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BOREL SUMMATION AND SPLITTING OF SEPARATRICES FOR THE HÉNON MAP

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Contents.

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1. Presentation.

1.1. Separatrices of the Hénon map.

The quadratic area-preserving map of $\mathbb{C}^2$ defined by

$$H : \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} u_1 = u + v - u^2 \\ v_1 = v - u^2 \end{pmatrix} \tag{1}$$

can be considered as a particular case of the celebrated Hénon map. The origin is a parabolic fixed point of H. We will be interested in two invariant curves W^+ and W^-, which we call "stable and unstable separatrices" for the discrete-time dynamical system defined by the iteration of H, because the first one is attracted and the other one repelled by the origin.

We will see that the curves W^+ and W^- can be naturally parametrized by a complex variable z, with well-defined asymptotics as $\operatorname{Re} z \to +\infty$ for W^+ and $\operatorname{Re} z \to -\infty$ for W^-. A single asymptotic series corresponds to both separatrices but in different domains of the complex plane. The intersection of these domains contains two connected components, where for $|\operatorname{Im} z| \to \infty$ the distance between the corresponding points is exponentially small. Our aim is to study this phenomenon asymptotically. Our approach is based on an application of Borel summation to the divergent asymptotic series.

When an invariant curve is given by a parametrization $(u(z), v(z))$, with z varying in some subset of $\mathbb{C}$, we will say that it is additively-parametrized if $H(u(z), v(z)) = ((u(z+1), v(z+1))$, which amounts to the following system of first-order finite-difference equations:

$$\begin{cases} u(z+1) = u(z) + v(z+1), \\ v(z+1) = v(z) - u^2(z). \end{cases} \tag{2}$$

The first equation, which reflects the identity $u_1 = u + v_1$ in (1), enables us to eliminate the v-component; the system then reduces to one second-order difference equation for the u-component:

$$u(z+1) - 2u(z) + u(z-1) = -u^2(z). \tag{3}$$

We will deal mostly with the solutions of Equation (3), and restore the corresponding v-components by $v(z) = u(z) - u(z-1)$ when willing to come back to additively-parametrized curves. Our main result on the separatrices of H can be formulated in terms of two special solutions of Equation (3).

THEOREM 1. — *Equation* (3) *admits a unique solution* u^+ *in* $\mathbb{R}^+$ *such that* $u^+(z) = -6z^{-2} + \mathcal{O}(z^{-4})$ *as* $z \to +\infty$, *and a unique solution* u^- *in* $\mathbb{R}^-$ *such that* $u^-(z) = -6z^{-2} + \mathcal{O}(z^{-4})$ *as* $z \to -\infty$. *These two functions are real-analytic and extend to entire functions of the complex variable* z. *There exists a complex constant* Θ, $\operatorname{Re}\Theta = 0$ *and* $\operatorname{Im}\Theta < 0$, *such that*

$$u^+(z) - u^-(z) = \mathrm{e}^{-2\pi \mathrm{i} z}\,\frac{z^4}{84}\,\big(\Theta + \mathcal{O}(z^{-2})\big) \quad \text{as } z \to -\mathrm{i}\infty. \tag{4}$$

Notice that in the unicity statements no assumption on smoothness of solutions is necessary.

We eliminated the freedom in the choice of the parametrization by the asymptotic condition. Indeed, whenever a function $u(z)$ is a solution of (3) the function $u(z + a(z))$ solves it too, provided a is a 1-periodic complex valued function. The absence of terms in z^{-3} eliminates this freedom.

The prescribed domain of definition $\mathbb{R}^+$ (*resp.* $\mathbb{R}^-$) can be replaced by any real interval $[z_0, +\infty[$ (*resp.* $]-\infty, -z_0]$), since the equation itself allows to propagate the definition of any solution towards the left or the right. Analyticity propagates as well. This can be used to obtain entire functions.

More can be said about the asymptotic behaviour of these special solutions: the expression $-6z^{-2}$ is nothing but the first non-trivial term of an infinite asymptotic expansion. This expansion is asymptotic for both u^+ and u^- at infinity in large sectors containing $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively. The intersection of the sectors has two connected components, one of which contains the negative part of the imaginary axis. The last statement of the theorem yields an exponentially small but nonvanishing asymptotic equivalent for the difference $u^+ - u^-$ along $\mathrm{i}\mathbb{R}^-$. We will be able to supplement the first term with an infinite asymptotic expansion — See Proposition 1 below. The way we normalized the constant Θ, dividing by 84, has no particular meaning at this stage. Later on Θ will be interpreted as a "splitting constant".

It is easy to check that the invariant curves associated to u^+ and u^- are the separatrices of H. Let us set $v^\pm(z) = u^\pm(z) - u^\pm(z-1)$: we obtain two additively-parametrized invariant curves of H defined by

$$\mathbf{p}^+(z) = \begin{pmatrix} u^+(z) \\ v^+(z) \end{pmatrix} \quad \text{and} \quad \mathbf{p}^-(z) = \begin{pmatrix} u^-(z) \\ v^-(z) \end{pmatrix},$$

with the property

$$u^\pm(z) = -6z^{-2} + \mathcal{O}(z^{-4}), \quad v^\pm(z) = 12z^{-3} + \mathcal{O}(z^{-4}), \qquad z \to \pm\infty.$$

For all $z \in \mathbb{C}$, the iterates $H^n(\mathbf{p}^{\pm}(z)) = \mathbf{p}^{\pm}(z+n)$ converge to the origin as $n \to \pm\infty$. This shows that the invariant curves we found are the "stable and unstable manifolds" of the origin. Their real parts form a cusp at the origin, with $\{v = 0,\ u < 0\}$ as tangent.

In fact we obtain much more information about the properties of formal and analytic solutions of Equation (3) than it is stated in Theorem 1. Our study leads to a beautiful analytical picture described in Section 1.3, which is of independent interest.

But let us evoke some of our motivations first.

1.2. Exponentially small splitting of separatrices.

The first motivation for our study is the appearance of Θ and other similar constants in the study of homoclinic phenomena in close-to-integrable Hamiltonian systems.

Let us consider the one-parameter family of quadratic area-preserving maps,

$$F_\varepsilon : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x_1 = x + \varepsilon y_1 \\ y_1 = y + \varepsilon x(1-x) \end{pmatrix},$$

where $\varepsilon > 0$ is a positive parameter. Any non-trivial quadratic diffeomorphism of the plane which preserves area and orientation and has two fixed points can be put, by a linear change of coordinates, into this one-parameter family.

For small positive ε the origin is a hyperbolic fixed point of F_ε and the corresponding stable and unstable manifolds are one-dimensional curves in the plane (x, y) (see Fig. 1). They look like the separatrix of the Hamiltonian vector field

$$\dot{x} = y = \partial_y h, \quad \dot{y} = x(1-x) = -\partial_x h,$$
$$h(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}y^3,$$

except that they may intersect transversally (the limit separatrix may "split"), but this important phenomenon is exponentially small with respect to ε (*cf.* [FS90]).

Any intersection of one of the separatrices with the horizontal axis is necessarily a homoclinic point due to a symmetry of the map (there is a linear reverser, which conjugates F_ε with F_ε^{-1} and exchanges stable and

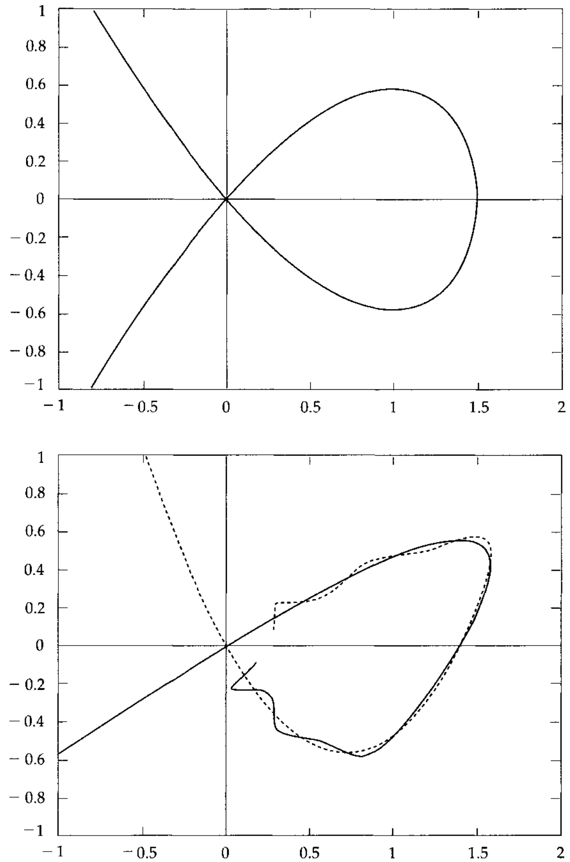

Figure 1. The limit separatrix (top) and the splitting of separatrices of the Hénon map (bottom). The unstable separatrix is drawn by the solid line, the stable one by the dashed line.

unstable curves). The angle of intersection α of the stable and unstable manifolds at the "first" such homoclinic point can be considered as a measure of the splitting. The asymptotic formula

$$\alpha = \frac{64\pi e^{-2\pi^2/\varepsilon}}{9\varepsilon^7}\left(|\Theta| + \mathcal{O}(\varepsilon)\right) \tag{5}$$

(with the same Θ as in Theorem 1) was proposed by one of the authors without a complete proof [Gel91]. The non-vanishing of Θ (and thus the transversality of the homoclinic intersection) was not proven there, but the numerical evaluation $|\Theta| \approx 2.474 \cdot 10^6$ was indicated.

The appearance of Θ as a splitting constant here is related to the fact that the map F_ε can be considered as a small perturbation of H. Indeed, in the coordinates $u = \varepsilon^2 x - \frac{1}{2}\varepsilon^2$, $v = \varepsilon^3 y$, it takes the form

$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} u_1 = u + v - u^2 + \varepsilon^4 \\ v_1 = v - u^2 + \varepsilon^4 \end{pmatrix}.$$

The detailed study of H is thus an important step in the derivation of Formula (5). But a subtle extra work is needed to complete the proof, which has not yet been written and which we will not address in the present paper.

Providing the rigorous proof of (5) is indeed a difficult analytical problem, comparable to the corresponding one for the Standard Map for which the task was recently achieved [Gel99], putting an end to the study initiated in [Laz84]. It is in the case of the Standard Map that the first definition of a splitting constant Θ_1 was proposed, by V.F. Lazutkin [Laz84], [LST89]. The papers [GLT91], [Gel91] contain numerical values of the splitting constants for quadratic, cubic and some other polynomial area-preserving maps.

Let us point out some other works closely related to the topic of our paper. V. Hakim and K. Mallick [HM93] proposed already to use the Borel summation for the study of the exponentially small splitting of separatrices of the Standard and Semistandard Maps. A more rigorous approach was developed by Y.B. Suris [Sur94] (for the Semistandard Map and for cubic maps). We mention also [Laz93], [Che98], [Tov94] and [TTJ98].

A formula similar to (5) describes the splitting of a small separatrix loop, created after a saddle-center bifurcation in a general family of analytic area-preserving maps [Gel00]. In this case each family of maps has its own splitting constant Θ. This number can be considered as a classification modulus of an area-preserving map near the parabolic fixed point which appears at the bifurcation.

1.3. Resurgence.

Another motivation for our study is the opportunity of illustrating Resurgence theory and contributing to it. "Resurgent functions" and the "alien calculus" which comes with them were discovered and studied by J. Écalle in the late 70s, particularly in relation with iteration of analytic maps [Eca81], but they have a much wider field of applications. A nice introduction to this theory can be found in the book [CNP93].

The present article will not assume any familiarity with Resurgence on the part of the reader. On the contrary, it may serve as initiation into this beautiful theory inasmuch as, in the situation at hand, we prove things with the tools of classical Complex Analysis and indicate their translation in terms of resurgent concepts.

One of the basic features of the theory is the tendency to consider formal solutions of a given problem and to try to extract as much information as possible from them. Thus we denote by $\mathbb{C}[[z^{-1}]]$ the space of all formal expansions in non-positive powers of z with complex coefficients (*i.e.* power series in z^{-1}), and by $\mathbb{C}[z][[z^{-1}]]$ the space of the sums of polynomials in z and power series in z^{-1}. We first give two lemmas to introduce formal series associated to Equation (3) and then state a theorem from which Theorem 1 follows.

LEMMA 1. — *Any nonzero solution of Equation* (3) *in* $\mathbb{C}[z][[z^{-1}]]$ *can be written in the form* $u(z+a)$, *where* $a \in \mathbb{C}$ *is arbitrary and* u *is the unique nonzero even solution in* $\mathbb{C}[z][[z^{-1}]]$,

$$u(z) = \sum_{k=1}^{\infty} \frac{a_k}{z^{2k}} = -6z^{-2} + \frac{15}{2} z^{-4} - \frac{663}{40} z^{-6} + \cdots .$$

The coefficients a_k *are real and* $(-1)^k a_k > 0$ *for all* $k \geq 1$.

Proof. — The second-order finite-difference operator in the left-hand side of (3) may be written

$$P = \mathrm{e}^{\partial_z} - 2 + \mathrm{e}^{-\partial_z}$$

(∂_z denotes differentiation with respect to z), so that the equation becomes $Pu = -u^2$. Expanding the equation in decreasing powers of z, one checks easily that the leading term of any formal nonzero solution must be $-6z^{-2}$, since $P(z^n) = n(n-1)z^{n-2} + \mathcal{O}(z^{n-4})$ for all $n \in \mathbb{Z}$. It turns out that the coefficient of z^{-3} is free whereas the following coefficients are determined inductively. Thus there is a unique nonzero even solution u, and since $u(z+a)$ is a solution for any $a \in \mathbb{C}$, no other non-trivial solution may exist. The resulting induction for the coefficients a_k is written in Section 6 (Formula (34)).

Now let us check that the series a_k is alternating. Consider the auxiliary series $U(x) = u(\mathrm{i}z) = \sum_{k\geq 1} (-1)^k a_k x^{-2k}$. It is the unique nonzero even solution in $\mathbb{C}[[x^{-1}]]$ of

$$-U(x+\mathrm{i}) + 2U(x) - U(x-\mathrm{i}) = \Big(4\sin^2 \frac{1}{2}\partial_x\Big)U = U^2.$$

But $\xi^2/4\sin^2(\frac{1}{2}\xi) = 1+\gamma(\xi)$, where $\gamma(\xi) = \sum_{k\geq 1}\gamma_k\xi^{2k}$ with $\gamma_k > 0$ for all $k \geq 1$ (these coefficients are proportional to the Bernoulli numbers). Rewriting the equation for U as

$$\partial_x^2 U = \big(1+\gamma(\partial_x)\big)U^2,$$

it is then easy to check that all coefficients of U are positive, and hence $(-1)^k a_k > 0$. □

We obtained a unique "formal separatrix" for the Hénon map H (up to the freedom of shifting the parameter), which is positively *and* negatively asymptotic to the parabolic fixed point:

$$\mathbf{p}(z) = \begin{pmatrix} u(z) \\ v(z) \end{pmatrix}, \quad v(z) = u(z) - u(z-1). \tag{6}$$

We will see that u and v, as power series of z^{-1}, have zero radius of convergence. The resurgent method consists precisely in analyzing this divergence with the help of the *formal Borel transform* $\mathcal{B}$. Also known as formal inverse Laplace transform, $\mathcal{B}$ is the linear transformation from $z^{-1}\mathbb{C}[[z^{-1}]]$ to $\mathbb{C}[[\zeta]]$ defined by $z^{-n-1} \mapsto \zeta^n/n!$ for all $n \geq 0$. For instance the Borel transform $\hat{u} = \mathcal{B}u$ of our unique nonzero even solution is

$$\hat{u}(\zeta) = \sum_{k=1}^{\infty} a_k \frac{\zeta^{2k-1}}{(2k-1)!}. \tag{7}$$

Since $\mathcal{B}$ turns multiplication of series into convolution, *i.e.*

$$\begin{aligned}&\widehat{\varphi} = \mathcal{B}\varphi,\ \widehat{\psi} = \mathcal{B}\psi\\ &\qquad\Longrightarrow\ \mathcal{B}(\varphi\psi)(\zeta) = (\widehat{\varphi} * \widehat{\psi})(\zeta) = \int_0^\zeta \widehat{\varphi}(\zeta')\widehat{\psi}(\zeta-\zeta')\,\mathrm{d}\zeta',\end{aligned}$$

and translation of z by a constant c into multiplication by $\mathrm{e}^{-c\zeta}$, we obtain that $\hat{u}$ is the unique nonzero odd solution in $\mathbb{C}[[\zeta]]$ of the transformed equation

$$(\mathrm{e}^{\zeta} - 2 + \mathrm{e}^{-\zeta})\hat{u}(\zeta) = -(\hat{u} * \hat{u})(\zeta). \tag{8}$$

In fact we will always deal with formal series $\varphi(z)$ whose Borel transforms $\widehat{\varphi}(\zeta)$ have nonzero radius of convergence, which means exactly that the coefficients of $\varphi(z)$ admit Gevrey-1 bounds. Observe that finite radius convergence for $\widehat{\varphi}(\zeta)$ implies divergence for $\varphi(z)$. This will be the case of $u(z)$, and our task will be to study the singularities of the analytic continuation of $\hat{u}(\zeta)$. In order to state the main result we have to introduce formal solutions of a linear equation which will allow to describe conveniently those singularities. This equation is nothing but the linearization of Equation (3) around u:

$$\varphi(z+1) - 2\varphi(z) + \varphi(z-1) = -2u(z)\varphi(z). \tag{9}$$

LEMMA 2. — *Equation* (9) *admits two particular formal solutions with real coefficients which can be written*

$$\varphi_1(z) = \sum_{k=1}^{\infty} \frac{b_k}{z^{2k+1}} = 12z^{-3} - 30z^{-5} + \frac{1989}{20}z^{-7} + \dots ,$$

$$\varphi_2(z) = \sum_{k=-2}^{\infty} \frac{d_k}{z^{2k}} = \frac{1}{84}z^4 + \frac{17}{840}z^2 - \frac{17}{2240} + \frac{3}{35}z^{-2} + \cdots .$$

Any solution of Equation (9) *in* $\mathbb{C}[z][[z^{-1}]]$ *is a linear combination of* φ_1 *and* φ_2.

