for all $m \geq 1$ and $(z_1', z_2', \ldots, z_m') \in Z^m$. Note that $\sigma[p]$ is a measurable strategy for player I available at x_n. Given measurable strategies σ and τ for players I and II available at x, the probability measure $P_{\sigma[p],\tau[p]} = P_{x_n,\sigma[p],\tau[p]}$ is easily seen to be a version of the $P_{\sigma,\tau}$ conditional distribution for $(z_{n+1}, z_{n+2}, \ldots)$ given $(z_1, z_2, \ldots, z_n)$. Thus, if $g: H \to R$ is bounded and universally measurable,

$$E_{\sigma,\tau}(g) = \int \{E_{\sigma[p_n(h)],\tau[p_n(h)]}(gp_n(h))\}\, dP_{\sigma,\tau}(h), \tag{2.1}$$

where for $p = p_n(h) = (z_1, z_2, \ldots, z_n)$, gp is the p-section of g defined on H by $(gp)(h') = (gp)(z_1', z_2', \ldots) = g(z_1, z_2, \ldots, z_n, z_1', z_2', \ldots)$. In the special case when $g(h) = u^*(h) = \limsup_n u(x_n)$, the function u^*p is just u^* and (2.1) simplifies to

$$E_{\sigma,\tau}(u^*) = \int \{E_{\sigma[p_n(h)],\tau[p_n(h)]}(u^*)\} \, dP_{\sigma,\tau}(h).$$

The upper value $\bar{V}(x)$ and the lower value $\underline{V}(x)$ of the game $\mathscr{N}(u)(x)$ are defined as follows:

$$\bar{V}(x) = \inf_\tau \sup_\sigma E_{\sigma,\tau}(u^*) \tag{2.2}$$

and

$$\underline{V}(x) = \sup_\sigma \inf_\tau E_{\sigma,\tau}(u^*), \tag{2.3}$$

where the suprema are over all measurable strategies σ for player I available at x and the infima over all measurable strategies τ for player II available at x.

We say that $(\sigma(x))_{x \in X}$ is a *universally measurable family of strategies* for player I if for every $x \in X$, $\sigma(x)$ is a measurable strategy for I available at x and, for every $n \geq 0$, $(\sigma(x))_n(z_1, z_2, \ldots, z_n)$ is a universally measurable mapping from $X \times Z^n$ to $P(A)$. One defines a *universally measurable family of strategies* for player II analogously.

A *stopping time* t is a mapping from H to $\{0, 1, \ldots\} \cup \{\infty\}$ such that, for $n = 0, 1, \ldots$, if $t(h) = n$ and h' agrees with h in the first n coordinates, then $t(h') = n$. [Notice that, if $t(h) = 0$ for some h, then t is identically zero.] A *stop rule* t is a stopping time which is everywhere finite. A stopping time (stop rule) t is *universally measurable* if, for every $n \geq 0$, the set $\{t \leq n\}$ is a universally measurable subset of H.

If t is a stopping time, $h = (z_1, z_2, \ldots) = ((a_1, b_1, x_1), (a_2, b_2, x_2), \ldots)$, and $t(h) < \infty$, we define the variables z_t, x_t, p_t to have values $z_{t(h)}, x_{t(h)}, p_t(h) = (z_1, z_2, \ldots, z_{t(h)})$ at h. Suppose now that t is a universally measurable stop rule. Then, it is not hard to verify that $P_{\sigma[p_t],\tau[p_t]} = P_{x_t,\sigma[p_t],\tau[p_t]}$ is a version of the $P_{\sigma,\tau}$ conditional distribution for $(z_{t+1}, z_{t+2}, \ldots)$ given $(z_1, z_2, \ldots, z_t)$ and (2.1) generalizes to

$$E_{\sigma,\tau}(g) = \int \{E_{\sigma[p_t],\tau[p_t]}(gp_t)\} \, dP_{\sigma,\tau}. \tag{2.4}$$

If t is a stop rule and $p = (z_1, z_2, \ldots, z_n)$ is a partial history, define $t[p]$ on H by

$$t[p](z_1', z_2', \ldots) = t(z_1, z_2, \ldots, z_n, z_1', z_2', \ldots) - n.$$

Notice that, if $t(z_1, z_2, \ldots, z_n, \ldots) \geq n$, then $t[p]$ is itself a stop rule, in which case $t[p]$ is called a *conditional stop rule given p*. If t is universally measurable, then so is $t[p]$. When $p = (z)$, we write z for p and $t[z] = t[p]$.

There is a natural way to associate with every stop rule t an ordinal number $j(t)$ called the *index* of t by setting $j(0) = 0$ and requiring, for $t > 0$, that

$$j(t) = \sup\{j(t[z]) + 1: z \in Z\}.$$

This definition of the index is equivalent to that of Dellacherie and Meyer [6], as was pointed out in [15, Proposition 4.1]. Furthermore, $j(t)$ is familiar to students of Dubins and Savage as being the structure of the finitary function z_t (cf. [8], Sections 2.7 and 2.9) except for the uninteresting case when Z is a singleton. Some of our proofs (Lemma 2.3 and Theorem 4.3) will use transfinite induction on $j(t)$ and it is important to notice that, for all $t > 0$ and all z, $j(t[z])$ is strictly less than $j(t)$.

Suppose $t > 0$ is a universally measurable stop rule and consider the special case of (2.1) where $n = 1$ and $g = u(x_t)$. Note that

$$(x_t z_1)(z_2, z_3, \dots) = x_t(z_1, z_2, \dots) = x_{t[z_1]}(z_2, z_3, \dots)$$

if we make the convention that $x_{t[z_1]}(z_2, z_3, \dots) = x_1$ when $t[z_1] = 0$. Thus (2.1) gives

$$E_{\sigma,\tau}(u(x_t)) = \int \left\{ E_{\sigma[z_1],\tau[z_1]}(u(x_{t[z_1]})) \, dP_{\sigma_0,\tau_0}(z_1). \right. \tag{2.5}$$

A *universally measurable* (or, just *measurable*) *policy* for player I available at x is a pair (σ, t) where σ is a measurable strategy for I available at x and t is a universally measurable stop rule. We say that $(\sigma(x), t(x))_{x \in X}$ is a *universally measurable family of policies* for player I if $(\sigma(x))_{x \in X}$ is a universally measurable family of strategies for player I, $t(x)$ is a stop rule for every $x \in X$ and $t(x)(h)$ is a universally measurable function on $X \times H$.

We conclude this section with some results which will be needed in the sequel.

Recall that a real-valued function φ on a Borel set Ω is upper analytic if the set $\{\varphi > c\}$ is analytic for every real c.

LEMMA 2.1. *Let φ be a function on a Borel subset Ω of a Polish space into $[0, 1]$. Then φ is upper analytic if and only if the set*

$$E = \{(\omega, c) \in \Omega \times [0, 1]: \varphi(\omega) > c\}$$

is analytic in $\Omega \times [0, 1]$.

PROOF. The "if" part follows from the fact that a section of an analytic set is analytic. For the converse, note that

$$E = \bigcup \left[\{\omega \in \Omega: \varphi(\omega) > r\} \times [0, r) \right],$$

where the union is over all rationals r in $[0, 1]$. Plainly, E is analytic. □

LEMMA 2.2. *Let $x \in X$, φ be a bounded, upper analytic function on X and μ be a probability measure on A such that $\mu(F(x)) = 1$. Define a function ψ on*

$G(x)$ by

$$\psi(b) = \iint \varphi(x_1) q(dx_1|x, a, b)\, \mu(da).$$

Then ψ is continuous on $G(x)$.

PROOF. Use [21, F3.9] and the dominated convergence theorem. □

For each $n \geq 1$, let $\mathscr{F}_n$ be the σ-field of subsets of H of the form $A \times Z \times Z \times \cdots$, where A is a universally measurable subset of Z^n, let $\mathscr{F}_0 = \{\phi, H\}$ and let $\mathscr{F}_\infty$ be the σ-field generated by $\bigcup_{n \geq 0} \mathscr{F}_n$. Suppose now that t is a universally measurable stopping time on H. Define $\mathscr{F}_t$ to be the collection of all sets $E \in \mathscr{F}_\infty$ such that $E \cap \{t \leq n\} \in \mathscr{F}_n$ for every $n \geq 0$. It is straightforward to verify that $\mathscr{F}_t$ is a σ-field.