Proof. — The first solution is obtained by differentiation:

$$\varphi_1 = \partial_z u,$$

hence $b_k = -2ka_k$ for all $k \geq 1$. The second solution can be found by substitution of the series into the equation. The resulting induction formulas for the d_k's are given in Section 6, Equation (36). We normalized φ_2 by choosing $d_{-2} = 1/84$ in order to have

$$\mathcal{W}_{\varphi_1,\varphi_2}(z) = 1,$$

where $\mathcal{W}_{\varphi_1,\varphi_2}$ is the *finite-difference Wronskian*

$$\mathcal{W}_{\varphi_1,\varphi_2}(z) = \det\begin{pmatrix} \varphi_1(z-1) & \varphi_2(z-1) \\ \varphi_1(z) & \varphi_2(z) \end{pmatrix}. \tag{10}$$

In the theory of linear finite-difference equations its role is similar to the role of the classical Wronskian for ordinary differential equations. In particular, if $\varphi(z)$ is a solution of (9), one checks easily that $c_1 = \mathcal{W}_{\varphi,\varphi_1}(z)$ and $c_2 = \mathcal{W}_{\varphi,\varphi_2}(z)$ must be 1-periodic in z and that $\varphi = c_2\varphi_1 - c_1\varphi_2$. But in the class of formal series periodicity holds for constants only. □

THEOREM 2. — *The formal Borel transform $\hat{u}$ is convergent at the origin and defines a holomorphic germ, which extends analytically along any path issuing from 0 and lying in $\mathbb{C} \setminus 2\pi\mathrm{i}\mathbb{Z}$ (except its origin), with exponential decay of type $-\infty$ along any non-vertical ray.*

There exist complex numbers Θ and μ such that the analytical continuation of $\hat{u}$ along the segment $]0,2\pi\mathrm{i}[$ can be written

$$\hat{u}(\zeta) = \frac{\Theta}{2\pi\mathrm{i}}\Big(\frac{4!\,d_{-2}}{(\zeta-2\pi\mathrm{i})^5} + \frac{2!\,d_{-1}}{(\zeta-2\pi\mathrm{i})^3} + \frac{d_0}{\zeta-2\pi\mathrm{i}}\Big) + \frac{1}{2\pi\mathrm{i}}h(\zeta-2\pi\mathrm{i})\log(\zeta-2\pi\mathrm{i}) + r(\zeta-2\pi\mathrm{i}), \tag{11}$$

where h and r are holomorphic at the origin, extend analytically along any path lying in $\mathbb{C} \setminus 2\pi\mathrm{i}\mathbb{Z}$, and

$$h(\zeta) = \Theta\sum_{k=1}^{\infty}\frac{d_k\zeta^{2k-1}}{(2k-1)!} + \mu\sum_{k=1}^{\infty}\frac{b_k\zeta^{2k}}{(2k)!}, \tag{12}$$

where b_k and d_k are defined by Lemma 2.

In the next section we will use the information about the Borel transform of u to recover by Laplace transform the solutions u^+ and u^- of Theorem 1. These solutions will have the same asymptotic expansion (namely $u(z)$), their difference will be related to the singularities of $\hat{u}$. The possibility of performing Laplace transform is guaranteed by the exponential decay of $\hat{u}$. The exponential type $-\infty$ implies that both solutions are entire.

The statement about the convergence and the analytic continuation of $\hat{u}$ can be considered as a definition of a resurgent function with $2\pi\mathrm{i}\mathbb{Z}$ as a lattice of singular points. This property looks quite natural in the general context of Écalle's theory, but to our knowledge this example of resurgence was not covered by the already existing results.

In particular, the holomorphic star of $\hat{u}$ is $\mathbb{C} \setminus \pm 2\pi\mathrm{i}[1,+\infty[$: we must remove the singular half-lines $2\pi\mathrm{i}[1,+\infty[$ and $-2\pi\mathrm{i}[1,+\infty[$ from the complex plane, and we may identify the resulting cut plane to the "first sheet" of the Riemann surface of $\hat{u}$. The analyticity of the Borel transform in the holomorphic star was proven by V. Chernov [Che98] by an adaptation of Lazutkin's method which gave the corresponding result for the Semistandard Map [Laz93]. We will use a different method and obtain the analytic continuation not only in the holomorphic star but also along paths which may cross the imaginary axis between singular points, *i.e.* we explore the other sheets of the Riemann surface of the multivalued analytic function $\hat{u}$.

We also provide the complete description of the singularity at $2\pi i$, which forms the second part of Theorem 2 and which amounts to a "resurgence relation" in Écalle's terminology — see Sections 4.2 and 5 for a translation in terms of alien calculus. (Of course the singularity at $-2\pi i$ can be deduced by real-analyticity of $\hat{u}$.) This singularity will appear to be directly responsible for the asymptotic equivalent (4) as $z \to -i\infty$, in which the exponential $e^{-2\pi i z}$ reflects the location of the singularity whereas the term in z^4 reflects its strength.

According to Theorem 2, the singularity is the sum of a polar part and a logarithmic term which we may identify to its variation h. Observe that the even part of h is simply the Borel transform of φ_1, up to a factor μ, whereas the odd part of h takes into account only the negative part of the expansion of φ_2, the polynomial part of φ_2 being reflected in the polar part of the singularity. This corresponds to the natural extension of the definition of the Borel transform to $\mathbb{C}[z][[z^{-1}]]$: the power series in z^{-1} were mapped to germs like $h(\zeta)$, which we now identify to the corresponding logarithmic singularities, and polynomials in z are mapped to polar singularities. (The extension can be pursued to deal with much more general formal objects and types of singularity [Ecalle93], [CNP93].)

In other words, the singularity of $\hat{u}$ at $2\pi i$ can be viewed as the Borel transform of $\Theta\varphi_2 + \mu\varphi_1$ — a special solution to the formal variational equation (9). Here Θ and μ are "transcendental" constants. On the other hand φ_1 and φ_2 represent the "elementary part"of the singularity at $2\pi i$: they are explicitly computable and closely related to u (one is the derivative of u and the other is its companion linear solution) thus to the Taylor series of $\hat{u}$ at 0.

This is a fundamental connection between the behaviour of $\hat{u}$ near its singular points and near 0. This self-reproduction property is at the origin of the name "resurgent".

1.4. Structure of the rest of the article.

In the next section we use Laplace transform to deduce Theorem 1 from Theorem 2. In fact a more refined asymptotic formula for the difference of the two separatrices is obtained.

Sections 3–5 are devoted to the proof of Theorem 2. They are somewhat technical, but it was part of our purpose to provide all the details of such a demonstration, which may hopefully help other searchers to understand and apply resurgent tools in concrete situations. Only the holomorphic star of $\hat{u}$, *i.e.* the first sheet of its Riemann surface, and the "nearby sheets" are discussed in Section 3. The singularity at $2\pi\mathrm{i}$ is then examined in Section 4. In fact this part of the work is sufficient to derive Theorem 1, but we give more insight of the resurgent structure of the problem and complete the proof of Theorem 2 in Section 5 where a stronger theorem is formulated.

In Section 6 we discuss numerical methods for evaluating the splitting constants Θ and μ. In [HM93] and [Sur94], the relation between them and the asymptotic behavior of the coefficients a_k of u was established; a more precise knowledge of $\hat{u}$ enables us to refine the estimates.

2. Asymptotics of the separatrices and of their difference.

Laplace transforms of $\hat{u}$. — Let us consider the two Laplace integrals

$$(13)\qquad u^+(z) = \int_0^{+\infty} \mathrm{e}^{-z\zeta}\hat{u}(\zeta)\,\mathrm{d}\zeta,\quad u^-(z) = \int_0^{-\infty} \mathrm{e}^{-z\zeta}\hat{u}(\zeta)\,\mathrm{d}\zeta.$$

Since $\hat{u} = \mathcal{B}u$, they can be called Borel-Laplace transforms of u. It follows from Theorem 2 that the integrals converge for all z and define two entire functions. According to classical properties of Laplace transform (see for instance [Ma]), they satisfy the original equation (3) and share the same asymptotic expansion

$$(14)\qquad \forall\delta\in\,]0,\pi[,\ \forall\rho>0,\qquad u^{\pm}(z) \stackrel{\text{as}}{=} \sum_{k\geq 1}^{\infty} \frac{a_k}{z^{2k}} \quad \text{in } S^{\pm}_{\delta,\rho},$$

where the sectors $S^+_{\delta,\rho}$ and $S^-_{\delta,\rho}$ are defined as

$$S^+_{\delta,\rho} = \big\{z\in\mathbb{C};\ |\arg z|\leq \pi-\delta \text{ and } |z|\geq\rho\big\},$$
$$S^-_{\delta,\rho} = \big\{z\in\mathbb{C};\ -z\in S^+_{\delta,\rho}\big\}.$$

The notation $\varphi(z) \stackrel{\text{as}}{=} \sum_{n=-n_0}^{\infty}\varphi_n z^{-n}$ in S, for a subset S of $\mathbb{C}$, a function φ defined in S and a formal series of $\mathbb{C}[z][[z^{-1}]]$, means that for any integer $N\geq -n_0$ the function

$$R_{\varphi,N}(z) = |z|^{N+1}\cdot\Big|\varphi(z) - \sum_{n=-n_0}^{N}\varphi_n z^{-n}\Big|$$

is bounded[(1)] in S.

Let us now form the difference

$$w(z) = u^+(z) - u^-(z) = \int_{-\infty}^{+\infty} \mathrm{e}^{-z\zeta}\hat{u}(\zeta)\,\mathrm{d}\zeta.$$

For large $|\operatorname{Im} z|$ the function under the integral oscillates rapidly. Since $\hat{u}$ extends analytically to the cut plane $\mathbb{C}\setminus \pm 2\pi\mathrm{i}[1,+\infty[$, still with exponential decay at infinity, it is easy to see that $w(z)$ is exponentially small with respect to $|\operatorname{Im} z|$.

Let $\delta, \rho > 0$ and pick z in $S^+_{\delta,\rho} \cap S^-_{\delta,\rho}$ with $\operatorname{Im} z < 0$ (the case of the symmetric connected component of the intersection, where $\operatorname{Im} z > 0$, could be treated similarly or by symmetry). We can push the path of integration upwards and apply the Cauchy theorem as long as we do not reach $2\pi\mathrm{i}$: choosing $\xi \in]0, 2\pi[$ and $\theta \in]\frac{1}{2}\pi - \delta, \frac{1}{2}\pi[$, we have

$$w(z) = \int_{\Gamma_{\xi,\theta}} \mathrm{e}^{-z\zeta}\hat{u}(\zeta)\,\mathrm{d}\zeta,$$

where $\Gamma_{\xi,\theta} = \{|\operatorname{Re}\zeta| = (-\xi + \operatorname{Im}\zeta)\cot\theta\}$ is formed by two symmetric half-lines meeting at $i\xi$; this path comes from $\mathrm{e}^{\mathrm{i}(\pi-\theta)}\infty$, goes straight to $\mathrm{i}\xi$ and then straight to $\mathrm{e}^{\mathrm{i}\theta}\infty$. The inequality $\operatorname{Re}(z\zeta) \geq \lambda\xi|\operatorname{Im} z| + (1-\lambda)|\operatorname{Im} z|\operatorname{Im}\zeta$, with $\lambda = \cot\delta\cot\theta \in]0,1[$, shows that $\mathrm{e}^{\tau|\operatorname{Im} z|}|w(z)|$ is bounded in $S^+_{\delta,\rho} \cap S^-_{\delta,\rho} \cap \{\operatorname{Im} z < 0\}$ with $\tau = \lambda\xi$ arbitrarily close to 2π.

But we can deform farther the path of integration, crossing the imaginary axis between $2\pi\mathrm{i}$ and $4\pi\mathrm{i}$ and decomposing w into the contribution of the singularity at $2\pi\mathrm{i}$ and an exponentially smaller term:

$$w(z) = \int_{\gamma_\theta} \mathrm{e}^{-z\zeta}\hat{u}(\zeta)\,\mathrm{d}\zeta + \int_{\Gamma_{\xi,\theta}} \mathrm{e}^{-z\zeta}\hat{u}(\zeta)\,\mathrm{d}\zeta,$$

with $\xi \in]2\pi, 4\pi[$, $\theta \in]\frac{1}{2}\pi - \delta, \frac{1}{2}\pi[$, and a path γ_θ coming from $\mathrm{e}^{\mathrm{i}(\pi-\theta)}\infty$, turning counterclockwise around $2\pi\mathrm{i}$ and going back to $\mathrm{e}^{\mathrm{i}(\pi-\theta)}\infty$.

By the same argument as above, the integral on $\Gamma_{\xi,\theta}$ is bounded by an expression Const.$\mathrm{e}^{-\tau|\operatorname{Im} z|}$, but this time $\tau > 2\pi$ can be made arbitrarily close to 4π. By virtue of Theorem 2, the integral on γ_θ can be written as

$$\Theta\mathrm{e}^{-2\pi\mathrm{i}z}(d_{-2}z^4 + d_{-1}z^2 + d_0) + \mathrm{e}^{-2\pi\mathrm{i}z}\int_0^{\mathrm{e}^{\mathrm{i}(\pi-\theta)}\infty} \mathrm{e}^{-z\xi}h(\xi)\,\mathrm{d}\xi,$$

i.e. the Borel-Laplace transform of $\Theta\varphi_2 + \mu\varphi_1$ up to a factor $\mathrm{e}^{-2\pi\mathrm{i}z}$. Thus we obtain the following proposition.

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(1) In fact this notation can be given a more precise meaning: these are Gevrey-1 asymptotic expansions, *i.e.* the function $R_{\varphi,N}(z)$ is bounded by $CL^N(N+n_0)!$ for some C and L independent of N.

PROPOSITION 1. — *For all* $\delta \in\,]0,\pi[$ *and* $\rho > 0$, *the following asymptotic formula holds in* $S^+_{\delta,\rho} \cap S^-_{\delta,\rho} \cap \{\operatorname{Im} z < 0\}$:

$$\mathrm{e}^{2\pi\mathrm{i}z}\big(u^+(z) - u^-(z)\big) \overset{\mathrm{as}}{=} \Theta\varphi_2(z) + \mu\varphi_1(z). \tag{15}$$

Coming back to the additively-parametrized invariant curves $\mathbf{p}^+(z)$ and $\mathbf{p}^-(z)$, we can work similarly with the v-components v^+ and v^- which are Borel-Laplace transforms of $\hat{v}(\zeta) = \mathrm{e}^{\zeta}\,\hat{u}(\zeta)$. Using the notations of Equation (6), we obtain the

PROPOSITION 2. — *For all* $\delta \in\,]0,\pi[$ *and* $\rho > 0$, *one has in* $S^{\pm}_{\delta,\rho}$:

$$\forall\delta \in\,]0,\pi[,\ \forall\rho > 0, \qquad \mathbf{p}^{\pm}(z) \overset{\mathrm{as}}{=} \mathbf{p}(z),$$

and in $S^+_{\delta,\rho} \cap S^-_{\delta,\rho} \cap \{\operatorname{Im} z < 0\}$:

$$\mathrm{e}^{2\pi\mathrm{i}z}\big(\mathbf{p}^+(z) - \mathbf{p}^-(z)\big) \overset{\mathrm{as}}{=} \Theta\mathbf{n}(z) + \mu\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}z}(z) \tag{16}$$

with $\mathrm{d}\mathbf{p}/\mathrm{d}z = \binom{\varphi_1}{\psi_1}$, $\mathbf{n} = \binom{\varphi_2}{\psi_2}$, $\psi_i(z) = \varphi_i(z) - \varphi_i(z-1)$.

(We have used the fact that $\varphi_1 = \partial_z u$, thus $\psi_1 = \partial_z v$.) Formula (16) describes the splitting of the complex curves W^+ and W^- which are exponentially close one to the other. One can say that the constant Θ describes the normal component of the splitting and μ the tangent one (notice that the symplectic 2-form $\mathrm{d}u \wedge \mathrm{d}v$ yields 1 when evaluated on $(\mathrm{d}\mathbf{p}/\mathrm{d}z(z), \mathbf{n}(z))$).