Suppose next that $\sigma, \sigma'(\tau, \tau')$ are measurable strategies for player I (II) available at x_0 and that t is a universally measurable stopping time. We say that the pair (σ, τ) *agrees with* (σ', τ') *prior to time* t if $n < t(h)$ implies

$$\sigma_n(p_n(h)) = \sigma'_n(p_n(h))$$

and

$$\tau_n(p_n(h)) = \tau'_n(p_n(h)).$$

LEMMA 2.3. *Let $\sigma, \sigma'(\tau, \tau')$ be measurable strategies for player I (II) available at x_0 and s a universally measurable stop rule such that (σ, τ) agrees with (σ', τ') prior to time s. Then $P_{\sigma,\tau} = P_{\sigma',\tau'}$ on $\mathscr{F}_s$.*

PROOF. The proof is by induction on the index $j(s)$. For $s = 0$, the result is trivial. So assume the result is true for all universally measurable stop rules of index less than ξ, where $\xi > 0$ is a fixed ordinal. Let s be a universally measurable stop rule of index ξ. Suppose $L \in \mathscr{F}_s$. Then, denoting by Lz_1 the z_1-section of L, we have:

$$\begin{aligned} P_{\sigma,\tau}(L) &= \int P_{\sigma[z_1],\tau[z_1]}(Lz_1)\, dP_{\sigma_0,\tau_0}(z_1), \\ &= \int P_{\sigma'[z_1],\tau'[z_1]}(Lz_1)\, dP_{\sigma'_0,\tau'_0}(z_1) \\ &= P_{\sigma',\tau'}(L), \end{aligned}$$

where the first and last equalities are by virtue of a variant of (2.1) and the intermediate equality uses the inductive hypothesis applied to $\sigma[z_1], \sigma'[z_1], \tau[z_1], \tau'[z_1], s[z_1]$, the fact that $Lz_1 \in \mathscr{F}_{s[z_1]}$ and the fact that $\sigma_0 = \sigma'_0$ and $\tau_0 = \tau'_0$ since $s \geq 1$. □

Lemma 2.3, in the gambling context, is proved in [25].

We now want to extend Lemma 2.3 to universally measurable stopping times. The next result is due to V. Pestien and S. Ramakrishnan.

LEMMA 2.4. *Let t be a universally measurable stopping time. Then $\mathscr{F}_t$ is the smallest σ-field $\mathscr{J}$ containing $\bigcup_{n \geq 0} \mathscr{F}_{t \wedge n}$, where $t \wedge n$ is the stop rule whose value at h is the smaller of the numbers $t(h)$ and n.*

PROOF. Let $n, k \geq 0$ and let $k > n$. We claim that

$$A \in \mathscr{F}_n \quad \text{implies} \quad A \cap \{t > k\} \in \mathscr{F}_{t \wedge k}. \tag{2.6}$$

To establish (2.6), we must prove for each $m \geq 0$ that

$$A \cap \{t > k\} \cap \{t \wedge k \leq m\} \in \mathscr{F}_m.$$

If $m < k$, then $A \cap \{t > k\} \cap \{t \wedge k \leq m\}$ is empty. If $m \geq k$, then $A \cap \{t > k\} \cap \{t \wedge k \leq m\} = A \cap \{t > k\}$. Since $A \in \mathscr{F}_n$, $\{t > k\} \in \mathscr{F}_k$ and $m \geq k > n$, it follows that $A \cap \{t > k\} \cap \{t \wedge k \leq m\} \in \mathscr{F}_m$, which proves (2.6).

Next, we observe that

$$A \in \mathscr{F}_\infty \quad \text{implies} \quad A \cap \{t = \infty\} \in \mathscr{J}. \tag{2.7}$$

Suppose $A \in \mathscr{F}_n$. Then

$$A \cap \{t = \infty\} = \bigcap_{k=n+1}^{\infty} A \cap \{t > k\},$$

so, by (2.6), $A \cap \{t = \infty\} \in \mathscr{J}$. Since $\bigcup_{n \geq 0} \mathscr{F}_n$ is a field generating $\mathscr{F}_\infty$, a monotone class argument establishes (2.7).

Now let $A \in \mathscr{F}_t$. Write

$$A = \left[\bigcup_{n \geq 0} A \cap \{t \leq n\} \right] \cup [A \cap \{t = \infty\}].$$

It is easy to see that $A \cap \{t \leq n\} \in \mathscr{F}_{t \wedge n} \subseteq \mathscr{J}$ for every $n \geq 0$. Hence, by (2.7), $A \in \mathscr{J}$. Thus, $\mathscr{F}_t \subseteq \mathscr{J}$.

For the reverse inclusion, observe that $\mathscr{F}_{t \wedge n} \subseteq \mathscr{F}_n$. Hence, if $A \in \mathscr{F}_{t \wedge n}$, then $A \cap \{t \leq m\} \in \mathscr{F}_m$ for $m \geq n$. If $A \in \mathscr{F}_{t \wedge n}$ and $m < n$, then $A \cap \{t \leq m\} = A \cap \{t \wedge n \leq m\} \in \mathscr{F}_m$. Hence $\mathscr{F}_{t \wedge n} \subseteq \mathscr{F}_t$ for every $n \geq 0$, so $\mathscr{J} \subseteq \mathscr{F}_t$. □

LEMMA 2.5. *Let $\sigma, \sigma'(\tau, \tau')$ be measurable strategies for player I (II) available at x_0 and t a universally measurable stopping time such that (σ, τ) agrees with (σ', τ') prior to time t. Then $P_{\sigma,\tau} = P_{\sigma',\tau'}$ on $\mathscr{F}_t$. In particular, if φ is a bounded, universally measurable function on X and s a universally measurable stop rule, then*

$$\int_{\{t \leq m\}} \varphi(x_t)\, dP_{\sigma,\tau} = \int_{\{t \leq m\}} \varphi(x_t)\, dP_{\sigma',\tau'} \tag{i}$$

for every $m \geq 0$, and

$$\int_{\{t = \infty\}} \varphi(x_s)\, dP_{\sigma,\tau} = \int_{\{t = \infty\}} \varphi(x_s)\, dP_{\sigma',\tau'}. \tag{ii}$$

PROOF. The first assertion is an immediate consequence of Lemma 2.3 and Lemma 2.4. For (i), observe that the function $\varphi(x_t)1_{\{t \le m\}}$ is $\mathscr{F}_t$-measurable. For (ii), note that $\varphi(x_s)$ is $\mathscr{F}_\infty$-measurable and that the σ-fields $\mathscr{F}_t$ and $\mathscr{F}_\infty$ coincide when restricted to the set $\{t = \infty\} \in \mathscr{F}_t \cap \mathscr{F}_\infty$, so that the function $\varphi(x_s)1_{\{t=\infty\}}$ is $\mathscr{F}_t$-measurable. □

3. Auxiliary one-day games. Consider an auxiliary one-day game $\mathscr{A}(\varphi)(x)$ starting from x, where φ is a bounded upper analytic function on X. In the game $\mathscr{A}(\varphi)(x)$, players I and II choose actions a, b simultaneously such that $a \in F(x)$, $b \in G(x)$ and the payoff from II to I is

$$\int \varphi(x_1) q(dx_1|x, a, b).$$

It follows from Lemma 2.2 and an old result of Fan [10, Theorem 2] that for each fixed $x \in X$, the game $\mathscr{A}(\varphi)(x)$ has a value, I has an ε-optimal strategy and II has an optimal strategy. The value of the game $\mathscr{A}(\varphi)(x)$ will be denoted by $(S\varphi)(x)$.

Here are some facts about the measurability of the value function and of ε-optimal (optimal) strategies of the players.

LEMMA 3.1. *Let Ω be a Borel subset of a Polish space. Suppose that v is a bounded, upper analytic function on $\Omega \times X$. Let ψ: $\Omega \times X \to R$ be defined by*

$$\psi(\omega, x) = (Sv(\omega, \cdot))(x).$$

Then ψ is upper analytic.

LEMMA 3.2. *Let φ be a bounded, upper analytic function on X. For each $\varepsilon > 0$, there is a universally measurable function f: $X \to P(A)$ such that for each $x \in X$, $f(x)(F(x)) = 1$ and*

$$\inf_{b \in G(x)} \iint \varphi(x_1) q(dx_1|x, a, b) f(x)(da) \ge (S\varphi)(x) - \varepsilon.$$

Furthermore, there is a universally measurable function g: $X \to P(B)$ such that for each $x \in X$, $g(x)(G(x)) = 1$ and

$$\sup_{a \in F(x)} \iint \varphi(x_1) q(dx_1|x, a, b) g(x)(db) \le (S\varphi)(x).$$

Lemmas 3.1 and 3.2 are straightforward consequences of Theorem 5.1 in Nowak [21]. We now record for later use two useful properties of the operator S.