Non-vanishing of Θ. — We will now see that

$$\mu \in \mathbb{R}, \quad \Theta \in \mathrm{i}\mathbb{R} \quad \text{and} \quad \operatorname{Im}\Theta < 0,$$

in particular u^+ and u^- do not coincide. Notice that, since $\hat{u}$ is odd,

$$u^-(z) = u^+(-z).$$

The non-vanishing of $u^+ - u^-$ can thus be interpreted as a defect of evenness, which turns out to be exponentially small for z tending to $-\mathrm{i}\infty$ (or to $+\mathrm{i}\infty$)[(2)] but the difference $u^+ - u^-$ is not small along the real axis.

Since the function $\hat{u}$ is real-analytic and odd, it is purely imaginary on the imaginary axis. In view of Theorem 2, since the coefficients b_k and d_k are real, this implies that the constant Θ is purely imaginary whereas μ is real.

(2) It is obvious on Equation (3) that $u(-z)$ is solution whenever $u(z)$ is solution. This symmetry property for the equation corresponds at the level of the map H to the existence of a "reverser"—an involutive transformation which conjugates H and H^{-1}.

Now suppose $\Theta = 0$: We will reach a contradiction. Indeed in that case it follows from Theorem 2 that $\hat{u}$ would be bounded on the segment $[0, 2\pi\mathrm{i}[$. Thus the function $\widehat{U}(\xi) = \mathrm{i}\hat{u}(\mathrm{i}\xi)$ would be bounded on the interval $[0, 2\pi[$. But it satisfies the convolution equation

$$\Big(4\sin^2 \frac{1}{2}\xi\Big)\widehat{U}(\xi) = (\widehat{U} * \widehat{U})(\xi)$$

which can easily be derived from Equation (8). Therefore the left-hand side of this equation would tend to zero when $\xi \longrightarrow 2\pi$. This is impossible, because according to the last statement of Lemma 1, $\widehat{U}$ is positive on $]0, 2\pi[$, so the right-hand side is positive too.

Moreover, since $\hat{u}(\mathrm{i}\xi) = -\mathrm{i}\widehat{U}(\xi)$ with $\widehat{U}$ positive, the coefficient $4!\,\Theta\, d_{-2}/2\pi\mathrm{i}$ of the leading term must be negative, hence $\operatorname{Im}\Theta < 0$.

Observe that in that chain of reasoning quite a precise information on the structure of the singularity at $2\pi\mathrm{i}$ was needed: we had to use the fact that the vanishing of the leading term would imply the vanishing of the whole polar part and even the boundedness of $\hat{u}$ near $2\pi\mathrm{i}$.

Unicity. — To complete the proof of Theorem 1 there remains only to check the unicity statement about u^+ and u^-. It is sufficient to treat the case of u^+ only.

Let $z_0 > 0$. For $n \in \mathbb{N}^*$, we define $\chi_n(z_0)$ to be the space of all functions $u : [z_0, +\infty[\to \mathbb{C}$ such that the quantity

$$\|u\|_n = \sup_{z \geq z_0} \{|z^n u(z)|\}$$

is finite. It is sufficient to show that, for z_0 large enough, any two solutions of Equation (3) such that $u(z) + 6z^{-2} \in \chi_4(z_0)$ must coincide.

LEMMA 3. — *For $n \in \mathbb{N}^*$, the formulas*

$$S_0 u(z) = -\sum_{k\geq 0} u(z+k), \quad S_1 u(z) = -\sum_{k\geq 1} u(z+k) \tag{17}$$

define two operators $S_0, S_1 \colon \chi_{n+1}(z_0) \longrightarrow \chi_n(z_0)$ such that, for all $v \in \chi_{n+1}(z_0)$, $u_0 = S_0 v$ and $u_1 = S_1 v$ are the unique solutions in $\chi_n(z_0)$ of the first-order finite-difference equations

$$u_1(z) - u_1(z-1) = v(z), \quad u_0(z+1) - u_0(z) = v(z). \tag{18}$$

Moreover

$$\|S_1 v\|_n \leq \frac{1}{n}\|v\|_{n+1}, \quad \|S_0 v\|_n \leq \Big(\frac{1}{z_0}+\frac{1}{n}\Big)\|v\|_{n+1}.$$

Proof. — Clearly the sums in (17) converge and define solutions of (18). Moreover, for $z \geq z_0$,

$$\sum_{k\geq 1}(z+k)^{-n-1} \leq \int_0^{+\infty} \frac{\mathrm{d}t}{(z+t)^{n+1}} = \frac{1}{nz^n}.$$

The result follows for S_1, and for S_0 (because $z^{-1} \geq z_0^{-1}$). □

Suppose now that u and u' solve Equation (3) and satisfy

$$u(z) = -6z^{-2} + \mathcal{O}(z^{-4}), \quad u'(z) = -6z^{-2} + \mathcal{O}(z^{-4}).$$

Consider the functions $v(z) = u(z) - u(z-1)$ and $v'(z) = u'(z) - u'(z-1)$, so that (u, v) and (u', v') are solutions of (2). We have

$$\delta u = u' - u \in \chi_4(z_0), \quad \delta v = v' - v \in \chi_4(z_0),$$

and

$$\delta u = S_1 \delta v, \quad \delta v = -S_0\big((u+u')\delta u\big).$$

Therefore, in view of Lemma 3,

$$\|\delta u\|_4 \leq \frac{1}{4}\Big(\frac{1}{5}+\frac{1}{z_0}\Big)\big\|(u+u')\delta u\big\|_6,$$

and $\|(u+u')\delta u\|_6 \leq \|u+u'\|_2 \cdot \|\delta u\|_4$ with $\|u+u'\|_2 \to 12$ as z_0 tends to infinity. We conclude that $\|\delta u\|_4 = 0$ if z_0 is large enough.

3. First sheets of the Riemann surface of $\hat{u}$.

We now begin the proof of Theorem 2. Let $\mathcal{R}$ be the Riemann surface consisting of all homotopy classes(3) of paths issuing from 0 and lying in $\mathbb{C} \setminus 2\pi\mathrm{i}\mathbb{Z}$ (except their origin). The natural projection $\zeta \in \mathcal{R} \mapsto \overset{\bullet}{\zeta} \in (\mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}) \cup \{0\}$ ($\overset{\bullet}{\zeta}$ is the extremity of any path representing ζ) is locally biholomorphic in a neighborhood of every point. One can also say that $\mathcal{R}$ is obtained by adding the origin to the main sheet of the universal covering of $\mathbb{C} \setminus 2\pi\mathrm{i}\mathbb{Z}$.

(3) When speaking of homotopy of paths, we always refer to homotopy with fixed extremities.

In this section we prove that $\hat{u}$ converges and extends analytically in the main sheet of $\mathcal{R}$ and its first half-sheets, *i.e.* the half-sheets which can be reached from the main sheet by crossing the imaginary axis exactly once.

Let us introduce notations for these subsets of $\mathcal{R}$. The main sheet $\mathcal{R}^{(0)}$, which is of course isomorphic to the cut plane $\mathbb{C} \setminus \pm 2\pi \mathrm{i}[1, +\infty[$, can be defined as the subset of homotopy classes of paths issuing from 0 and lying in $\mathbb{C} \setminus \pm 2\pi \mathrm{i}[1, +\infty[$. The union of $\mathcal{R}^{(0)}$ and the "nearby half-sheets" will be denoted by $\mathcal{R}^{(1)}$: it is the set of homotopy classes of paths issuing from 0, lying in $\mathbb{C} \setminus 2\pi\mathrm{i}\mathbb{Z}$ and crossing the imaginary axis at most once. We arrive to a nearby half-sheet when we follow a path which crosses the imaginary axis between two singular points. We arrive to different sheets of the Riemann surface when we pass between different singularities. Thus there are infinitely many nearby half-sheets.

For a given analytic germ at the origin, saying that it extends to an analytic function on $\mathcal{R}$ (*resp.* $\mathcal{R}^{(0)}$, *resp.* $\mathcal{R}^{(1)}$) amounts to saying that any path which represents an element of $\mathcal{R}$ (*resp.* $\mathcal{R}^{(0)}$, *resp.* $\mathcal{R}^{(1)}$) is a path of analytic continuation for it.

Our first goal is thus to prove the

PROPOSITION 3. — *The formal Borel transform $\hat{u}$ is convergent at the origin and defines a holomorphic germ which extends analytically to $\mathcal{R}^{(1)}$ with exponential decay at infinity on each half-sheet of $\mathcal{R}^{(1)}$.*

The series $\hat{u}(\zeta) \in \mathbb{C}[[\zeta]]$ is defined by Formula (7) as the formal Borel transform of u. But in order to study it outside its disk of convergence, the inductive computation of its coefficients a_k does not help much. This is why we will use an alternative representation of $\hat{u}$, expressing it as the limit of some iterative scheme at each step of which properties of analyticity can be checked in $\mathcal{R}^{(1)}$. The proof of Proposition 3 will follow from the uniform convergence of the scheme (on a system of subsets, the union of which covers $\mathcal{R}^{(1)}$).

3.1. Iterative scheme.

One can guess that the lattice of singular points $2\pi\mathrm{i}\mathbb{Z}$ stems from the division by $\mathrm{e}^{\zeta} - 2 + \mathrm{e}^{-\zeta} = 4\sinh^2 \frac{1}{2}\zeta$ in Equation (8). To exploit that idea, we first define a new unknown series $\hat{v}$ by

$$\hat{u}(\zeta) = -6\zeta + \hat{v}(\zeta).$$

It follows from Lemma 1 that $\hat{v}$ is the unique series in $\zeta^3\mathbb{C}[[\zeta]]$ such that $-6\zeta + \hat{v}(\zeta)$ solves (8), *i.e.* such that $\hat{v}$ solves the convolution equation

$$\alpha\hat{v} - 12\zeta * \hat{v} = \widehat{w}_0 - \hat{v} * \hat{v}, \tag{19}$$

where $\alpha(\zeta) = 4\sinh^2 \frac{1}{2}\zeta$ and $\widehat{w}_0(\zeta) = 6[\zeta\alpha(\zeta) - \zeta^3]$.

Let us introduce some auxiliary meromorphic functions:

$$\begin{aligned} \beta(\zeta) &= \frac{1}{\alpha(\zeta)} && \in \zeta^{-2}\mathbb{C}\{\zeta\}, \\ Y(\zeta) &= \frac{3\cosh\frac{1}{2}\zeta(3 + 2\cosh^2\frac{1}{2}\zeta)}{2\sinh^5\frac{1}{2}\zeta} && \in \zeta^{-5}\mathbb{C}\{\zeta\}, \\ Z(\zeta) &= -\zeta Y(\zeta) + \frac{4 + 11\cosh^2\frac{1}{2}\zeta}{\sinh^4\frac{1}{2}\zeta} && \in \zeta^{2}\mathbb{C}\{\zeta\}. \end{aligned}$$

We will denote by $\int_0 f$ the formal series or the function $\zeta \mapsto \int_0^\zeta f(\zeta_1)\,\mathrm{d}\zeta_1$, whenever f is a formal series ($f \in \mathbb{C}[[\zeta]]$) or a holomorphic function ($f \in \mathbb{C}\{\zeta\}$).

LEMMA 4 (The operator E). — *The operator*

$$\widehat{V} \in \zeta^3\mathbb{C}[[\zeta]] \longmapsto \widehat{W} = \alpha\widehat{V} - 12\zeta * \widehat{V} \in \zeta^5\mathbb{C}[[\zeta]]$$

is invertible and its inverse E can be expressed as

$$\widehat{W} \in \zeta^5\mathbb{C}[[\zeta]] \longmapsto E\cdot\widehat{W} = \beta\widehat{W} + \frac{1}{12}Z\int_0 Y\widehat{W} - \frac{1}{12}Y\int_0 Z\widehat{W}.$$

If $\widehat{W} \in \zeta^n\mathbb{C}[[\zeta]]$ with $n \geq 5$, $E\cdot\widehat{W} \in \zeta^{n-2}\mathbb{C}[[\zeta]]$.

If $\widehat{W} \in \zeta^5\mathbb{C}\{\zeta\}$ and if the germ defined by $\widehat{W}$ extends analytically to $\mathcal{R}^{(1)}$ (resp. to $\mathcal{R}$), $E\cdot\widehat{W} \in \zeta^3\mathbb{C}\{\zeta\}$ and the germ defined by $E\cdot\widehat{W}$ extends analytically to $\mathcal{R}^{(1)}$ too (resp. to $\mathcal{R}$).

Proof. — Let $\widehat{W} \in \zeta^5\mathbb{C}[[\zeta]]$. In order to find $\widehat{V}$, we use the change of unknown function $\widehat{V} = \beta F$ and we differentiate twice the operator that we want to invert: $\widehat{V} \in \zeta^3\mathbb{C}[[\zeta]]$ is solution of

$$\alpha\widehat{V} - 12\zeta * \widehat{V} = \widehat{W}$$

if and only if

$$F = \alpha\widehat{V} \in \zeta^5\mathbb{C}[[\zeta]] \quad\text{and}\quad F'' - 12\beta F = \widehat{W}''. \tag{20}$$

One checks easily that $y = \alpha Y/12 \in \zeta^{-3}\mathbb{C}\{\zeta\}$ and $z = \alpha Z/12 \in \zeta^4\mathbb{C}\{\zeta\}$ are independent solutions of the corresponding homogeneous equation $f''-12\beta f=0$, with Wronskian $yz'-y'z=1$ (in fact $z=y\int_0 y^{-2}$). Thus, whenever $g\in\zeta^3\mathbb{C}[[\zeta]]$, the solutions of $f''-12\beta f=g$ are the series $f=-y\int_0 zg+z\int_0 yg+c_1y+c_2g$, and among them only $f=-y\int_0 zg+z\int_0 yg$ lies in $\zeta^5\mathbb{C}[[\zeta]]$.

Hence a unique solution for (20):

$$F=-y\int_0 z\widehat{W}''+z\int_0 y\widehat{W}''=-y\int_0 Z\widehat{W}+z\int_0 Y\widehat{W}+\widehat{W}$$

(the last identity stems from a double integration by part). Multiplying by β, we obtain the desired formula for $\widehat{V}$.

The property of decreasing the valuation by 2 at most is easily checked.

If $\widehat{W}$ is a convergent power-series, so is $\widehat{V}$. The analyticity in $\mathcal{R}^{(1)}$ or $\mathcal{R}$ is preserved because Y and Z are meromorphic with poles in $2\pi i\mathbb{Z}$ only. □

LEMMA 5 (Algorithm for the $\hat{v}_n'$s). — *The formulas*

- $\widehat{w}_0=6[\zeta\alpha(\zeta)-\zeta^3]\in\zeta^5\mathbb{C}[[\zeta]]$;
- $\hat{v}_n=E\cdot\widehat{w}_n,\quad n\geq 0$;
- $\widehat{w}_n=-\sum_{n_1+n_2=n-1}\hat{v}_{n_1}*\hat{v}_{n_2},\quad n\geq 1$,

define inductively two sequences of formal series satisfying

$$\forall n\geq 0,\quad \hat{v}_n\in\zeta^{2n+3}\mathbb{C}[[\zeta]],\quad \widehat{w}_n\in\zeta^{2n+5}\mathbb{C}[[\zeta]],$$

and such that the unique nonzero odd solution of (8) *is*

$$\hat{u}(\zeta)=-6\zeta+\sum_{n\geq 0}\hat{v}_n(\zeta).$$

Proof. — The properties of the operator E ensure that the series $\hat{u}_n$ and $\hat{v}_n$ are well defined by induction, with valuations bounded from below as indicated in Lemma 5. Thus the series of formal series

$$\hat{v}=\sum_{n\geq 0}\hat{v}_n\quad\text{and}\quad\widehat{w}=\sum_{n\geq 0}\widehat{w}_n$$

are convergent in $\mathbb{C}[[\zeta]]$. We have

$$\hat{v}\in\zeta^3\mathbb{C}[[\zeta]],\quad \alpha\hat{v}-12\zeta*\hat{v}=\widehat{w}\quad\text{and}\quad\widehat{w}=\widehat{w}_0-\hat{v}*\hat{v}$$

by construction, hence the result follows. □

It is a well-known result of Resurgence theory that, if two germs extend analytically to $\mathcal{R}$, their convolution product has the same property. (We will recall the reason why this is so in Section 3.3.) This fact and the last part of Lemma 4 show that each power-series $\hat{v}_n$ or $\widehat{w}_n$ has nonzero radius of convergence and defines a germ which extends analytically to $\mathcal{R}$, since we start with $\widehat{w}_0$ which converges to an entire function. We won't try to prove the convergence of the series $\sum \hat{v}_n$ in the whole Riemann surface $\mathcal{R}$ now, but we retain that each term extends analytically to $\mathcal{R}^{(1)}$.

In order to prove Proposition 3, it is thus sufficient to study the convergence of $\sum \hat{v}_n$ as a series of holomorphic functions in $\mathcal{R}^{(1)}$.

3.2. Analytic continuation in the main sheet.

For $\rho \in \left]0, \frac{1}{2}\pi\right[$, we define $\mathcal{D}_\rho$ to be a closed subset of $\mathbb{C}$ obtained by removing the open disks of center $\pm 2\pi\mathrm{i}$ and radius ρ and all the points which are "hidden" by those disks from an observer based at the origin:

$$\mathcal{D}_\rho = \mathbb{C} \setminus \left\{ t\zeta,\ t \in \left]1, +\infty\right[,\ \zeta \in D(\pm 2\pi\mathrm{i}, \rho) \right\}.$$

The main sheet of $\mathcal{R}$ obviously coincides with the union of all these sets.