LEMMA 3.3. *Let $\varphi_1 \le \varphi_2 \le \cdots$ be uniformly bounded, upper analytic functions on X. Then* (a) $S\varphi_1 \le S\varphi_2$ *and* (b) $\lim_n S\varphi_n = S(\lim_n \varphi_n)$.

PROOF. (a) is obvious. For (b), set $\varphi = \lim_n \varphi_n$, so φ is a bounded, upper analytic function. Now let $\varepsilon > 0$ and fix x. Choose $\mu \in P(A)$ such that $\mu(F(x)) = 1$ and

$$\iint \varphi(x_1) q(dx_1|x, a, b)\mu(da) > (S\varphi)(x) - \varepsilon/2$$

for all $b \in G(x)$. By the monotone convergence theorem,

$$\iint \varphi_n(x_1) q(dx_1|x, a, b)\mu(da) \uparrow \iint \varphi(x_1) q(dx_1|x, a, b)\mu(da) \tag{3.1}$$

for all $b \in G(x)$.

Now the functions in (3.1) are continuous on $G(x)$ by virtue of Lemma 2.2, so, by Dini's Theorem [9, page 190], the convergence in (3.1) is uniform on $G(x)$. Consequently, for sufficiently large n and all $b \in G(x)$,

$$\iint \varphi_n(x_1) q(dx_1|x, a, b)\mu(da) \geq (S\varphi)(x) - \varepsilon.$$

Hence

$$\inf_{\nu \in P(G(x))} \iiint \varphi_n(x_1) q(dx_1|x, a, b)\mu(da)\nu(db) \geq (S\varphi)(x) - \varepsilon$$

for all sufficiently large n, and so

$$\sup_{\lambda \in P(F(x))} \inf_{\nu \in P(G(x))} \iiint \varphi_n(x_1) q(dx_1|x, a, b)\lambda(da)\nu(db)$$
$$\geq (S\varphi)(x) - \varepsilon.$$

It follows that

$$(S\varphi_n)(x) \geq (S\varphi)(x) - \varepsilon$$

for all sufficiently large n. This completes the proof. □

4. Leavable games. Let u be a bounded, upper analytic function on X. Then u and an initial position x determine a leavable game $\mathscr{L}(u)(x)$, which is played exactly like the game $\mathscr{N}(u)(x)$ introduced in Section 1, except that now I gets to terminate the game unilaterally at any time of his choice and the payoff to I from II is the value of u at the terminal state. More formally, I chooses a measurable strategy σ available at x and a measurable stop rule t, player II chooses a measurable strategy τ available at x, and the expected payoff to I from II is $E_{\sigma,\tau}(u(x_t))$. Here we allow $t = 0$ and require $x_0 = x$.

Define inductively

$$U_0 = u \tag{4.1}$$

and for $n \geq 0$,

$$U_{n+1} = u \vee SU_n. \tag{4.2}$$

Here $a \vee b$ is the maximum of a and b. Let

$$U = \sup_n U_n. \tag{4.3}$$

LEMMA 4.1. (a) *For every* $n \geq 0$, U_n *is upper analytic and* $U_n \leq U_{n+1}$.

(b) *The functions* U_n *are uniformly bounded.*

(c) U *is upper analytic and* $\sup|U| \leq \sup|u|$.

(d) U *is the least, bounded, upper analytic function* φ *on* X *such that* (i) $\varphi \geq u$ *and* (ii) $S\varphi \leq \varphi$.

(e) $U = u \vee SU$.

PROOF. To prove (a), use induction, Lemma 3.1 and Lemma 3.3. Next, observe that if φ is bounded and upper analytic on X, then $\sup|S\varphi| \leq \sup|\varphi|$. It follows from this and induction that $\sup|U_n| \leq \sup|u|$ for every $n \geq 0$. This establishes (b) and also that $\sup|U| \leq \sup|u|$. The other assertion in (c) is immediate from (a). For (d), assume that φ is bounded, upper analytic on X and satisfies (i) and (ii). So $\varphi \geq U_0 = u$. Suppose that $\varphi \geq U_n$. Then $\varphi \geq S\varphi \geq SU_n$ by Lemma 3.3(a) and, so $\varphi \geq u \vee SU_n = U_{n+1}$. Hence, $\varphi \geq U_n$ for all n and so $\varphi \geq U$. On the other hand, $U \geq u$, and, by Lemma 3.3(b), $SU = S(\lim_n U_n) = \lim_n SU_n \leq \lim_n U_{n+1} = U$. This proves (d). It follows that $U \geq u \vee SU$. For the opposite inequality, fix x and suppose that $u(x) < U(x)$. Then, for n sufficiently large, $u(x) < U_n(x)$ and so $U(x) = \lim_n U_{n+1}(x) = \lim_n (SU_n)(x) = (SU)(x)$ by Lemma 3.3(b). This completes the proof of (e). □

LEMMA 4.2. *For every* $n \geq 0$ *and* $\varepsilon > 0$, *player* I *has a universally measurable family of policies* $(\sigma_\varepsilon^n(x), t^n(x))_{x \in X}$ *such that* $t^n(x) \leq n$, $x \in X$, *and such that for any measurable strategy* τ *of player* II *available at* x,

$$E_{\sigma_\varepsilon^n(x), \tau}\big(u(x_{t^n(x)})\big) \geq U_n(x) - \varepsilon \tag{4.4}$$

for every $x \in X$.

PROOF. First, fix a universally measurable function f^* on X into $P(A)$ such that $f^*(x)(F(x)) = 1$ for every $x \in X$. The existence of f^* is a consequence of the von Neumann selection theorem [19, 4E.9, page 240].

For $n = 0$, let $t^0(x) \equiv 0$ and $\sigma_\varepsilon^0(x)$ be the strategy which uses f^* every day. Then (4.4) is a triviality for $n = 0$. Assume next that the result is true for n. Define

$$\begin{aligned} t^{n+1}(x)(h) &\equiv 0 \quad \text{if } u(x) \geq U_{n+1}(x), \\ &= t^n(x_1)(z_2, z_3, \dots) + 1 \quad \text{if } u(x) < U_{n+1}(x). \end{aligned}$$

Plainly, $t^{n+1}(x) \leq n + 1$. If $u(x) \geq U_{n+1}(x)$, let $\sigma_\varepsilon^{n+1}(x)$ be the strategy which uses f^* every day. Suppose, next, that $u(x) < U_{n+1}(x)$. Fix a universally measurable function $f\colon X \to P(A)$ such that for each $x \in X$, $f(F(x)) = 1$ and

$$(4.5) \qquad \inf_{b \in G(x)} \iint U_n(x_1) q(dx_1 | x, a, b) f(x)(da) \geq (SU_n)(x) - \varepsilon/2.$$

The existence of f follows from Lemma 3.2 by taking $\varphi = U_n$. Define

$$\begin{aligned} \sigma_\varepsilon^{n+1}(x)_0 &= f(x), \\ \sigma_\varepsilon^{n+1}(x)_i(z_1, z_2, \dots, z_i) &= \sigma_{\varepsilon/2}^n(x_1)_{i-1}(z_2, z_3, \dots, z_i) \quad \text{if } 1 \leq i \leq n, \\ &= f^*(x_i) \quad \text{if } n+1 \leq i. \end{aligned}$$

It is easily verified that $(\sigma_\varepsilon^{n+1}(x), t^{n+1}(x))_{x \in X}$ is a universally measurable family of policies. It remains to verify (4.4) for $n+1$. If $u(x) \geq U_{n+1}(x)$, once again (4.4) is trivial. So suppose that $u(x) < U_{n+1}(x)$. Let τ be a measurable strategy for player II available at x. Then, for fixed $x \in X$,

$$\begin{aligned} E_{\sigma_\varepsilon^{n+1}(x),\tau}\big(u(x_{t^{n+1}(x)})\big) &= \iint u(x_{t^{n+1}(x)[z_1]})\, dP_{\sigma_\varepsilon^{n+1}(x)[z_1],\tau[z_1]}\, dP_{\sigma^{n+1}(x),\tau} \\ &= \iint u(x_{t^n(x_1)})\, dP_{\sigma_{\varepsilon/2}^n(x_1),\tau[z_1]}\, dP_{\sigma^{n+1}(x),\tau} \\ &\geq \iiint U_n(x_1) q(dx_1 | x, a, b) f(x)(da) \tau_0(db) - \varepsilon/2 \\ &\geq (SU_n)(x) - \varepsilon \\ &= U_{n+1}(x) - \varepsilon, \end{aligned}$$

where the first inequality is by virtue of the induction hypothesis and the second by virtue of (4.5). Thus, (4.4) is established for $n+1$. □

THEOREM 4.3. *The game $\mathscr{L}(u)(x)$ has value $U(x)$ for every $x \in X$. For every $\varepsilon > 0$, I has a universally measurable family of ε-optimal policies. Player II has a universally measurable family of optimal strategies.*

PROOF. For $x \in X$, define $n(x)$ to be the least $k \geq 0$ such that $U_k(x) > U(x) - \varepsilon/2$. Then $n(x)$ is a universally measurable function of x. Let $(\sigma_{\varepsilon/2}^n(x), t^n(x))_{x \in X}$, $n \geq 0$, be as in the statement of Lemma 4.2. Set

$$\bar{\sigma}(x) = \sigma_{\varepsilon/2}^{n(x)}(x)$$

and

$$\bar{t}(x) = t^{n(x)}(x).$$

Then $(\bar{\sigma}(x), \bar{t}(x))_{x \in X}$ is a universally measurable family of policies. Moreover, by Lemma 4.2, for any measurable strategy τ for player II available at x,

$$\begin{aligned} E_{\bar{\sigma}(x),\tau}\big(u(x_{\bar{t}(x)})\big) &= E_{\sigma_{\varepsilon/2}^{n(x)}(x),\tau}\big(u(x_{t^{n(x)}(x)})\big) \\ (4.6) \qquad &\geq U_{n(x)}(x) - \varepsilon/2 \\ &\geq U(x) - \varepsilon \end{aligned}$$

for every $x \in X$.

Using Lemma 3.2, fix a universally measurable function $g\colon X \to P(B)$ such that for each $x \in X$, $g(x)(G(x)) = 1$ and

$$\sup_{a \in F(x)} \iint U(x_1) q(dx_1|s, a, b) g(x)(db) \le (SU)(x). \tag{4.7}$$

Define

$$\begin{aligned} \tau(x)_0 &= g(x), \\ \tau(x)_n(z_1, z_2, \dots, z_n) &= g(x_n), \qquad n \ge 1. \end{aligned} \tag{4.8}$$

Then $(\tau(x))_{x \in X}$ is a universally measurable family of strategies for II.

We will now prove that if (σ, τ) is any measurable policy of player I available at x, then

$$E_{\sigma, \tau(x)}(u(x_t)) \le U(x) \tag{4.9}$$

for every $x \in X$.

We prove (4.9) by induction on $j(t)$. The inequality is obvious when $j(t) = 0$, that is, when $t = 0$. Let t be a measurable stop rule with index $j(t) = \alpha > 0$ and assume that (4.9) holds for all measurable strategies σ of player I available at x, all $x \in X$ and all measurable stop rules of index less than α. Then, by (4.7) and Lemma 4.1(d),

$$\begin{aligned} E_{\sigma, \tau(x)}(u(x_t)) &= \iint u(x_{t[z_1]})\, dP_{\sigma[z_1], \tau(x)[z_1]}\, dP_{\sigma, \tau(x)} \\ &= \iint u(x_{t[z_1]})\, dP_{\sigma[z_1], \tau(x_1)}\, dP_{\sigma, \tau(x)} \\ &\le \int U(x_1)\, dP_{\sigma, \tau(x)} \\ &= \iiint U(x_1) q(dx_1|x, a, b) \sigma_0(da) g(x)(db) \\ &\le (SU)(x) \\ &\le U(x). \end{aligned}$$

This terminates the proof of the theorem. □

Consider now a modification $\mathscr{L}^*(u)(x)$ of the leavable game in which player I chooses a measurable strategy σ available at x and a measurable stop rule $t \ge 1$, player II chooses a measurable strategy τ available at x and, as before, II pays I the quantity $E_{\sigma,\tau}(u(x_t))$. The only difference is that player I is not allowed to choose $t = 0$.

THEOREM 4.4. *The game $\mathscr{L}^*(u)(x)$ has value $(SU)(x)$ for every $x \in X$. For every $\varepsilon > 0$, I has a universally measurable family of ε-optimal policies. Player II has a universally measurable family of optimal strategies.*

PROOF. Let $\bar{X}$ be a homeomorphic copy of X and disjoint with X. If $x \in X$, its copy in $\bar{X}$ will be denoted by $\bar{x}$. Consider a new problem with state space $X \cup \bar{X}$, the same action sets F, G, the same utility u and the same law of motion q on X and extended to $\bar{X}$ as follows:

$$F(\bar{x}) = F(x); \qquad G(\bar{x}) = G(x),$$
$$u(\bar{x}) = \inf\{u(y) : y \in X\} - 1$$

and

$$q(\cdot|\bar{x}, a, b) = q(\cdot|x, a, b).$$

Notice that, for any $x \in X$, the leavable game $\mathscr{L}(u)(\bar{x})$ is equivalent to $\mathscr{L}^*(u)(x)$ because I will not have any incentive to use $t = 0$ if the starting state is $\bar{x}$. Consequently, $U(\bar{x})$, the value of the game $\mathscr{L}(u)(\bar{x})$, will also be the value of the game $\mathscr{L}^*(u)(x)$. By Lemma 4.1(e),

$$U(\bar{x}) = u(\bar{x}) \vee (SU)(\bar{x}) = (SU)(\bar{x}) = (SU)(x).$$

Hence the value of $\mathscr{L}^*(u)(x)$ is $(SU)(x)$, as was to be proved. For any $0 < \varepsilon < 1$, by Theorem 4.3, player I has a universally measurable family of ε-optimal policies $(\sigma(\bar{x}), t(\bar{x}))_{x \in X}$ in the games $\mathscr{L}(u)(\bar{x})$, $\bar{x} \in \bar{X}$. But then $t(\bar{x}) \geq 1$ and $\sigma(\bar{x})$ is available to I at x for every $x \in X$, so this family of policies will be ε-optimal in the games $\mathscr{L}^*(u)(x)$, $x \in X$. Finally, let $\tau(x)$ be defined by (4.7) and (4.8). Then, as was observed in the course of the proof of Theorem 4.3, for any measurable policy (σ, t) available to I at x with $t \geq 1$,

$$E_{\sigma, \tau(x)}(u(x_t)) \leq (SU)(x).$$

Hence, $(\tau(x))_{x \in X}$ is a measurable family of optimal strategies for II in $\mathscr{L}^*(u)(x)$, $x \in X$. This completes the proof. □

5. Inductive definability. In order to prove that the games $\mathscr{N}(u)$ have a value and that the value function is measurable, it will be necessary to iterate the $\mathscr{L}^*$ games of the previous section a transfinite number of times, ensuring at the same time that these iterated games have value functions that are measurable. A result of Moschovakis from the theory of inductive definability is tailor made to handle these problems of measurability. To formulate the result, let Y be an infinite set and Φ a mapping from the power set of Y to the power set of Y. Say that Φ is a *monotone operator* if, whenever $E_1 \subseteq E_2 \subseteq Y$, then $\Phi(E_1) \subseteq \Phi(E_2)$. Define the iterates of Φ by transfinite induction as follows:

$$\Phi^{\xi} = \Phi\left(\bigcup_{\eta > \xi} \Phi^{\eta}\right), \tag{5.1}$$

where ξ is any ordinal. So, in particular, $\Phi^0 = \Phi(\phi)$. It is easy to verify that Φ^{∞}, the least fixed point of Φ, is given by $\bigcup\{\Phi^{\eta}: \eta < \kappa\}$, where κ is the least cardinal greater than the cardinality of Y.

Suppose that Y is a Borel subset of a Polish space and Φ is a monotone operator on Y. We say Φ *respects coanalytic sets* if, whenever Ω is a Polish space and C is a coanalytic subset of $\Omega \times Y$, then the set

$$C^* = \{(\omega, y) \in \Omega \times Y: y \in \Phi(C_\omega)\} \tag{5.2}$$

is also coanalytic. (Here $C_\omega = \{y \in Y: (\omega, y) \in C\}$.)

THEOREM 5.1. *Let Φ be a monotone operator on a Borel subset Y of a Polish space and suppose that Φ respects coanalytic sets. Then*:

(a) *Φ^∞ is a coanalytic subset of Y,*

(b) *$\Phi^\infty = \bigcup_{\xi < \omega_1} \Phi^\xi$, where ω_1 is the first uncountable ordinal,*

(c) *if E is a Borel set contained in Φ^∞, then there is $\xi < \omega_1$ such that $E \subseteq \Phi^\xi$.*

Part (a) of the theorem is a special case of a very general result of Moschovakis [19, 7C.8, page 414]. Parts (b) and (c) are not stated explicitly in [19], but they can be deduced from results there and this deduction is carried out by Louveau [14]. Zinsmeister [26] also gives a nice exposition of Moschovakis's theorem.