LEMMA 6 (Initial bounds). — *For any $\rho \in \left]0, \frac{1}{2}\pi\right[$, there exist positive numbers c, c_0 such that*

$$\forall \zeta \in \mathcal{D}_\rho \setminus \{0\}, \qquad \begin{cases} |\beta(\zeta)| \le c^2 |\zeta|^{-2}, \\ |Y(\zeta)| \le c|\zeta|^{-5}, \\ |Z(\zeta)| \le c|\zeta|^2, \end{cases}$$

and

$$\forall \zeta \in \mathcal{D}_\rho, \quad \left|\hat{v}_0(\zeta)\right| \le c_0 \frac{|\zeta|^3}{3!}.$$

Proof. — Let $\rho \in \left]0, \frac{1}{2}\pi\right[$. We observe that $\operatorname{Re}\zeta \ge \left(\rho/\sqrt{\rho^2 + 4\pi^2}\,\right)|\zeta|$ for all $\zeta \in \mathcal{D}_\rho$. Let us first consider the functions β, Y and Z: they are analytic in $\mathcal{D}_\rho$, except at the origin for β and Y which have poles of order 2 and 5 there, whereas Z has a zero of order 2 at the origin. On the other hand these functions decay exponentially when $|\zeta|$ tends to infinity (with ζ remaining in $\mathcal{D}_\rho$), because

$$\lim_{\operatorname{Re}\zeta \to \pm\infty} \mathrm{e}^{\pm\zeta} \beta(\zeta) = 1, \qquad \lim_{\operatorname{Re}\zeta \to \pm\infty} \mathrm{e}^{\pm\zeta}\, Y(\zeta) = 12,$$

$$\lim_{\operatorname{Re}\zeta \to \pm\infty} \zeta^{-1} \mathrm{e}^{\pm\zeta}\, Z(\zeta) = -12,$$

and exponential decay with respect to $|\operatorname{Re}\zeta|$ in $\mathcal{D}_\rho$ means exponential decay with respect to $|\zeta|$, hence the result follows.

Now $\hat{v}_0 = \beta\widehat{w}_0 + \frac{1}{12} Z \int_0 Y\widehat{w}_0 - \frac{1}{12} Y \int_0 Z\widehat{w}_0$, where $\widehat{w}_0$ is an entire function of order 1 satisfying

$$\lim_{\mathrm{Re}\zeta\to\pm\infty} \zeta^{-1}\mathrm{e}^{-(\pm\zeta)}\,\widehat{w}_0(\zeta) = 6.$$

Thus $\widehat{w}_0(\zeta) \leq$ Const. $|\zeta|\mathrm{e}^{|\mathrm{Re}\zeta|}$ for $|\zeta| > \rho$. On the other hand $\widehat{w}_0(\zeta) = \mathcal{O}(\zeta^5)$ near the origin. From that we deduce inequalities

$$|Y\widehat{w}_0| \leq \text{Const.}\,\big(1+|\zeta|\big) \quad \text{and} \quad |Z\widehat{w}_0| \leq \text{Const.}\,\big(|\zeta|^2+|\zeta|^7\big) \text{ in } \mathcal{D}_\rho$$

which show that $\hat{v}_0(\zeta) \leq$ Const. $|\zeta|^3$ for $\zeta > \rho$. And the proof is complete since $\hat{v}_0(\zeta) = \mathcal{O}(\zeta^3)$. □

LEMMA 7 (Bounds in the main sheet). — *Let $\rho \in\,]0, \frac{1}{2}\pi[$.*

(a) *If $\widehat{F}$ and $\widehat{G}$ are holomorphic functions in $\mathcal{D}_\rho$ which satisfy*

$$\forall\zeta\in\mathcal{D}_\rho,\quad \big|\widehat{F}(\zeta)\big| \leq \mathcal{F}\big(|\zeta|\big) \text{ and } \big|\widehat{G}(\zeta)\big| \leq \mathcal{G}\big(|\zeta|\big),$$

where $\mathcal{F}$ and $\mathcal{G}$ are continuous functions on $\mathbb{R}^+$, their convolution product $\widehat{F} * \widehat{G}$ is holomorphic in $\mathcal{D}_\rho$ and satisfies

$$\forall\zeta\in\mathcal{D}_\rho,\quad \big|(\widehat{F} * \widehat{G})(\zeta)\big| \leq (\mathcal{F} * \mathcal{G})\big(|\zeta|\big).$$

(b) *If $\widehat{W}$ is holomorphic in $\mathcal{D}_\rho$ and satisfies*

$$\forall\zeta\in\mathcal{D}_\rho,\quad \big|\widehat{W}(\zeta)\big| \leq C|\zeta|^\nu$$

for some real $C > 0$ and integer $\nu \geq 5$, the function $E\cdot\widehat{W}$ is holomorphic in $\mathcal{D}_\rho$ and satisfies

$$\forall\zeta\in\mathcal{D}_\rho,\quad \big|(E\cdot\widehat{W})(\zeta)\big| \leq 2c^2C|\zeta|^{\nu-2}$$

with c as in Lemma 6.

Proof. — Part (a) is quite obvious since

$$(\widehat{F} * \widehat{G})(\zeta) = \zeta\int_0^1 \widehat{F}(t\zeta)\widehat{G}\big((1-t)\zeta\big)\,\mathrm{d}t.$$

Let $\rho \in\,]0, \frac{1}{2}\pi[$ and $\widehat{W}$, C, ν as in Part *(b)*. For $\zeta\in\mathcal{D}_\rho$, we can write $(E\cdot\widehat{W})(\zeta)$ as the sum of three terms:

$$(E\cdot\widehat{W})(\zeta) = \beta(\zeta)\widehat{W}(\zeta) + \frac{Z(\zeta)}{12}\int_0^1 \zeta Y(t\zeta)\widehat{W}(t\zeta)\,\mathrm{d}t - \frac{Y(\zeta)}{12}\int_0^1 \zeta Z(t\zeta)\widehat{W}(t\zeta)\,\mathrm{d}t.$$

By virtue of the previous lemma, the first term is bounded by $c^2C|\zeta|^{\nu-2}$, the second one by $(c^2C/12(\nu-4))|\zeta|^{\nu-2}$ and the third one by $(c^2C/12(\nu+3))|\zeta|^{\nu-2}$, hence the result follows. □

LEMMA 8 (Convergence in the main sheet). — *Let $\rho \in \left]0, \frac{1}{2}\pi\right[$ and $c, c_0 > 0$ as in Lemma 6. The formulas*

$$c'_n = \sum_{n_1+n_2=n-1} c_{n_1} c_{n_2}, \quad c_n = \frac{c^2}{21} c'_n, \qquad n \geq 1$$

define inductively two sequences of positive numbers satisfying

$$\forall \zeta \in \mathcal{D}_\rho, \quad \left|\hat{v}_n(\zeta)\right| \leq c_n \frac{|\zeta|^{2n+3}}{(2n+3)!} \quad \text{and} \quad \left|\widehat{w}_n(\zeta)\right| \leq c'_n \frac{|\zeta|^{2n+5}}{(2n+5)!}.$$

The series of functions $\sum \hat{v}_n$ converges uniformly in $\mathcal{D}_\rho$ to a holomorphic function $\hat{v}$ and $\hat{u} = -6\zeta + \hat{v}$ has exponential decay at infinity in $\mathcal{D}_\rho$.

Proof. — Taking into account the bound for v_0 which is provided by Lemma 6, we proceed by induction and suppose that $\hat{v}_0, \ldots, \hat{v}_{n-1}$ are bounded as indicated in Lemma 8 for some $n \geq 1$. The desired bound for $\widehat{w}_n$ is obtained by Part (a) of Lemma 7, since

$$n_1 + n_2 = n - 1 \implies \frac{\zeta^{2n_1+3}}{(2n_1+3)!} * \frac{\zeta^{2n_2+3}}{(2n_2+3)!} = \frac{\zeta^{2n+5}}{(2n+5)!}.$$

Then we derive the bound for $\hat{v}_n$ by Part (b) of Lemma 7, since $(2n+4)(2n+5) \geq 42$.

Let $\lambda = 4c^2/21$. The generating series $c(X) = \sum_{n \geq 0} c_n X^n$ is easily computed: $c(X) = c_0 + \frac{\lambda}{4} X c(X)^2$, thus

$$c(X) = 2\frac{1 - (1 - c_0 \lambda X)^{1/2}}{\lambda X}.$$

It defines a holomorphic function on the open disk of center 0 and radius $(c_0\lambda)^{-1}$, which is bounded on the closure of that disk, therefore $c_n \leq \text{Const.}\,(c_0\lambda)^n$ for all $n \geq 0$. From that we deduce the uniform convergence of the series of analytic functions $\sum \hat{v}_n$ in $\mathcal{D}_\rho$ and an exponential bound for the sum:

$$\forall \zeta \in \mathcal{D}_\rho, \quad \left|\hat{v}(\zeta)\right| \leq \text{Const.}\, \mathrm{e}^{(c_0\lambda)^{1/2}|\zeta|}.$$

For $\hat{u}(\zeta) = -6\zeta + \hat{v}(\zeta)$ we can choose $\tau = (c_0\lambda)^{1/2}$ and write $|\hat{u}(\zeta)| \leq \text{Const.}\,|\zeta|\,\mathrm{e}^{\tau|\zeta|}$ in $\mathcal{D}_\rho$ (since $\hat{u}(\zeta) = \mathcal{O}(\zeta)$). But we can improve this bound by considering the equation we started with:

$$\forall \zeta \in \mathcal{D}_\rho \setminus \{0\}, \quad \hat{u}(\zeta) = -\beta(\zeta)(\hat{u} * \hat{u})(\zeta).$$

We know indeed that, if $|\zeta| > \rho$, $|\beta(\zeta)| \leq \text{Const.}\, \mathrm{e}^{-|\mathrm{Re}\,\zeta|}$, and $|\operatorname{Re}\zeta| \geq 2\delta|\zeta|$ in $\mathcal{D}_\rho$ with $\delta = \frac{1}{2}(\rho^2+4\pi^2)^{-1/2}\rho$. Let us introduce a number $C > 0$ such that

$$\forall \zeta \in \mathcal{D}_\rho, \quad |\zeta|^2 \cdot |\beta(\zeta)| \leq C\mathrm{e}^{-\delta|\zeta|}.$$

We now see that any exponential bound

$$\forall \zeta \in \mathcal{D}_\rho, \quad |\hat{u}(\zeta)| \leq C_0|\zeta|\mathrm{e}^{\tau|\zeta|},$$

with $C_0 > 0$ and $\tau \in \mathbb{R}$, implies $|(\hat{u} * \hat{u})(\zeta)| \leq \frac{C_0}{3!}|\zeta|^3\mathrm{e}^{\tau|\zeta|}$, and thus

$$\forall \zeta \in \mathcal{D}_\rho, \quad |\hat{u}(\zeta)| \leq \frac{C_0 C}{3!}|\zeta|\mathrm{e}^{(\tau-\delta)|\zeta|}.$$

This allows to decrease the exponential type τ indefinitely, and we conclude that *for all* $\tau \in \mathbb{R}$, the function $|\zeta|^{-1}\mathrm{e}^{-\tau|\zeta|}\,|\hat{u}(\zeta)|$ is bounded in $\mathcal{D}_\rho$. □

3.3. Analytic continuation in the nearby sheets.

We now explore farther the Riemann surface $\mathcal{R}$, but still progressively. With respect to Section 3.2, some more geometrical facts are involved, but the analysis is quite similar.

Let $M \in \mathbb{N}^*$ and $\rho \in\,]0, 2\pi/(2M+1)[$. We define the disks $D_1, \ldots, D_{M+1}$ and the opposite disks $D_{-1}, \ldots, D_{-M-1}$ by

$$D_m = D(2\pi \mathrm{i} m, m\rho), \quad D_{-m} = D(-2\pi\mathrm{i} m, m\rho), \quad m = 1, \ldots, M.$$

We define $\mathcal{D}_{\rho,M}$ to be the closed set obtained by removing from $\mathbb{C}$ all these disks:

$$\mathcal{D}_{\rho,M} = \mathbb{C} \setminus \Big(\bigcup_{\substack{-M \leq m \leq M \\ m \neq 0}} D_m \Big).$$

We define $\mathcal{R}^{(1)}_{\rho,M}$ to be the subset of $\mathcal{R}^{(1)}$ consisting of all the points ζ which can be represented by a path contained in $\mathcal{D}_{\rho,M}$ and such that the shortest such path γ_ζ is either

1) a straight segment;

2) or the union of a straight segment issuing from the origin and tangent to some disk D_m $(-M \leq m \leq M,\ m \neq 0)$ and of an arc of the circle ∂D_m ending at $\overset{\bullet}{\zeta}$, and we require in that situation that the half-line $L(\zeta)$ tangent to γ_ζ at $\overset{\bullet}{\zeta}$ and going backwards be contained in $\mathcal{D}_{\rho,M}$;

3) or the union of a straight segment issuing from the origin and tangent to some disk D_m ($-M \leq m \leq M$, $m \neq 0$), of an arc of the circle ∂D_m, and of a straight segment $S(\zeta)$ tangent to D_m, ending at $\overset{\bullet}{\zeta}$ and such that the half-line $L(\zeta)$ which extends $S(\zeta)$ backwards from $\overset{\bullet}{\zeta}$ be contained in $\mathcal{D}_{\rho,M}$.

In the first case ζ lies in the main sheet $\mathcal{R}^{(0)}$, but in the last case it lies in the half-sheet contiguous to $\mathcal{R}^{(0)}$ corresponding to one crossing of $]2\pi\mathrm{i}m, 2\pi\mathrm{i}(m+1)[$ if $m \geq 1$ (of $]2\pi\mathrm{i}(m-1), 2\pi\mathrm{i}m[$ if $m \leq -1$). In fact only a sector of this half-sheet is accessible because of the restriction $L(\zeta) \subset \mathcal{D}_{\rho,M}$ (see Fig. 2).

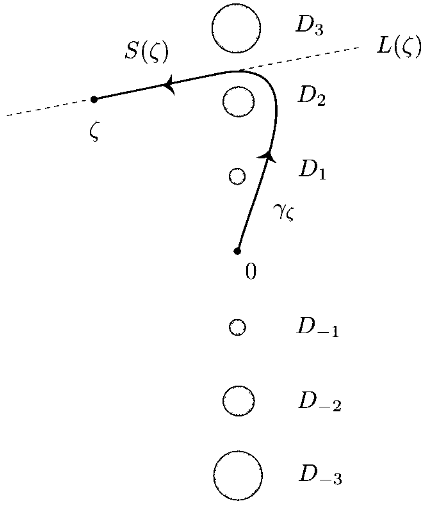

Figure 2. The paths γ_ζ

We have by construction

$$\mathcal{R}^{(1)} = \bigcup_{\substack{\rho \in]0, \frac{1}{2}\pi[\\ M \in \mathbb{N}^*}} \mathcal{R}^{(1)}_{\rho,M}.$$

We now fix for the rest of this section $M \in \mathbb{N}^*$ and $\rho \in]0, 2\pi/(2M+1)[$. Our goal is to prove the uniform convergence of the series $\sum \hat{v}_n$ in $\mathcal{R}^{(1)}_{\rho,M}$.

We need to recall how one follows the analytic continuation of the convolution product of two holomorphic functions of $\mathcal{R}$, and to exhibit bounds which generalize Part (a) of Lemma 7. To that end we define, for each $\zeta \in \mathcal{R}^{(1)}_{\rho,M}$, a particular path Γ_ζ which is homotopic to γ_ζ and represents thus the same point ζ.

The path Γ_ζ is obtained by a deformation of γ_ζ which makes it symmetrically contractile. One can visualize its construction by letting a point ζ_1 move along γ_ζ from the origin to ζ, the point ζ_1 remaining connected to the origin by an extensible thread, and imagining fixed nails pointing upwards at the points of $2\pi\mathrm{i}\mathbb{Z}$, with diameter $2|m|\rho$ for the nail at $2m\mathrm{i}\pi$, and moving nails pointing downwards at the points of $\zeta_1 + 2\pi\mathrm{i}\mathbb{Z}$ (with diameter $2|m|\rho$ for the nail at $\zeta_1 - 2m\mathrm{i}\pi$) between which the thread is stretched progressively when ζ_1 moves along γ_ζ: at the end of the process ζ_1 has reached ζ and Γ_ζ is the thread under its final form. (One can think that the fixed nails remain on a fixed rule, and the moving nails are fastened to another rule which is parallel to the first one with reverse orientation and which is trailed by ζ_1 in its motion.) Notice that at each moment of the process the thread between the origin and ζ_1 remains symmetric with respect to its midpoint, thus Γ_ζ is symmetric and symmetrically contractile.

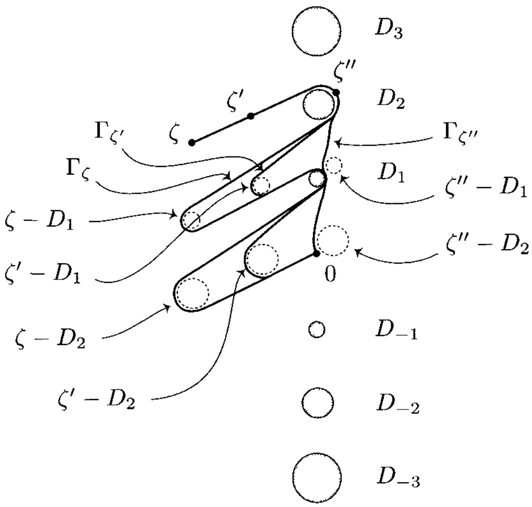

Figure 3. The paths Γ_ζ

The previous construction applies to paths which are more general than the paths γ_ζ and which lead to points lying in $\mathcal{R}$ but not necessarily in $\mathcal{R}^{(1)}_{\rho,M}$. In our case, for a given point $\zeta \in \mathcal{R}^{(1)}_{\rho,M}$, the resulting path Γ_ζ is easily described according to the three possible shapes of γ_ζ (see Figure 3):

- in case 1 above, Γ_ζ coincides with γ_ζ;

- in case 3, if $m \geq 1$, the path Γ_ζ starts from the origin by a straight segment, meanders between the disks $\zeta - D_m$, $D_1, \ldots, \zeta - D_{m-k}$,

$D_{k+1}, \dots, \zeta - D_1$, D_m (in that order) and ends by a straight segment leading to ζ; moreover it is the shortest such path (if $m \le -1$, D_{m-k} must be replaced by D_{m+k} ($1 \le k \le m-1$) in the previous sentence); it is thus a succession of straight segments and arcs of circle;

• in case 2, the description is the same as in the previous case except that there is no straight segment from D_k to $\zeta - D_{m-k}$ for $k = 0, \dots, m$ because of tangencies (with the convention $D_0 = \{0\}$).

The paths γ_ζ and Γ_ζ can be viewed as subsets of $\mathcal{R}^{(1)}_{\rho,M}$ rather than subsets of $\mathcal{D}_{\rho,M}$ (*i.e.* we identify them with their lifts in $\mathcal{R}$). Since Γ_ζ is symmetrically contractile, one can follow the analytic continuation at ζ of the convolution product $\widehat{F} * \widehat{G}$ of two germs $\widehat{F}, \widehat{G}$ which extend analytically to $\mathcal{R}^{(1)}_{\rho,M}$, and write it as

$$(\widehat{F} * \widehat{G})(\zeta) = \int_{\Gamma_\zeta} \widehat{F}(\zeta_1)\widehat{G}(\zeta_2)\,\mathrm{d}\zeta_1,$$

where ζ_2 is determined as the symmetric point of ζ_1 on Γ_ζ. Let us denote by s_ζ the curvilinear abscissa on Γ_ζ, by M_ζ the corresponding parameterization of Γ_ζ and by $\ell(\zeta)$ the length of Γ_ζ: we have $\ell(\zeta) = s_\zeta(\zeta)$ and the maps

$$\begin{aligned} \Gamma_\zeta &\longrightarrow \big[0, \ell(\zeta)\big], &\qquad \zeta_1 &\longmapsto s_\zeta(\zeta_1), \\ \big[0, \ell(\zeta)\big] &\longrightarrow \Gamma_\zeta, & s &\longmapsto M_\zeta(s) \end{aligned}$$

are mutually reciprocal. The formula for the analytic continuation of the convolution product may be written

$$(\widehat{F} * \widehat{G})(\zeta) = \int_0^{\ell(\zeta)} \widehat{F}\big(M_\zeta(s)\big)\widehat{G}\big(M_\zeta(\ell(\zeta) - s)\big)\Big(\frac{\mathrm{d}M_\zeta}{\mathrm{d}s}\Big)\mathrm{d}s.$$

Lemma 9 (New bounds for the convolution). — *If $\widehat{F}$ and $\widehat{G}$ are holomorphic functions in $\mathcal{R}^{(1)}_{\rho,M}$ which satisfy*

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad \big|\widehat{F}(\zeta)\big| \le \mathcal{F}\big(\ell(\zeta)\big) \quad \textit{and} \quad \big|\widehat{G}(\zeta)\big| \le \mathcal{G}\big(\ell(\zeta)\big),$$

where $\mathcal{F}$ and $\mathcal{G}$ are continuous increasing functions on $\mathbb{R}^+$, their product of convolution $\widehat{F} * \widehat{G}$, which is holomorphic in $\mathcal{R}^{(1)}_{\rho,M}$, satisfies

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad \big|(\widehat{F} * \widehat{G})(\zeta)\big| \le (\mathcal{F} * \mathcal{G})\big(\ell(\zeta)\big).$$

Proof. — The description of Γ_ζ given above allows one to check that

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad \Gamma_\zeta \subset \mathcal{R}^{(1)}_{\rho,M}$$

and

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M},\ \forall \zeta_1 \in \Gamma_\zeta, \quad \ell(\zeta_1) \le s_\zeta(\zeta_1). \tag{21}$$

The conclusion then comes easily: since $\mathcal{F}$ and $\mathcal{G}$ are increasing, for $\zeta \in \mathcal{R}^{(1)}_{\rho,M}$ we have

$$\begin{aligned} \big|(\widehat{F} * \widehat{G})(\zeta)\big| &\le \int_0^{\ell(\zeta)} \mathcal{F}\big(\ell(M_\zeta(s))\big)\mathcal{G}\big(\ell(M_\zeta(\ell(\zeta)-s))\big)\mathrm{d}s \\ &\le \int_0^{\ell(\zeta)} \mathcal{F}\big(s_\zeta(M_\zeta(s))\big)\mathcal{G}\big(s_\zeta(M_\zeta(\ell(\zeta)-s))\big)\mathrm{d}s \\ &= (\mathcal{F} * \mathcal{G})(\ell(\zeta)). \end{aligned}$$ □

Note that in this approach the inequality (21) is essential, but we know how to check such an inequality only for points in $\mathcal{R}^{(1)}_{\rho,M}$. One may then wonder whether it is possible to explore farther the Riemann surface $\mathcal{R}$ by a similar method or whether we are confined to the nearby sheets; we show in Section 5 how to bypass the difficulty in order to explore every sheet of $\mathcal{R}$.

LEMMA 10 (New bounds for the operator E). — *There exist $c, c_0 > 0$ such that*

(a) *for all $\zeta \in \mathcal{R}^{(1)}_{\rho,M}$,*

$$|\beta(\zeta)| \le c^2\ell(\zeta)^{-2}, \quad |Y(\zeta)| \le c\,\ell(\zeta)^{-5},$$

$$|Z(\zeta)| \le c\,\ell(\zeta)^2, \quad \hat{v}_0(\zeta) \le c_0\frac{\ell(\zeta)^3}{3!};$$

(b) *if $\widehat{W}$ is holomorphic in $\mathcal{R}^{(1)}_{\rho,M}$ and satisfies*

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad \big|\widehat{W}(\zeta)\big| \le C\ell(\zeta)^\nu$$

for some real $C > 0$ and integer $\nu \ge 5$, the function $E \cdot \widehat{W}$ is holomorphic in $\mathcal{R}^{(1)}_{\rho,M}$ and satisfies

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad \big|(E \cdot \widehat{W})(\zeta)\big| \le 2c^2C\ell(\zeta)^{\nu-2}.$$

Proof. — One checks the existence of a number $\kappa > 0$ such that

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad \ell(\zeta) \le \kappa(|\overset{\bullet}{\zeta}| + 1).$$

On the other hand, for $\zeta \in \mathcal{R}^{(1)}_{\rho,M}$ we have $|\overset{\bullet}{\zeta}| \le \ell(\zeta)$ and

$$|\overset{\bullet}{\zeta}| \le \rho \implies \zeta \in \mathcal{R}^{(0)} \implies \ell(\zeta) = |\overset{\bullet}{\zeta}|.$$

Thus Part (a) of Lemma 10 follows from Lemma 6.

Let $\widehat{W}$, C, ν as in Part (b). The formula for the analytic continuation of $E \cdot \widehat{W}$ at a point ζ of $\mathcal{R}^{(1)}_{\rho,M}$ may be written

$$\begin{aligned}(E \cdot \widehat{W})(\zeta) = \beta(\zeta)\widehat{W}(\zeta) \\ &+ \frac{1}{12} Z(\zeta) \int_0^{\ell(\zeta)} (Y\widehat{W})(M_\zeta(s)) \Big(\frac{\mathrm{d}M_\zeta}{\mathrm{d}s}\Big) \mathrm{d}s \\ &- \frac{1}{12} Y(\zeta) \int_0^{\ell(\zeta)} (Z\widehat{W})(M_\zeta(s)) \Big(\frac{\mathrm{d}M_\zeta}{\mathrm{d}s}\Big) \mathrm{d}s.\end{aligned}$$

Let us treat separately these three terms, using the inequalities of Part (a):

- the first term is bounded by $c^2 C \ell(\zeta)^{\nu-2}$;
- we observe that

$$\big|(Y\widehat{W})(M_\zeta(s))\big| \le cC\ell(M_\zeta(s))^{\nu-5} \le cCs^{\nu-5}$$

because of the inequality (21) and the hypothesis $\nu \ge 5$, thus the second term is bounded by $c^2C/(12(\nu-4))\,\ell(\zeta)^{\nu-2}$;

- analogously

$$\big|(Z\widehat{W})(M_\zeta(s))\big| \le cC\ell(M_\zeta(s))^{\nu+2} \le cCs^{\nu+2}$$

thus the third term is bounded by $c^2C/(12(\nu+3))\ell(\zeta)^{\nu-2}$.

Hence the desired bound for $|(E \cdot \widehat{W})(\zeta)|$ follows. □

LEMMA 11 (Convergence in the nearby sheets). — *Let $c, c_0 > 0$ as in the previous lemma. The formulas*

$$c'_n = \sum_{n_1+n_2=n-1} c_{n_1} c_{n_2}, \quad c_n = \frac{c^2}{21} c'_n, \qquad n \ge 1$$

define inductively two sequences of positive numbers satisfying

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad \big|\hat{v}_n(\zeta)\big| \le c_n \frac{\ell(\zeta)^{2n+3}}{(2n+3)!} \quad \text{and} \quad \big|\widehat{w}_n(\zeta)\big| \le c'_n \frac{\ell(\zeta)^{2n+5}}{(2n+5)!}.$$

The series of functions $\sum \hat{v}_n$ converges uniformly in $\mathcal{R}^{(1)}_{\rho,M}$ to a holomorphic function $\hat{v}$ and $\hat{u} = -6\zeta + \hat{v}$ has exponential decay at infinity in $\mathcal{R}^{(1)}_{\rho,M}$.

Proof. — The desired inequalities are obtained exactly in the same way as those of Lemma 8. This proves the convergence of the series of functions $\sum \hat{v}_n$, and $\hat{u}$ is thus holomorphic in $\mathcal{R}^{(1)}_{\rho,M}$ with an exponential bound

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad \big|\hat{u}(\zeta)\big| \le \text{Const.}\, \ell(\zeta)\, \mathrm{e}^{\tau\ell(\zeta)},$$

where $\tau = (4c_0c^2/21)^{1/2}$. As in the end of the proof of Lemma 8, we can improve this bound and decrease the exponential type τ, but this time the implication

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad \big|\hat{u}(\zeta)\big| \le C_0\, \ell(\zeta)\mathrm{e}^{\tau\ell(\zeta)}$$
$$\Longrightarrow \forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad \big|(\hat{u} * \hat{u})(\zeta)\big| \le \frac{C_0}{3!}\, \ell(\zeta)^3\mathrm{e}^{\tau\ell(\zeta)}$$

is ensured by Lemma 9 only for $\tau \ge 0$; introducing numbers $\delta, C > 0$ such that

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad \ell(\zeta)^2\big|\beta(\zeta)\big| \le C\mathrm{e}^{-\delta\ell(\zeta)},$$

we thus can reach $|\hat{u}(\zeta)| \le \text{Const.}\, \ell(\zeta)\mathrm{e}^{(\tau-\delta)\ell(\zeta)}$ with $\tau > 0$ and $\tau - \delta < 0$, but we must then stop. (In fact, it is a consequence of the resurgent properties of $\hat{u}$ explained in Section 5 that it has exponential type $-\infty$ in $\mathcal{R}^{(1)}_{\rho,M}$ too.) □

This ends the proof of Proposition 3.

4. First singularity of $\hat{u}$.

The aim of this section is to prove the statement concerning the singularity at $2\pi\mathrm{i}$ of $\hat{u}$ in Theorem 2 (Formulas (11) and (12)).

4.1. Shape of the singularity.

PROPOSITION 4. — *There exist $A_5, A_3, A_1 \in \mathbb{C}$ and $h(\xi), r(\xi) \in \mathbb{C}\{\xi\}$ such that $h(0) = 0$ and*

$$\hat{u}(2\pi i + \xi) = \frac{A_5}{\xi^5} + \frac{A_3}{\xi^3} + \frac{A_1}{\xi} + \frac{1}{2\pi i} h(\xi) \log \xi + r(\xi),$$

for $\zeta = 2\pi i + \xi$ close to $2\pi i$ on the main sheet.

For the proof of this proposition, we will use the same notations as in the previous section: $\hat{u}(\zeta) = -6\zeta + \hat{v}(\zeta) = -6\zeta + \sum \hat{v}_n(\zeta)$, and obtain by induction the shape of the singularity at $2\pi i$ for each $\hat{v}_n$. The property of convergence established in the previous section will then yield the result. Let us introduce a definition designed for our purpose:

- We say that a germ $\widehat{F} \in \mathbb{C}\{\zeta\}$ is of type (-1) if it is odd and of valuation 5 at least, and if it extends analytically to $\mathcal{R}^{(1)}$ and can be written

$$\widehat{F}(\zeta) = \frac{B}{\zeta - 2\pi i} + \frac{\widehat{H}(\zeta - 2\pi i)}{2\pi i} \log(\zeta - 2\pi i) + \widehat{R}(\zeta - 2\pi i)$$

in a neighborhood of $2\pi i$ on the main sheet, where $B \in \mathbb{C}$, and $\widehat{H}$ and $\widehat{R}$ are holomorphic at the origin with $\widehat{H}(\xi) = C\xi + D\xi^3 + \mathcal{O}(\xi^5)$ for some $C, D \in \mathbb{C}$.

- We say that a germ $\widehat{F} \in \mathbb{C}\{\zeta\}$ is of type (-5) if it is odd and of valuation 3 at least, and if it extends analytically to $\mathcal{R}^{(1)}$ and can be written

$$\widehat{F}(\zeta) = \frac{B}{(\zeta - 2\pi i)^5} + \frac{C}{(\zeta - 2\pi i)^3} + \frac{D}{\zeta - 2\pi i} + \frac{\widehat{H}(\zeta - 2\pi i)}{2\pi i} \log(\zeta - 2\pi i) + \widehat{R}(\zeta - 2\pi i)$$

in a neighborhood of $2\pi i$ on the main sheet, where $B, C, D \in \mathbb{C}$, and $\widehat{H}$ and $\widehat{R}$ are holomorphic at the origin with $\widehat{H}(\xi) = \mathcal{O}(\xi)$.

Remark. — One can rephrase the above definition using the *alien derivation* $\Delta_{2\pi i}$ of Resurgence theory. More details will be given in Section 5, but we can already mention that this operator measures the singularity at $2\pi i$ of the Borel transform of a given series (using the Borel transform, with the extended conventions indicated at the end of Section 1.3, to encode the singularity): an odd germ $\widehat{F} \in \mathbb{C}\{\zeta\}$ corresponds to an even formal series $\widetilde{F}(z) \in \mathbb{C}[[z^{-1}]]$ via formal Borel-Laplace transform, and the requirements on the shape of the singularity at $2\pi\mathrm{i}$ amount respectively to the conditions

$$\Delta_{2\pi\mathrm{i}}\widetilde{F} = 2\pi\mathrm{i}B + \widetilde{H}(z),$$
$$\widetilde{H}(z) = Cz^{-2} + 3!Dz^{-4} + \mathcal{O}(z^{-6}) \in \mathbb{C}[z][[z^{-1}]],$$

and

$$\Delta_{2\pi\mathrm{i}}\widetilde{F} = \frac{2\pi\mathrm{i}}{4!}Bz^4 + \frac{2\pi\mathrm{i}}{2!}Cz^2 + \frac{2\pi\mathrm{i}}{0!}D + \widetilde{H}(z),$$
$$\widetilde{H}(z) \in z^{-2}\mathbb{C}[[z^{-1}]].$$

LEMMA 12 (Transformation of singularities). — *The convolution product of two germs of type (-5) is of type (-1), and the image by the operator E of a germ of type (-1) is of type (-5).*

Proof. — Let us consider two germs $\widehat{F}_1$ and $\widehat{F}_2$ of type (-5): their convolution product $\widehat{G}$ is odd and of valuation 7 at least, and it extends analytically to $\mathcal{R}^{(1)}$. One checks that its singularity at $2\pi\mathrm{i}$ has the desired form by a direct analysis of the convolution integral, writing it as

$$\widehat{G}(\zeta) = \int_{\zeta/2}^{\zeta} \widehat{F}_1(\zeta_1)\widehat{F}_2(\zeta - \zeta_1)\,\mathrm{d}\zeta_1 + \int_{\zeta/2}^{\zeta} \widehat{F}_1(\zeta - \zeta_2)\widehat{F}_2(\zeta_2)\,\mathrm{d}\zeta_2$$

like in the proof of Proposition 5 below.