Without loss of generality, we assume that the bounded, upper analytic utility function u takes values in $[0, 1]$. Set

$$B = \{(x, c) \in X \times [0, 1]: u(x) > c\},$$

so B is an analytic subset of $X \times [0, 1]$ by virtue of Lemma 2.1.

Let E be a subset of $X \times [0, 1]$. Define φ_E: $X \to [0, 1]$ by

$$\varphi_E(x) = \sup\{c \in [0, 1]: (x, c) \in E^c \cap B\},$$

where $\sup(\phi) = 0$.

Suppose, next, that w: $X \to [0, 1]$ and let

$$E(w) = \{(x, c) \in X \times [0, 1]: w(x) \le c\}.$$

Then, as is easy to see, $\varphi_{E(w)} = u \wedge w$, where $(u \wedge w)(x) =$ the minimum of $u(x)$ and $w(x)$.

We now extend the definition of the operator S, introduced in Section 3, to all functions on X onto $[0, 1]$. For w: $X \to [0, 1]$, define

$$(Sw)(x) = \inf_{\nu \in P(G(x))} \sup_{\mu \in P(F(x))} \int^* \int^* w(x') q(dx'|x, a, b)(\mu \times \nu)(da \times db),$$

where $\int^*$ stands for the outer integral (see [1], page 273, for a definition).

Set

$$w_1 = Sw \vee w,$$

$$w_{n+1} = Sw_n \vee w, \qquad n \ge 1$$

and

$$w_\infty = \sup_n w_n.$$

Finally, define a new operator T by setting

$$Tw = Sw_\infty.$$

Notice that, when w is the function u of Section 4, $w_\infty = U$ is the value of the game $\mathscr{L}(u)$ and $Tw = SU$ is the value of $\mathscr{L}^*(u)$.

We are now ready to define a monotone operator on $X \times [0,1]$. For $E \subseteq X \times [0,1]$, let

$$\Phi(E) = \{(x,c) \in X \times [0,1]: (T\varphi_E)(x) \le c\}. \tag{5.3}$$

It is easy to verify that Φ is monotone.

LEMMA 5.2. *Φ respects coanalytic sets.*

PROOF. Let Ω be Polish and let C be a coanalytic subset of $\Omega \times X \times [0,1]$. We have to prove that the set

$$C^* = \{(\omega, x, c) \in \Omega \times X \times [0,1]: (x,c) \in \Phi(C_\omega)\}$$

is coanalytic.

Define v: $\Omega \times X \to [0,1]$ by

$$v(\omega, x) = \sup\{c \in [0,1]: (x,c) \in C_\omega^c \cap B\},$$

that is, $v(\omega, x) = \varphi_{C_\omega}(x)$. The function v is upper analytic, for

$$v(\omega, x) > a \leftrightarrow (\exists\, c)(c > a \text{ and } (\omega, x, c) \in C^c \cap (\Omega \times B)),$$

which is an analytic condition in ω, x, a. Hence, by Lemma 3.1, the function v_1 defined by

$$v_1(\omega, x) = v(\omega, x) \vee (Sv(\omega, \cdot))(x)$$

is upper analytic. So, by induction and Lemma 3.1, the functions

$$v_{n+1}(\omega, x) = v(\omega, x) \vee (Sv_n(\omega, \cdot))(x)$$

are upper analytic as well. It follows that so is the function

$$v_\infty(\omega, x) = \sup_n v_n(\omega, x).$$

Using Lemma 3.1 one more time, we see that the function

$$(Tv(\omega, \cdot))(x) = (Sv_\infty(\omega, \cdot))(x)$$

is upper analytic on $\Omega \times X$. Since

$$C^* = \{(\omega, x, c) \in \Omega \times X \times [0,1]: (Tv(\omega, \cdot))(x) \le c\},$$

it follows from Lemma 2.1 that C^* is coanalytic. □

LEMMA 5.3. *If w is a function on X into* $[0,1]$, *then* $\Phi(E(w)) = E(T(u \wedge w))$.

PROOF. This is immediate from the definitions of Φ and T. □

Define by transfinite induction the functions Q_ξ on X as follows:

(5.4) $$Q_0 = Tu$$

and for $\xi > 0$,

(5.5) $$Q_\xi = T\Big(u \wedge \inf_{\eta<\xi} Q_\eta\Big).$$

LEMMA 5.4. (a) $\Phi^\xi = E(Q_\xi)$, $\xi < \omega_1$.
(b) *For every* $\xi < \omega_1$, Q_ξ *is upper analytic.*

PROOF. If $E = \phi$, then $\varphi_E = u$. Hence

$$\begin{aligned}\Phi^0 &= \Phi(\phi)\\ &= \{(x,c) \in X \times [0,1]: (Tu)(x) \le c\}\\ &= \{(x,c) \in X \times [0,1]: Q_0(x) \le c\}\\ &= E(Q_0).\end{aligned}$$

Assume next that $\xi > 0$ and that (a) is true for all $\eta < \xi$. Let

$$\Phi^{<\xi} = \bigcup_{\eta<\xi} \Phi^\eta \quad \text{and} \quad E^\xi = E\Big(\inf_{\eta<\xi} Q_\eta\Big).$$

It is easy to verify by using the inductive hypothesis that

$$\varphi_{E^\xi} = \varphi_{\Phi^{<\xi}}.$$

But

$$\varphi_{E^\xi} = u \wedge \inf_{\eta<\xi} Q_\eta,$$

so

$$T\Big(u \wedge \inf_{\eta<\xi} Q_\eta\Big) = T\varphi_{\Phi^{<\xi}}.$$

Consequently,

$$\begin{aligned}\Phi^\xi &= \Phi(\Phi^{<\xi})\\ &= \{(x,c) \in X \times [0,1]: (T\varphi_{\Phi^{<\xi}})(x) \le c\}\\ &= \Big\{(x,c) \in X \times [0,1]: T\Big(u \wedge \inf_{\eta<\xi} Q_\eta\Big) \le c\Big\}\\ &= \{(x,c) \in X \times [0,1]: Q_\xi(x) \le c\}\\ &= E(Q_\xi).\end{aligned}$$

For (b), first observe that it follows from Lemma 5.2 and induction on ξ that Φ^ξ is coanalytic for all $\xi < \omega_1$. It then follows from (a) and Lemma 2.1 that the functions Q_ξ, $\xi < \omega_1$, are upper analytic. □

Let

(5.6) $$Q = \inf_{\xi<\omega_1} Q_\xi.$$

THEOREM 5.5. *The function Q is upper analytic and $T(u \wedge Q) = Q$.*

PROOF. First, note that the monotone operator Φ defined by (5.3) satisfies the hypothesis of Theorem 5.1 by virtue of Lemma 5.2. So, according to Theorem 5.1(b),

$$\Phi^{\infty} = \bigcup_{\xi<\omega_1} \Phi^{\xi}.$$

Hence, by Lemma 5.4(a),

$$\begin{aligned}\Phi^{\infty} &= \bigcup_{\xi<\omega_1} \{(x,c) \in X \times [0,1]: Q_{\xi}(x) \le c\} \\ &= \Big\{(x,c) \in X \times [0,1]: \Big(\inf_{\xi<\omega_1} Q_{\xi}\Big)(x) \le c\Big\} \qquad (5.7) \\ &= E(Q).\end{aligned}$$

Hence, by Theorem 5.1(a) and Lemma 2.1, Q is upper analytic. Moreover, from (5.7) and Lemma 5.3, we have

$$\begin{aligned}\Phi^{\infty} &= \Phi(\Phi^{\infty}) \\ &= \Phi(E(Q)) \\ &= E(T(u \wedge Q)).\end{aligned}$$

Hence $T(u \wedge Q) = Q$. □

6. Nonleavable games. In this section we prove that the game $\mathscr{N}(u)(x)$ of Section 1 has value $Q(x)$. Let $\overline{V}(x), \underline{V}(x)$ be, respectively, the upper value and lower value of the game $\mathscr{N}(u)(x)$ as defined by (2.2) and (2.3). The next result shows that $\underline{V} \ge Q$.