Alternatively one can use the framework of Resurgence and the important fact that the operator $\Delta_{2\pi\mathrm{i}}$ satisfies the Leibniz rule: the formal series $\widetilde{G}(z)$ associated with $\widehat{G}(\zeta)$ is the product of the formal series $\widetilde{F}_1$ and $\widetilde{F}_2$ associated with our germs,

$$\Delta_{2\pi\mathrm{i}}\widetilde{G} = \widetilde{F}_1\Delta_{2\pi\mathrm{i}}\widetilde{F}_2 + \widetilde{F}_2\Delta_{2\pi\mathrm{i}}\widetilde{F}_1,$$

and for $j = 1, 2$ we have $\widetilde{F}_j(z) = \mathcal{O}(z^{-4})$, even, whereas $\Delta_{2\pi\mathrm{i}}\widetilde{F}_j(z) = B_j z^4 + C_j z^2 + D_j + \mathcal{O}(z^{-2})$ for some complex numbers B_j, C_j, D_j, hence the result follows.

Let us now consider a germ $\widehat{F}$ of type (-1). We have already noticed that $\widehat{G} = E \cdot \widehat{F}$ is of valuation 3 at least and extends analytically to $\mathcal{R}^{(1)}$; it is easily seen to be odd. Let us study its singularity at $2\pi\mathrm{i}$. We use the expression

$$\widehat{G} = Y \int_0 \Big(y^{-2} \int_0 y\widehat{F}''\Big) \quad \text{with } y = \frac{\alpha Y}{12},$$

which can be checked from the proof of Lemma 4. For ξ small and such that $2\pi\mathrm{i} + \xi$ lies in the main sheet $\mathcal{R}^{(0)}$, we can write

$$\widehat{F}''(2\pi\mathrm{i} + \xi) = *\,\xi^{-3} + *\,\xi^{-1} + \xi\big(* + \mathcal{O}(\xi^2)\big)\log\xi + \mathrm{reg}(\xi),$$

where the stars $*$ stand for some complex numbers and $\mathrm{reg}(\xi)$ denotes some regular germ. But $y(2\pi\mathrm{i} + \xi) = *\,\xi^{-3}(1 + \mathcal{O}(\xi^2))$ is odd, thus

$$\begin{aligned}(y\widehat{F}'')(2\pi\mathrm{i} + \xi) = *\,\xi^{-6} + *\,\xi^{-4} + *\,\xi^{-3} + *\,\xi^{-2} + *\,\xi^{-1}\\ + \big(*\,\xi^{-2} + \mathrm{reg}(\xi)\big)\log\xi + \mathrm{reg}(\xi)\end{aligned}$$

and

$$\Big(\int_0 y\widehat{F}''\Big)(2\pi\mathrm{i} + \xi) = \xi^{-5}(* + \mathcal{O}(\xi^2)) + \xi^{-1}\mathrm{reg}(\xi)\log\xi.$$

Now $y^{-2}(2\pi\mathrm{i} + \xi) = *\,\xi^6(1 + \mathcal{O}(\xi^2))$ is even, thus $(y^{-2}\int_0 y\widehat{F}'')(2\pi\mathrm{i} + \xi) = \xi(* + \mathcal{O}(\xi^2)) + \xi^5\mathrm{reg}(\xi)\log\xi$,

$$\Big[\int_0\Big(y^{-2}\int_0 y\widehat{F}''\Big)\Big](2\pi\mathrm{i} + \xi) = * + *\,\xi^2 + \mathcal{O}(\xi^4) + \xi^6\mathrm{reg}(\xi)\log\xi,$$

and since $Y(2\pi\mathrm{i} + \xi) = *\,\xi^{-5}(1 + \mathcal{O}(\xi^2))$ is even, we conclude that

$$\widehat{G}(2\pi\mathrm{i} + \xi) = *\,\xi^{-5} + *\,\xi^{-3} + *\,\xi^{-1} + \xi\mathrm{reg}(\xi)\log\xi + \mathrm{reg}(\xi)$$

as required. □

Since $\widehat{w}_0$ extends to an entire function, it follows easily by induction that each $\hat{v}_n$ is of type (-5) and that each $\widehat{w}_n$ is of type (-1). Thus there exist sequences of numbers $(A_5^{(n)})$, $(A_3^{(n)})$ $(A_1^{(n)})$ and sequences of functions $(h^{(n)})$, $(r^{(n)})$ holomorphic near the origin such that, for all $n \geq 0$,

$$\hat{v}_n(2\pi\mathrm{i} + \xi) = \frac{A_5^{(n)}}{\xi^5} + \frac{A_3^{(n)}}{\xi^3} + \frac{A_1^{(n)}}{\xi} + \frac{1}{2\pi\mathrm{i}}\,h^{(n)}(\xi)\log\xi + r^{(n)}(\xi),$$

for $\zeta = 2\pi\mathrm{i} + \xi$ close to $2\pi\mathrm{i}$ on the main sheet, with $h^{(n)}(0) = 0$.

For any $n \geq 0$, the function $h^{(n)}$ is nothing but the "variation" (or monodromy) of the singularity of $\hat{v}_n$ around $2\pi\mathrm{i}$:

$$h^{(n)}(\xi) = \hat{v}_n(2\pi\mathrm{i} + \xi) - \hat{v}_n(2\pi\mathrm{i} + \xi\cdot \mathrm{e}^{-2\pi\mathrm{i}})$$

for $\zeta = 2\pi\mathrm{i} + \xi$ close to $2\pi\mathrm{i}$ on the main sheet, if we denote by $2\pi\mathrm{i} + \xi\cdot\mathrm{e}^{-2\pi\mathrm{i}}$ the point of $\mathcal{R}$ with the same projection onto $\mathbb{C}$ but lying in the sheet immediately "below" the main one (*i.e.* ζ is represented by the segment $[0,\overset{\bullet}{\zeta}]$, but $2\pi\mathrm{i} + \xi \cdot \mathrm{e}^{-2\pi\mathrm{i}}$ is represented by the path which begins by the straight segment and continues by a clockwise-oriented circle around $2\pi\mathrm{i}$). But then Lemma 11 implies the uniform convergence of the series $\sum h^{(n)}$ in a disk $D(0,\rho_0)$ centered at the origin and of sufficiently small radius ρ_0:

$$h = \sum_{n\geq 0} h^{(n)}$$

is holomorphic at the origin and satisfies $h(0) = 0$.

Now consider the functions

$$\hat{v}_n^*(\xi) = \frac{A_5^{(n)}}{\xi^5} + \frac{A_3^{(n)}}{\xi^3} + \frac{A_1^{(n)}}{\xi} + r^{(n)}(\xi)$$

for $n \geq 0$: they are holomorphic in the pointed disk $\{0 < |\xi| < \rho_0\}$ and the series $\sum \hat{v}_n^*$ is uniformly convergent in the annulus $D(0,\rho_0) \setminus D(0,\rho)$ for all $\rho \in \,]0,\rho_0[$; its sum

$$\hat{v}^*(\xi) = \sum_{n\geq 0} \hat{v}_n^*(\xi) = \hat{v}(2\pi\mathrm{i} + \xi) - \frac{1}{2\pi\mathrm{i}} h(\xi) \log \xi$$

is holomorphic in the pointed disk $D(0,\rho_0) \setminus \{0\}$. Writing the coefficients $A_5^{(n)}, A_3^{(n)}, A_1^{(n)}$ as Cauchy integrals involving $\hat{v}_n^*$, we thus deduce that the series

$$A_5 = \sum_{n\geq 0} A_5^{(n)}, \quad A_3 = \sum_{n\geq 0} A_3^{(n)}, \quad A_1 = \sum_{n\geq 0} A_1^{(n)}$$

are convergent. Observing that the function $\xi \mapsto \hat{v}^*(\xi) - A_5\xi^{-5} - A_3\xi^{-3} - A_1\xi^{-1}$ is regular at the origin, we conclude that $\hat{v}$ itself is a germ of type (-5). This ends the proof of Proposition 4.

4.2. First resurgent relations.

PROPOSITION 5. — *Let*

$$\varphi = \frac{2\pi\mathrm{i}}{4!} A_5 z^4 + \frac{2\pi\mathrm{i}}{2!} A_3 z^2 + \frac{2\pi\mathrm{i}}{0!} A_1 + \mathcal{B}^{-1} h \in \mathbb{C}[z][[z^{-1}]],$$

where $A_5, A_3, A_1 \in \mathbb{C}$ and $h(\xi) \in \mathbb{C}\{\xi\}$ are defined by Proposition 4. There exist complex numbers Θ and μ such that $\varphi = \Theta\varphi_2 + \mu\varphi_1$, i.e. φ is a linear combination of the formal series defined in Lemma 2.

This statement amounts exactly to Formulas (11) and (12) of Theorem 2: the singularity of $\hat{u}$ at $2\pi\mathrm{i}$ is the Borel transform of $\Theta\varphi_2 + \mu\varphi_1$. In the framework of Resurgence theory, this statement can be proved very quickly: according to the remark following Proposition 4, the formal series $\varphi \in \mathbb{C}[z][[z^{-1}]]$ is nothing but $\Delta_{2\pi\mathrm{i}}u$; as already mentioned the operator $\Delta_{2\pi\mathrm{i}}$ obeys the Leibniz rule, moreover it commutes with translations of step 1; applying $\Delta_{2\pi\mathrm{i}}$ to both sides of Equation (3), we get

$$\varphi(z+1) - 2\varphi(z) + \varphi(z-1) = -2u(z)\varphi(z),$$

whence the result follows by virtue of Lemma 2. The formula

$$\Delta_{2\pi\mathrm{i}}u = \Theta\varphi_2 + \mu\varphi_1,$$

which links $\varphi = \Delta_{2\pi\mathrm{i}}u$ with $\varphi_1 = \partial_z u$ and its companion linear solution φ_2, is called a *resurgent relation.*

To give to the reader an idea of the analysis which is involved in the resurgent formalism, we now provide an elementary proof of Proposition 5 which makes use of the classical Complex Analysis only. Let $\zeta \in \mathcal{R}^{(0)}$, close to $2\pi\mathrm{i}$. We note that

$$(\hat{u} * \hat{u})(\zeta) = 2\int_{\zeta/2}^{\zeta} \hat{u}(\zeta - \zeta')\hat{u}(\zeta')\,\mathrm{d}\zeta',$$

where the integral is taken over a rectilinear segment. In this way we separate the singular and the regular factors: when ζ is close to $2\pi\mathrm{i}$ the argument of the first function, $\hat{u}(\zeta-\zeta')$, remains far from the singularity. The convolution equation (8) takes the form,

$$4\sinh^2\frac{\zeta}{2}\ u(\zeta) = -2\int_{\zeta/2}^{\zeta} \hat{u}(\zeta - \zeta')\hat{u}(\zeta')\,\mathrm{d}\zeta'. \tag{22}$$

Now we use the first two terms of expansions (7) and (11) to evaluate $\hat{u}(\zeta-\zeta')$ and $\hat{u}(\zeta')$ respectively. We have

$$\begin{aligned} -2\int_{\zeta/2}^{\zeta} \hat{u}(\zeta - \zeta')\hat{u}(\zeta')\mathrm{d}\zeta' &= -\frac{a_1A_5}{6}\,\frac{1}{\xi^3} - \Big(a_1A_3 + \frac{a_2A_5}{12}\Big)\frac{1}{\xi} + \mathcal{O}(1) \\ &= \frac{A_5}{\xi^3} + \Big(6A_3 - \frac{15A_5}{24}\Big)\frac{1}{\xi} + \mathcal{O}(1), \end{aligned}$$

where $\xi = \zeta - 2\pi\mathrm{i}$. On the other hand,

$$4\sinh^2\frac{\zeta}{2}\ u(\zeta) = \frac{A_5}{\xi^3} + \Big(A_3 + \frac{A_5}{12}\Big)\frac{1}{\xi} + \mathcal{O}(\xi).$$

Comparing the last two equations we conclude that (22) implies

$$A_3 = \frac{17A_5}{120}\cdot$$

We obtain the third polar coefficients A_1 and the function h from the analysis of the variation (monodromy) of $\hat{u}$.

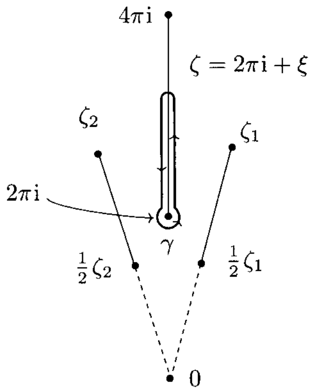

Figure 4. Paths of integration

Let two points ζ_1 and ζ_2 converge to the imaginary axis just above $2\pi\mathrm{i}$, from the right-hand side and from the left-hand side respectively. Let $\zeta = 2\pi\mathrm{i} + \xi$ denote the limit point (see Fig. 4). Then the prelogarithmic factor of (11) is given by

$$h(\xi) = \lim_{\substack{\zeta_1 \to 2\pi\mathrm{i}+\xi+0 \\ \zeta_2 \to 2\pi\mathrm{i}+\xi-0}} u(\zeta_1) - u(\zeta_2).$$

In order to evaluate the limit we take the difference of the two copies of the convolution equation,

$$4\sinh^2\frac{\zeta_k}{2}\ u(\zeta_k) = -2\int_{\zeta_k/2}^{\zeta_k} \hat{u}(\zeta_k - \zeta')\hat{u}(\zeta')\,\mathrm{d}\zeta',$$

and pass to the limit:

$$\begin{aligned} 4\sinh^2\frac{\xi}{2}\ h(\xi) &= -2\int_\gamma \hat{u}(\zeta - \zeta')\hat{u}(\zeta')\mathrm{d}\zeta' \\ &= -2\int_{\xi'+2\pi\mathrm{i}\in\gamma} \hat{u}(\xi - \xi')\hat{u}(2\pi\mathrm{i} + \xi')\mathrm{d}\xi'. \end{aligned}$$

Now we substitute the convergent expansion (11) instead of $\hat{u}(2\pi\mathrm{i}+\xi)$:

$$\begin{aligned} 4\sinh^2\frac{\xi}{2}\ h(\xi) &= -2\int_{\xi'+2\pi\mathrm{i}\in\gamma} \hat{u}(\xi-\xi')\frac{\log\xi'}{2\pi\mathrm{i}}h(\xi')\mathrm{d}\xi' \\ &\qquad - 4\pi\mathrm{i}\operatorname*{Res}_{\xi'=0}\Big[\hat{u}(\xi-\xi')\Big(\frac{A_5}{\xi'^5}+\frac{A_3}{\xi'^3}+\frac{A_1}{\xi'}\Big)\Big] \\ &= -2\int_0^{\xi}\hat{u}(\xi-\xi')h(\xi')\mathrm{d}\xi' \\ &\qquad - 4\pi\mathrm{i}\Big(A_5\frac{\partial_\xi^4\hat{u}(\xi)}{4!}+A_3\frac{\partial_\xi^2\hat{u}(\xi)}{2!}+A_1\hat{u}(\xi)\Big). \end{aligned}$$

In this way we obtain the following equation for the singularity of $\hat{u}$:

$$4\sinh^2\frac{\xi}{2}\ h(\xi) = -2(\hat{u}*h)(\xi) - f(\xi), \tag{23}$$

where

$$f(\xi) = 4\pi\mathrm{i}\Big(A_5\frac{\partial_\xi^4\hat{u}(\xi)}{4!}+A_3\frac{\partial_\xi^2\hat{u}(\xi)}{2!}+A_1\hat{u}(\xi)\Big).$$

This is a linear nonhomogeneous equation for h. Since $h(0)=\hat{u}(0)=0$ the corresponding terms of the equation are cubic at zero. Consequently $f(0)=f'(0)=f''(0)=0$. The equality $f'(0)=0$ implies

$$\frac{A_5a_3}{4!}+\frac{A_3a_2}{2!}+A_1a_1=0.$$

This implies $A_1=-\frac{17}{640}A_5$.

To finish the proof of Proposition 5, it is sufficient to check that Equation (23) generates exactly the same recurrence rule for h_k as the formal homogeneous variational equation (9), which is not too difficult.

5. Description of the whole resurgent structure.

Gathering the results of Propositions 3, 4 and 5, we see that in order to complete the proof of Theorem 2 there remains only to check that $\hat{u}$ extends analytically to the whole of $\mathcal{R}$. Notice that the derivation of Theorem 1 in Section 2 does not require this fact: the three propositions are sufficient for it.

We will now show a theorem (Theorem 3 below) which contains Theorem 2 and gives an overview of the resurgent structure of all the formal series which have been introduced in this article. But its proof will rely on Resurgence theory, and we will try to acquaint the reader with the tools to be used.

5.1. A (more) general formal solution.

We first extend the notion of formal solution for the equation (3).

PROPOSITION 6 (Normalized general solution). — *There is a unique sequence of nonzero even series* $(\tilde{u}_n)_{n\in\mathbb{N}}$ *in* $\mathbb{C}[z][[z^{-1}]]$ *such that*

- *the series*

$$\tilde{u}(z,b) = \sum_{n\geq 0} b^n \tilde{u}_n(z)$$

satisfies formally (3) *when expanding both sides of the equation in powers of* b *and then in powers of* z;

- $\tilde{u}_1(z) = z^4 + \mathcal{O}(z^2)$;
- *for all* $n \geq 2$, *the coefficient of* z^4 *in* $\tilde{u}_n$ *is zero.*

Remark. — A more general formal solution is obtained by considering

$$\tilde{u}\big(z + a(z), b(z)\big) = \sum_{n\geq 0} b(z)^n \tilde{u}_n\big(z + a(z)\big)$$

where $a(z)$ and $b(z)$ are 1-periodic objects, *e.g.* formal expansions in powers of $\mathrm{e}^{2\pi\mathrm{i}z}$ (or of $\mathrm{e}^{-2\pi\mathrm{i}z}$, but not both at the same time).

Proof. — Let us introduce notations for difference operators:

$$S : f(z) \longmapsto f(z+1) - f(z), \quad P : f(z) \longmapsto f(z+1) - 2f(z) + f(z-1).$$

When a formal Laurent series $f \in \mathbb{C}[z][[z^{-1}]]$ is given, it admits a primitive in $\mathbb{C}[z][[z^{-1}]]$ if and only if its residuum (the coefficient of z^{-1} in f) vanishes; in that case we denote by $\partial_z^{-1} f$ the unique primitive of f without constant term. The invertibility of S is easily studied:

LEMMA 13. — *A formal Laurent series* $f \in \mathbb{C}[z][[z^{-1}]]$ *admits a preimage by* S *in* $\mathbb{C}[z][[z^{-1}]]$ *if and only if its residuum vanishes. In that case the unique preimage of* f *without constant term can be obtained as*

$$S^{-1} f = B(\partial_z)\partial_z^{-1} f,$$

where

$$B(X) = \frac{X}{\mathrm{e}^X - 1} = 1 - \frac{X}{2} + \sum_{\ell\geq 1} (-1)^{\ell+1} B_\ell \frac{X^{2\ell}}{(2\ell)!}.$$

(The proof is straightforward.)

When substituting $\tilde{u}(z,b)$ inside (3) and expanding with respect to b both sides of equation, we find

- $P\tilde{u}_0 = -\tilde{u}_0^2$;
- $P\tilde{u}_1 = -2\tilde{u}_0\tilde{u}_1$;
- $P\tilde{u}_n = -\sum_{n_1+n_2=n} \tilde{u}_{n_1}\tilde{u}_{n_2}, \quad n \geq 2.$

We already know that the first of these equations admits a unique nonzero even solution $\tilde{u}_0$, which is nothing but the series called u in the rest of the paper:

$$\tilde{u}_0(z) = \sum_{k\geq 1} a_k z^{-2k} = -6z^{-2} + \frac{15}{2}z^{-4} - \frac{663}{40}z^{-6} + \frac{43647}{800}z^{-8} + \mathcal{O}(z^{-10}).$$

The second equation coincides with the variational equation (9) whose fundamental system of solutions (φ_1, φ_2) was introduced in Lemma 2, according to which there is only one possibility for $\tilde{u}_1$:

$$\tilde{u}_1(z) = 84\varphi_2(z) = z^4 + \frac{17}{10}z^2 - \frac{51}{80} + \frac{36}{5}z^{-2} + \mathcal{O}(z^{-4}). \tag{24}$$

We recall that

$$\varphi_1 = \partial_z \tilde{u}_0 = \sum_{k\geq 1} b_k z^{-2k-1}, \tag{25}$$

whereas $\varphi_2 = \sum_{k\geq -2} d_k z^{-2k}$ could be found directly. But one could also use a method which is the finite-difference analogue of the classical method of variation of parameters for second-order ordinary differential equations (see *e.g.* [Gel99] for detailed explanations); this leads to

$$\varphi_2 = \varphi_1 S^{-1}\chi, \quad \chi(z) = \frac{1}{\varphi_1(z)\varphi_1(z+1)} \tag{26}$$

(it can be checked that χ has no residuum since φ_1 is odd).

The next equations can be considered as linear non-homogeneous finite-difference equations: for $n \geq 2$, the series $\tilde{u}_n$ is required to satisfy

$$(P + 2\tilde{u}_0).\tilde{u}_n = \tilde{v}_n \tag{27}$$

with a right-hand side

$$\tilde{v}_n = -\sum_{k=1}^{n-1} \tilde{u}_k \tilde{u}_{n-k}$$

determined by the previous terms $\tilde{u}_0, \ldots, \tilde{u}_{n-1}$.

LEMMA 14. — *If a Laurent series $\psi \in \mathbb{C}[z^2][[z^{-2}]]$ is given such that $\varphi_1\psi$ has no residuum, the linear non-homogeneous equation*

$$(P + 2\tilde{u}_0).\varphi = \psi$$

admits a unique solution φ in $\mathbb{C}[z^2][[z^{-2}]]$ whose coefficient of z^4 vanishes. This solution can be written

$$\varphi(z) = \frac{1}{2}\Big((\Phi(z) + \Phi(-z)\Big) + c\,\varphi_2(z),$$

where c is some complex number and

$$\Phi = -\varphi_1 S^{-1}(\varphi_2\psi) + \varphi_2 S^{-1}(\varphi_1\psi).$$

Proof of the lemma. — Since the "Wronskian" of (φ_1, φ_2) is equal to 1, one can check that $\Phi = \alpha\varphi_1 + \beta\varphi_2$ is solution of the non-homogeneous equation as soon as $S\alpha = -\varphi_2\psi$ and $S\beta = \varphi_1\psi$. By hypothesis the series $\varphi_1\psi$ has no residuum, and the same is true for $\varphi_2\psi$ because φ_2 and ψ are even; we can thus choose $\alpha = -S^{-1}(\varphi_2\psi)$ and $\beta = S^{-1}(\varphi_1\psi)$, and we obtain a first solution Φ. But since $\tilde{u}_0$ and ψ are even, $\Phi(-z)$ is also solution of the non-homogeneous equation, therefore the odd series $\Phi(z) - \Phi(-z)$ satisfies the homogeneous equation and can be written $c_1\varphi_1(z) + c_2\varphi_2(z)$ with $c_1, c_2 \in \mathbb{C}$. Now $c_2 = 0$ because of oddness and

$$\Phi(z) = \frac{1}{2}\Big(\Phi(z) + \Phi(-z)\Big) + \frac{c_1}{2}\varphi_1(z).$$

The unique even solution φ without coefficient in front of z^4 is obtained by removing $\frac{1}{2}c_1\varphi_1(z)$ and adding the appropriate multiple of $\varphi_2(z)$. □

We now proceed by induction in order to solve the equations (27). Let us suppose that, for some $n \geq 2$, the series $\tilde{u}_0, \dots, \tilde{u}_{n-1}$ have been determined in $\mathbb{C}[z^2][[z^{-2}]]$. The series $\tilde{v}_n = -\sum_{k=1}^{n-1} \tilde{u}_k\tilde{u}_{n-k}$ belongs to that space too, and we only have to check that $\varphi_1\tilde{v}_n = \tilde{v}_n\partial_z\tilde{u}_0$ has no residuum. This results from the identity[4]

$$\tilde{v}_n\partial_z\tilde{u}_0 = -\frac{1}{2}\partial_z\Big[\tilde{u}_0\sum_{k=1}^{n-1}\tilde{u}_k\tilde{u}_{n-k}\Big] \\ - \frac{1}{2}\sum_{k=1}^{n-1}\Big[(\partial_z\tilde{u}_k)(P\tilde{u}_{n-k}) + (\partial_z\tilde{u}_{n-k})(P\tilde{u}_k)\Big],$$

[4] A compact way of deriving this identity consists in introducing the generating series $U = \sum_{n\geq 1} b^n\tilde{u}_n$ and $V = \sum_{n\geq 2} b^n\tilde{v}_n$, and checking that

$$\partial_z\Big(\int_0^{U(z,b)}\Big[\tilde{u}_0^2(z) - (\tilde{u}_0(z) + X)^2\Big]\mathrm{d}X\Big) = (P\cdot U)\partial_z U + (\partial_z\tilde{u}_0)V.$$

since the derivative of a Laurent series has no residuum and, for any two Laurent series f and g,

$$(\partial_z f)(Pg)+(Pf)(\partial_z g) = \sum_{m\geq 0} \frac{2}{(2m+2)!}\Big[(\partial_z f)(\partial_z^{2m+2} g)+(\partial_z^{2m+2} f)(\partial_z g)\Big]$$

has no residuum due to

$$F(\partial_z^{2m+1} G) + (\partial_z^{2m+1} F)G = \partial_z\Big[\sum_{\ell=0}^{2m}(-1)^\ell(\partial_z^\ell F)(\partial_z^{2m-\ell} G)\Big]. \qquad \square$$

Remark. — In fact $\tilde{u}(z,b) = \sum_{m\geq 0} z^{-2m-2} U_m(bz^6)$, with a family of formal series $U_m(X) \in \mathbb{Q}[[X]]$. For instance $U_0(X)$ is the generating series for the leading terms:

$$U_0(X) = \sum_{n\geq 0} c_n X^n \quad \text{with} \quad \forall n \geq 0, \quad \tilde{u}_n(z) = c_n z^{6n-2} + \mathcal{O}(z^{6n-4}).$$

Its coefficients can be computed inductively since it is the unique nonzero formal solution of the equation

$$(6b\partial_b - 2)(6b\partial_b - 3)U_0 = -U_0^2.$$

It can be checked that all the series U_m have positive radius of convergence.

5.2. Resurgent properties of the general formal solution.

Now comes the essential result of this section, which contains in fact Theorem 2. The following theorem is formulated in the language of Resurgence theory; we will provide explanations on its meaning after its statement.

THEOREM 3 (Resurgence of the general solution). — *For each integer $n \geq 0$, the formal series $\tilde{u}_n$ is a simply ramified resurgent function whose minor extends analytically to $\mathcal{R}$, with a growth of exponential type $-\infty$ along the non-vertical rays of each half-sheet of $\mathcal{R}$.*

There exist two families of formal series

$$A_\omega(b) = \sum_{n\geq 0} A_{\omega||n} b^n, \quad B_\omega(b) = \sum_{n\geq 0} B_{\omega||n} b^n, \qquad \omega \in 2\pi i\mathbb{Z}^*,$$

such that the Bridge equation holds:

$$\forall \omega \in 2\pi \mathrm{i}\mathbb{Z}^*, \quad \Delta_\omega \tilde{u}(z,b) = \big(A_\omega(b)\partial_b + B_\omega(b)\partial_z\big)\tilde{u}(z,b). \tag{28}$$

This equation must be understood as a compact writing of the resurgence relations

(29) $\forall \omega \in 2\pi \mathrm{i}\mathbb{Z}^*$, $\forall n \geq 0$,

$$\Delta_\omega \tilde{u}_n = \sum_{n_1+n_2=n} \big[(n_2+1)A_{\omega||n_1}\tilde{u}_{n_2+1} + B_{\omega||n_1}\partial_z \tilde{u}_{n_2}\big]$$

which allow one to compute all the alien derivatives of the resurgent functions $\tilde{u}_n$.

5.3. Explanation of resurgent terminology and remarks.

a) *Simply ramified resurgent functions*

If a formal Laurent series $\widetilde{\varphi}(z) \in \mathbb{C}[z][[z^{-1}]]$ is given, we can isolate the polynomial part $\psi(z)$ and compute the formal Borel transform of the remainder $\widetilde{\varphi}_0(z)$ according to the usual rule $\mathcal{B} : z^{-n-1} \mapsto \zeta^n/n!$:

$$\widetilde{\varphi} = \psi + \varphi_0,$$

$$\psi \in \mathbb{C}[z], \quad \widetilde{\varphi}_0 \in z^{-1}\mathbb{C}[[z^{-1}]], \quad \widehat{\varphi} = \mathcal{B}\widetilde{\varphi}_0 \in \mathbb{C}[[\zeta]].$$

Let us suppose that $\widehat{\varphi}$ is a convergent power-series which defines a germ of analytic function which extends analytically to $\mathcal{R}$. In that situation, $\widetilde{\varphi}$ is said to be a *resurgent function* and $\widehat{\varphi}$ is called its *minor.* Thus the first assertion in Theorem 3 constitutes a generalization of Proposition 3.

If moreover, when following the analytic continuation of the minor $\widehat{\varphi}$, the only encountered singularities are of the form

$$\{\text{polar part}\} + \{\text{logarithmic singularity}\},$$

the resurgent function $\widetilde{\varphi}$ is said to be *simply ramified* (we mean that, if $\widehat{\varphi}_\Gamma$ denotes the determination of $\widehat{\varphi}$ obtained by following some path Γ of analytic continuation which leads close to some point ω of $2\pi\mathrm{i}\mathbb{Z}$, we can write

$$\widehat{\varphi}_\Gamma(\omega+\xi) = \mathrm{pol}(\xi^{-1}) + \frac{1}{2\pi i}\mathrm{var}(\xi)\log\xi + \mathrm{reg}(\xi),$$

with $\mathrm{pol}(X) \in \mathbb{C}[X]$ and $\mathrm{var}(\xi), \mathrm{reg}(\xi) \in \mathbb{C}\{\xi\}$). The situation described in Proposition 4 was a particular case of this kind of singularity.

Simply ramified resurgent functions whose minors extends analytically to $\mathcal{R}$ form a subalgebra RES of $\mathbb{C}[z][[z^{-1}]]$.

b) *Alien derivations*

Let $\omega \in 2\pi\mathrm{i}\mathbb{Z}^*$. The *alien derivation of index* ω is a particular linear operator Δ_ω of RES which satisfies the Leibniz rule:

$$\forall \widetilde{\varphi}_1, \widetilde{\varphi}_2 \in \mathrm{RES}, \quad \Delta_\omega(\widetilde{\varphi}_1\widetilde{\varphi}_2) = (\Delta_\omega\widetilde{\varphi}_1)\widetilde{\varphi}_2 + \widetilde{\varphi}_1(\Delta_\omega\widetilde{\varphi}_2).$$

For definiteness, let us first consider the case where $\omega = 2\pi\mathrm{i}r$ with $r \geq 1$: if $\widetilde{\varphi}$ is given in RES, we may consider the 2^{r-1} determinations of the minor $\widehat{\varphi}$ in the segment $]2\pi\mathrm{i}(r-1), 2\pi\mathrm{i}r[$ which are obtained by following its analytic continuation along the half-line $\mathrm{i}\mathbb{R}^+$ and circumventing the intermediary singular points $2\pi\mathrm{i}, \dots, 2\pi\mathrm{i}(r-1)$ to the left or to the right; we denote them

$$\widehat{\varphi}^{\varepsilon_1,\dots,\varepsilon_{r-1}},$$

each ε_ℓ being a plus sign or a minus sign indicating whether $2\pi\mathrm{i}\ell$ was circumvented to the left or to the right. For $\zeta \in]-2\pi\mathrm{i}, 0[$, we set

$$\check{\chi}(\zeta) = \sum_{\varepsilon_1,\dots,\varepsilon_{r-1}} \frac{p(\varepsilon)!\, q(\varepsilon)!}{r!} \varphi^{\varepsilon_1,\dots,\varepsilon_{r-1}}(2\pi\mathrm{i}r + \zeta),$$

where the integers $p(\varepsilon)$ and $q(\varepsilon) = r - 1 - p(\varepsilon)$ denote the numbers of plus signs and of minus signs in the sequence $(\varepsilon_1, \dots, \varepsilon_{r-1})$. According to our hypothesis on the shape of the singularities of the minor $\widehat{\varphi}$, the function $\check{\chi}$ must take the form

$$\check{\chi}(\zeta) = A_N\zeta^{-N} + \cdots + A_1\zeta^{-1} + \frac{1}{2\pi\mathrm{i}}\widehat{\chi}(\zeta)\log\zeta + \mathrm{reg}(\zeta),$$

where $A_1, \dots, A_N$ are some complex numbers and $\widehat{\chi}(\zeta), \mathrm{reg}(\zeta) \in \mathbb{C}\{\zeta\}$. We define

$$\Delta_\omega\widetilde{\varphi} = 2\pi\mathrm{i}\Big[\frac{(-1)^{N-1}}{(N-1)!}A_N z^{N-1} + \cdots + A_1\Big] + \mathcal{B}^{-1}\widehat{\chi}.$$

It can be checked that $\Delta_\omega\widetilde{\varphi}$ is a well-defined element of RES; observe that its minor $\widehat{\chi}$ can be computed in the segment $]0, 2\pi\mathrm{i}[$ according to the formula

$$\widehat{\chi}(\zeta) = \sum_{\varepsilon_1,\dots,\varepsilon_{r-1}} \frac{p(\varepsilon)!\, q(\varepsilon)!}{r!} \big[\varphi^{\varepsilon_1,\dots,\varepsilon_{r-1},+}(2\pi\mathrm{i}r+\zeta) - \varphi^{\varepsilon_1,\dots,\varepsilon_{r-1},-}(2\pi\mathrm{i}r+\zeta)\big].$$

If $\omega = -2\pi i r$ with $r \geq 1$, the operator Δ_ω is defined in a similar fashion. If a simply ramified resurgent function $\widetilde{\varphi}$ has only real coefficients, *i.e.* $\widetilde{\varphi}(z) \in \mathrm{RES} \cap \mathbb{R}[z][[z^{-1}]]$, one checks that

$$\Delta_{-2\pi i r}\widetilde{\varphi}(z) = -\overline{(\Delta_{2\pi i r}\widetilde{\varphi})(\overline{z})}.$$

On the other hand, if $\widetilde{\varphi}$ is even with respect to z,

$$\Delta_{-2\pi i r}\widetilde{\varphi}(z) = (\Delta_{2\pi i r}\widetilde{\varphi})(-z).$$

The fact that the operators Δ_ω are derivations is essential in Resurgence theory. They are called alien derivations by contrast with the natural derivation ∂_z. There is a relation

$$\Delta_\omega \circ \partial_z = (\partial_z - \omega) \circ \Delta_\omega,$$

but no relation between the Δ_ω's themselves: they generate a free Lie algebra.