THEOREM 6.1. *Let $\varepsilon > 0$. Then there is a universally measurable family of strategies $(\sigma(x))_{x \in X}$ for player* I *such that for any measurable strategy τ of player* II *available at x,*

$$E_{\sigma(x),\tau}(u^*) \ge Q(x) - \varepsilon \qquad (6.1)$$

for every $x \in X$. Consequently, $\underline{V} \ge Q$.

PROOF. Consider the game $\mathscr{L}^*(u \wedge Q)(x)$. By Theorem 4.4 and Theorem 5.5 this game has value $(T(u \wedge Q))(x) = Q(x)$. Moreover, by virtue of Theorem 4.4, we can choose, for each $\delta > 0$, a universally measurable family $(\bar{\sigma}(x,\delta), \bar{t}(x,\delta))$ of δ-optimal policies for player I in the game $\mathscr{L}^*(u \wedge Q)(x)$.

Let $\delta_0, \delta_1, \dots$ be positive numbers such that $\Sigma_n \delta_n < \varepsilon$. For each $x \in X$ and $n \ge 0$, set $\sigma^n(x) = \bar{\sigma}(x, \delta_n)$, $t_n(x) = t(x, \delta_n)$. For $h = (z_1, z_2, \dots, z_n, \dots)$, define

$$\begin{aligned}s_0(x)(h) &= t_0(x)(h), \\ s_{n+1}(x)(h) &= s_n(x)(h) + t_{n+1}(x_{s_n(x)})(z_{s_n(x)+1}, z_{s_n(x)+2}, \dots)\end{aligned}$$

and

$$\begin{aligned}\sigma(x)_0 &= \sigma^0(x)_0,\\ \sigma(x)_n(z_1, z_2, \dots, z_n) &= \sigma^0(x)_n(z_1, z_2, \dots, z_n) \quad \text{if } n < s_0(x)(h)\\ &= \sigma^{k+1}(x_{s_k(x)})_{n-s_k(x)}(z_{s_k(x)+1}, z_{s_k(x)+2}, \dots, z_n)\\ &\qquad \text{if } s_k(x)(h) \le n < s_{k+1}(x)(h).\end{aligned}$$

Plainly, $(\sigma(x))_{x \in X}$ is a universally measurable family of strategies.

We shall now verify (6.1). So fix $x_0 \in X$ and a measurable strategy τ for player II available at x_0, and let $P = P_{\sigma(x_0), \tau}$. The expectations and conditional expectations below are all with respect to P. Recall that, for a stop rule t and a history $h = (z_1, z_2, \dots) = ((a_1, b_1, x_1), (a_2, b_2, x_2), \dots)$, we define $x_t(h) = x_{t(h)}$ and $p_t(h) = p_{t(h)}(h) = (z_1, z_2, \dots, z_{t(h)})$.

Set

$$Y_n = (u \wedge Q)(x_{s_n(x_0)}), \qquad n \ge 0.$$

By assumption,

$$E(Y_0) \ge Q(x_0) - \delta_0$$

and for $n \ge 1$,

$$E(Y_n | p_{s_{n-1}(x_0)}) \ge Q(x_{s_{n-1}(x_0)}) - \delta_n \quad \text{almost surely } (P),$$

where the conditional expectation is with respect to the σ-field $\mathscr{F}_{s_{n-1}(x_0)}$.

So, for $n \ge 1$,

$$\begin{aligned}E(Y_n) &\ge E(Q(x_{s_{n-1}(x_0)})) - \delta_n\\ &\ge E(Y_{n-1}) - \delta_n.\end{aligned}$$

By iterating the last inequality, we get, for $n \ge 0$,

$$\begin{aligned}E(Y_n) &\ge E(Y_0) - (\delta_1 + \delta_2 + \cdots + \delta_n)\\ &\ge Q(x_0) - (\delta_0 + \delta_1 + \cdots + \delta_n)\\ &\ge Q(x_0) - \varepsilon.\end{aligned}$$

Hence

$$\limsup_n E(Y_n) \ge Q(x_0) - \varepsilon.$$

But

$$\begin{aligned}E(u^*) &= E\Big(\limsup_n u(x_n)\Big)\\ &\ge E\Big(\limsup_n u(x_{s_n(x_0)})\Big)\\ &\ge \limsup_n E(u(x_{s_n(x_0)}))\\ &\ge \limsup_n E(Y_n)\\ &\ge Q(x_0) - \varepsilon,\end{aligned}$$

which verifies (6.1). □

Now we have to prove that $\bar{V} \le Q$. We need a result which may be viewed as a measurable version of one-half of the Fatou equation. The proof is adapted from [24].

LEMMA 6.2. *Let $\varepsilon > 0$. Then there is a universally measurable function $t^{\varepsilon}(\mu, h) = t(\mu, h)$ from $P(H) \times H$ to $N = \{1, 2, \dots\}$ such that* (a) *for fixed μ, $t(\mu, \cdot\,)$ is a stop rule greater than or equal to* 1, *and* (b) *for every μ,*

$$E_{\mu}(u^*) \le E_{\mu}\big(u(x_{t(\mu,\,\cdot)})\big) + \varepsilon,$$

where E_{μ} is the expectation operator under μ.

PROOF. For each $n \ge 1$, let $\mu[z_1, z_2, \dots, z_n]$ be a Borel measurable function from $P(H) \times Z^n$ to $[0, 1]$ such that $\mu[z_1, z_2, \dots, z_n]$ is a version of the conditional distribution under μ of $z_{n+1}, z_{n+2}, \dots$ given $z_1, z_2, \dots, z_n$. The existence of such a function is established in [16]. Consequently,

$$E_{\mu}(u^* | z_1, z_2, \dots, z_n) = \int u^* \mu[z_1, z_2, \dots, z_n](dh)$$

is a universally measurable function from $P(H) \times Z^n$ to $\mathfrak{R}$ ([1], page 180). By the Levy 0-1 law ([20], page 133),

$$E_{\mu}(u^* | z_1, z_2, \dots, z_n) \to u^*(x_1, x_2, \dots)$$

almost surely (μ).

Let $s(\mu, h)$ be the least $k \ge 1$ such that

$$E_{\mu}(u^* | z_1, z_2, \dots, z_k) < u(x_k) + \delta,$$

if such a k exists. Otherwise, set $s(\mu, h) = \infty$.

Then s is a universally measurable function on $P(H) \times H$ such that for each fixed μ, $s(\mu, \cdot\,)$ is a stopping time and $\mu(\{s(\mu, \cdot\,) < \infty\}) = 1$. Let

$$g_m(\mu) = \mu(\{s(\mu, \cdot) \le m\}), \qquad m \ge 1,$$

so g_m is universally measurable on $P(H)$ ([1], page 177). Let $N(\mu)$ be the least m such that

$$g_m(\mu) > 1 - \delta,$$

so $N(\mu)$ is also universally measurable.

Let

$$t(\mu, h) = s(\mu, h) \wedge N(\mu).$$

Plainly, t is universally measurable and, for each fixed μ, $t(\mu, \cdot\,)$ is a stop rule. Moreover,

$$\mu(\{t(\mu, \cdot) = s(\mu, \cdot)\}) > 1 - \delta$$

for every $\mu \in P(H)$. Hence,

$$\begin{aligned}
&E_\mu(u^*) - E_\mu(u(x_{t(\mu,\cdot)})) \\
&\quad = \int \left[E_\mu(u^* | p_{t(\mu,\cdot)}) - u(x_{t(\mu,\cdot)}) \right] d\mu \\
&\quad = \int_{\{s(\mu,\cdot) \le N(\mu)\}} \left[E_\mu(u^* | p_{t(\mu,\cdot)}) - u(x_{t(\mu,\cdot)}) \right] d\mu \\
&\qquad + \int_{\{s(\mu,\cdot) > N(\mu)\}} \left[E_\mu(u^* | p_{t(\mu,\cdot)}) - u(x_{t(\mu,\cdot)}) \right] d\mu \\
&\quad \le \delta + 2\|u\|\delta \\
&\quad = \delta(1 + 2\|u\|),
\end{aligned}$$

where $\|u\| = \sup_{x \in X} |u(x)|$.

Now choose δ so that $\delta(1 + 2\|u\|) = \varepsilon$ to complete the proof. □

THEOREM 6.3. *For every $\varepsilon > 0$ and $\xi < \omega_1$, player* II *has a universally measurable family of strategies $(\tau^{\xi,\varepsilon}(x))_{x \in X}$ such that for any measurable strategy σ for player* I *available at x,*

$$E_{\sigma,\tau^{\xi,\varepsilon}(x)}(u^*) \le Q_\xi(x) + \varepsilon \tag{6.2}$$

for every $x \in X$.

PROOF. The proof is by induction on ξ. So consider the case when $\xi = 0$. By Theorem 4.4, choose a universally measurable family $(\bar{\tau}(x))_{x \in X}$ of optimal strategies for player II in the games $\mathscr{L}^*(u)(x)$, $x \in X$. Set

$$\tau^{0,\varepsilon}(x) = \bar{\tau}(x), \qquad x \in X.$$

Now fix $x_0 \in X$ and let σ be a measurable strategy for I available at x_0. Let $P = P_{\sigma,\bar{\tau}(x_0)}$. Use Lemma 6.2 to choose a universally measurable stop rule $t \ge 1$ such that

$$E_P(u^*) \le E_P(u(x_t)) + \varepsilon.$$

Then

$$\begin{aligned}
E_P(u^*) &\le E_P(u(x_t)) + \varepsilon \\
&\le Q_0(x_0) + \varepsilon,
\end{aligned}$$

since $\bar{\tau}(x_0)$ is optimal in $\mathscr{L}^*(u)(x_0)$.

For the inductive step, let $\xi > 0$ and assume that the result is true for all $\eta < \xi$. By Theorem 4.4, we can find an optimal family $(\bar{\tau}(x))_{x \in X}$ of universally measurable strategies for II in the games $\mathscr{L}^*(u \wedge \inf_{\eta < \xi} Q_\eta)(x)$, $x \in X$. Let

$$\lambda(h) = \inf\left\{ k \ge 1 \colon u(x_k) > \left(\inf_{\eta < \xi} Q_\eta \right)(x_k) \right\},$$

where $\inf(\phi) = \infty$, so that λ is a universally measurable stopping time. Let, for $\eta < \xi$,

$$C_\eta = \left\{x \in X: Q_\eta(x) < \left(\inf_{\zeta<\xi} Q_\zeta\right)(x) + \varepsilon/8\right\}.$$

Note that the sets C_η are universally measurable. For $h = (z_1, z_2, \dots, z_n, \dots)$, we now define $\tau^{\xi,\varepsilon}(x)$ as follows:

$$\tau^{\xi,\varepsilon}(x)_0 = \bar{\tau}(x)_0$$

$$\begin{aligned}\tau^{\xi,\varepsilon}(x)_n(z_1, z_2, \dots, z_n) &= \bar{\tau}(x)_n(z_1, z_2, \dots, z_n) \quad \text{if } n < \lambda(h)\\ &= \tau^{\eta,\varepsilon/8}(x_\lambda)_{n-\lambda}(z_{\lambda+1}, z_{\lambda+2}, \dots, z_n)\\ &\qquad \text{if } n > \lambda(h) \text{ and } x_\lambda \in C_\eta - \bigcup_{\eta'<\eta} C_{\eta'}.\end{aligned}$$

Plainly, $(\tau^{\xi,\varepsilon}(x))_{x\in X}$ is a universally measurable family of strategies.

Fix $x_0 \in X$ and let σ be a measurable strategy for I available at x_0. Set $P = P_{\sigma,\tau^{\xi,\varepsilon}(x_0)}$. Choose $m \geq 1$ such that

$$P(\{\lambda < \infty\}) \leq P(\{\lambda \leq m\}) + \frac{\varepsilon}{8(\|u\| + 1)}. \tag{6.3}$$

Since $\tau^{\xi,\varepsilon}(x_0)$ and $\bar{\tau}(x_0)$ agree prior to time λ, it follows by virtue of Lemma 2.5 that

$$P_{\sigma,\bar{\tau}(x_0)}(\{\lambda < \infty\}) \leq P_{\sigma,\bar{\tau}(x_0)}(\{\lambda \leq m\}) + \frac{\varepsilon}{8(\|u\| + 1)}. \tag{6.4}$$

Now define a function s as follows:

$$\begin{aligned}s(h) &= (\lambda \wedge m)(h) \quad \text{if } \lambda(h) \leq m\\ &= m + t^{\varepsilon/4}(P[p_m(h)], (h_{m+1}, h_{m+2}, \dots)) \quad \text{if } \lambda(h) > m,\end{aligned}$$

where the function $t^{\varepsilon/4}$ is as in the statement of Lemma 6.2 and $P[p_m(h)]$ abbreviates $P_{\sigma[p_m(h)],\tau^{\xi,\varepsilon}[p_m(h)]}$. It is easily verified that s is a universally measurable stop rule.

In the calculations below, expectations and conditional expectations will be with respect to the probability measure P. First, write

$$E(u^*) = \int_{\{\lambda\leq m\}} u^*\, dP + \int_{\{\lambda>m\}} u^*\, dP. \tag{6.5}$$

We will now obtain bounds on the two terms on the right-hand side. For the first term, condition on $p_{\lambda \wedge m}$ and calculate

$$\begin{aligned}\int_{\{\lambda \le m\}} u^* \, dP &= \int_{\{\lambda \le m\}} \left[\int u^* \, dP_{\sigma[p_\lambda], \tau^{\xi,\varepsilon}(x_0)[p_\lambda]} \right] dP \\ &\le \int_{\{\lambda \le m\}} \left(\inf_{\eta<\xi} Q_\eta \right)(x_\lambda) \, dP + \varepsilon/4 \\ &= \int_{\{\lambda \le m\}} \left(u \wedge \inf_{\eta<\xi} Q_\eta \right)(x_\lambda) \, dP + \varepsilon/4 \qquad (6.6) \\ &= \int_{\{\lambda \le m\}} \left(u \wedge \inf_{\eta<\xi} Q_\eta \right)(x_\lambda) \, dP_{\sigma, \bar{\tau}(x_0)} + \varepsilon/4 \\ &= \int_{\{\lambda \le m\}} \left(u \wedge \inf_{\eta<\xi} Q_\eta \right)(x_s) \, dP_{\sigma, \bar{\tau}(x_0)} + \varepsilon/4,\end{aligned}$$

where the first inequality uses the definition of $\tau^{\xi,\varepsilon}(x_0)[p_\lambda]$ and the inductive hypothesis, the third equality is by virtue of the fact that $\tau^{\xi,\varepsilon}(x_0)$ and $\bar{\tau}(x_0)$ agree prior to time λ and Lemma 2.5(i), and the final equality is by the definition of s.

For the second term on the right side of (6.5), we condition on p_m and calculate

$$\begin{aligned}\int_{\{\lambda > m\}} u^* \, dP &= \int_{\{\lambda > m\}} \left[\int u^* \, dP[p_m] \right] dP \\ &\le \int_{\{\lambda > m\}} \left[\int u(x_{t^{\varepsilon/4}(P[p_m], \cdot)}) \, dP[p_m] \right] dP + \varepsilon/4 \\ &= \int_{\{\lambda > m\}} E(u(x_s) | p_m) \, dP + \varepsilon/4 \\ &= \int_{\{\lambda > m\}} u(x_s) \, dP + \varepsilon/4 \\ &= \int_{\{m < \lambda < \infty\}} u(x_s) \, dP + \int_{\{\lambda = \infty\}} u(x_s) \, dP + \varepsilon/4 \qquad (6.7) \\ &\le \int_{\{m < \lambda < \infty\}} \left(u \wedge \inf_{\eta<\xi} Q_\eta \right)(x_s) \, dP + \varepsilon/4 \\ &\quad + \int_{\{\lambda = \infty\}} \left(u \wedge \inf_{\eta<\xi} Q_\eta \right)(x_s) \, dP + \varepsilon/4 \\ &\le \int_{\{m < \lambda < \infty\}} \left(u \wedge \inf_{\eta<\xi} Q_\eta \right)(x_s) \, dP_{\sigma, \bar{\tau}(x_0)} + 2\varepsilon/4 \\ &\quad + \int_{\{\lambda = \infty\}} \left(u \wedge \inf_{\eta<\xi} Q_\eta \right)(x_s) \, dP_{\sigma, \bar{\tau}(x_0)} + \varepsilon/4 \\ &= \int_{\{\lambda > m\}} \left(u \wedge \inf_{\eta<\xi} Q_\eta \right)(x_s) \, dP_{\sigma, \bar{\tau}(x_0)} + 3\varepsilon/4,\end{aligned}$$

where the first inequality is by virtue of Lemma 6.2, the second equality is by the definition of the stop rule s, the second inequality is by virtue of (6.3) and the facts that $\|Q_\eta\| \le \|u\|$ and $u(x_s) \le (\inf_{\eta<\xi} Q_\eta)(x_s)$ on $\{\lambda > s\}$ and, the final inequality is by virtue of (6.4), the fact that $\tau^{\xi,\varepsilon}(x_0)$ agrees with $\bar{\tau}(x_0)$ prior to time λ and Lemma 2.5(ii).

Hence, by (6.5), (6.6) and (6.7),

$$E(u^*) \le \int \Big(u \wedge \inf_{\eta<\xi} Q_\eta\Big)(x_s)\, dP_{\sigma, \bar{\tau}(x_0)} + \varepsilon$$

$$\le Q_\xi(x_0) + \varepsilon,$$

the last inequality being justified by the fact that $\bar{\tau}(x_0)$ is optimal for player II in the game $\mathscr{L}^*(u \wedge \inf_{\eta<\xi} Q_\eta)(x_0)$, which has value $Q_\xi(x_0)$. Thus, we have verified (6.2) and the proof is complete. □

COROLLARY 6.4. $\bar{V} \le Q$.

PROOF. It follows from Theorem 6.3 that $\bar{V} \le Q_\xi$ for every $\xi < \omega_1$. The conclusion now follows from (5.6). □

COROLLARY 6.5. *For every* $\varepsilon > 0$, *player* II *has a universally measurable family of strategies* $(\tau(x))_{x\in X}$ *such that for every measurable strategy* σ *of player* I *available at* x,

$$E_{\sigma, \tau(x)}(u^*) \le Q(x) + \varepsilon$$

for every $x \in X$.

PROOF. Assume without loss of generality that $0 \le u \le 1$. Let $C_\xi = \{x \in X: Q_\xi(x) = Q(x)\}$, $\xi < \omega_1$. Then, as is easy to see, the sets C_ξ are universally measurable and $\bigcup_{\xi<\omega_1} C_\xi = X$. We define

$$\tau(x) = \tau^{\xi,\varepsilon}(x) \quad \text{if } x \in C_\xi - \bigcup_{\eta<\xi} C_\eta,$$

where $\tau^{\xi,\varepsilon}$ is defined by Theorem 6.3. In order to prove that $(\tau(x))_{x\in X}$ is universally measurable, we have to verify that, for each $n \ge 0$, $\tau(x)_n(z_1, z_2, \dots, z_n)$ is universally measurable in $x, z_1, z_2, \dots, z_n$. So let μ be a probability measure on the Borel sets of $X \times Z^n$. Let μ_0 be the marginal of μ on X. Then Q is μ_0-measurable, so there is a Borel function Q': $X \to [0,1]$ such that $Q' \ge Q$ and $Q' = Q$ a.s. (μ_0). Plainly, the set $\{(x,c) \in X \times [0,1]: Q'(x) \le c\}$ is Borel and is contained in $E(Q) = \{(x,c) \in X \times [0,1]: Q(x) \le c\}$. But, by (5.7), $E(Q) = \Phi^\infty$, where the monotone operator Φ is defined by (5.3). Consequently, by Theorem 5.1(c), there is $\xi < \omega_1$ such that $\{(x,c) \in X \times [0,1]: Q'(x) \le c\} \subseteq \Phi^\xi$. So, by Lemma 5.4(a), it follows that $Q_\xi \le Q'$, so that $Q_\xi = Q$ a.s. (μ_0). Thus, $\mu((X - C_\xi) \times Z^n) = 0$. Now it is easy to verify directly from the definition that, restricted to the universally measurable set $C_\xi \times Z^n$, the function $\tau(x)_n(z_1, z_2, \dots, z_n)$ is universally measurable, hence it follows that it is μ-measurable on $X \times Z^n$. Since μ was an arbitrary probability measure on $X \times Z^n$, this proves that $\tau(x)_n(z_1, z_2, \dots, z_n)$ is universally measurable.

Finally, fix $x_0 \in X$ and let σ be a measurable strategy for I available at x_0. If $x_0 \in C_\xi - \bigcup_{\eta < \xi} C_\eta$ for $\xi < \omega_1$, then

$$\begin{aligned} E_{\sigma, \tau(x_0)}(u^*) &= E_{\sigma, \tau^{\xi, \varepsilon}(x_0)}(u^*) \\ &\leq Q_\xi(x_0) + \varepsilon \\ &= Q(x_0) + \varepsilon, \end{aligned}$$

where the inequality is by virtue of Theorem 6.3. This completes the proof. □

Theorem 1.1 now falls out of Theorem 6.1, Corollary 6.4 and Corollary 6.5.

REFERENCES

[1] BERTSEKAS, D. P. and SHREVE, S. E. (1978). *Stochastic Optimal Control: The Discrete Time Case*. Academic, New York.

[2] BEWLEY, T. and KOHLBERG, E. (1976). The asymptotic theory of stochastic games. *Math. Oper. Res.* **1** 197–208.

[3] BLACKWELL, D. (1969). Infinite G_δ games with imperfect information. *Zastos. Mat.* **10** 99–101.

[4] BLACKWELL, D. (1989). Operator solution of infinite G_δ games of imperfect information. In *Probability, Statistics and Mathematics. Papers in Honor of S. Karlin* (T. W. Anderson, K. B. Athreya and D. L. Iglehart, eds.) 83–87. Academic, New York.

[5] BLACKWELL, D. and FERGUSON, T. S. (1968). The big match. *Ann. Math. Statist.* **39** 159–163.

[6] DELLACHERIE, C. and MEYER, P. A. (1975). Ensembles analytiques et temps d'arrêt. *Séminaire de Probabilités IX. Lecture Notes in Math.* **465** 373–389. Springer, Berlin.

[7] DUBINS, L., MAITRA, A., PURVES, R. and SUDDERTH, W. (1989). Measurable, nonleavable gambling problems. *Israel J. Math.* **67** 257–271.

[8] DUBINS, L. E. and SAVAGE, L. J. (1976). *Inequalities for Stochastic Processes*. Dover, New York.

[9] ENGELKING, R. (1977). *General Topology. Monografie Matematyczne* **60**. PWN, Warsaw.

[10] FAN, KY (1953). Minimax theorems. *Proc. Nat. Acad. Sci. U.S.A.* **39** 42–47.

[11] GILLETTE, D. (1957). Stochastic games with zero-stop probabilities. In *Contributions to the Theory of Games III. Ann. Math. Studies* **39** 179–187. Princeton Univ. Press.

[12] HOFFMAN, A. J. and KARP, R. M. (1966). On nonterminating stochastic games. *Management Sci.* **12** 359–370.

[13] KOHLBERG, E. (1974). Repeated games with absorbing states. *Ann. Statist.* **2** 724–738.

[14] LOUVEAU, A. (1981–82). Capacitabilité et selections Boreliennes. In *Séminaire Initiation à l'Analyse* **21** 19-01–19-21. *Publications Mathématiques de l'Université Pierre et Marie Curie* **54**.

[15] MAITRA, A., PESTIEN, V. and RAMAKRISHNAN, S. (1990). Domination by Borel stopping times and some separation properties. *Fund. Math.* **135** 189–201.

[16] MAITRA, A., PURVES, R. and SUDDERTH, W. (1990). Leavable gambling problems with unbounded utilities. *Trans. Amer. Math. Soc.* **320** 543–567.

[17] MAITRA, A. and SUDDERTH, W. (1990). An operator solution of stochastic games. *Israel J. Math.* To appear.

[18] MERTENS, J.-F. and NEYMAN, A. (1981). Stochastic games. *Internat. J. Game Theory* **10** 53–66.

[19] MOSCHOVAKIS, Y. N. (1980). *Descriptive Set Theory*. North-Holland, Amsterdam.

[20] NEVEU, J. (1970). *Bases Mathématiques du Calcul des Probabilités*. Masson and Cie, Paris.

[21] NOWAK, A. S. (1985). Universally measurable strategies in zero-sum stochastic games. *Ann. Probab.* **13** 269–287.

[22] PARTHASARATHY, K. R. (1967). *Probability Measures on Metric Spaces.* Academic, New York.

[23] SHAPLEY, L. (1953). Stochastic games. *Proc. Nat. Acad. Sci. U.S.A.* **39** 1095–1100.

[24] SUDDERTH, W. (1971). A "Fatou equation" for randomly stopped variables. *Ann. Math. Statist.* **42** 2143–2146.

[25] SUDDERTH, W. (1969). On the existence of good stationary strategies. *Trans. Amer. Math. Soc.* **135** 399–414.

[26] ZINSMEISTER, M. (1989). Les dérivations analytiques. *Séminaire de Probabilités XXIII. Lecture Notes in Math.* **1372** 21–46. Springer, Berlin.

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