These operators encode in fact the whole singular behavior of the minors. Given a sequence $\omega_1, \ldots, \omega_n$ in $2\pi i\mathbb{Z}^*$, the composed operator $\Delta_{\omega_n} \circ \cdots \circ \Delta_{\omega_1}$ gives information on the singularities over the point $\omega_1 + \cdots + \omega_n$.

The point of view on Resurgence theory that we have indicated is rather restrictive and we refer the interested reader to [Eca81], [Eca93], [CNP93], [BSSV98] for further properties and more general definitions.

c) *Bridge equation*

The so-called *Bridge equation* (28) is an example of a general phenomenon which is at the origin of the name *Resurgent function*: the definition of a general resurgent function $\widetilde{\varphi}$ *a priori* does not force any relationship between $\widetilde{\varphi}$ and its alien derivatives, but for the resurgent functions of natural origin (*i.e.* solutions of some analytic problem) it is observed that the alien derivatives obey particular relations depending on the problem under consideration.

The equation (28) can be viewed as a bridge between alien calculus and ordinary differential calculus in the case of the formal solution $\tilde{u}$, hence its name. The families of complex numbers $A_{\omega||n}$ and $B_{\omega||n}$ which determine the differential operator in the right-hand side represent all the "transcendental" part of the information that is needed to describe the singular structure in the Borel plane (whereas the series $\tilde{u}_n$ themselves represent the "elementary" part).

In our case, the realness of the coefficients of the series $\tilde{u}_n$ implies that

$$\forall\omega \in 2\pi\mathrm{i}\mathbb{Z}^*,\quad A_{-\omega}(b) = -\overline{A_\omega(\,\overline{b}\,)},\quad B_{-\omega}(b) = -\overline{B_\omega(\,\overline{b}\,)},$$

and since $\tilde{u}(z,b)$ is even with respect to z, we have also

$$\forall\omega \in 2\pi\mathrm{i}\mathbb{Z}^*,\quad A_{-\omega}(b) = A_\omega(b),\quad B_{-\omega}(b) = -B_\omega(b).$$

Therefore we conclude that

$$\forall\omega \in 2\pi\mathrm{i}\mathbb{Z}^*,\ \forall n \in \mathbb{N},\quad A_{\omega||n} = A_{-\omega||n} \in i\mathbb{R},\quad B_{\omega||n} = -B_{-\omega||n} \in \mathbb{R}.$$

The Bridge equation (28) provides the decomposition of

$$\Delta_\omega \tilde{u}(z,b) = \sum_{n\geq 0} b^n \Delta_\omega \tilde{u}_n(z)$$

as sum of its even part $A_\omega(b)\partial_b\tilde{u}(z,b)$ and its odd part $B_\omega(b)\partial_z\tilde{u}(z,b)$ (even and odd with respect to z). Note that all the successive alien derivatives may be computed by iteration of the Bridge equation:

$$\begin{aligned}&\Delta_{\omega_1} \circ \Delta_{\omega_2}\tilde{u}(z,b)\\ &\qquad = \big(A_{\omega_2}(b)\partial_b + B_{\omega_2}(b)(\partial_z - \omega_2)\big)\big(A_{\omega_1}(b)\partial_b + B_{\omega_1}(b)\partial_z\big)\tilde{u}(z,b)\end{aligned}$$

(beware of the inversion of indices), *etc.* Therefore, in principle, the singularities of each determination of the minors $\hat{u}_n$ can be expressed in terms of the numbers A_ω and B_ω and of the coefficients of the series $\tilde{u}_n$.

d) *First singularity of the first minor*

Here are the resurgence relations for the first series:

$$\forall\omega \in 2\pi\mathrm{i}\mathbb{Z}^*,\quad \Delta_\omega\tilde{u}_0 = A_{\omega||0}\tilde{u}_1 + B_{\omega||0}\frac{\mathrm{d}\tilde{u}_0}{\mathrm{d}z}\cdot$$

On the other hand, in the case of $\omega = 2\pi\mathrm{i}$, we can rephrase Proposition 4:

$$\Delta_{2\pi\mathrm{i}}\tilde{u}_0 = \frac{\Theta}{84}\tilde{u}_1 + \mu\frac{\mathrm{d}\tilde{u}_0}{\mathrm{d}z}$$

(using (24) and (25)). Therefore,

$$A_{2\pi\mathrm{i}||0} = \frac{1}{84}\Theta,\quad B_{2\pi\mathrm{i}||0} = \mu.$$

5.4. Idea of the proof of Theorem 3.

a) We already know, from Section 3, that the minor $\hat{u}_0$ of the first formal series $\tilde{u}_0$ converges at the origin and extends analytically to $\mathcal{R}^{(1)}$.

The same is true for the minors of φ_1 and φ_2. Indeed, in the case of φ_1, the relation (25) can be translated into a relation between the minors: $\widehat{\varphi}_1(\zeta) = -\zeta\hat{u}_0(\zeta)$, which shows that $\widehat{\varphi}_1$ extends analytically to $\mathcal{R}^{(1)}$. In the case of φ_2, consider the relation (26): the series $\varphi_1(z+1)$ admits a minor $\mathrm{e}^{-\zeta}\widehat{\varphi}_1(\zeta)$ which extends analytically to $\mathcal{R}^{(1)}$, and it can be checked that the series χ which involves the multiplicative inverses of $\varphi_1(z)$ and $\varphi_1(z+1)$ has also a minor $\widehat{\chi}$ which is holomorphic in $\mathcal{R}^{(1)}$; then the operator S^{-1}, which simply amounts to division by $\mathrm{e}^{-\zeta}-1$ for the minors, and the multiplication of series preserve the property of having a minor which extends analytically to $\mathcal{R}^{(1)}$.

We then obtain by induction that, for each $n \geq 0$ the minor $\hat{u}_n$ of $\tilde{u}_n$ converges at the origin and extends analytically to $\mathcal{R}^{(1)}$. Indeed we recall that

$$\tilde{u}_n(z) = \frac{1}{2}\big(\widetilde{\Phi}_n(z) + \widetilde{\Phi}_n(-z)\big) + \mathrm{Const.}\,\varphi_2(z)$$

with $\tilde{v}_n = -\sum_{k=1}^{n-1}\tilde{u}_k\tilde{u}_{n-k}$ and $\widetilde{\Phi}_n = -\varphi_1 S^{-1}(\varphi_2\tilde{v}_n) + \varphi_2 S^{-1}(\varphi_1\tilde{v}_n)$, and the same arguments as above apply.

b) In order to prove the analyticity of the $\hat{u}_n$'s in every sheet of the Riemann surface $\mathcal{R}$, we will use the alien derivations as a tool to "propagate" analyticity from one sheet to some nearby sheets. We first define an infinite decreasing sequence $\mathrm{RES}^{(1)}, \mathrm{RES}^{(2)}, \ldots$ of subalgebras of $\mathbb{C}[z][[z^{-1}]]$, whose intersection is nothing but the algebra RES of simply ramified resurgent functions. Then we will explain how one can check that the $\tilde{u}_n$'s belong to each algebra $\mathrm{RES}^{(N)}$ and therefore to RES.

Let $\mathrm{RES}^{(1)}$ be the subspace of $\mathbb{C}[z][[z^{-1}]]$ consisting of all the Laurent series $\widetilde{\varphi}$ whose minor $\widehat{\varphi}$ satisfy the following properties:

- $\widehat{\varphi}$ converges at the origin and extends analytically to $\mathcal{R}^{(0)}$ (the main sheet of $\mathcal{R}$);

- $\widehat{\varphi}$ extends analytically also along the paths which issue from the origin and end on $(\mathrm{i}\mathbb{R}) \setminus 2\pi\mathrm{i}\mathbb{Z}$ without crossing the imaginary axis (this allows to define "lateral" continuations of $\widehat{\varphi}$ between the singular points);

- $\widehat{\varphi}$ has at worst *ramified singularities* at $2\pi\mathrm{i}$ and $-2\pi\mathrm{i}$, *i.e.* singularities of the form $\mathrm{pol}(\zeta^{-1}) + \frac{1}{2\pi\mathrm{i}}\mathrm{var}(\zeta)\log\zeta$ with $\mathrm{pol}(X) \in \mathbb{C}[X]$ and $\mathrm{var}(\zeta) \in \mathbb{C}[\zeta^{-1}]\{\zeta\}$.

One can check that $\mathrm{RES}^{(1)}$ is a subalgebra of $\mathbb{C}[z][[z^{-1}]]$ which contains RES, and on which operators $\Delta_{2\pi\mathrm{i}}$, $\Delta_{-2\pi\mathrm{i}}$ may be defined as previously and still satisfy the Leibniz rule; but these operators take their values in a space of formal series which will be larger than $\mathbb{C}[z][[z^{-1}]]$ (these formal series may involve $\log z$).

Now consider the subspace $\mathrm{RES}^{(2)}$ consisting of all the elements $\widetilde{\varphi}$ of $\mathrm{RES}^{(1)}$ such that $\Delta_{2\pi\mathrm{i}}\widetilde{\varphi}$ and $\Delta_{-2\pi\mathrm{i}}\widetilde{\varphi}$ belong to $\mathrm{RES}^{(1)}$, and the lateral continuations of $\widehat{\varphi}$ have ramified singularities at $\pm 4\pi\mathrm{i}$. It is stable by multiplication too, and not only $\Delta_{\pm 2\pi\mathrm{i}} \circ \Delta_{\pm 2\pi\mathrm{i}}$ are defined on it, but also operators $\Delta_{4\pi\mathrm{i}}$, $\Delta_{-4\pi\mathrm{i}}$ which extend the alien derivations $\Delta_{\pm 4\pi\mathrm{i}}$ of RES and still satisfy the Leibniz rule.

Let us try to indicate the idea behind the definition of $\mathrm{RES}^{(2)}$. The condition $\widetilde{\varphi} \in \mathrm{RES}^{(1)}$ implies that the minor $\widehat{\varphi}$ extends to $\mathcal{R}^{(0)}$, and even until $]2\pi\mathrm{i}, 4\pi\mathrm{i}[$ if $2\pi\mathrm{i}$ is circumvented to the left or to the right: let us denote by $\widehat{\varphi}^{+}(\zeta)$ and $\widehat{\varphi}^{-}(\zeta)$ the two corresponding determinations of $\widehat{\varphi}$ at a point ζ of $]2\pi\mathrm{i}, 4\pi\mathrm{i}[$. The minor of $\Delta_{2\pi\mathrm{i}}\widetilde{\varphi}$ is nothing but

$$\widehat{\chi}(\zeta) = \widehat{\varphi}^{+}(2\pi\mathrm{i} + \zeta) - \widehat{\varphi}^{-}(2\pi\mathrm{i} + \zeta) \quad \text{for} \quad \zeta \in]0, 2\pi\mathrm{i}[.$$

Now the condition $\Delta_{2\pi\mathrm{i}}\widetilde{\varphi} \in \mathrm{RES}^{(1)}$ implies that $\widehat{\chi}$ extends to $\mathcal{R}^{(0)}$, and thus $\widehat{\varphi}^{+}$ extends to the whole half-sheet contiguous to $\mathcal{R}^{(0)}$ defined by paths which cross $]2\pi\mathrm{i}, 4\pi\mathrm{i}[$ from right to left, since we can write for the points ζ in that half-sheet

$$\widehat{\varphi}^{+}(\zeta) = \widehat{\varphi}(\overset{\bullet}{\zeta}) + \widehat{\chi}(\overset{\bullet}{\zeta} - 2\pi\mathrm{i}).$$

This is the key-point: the determination of $\widehat{\varphi}$ in that half-sheet may be expressed in terms of the functions $\widehat{\varphi}$ and $\widehat{\chi}$ in the main sheet. Similarly, $\widehat{\varphi}^{-}$ extends to the symmetric half-sheet, according to the formula

$$\widehat{\varphi}^{-}(\zeta) = \widehat{\varphi}(\overset{\bullet}{\zeta}) - \widehat{\chi}(\overset{\bullet}{\zeta} - 2\pi\mathrm{i}).$$

Therefore, our requirement on $\Delta_{2\pi\mathrm{i}}\widetilde{\varphi}$, which deals with analyticity in the main sheet for its minor, can be interpreted as a property of analyticity for $\widehat{\varphi}$ in other sheets of $\mathcal{R}$.

One can go on and define inductively $\mathrm{RES}^{(3)}, \mathrm{RES}^{(4)}, \ldots$, by requiring at each level that all the "computable" alien derivatives of the previous level lie in $\mathrm{RES}^{(1)}$ and adding a condition about the shape of the singularity of the minor one step farther. In fact the algebra $\mathrm{RES}^{(N)}$ at a level $N \geq 1$ is characterized by the possibility of defining on it all the operators

$$\Delta_{2\pi\mathrm{i}N_\ell} \circ \cdots \circ \Delta_{2\pi\mathrm{i}N_1}, \quad \ell \in \mathbb{N}^*, \quad N_1, \ldots, N_\ell \in \mathbb{Z}^*, \quad |N_1| + \cdots + |N_\ell| \leq N.$$

By definition $\mathrm{RES}^{(N+1)} \subset \mathrm{RES}^{(N)}$, and $\mathrm{RES} = \bigcap_{N\geq 1} \mathrm{RES}^{(N)}$.

c) We have already seen that

$$\forall n \geq 0, \quad \tilde{u}_n \in \mathrm{RES}^{(1)}.$$

The main part of the work was done in the case of $\tilde{u}_0$, and in fact the arguments of Section 3 would allow one to check that the lateral continuations of $\hat{u}_0$ have only ramified singularities.

Taking the alien derivatives at $\pm 2\pi\mathrm{i}$ of the equations that the series $\tilde{u}_n$ satisfy and using the fact that $\Delta_{\pm 2\pi\mathrm{i}} \circ P = P \circ \Delta_{\pm 2\pi\mathrm{i}}$, we obtain a system of linear equations for the series $\Delta_{\pm 2\pi\mathrm{i}}\tilde{u}_n$, which can be written as a single equation for $\Delta_{\pm 2\pi\mathrm{i}}\tilde{u} = \sum_{n\geq 0} b^n \Delta_{\pm 2\pi\mathrm{i}}\tilde{u}_n$:

$$P(\Delta_{\pm 2\pi\mathrm{i}}\tilde{u}) = -2\tilde{u}\Delta_{\pm 2\pi\mathrm{i}}\tilde{u}.$$

We have at our disposal independent solutions of this linear equation: $\partial_b\tilde{u}$ and $\partial_z\tilde{u}$, and this allows us to prove the existence of resurgence relations

$$\Delta_{\pm 2\pi\mathrm{i}}\tilde{u}_n = \sum_{n_1+n_2=n} \big[(n_2+1)A_{\pm 2\pi\mathrm{i}||n_1}\tilde{u}_{n_2+1} + A_{\pm 2\pi\mathrm{i}||n_1}\partial_z\tilde{u}_{n_2}\big].$$

With this, and with the help of a verification on the shape of the singularities of the minors $\hat{u}_n$ at $\pm 4\pi\mathrm{i}$, we deduce that

$$\forall n \geq 0, \quad \tilde{u}_n \in \mathrm{RES}^{(2)}.$$

We can then proceed by induction and prove that

$$\forall N \geq 1,\ \forall n \geq 0, \quad \tilde{u}_n \in \mathrm{RES}^{(N)},$$

by iterating the previous arguments: at each level N, applying the alien derivations $\Delta_{\pm 2\pi\mathrm{i}N}$ (which commute with P) to the equations that the series $\tilde{u}_n$ satisfy, we obtain that the series $\Delta_{\pm 2\pi\mathrm{i}N}\tilde{u}_n$ (which could involve $\log z$) satisfy linear equations from which we deduce that they are linear combinations of the $\tilde{u}_n$'s and $\partial_z\tilde{u}_n$'s:

$$\Delta_{\pm 2\pi\mathrm{i}N}\tilde{u}_n = \sum_{n_1+n_2=n} \big[(n_2+1)A_{\pm 2\pi\mathrm{i}N||n_1}\tilde{u}_{n_2+1} + A_{\pm 2\pi\mathrm{i}N||n_1}\partial_z\tilde{u}_{n_2}\big]$$

(thus they are Laurent series and do *not* involve $\log z$), therefore all the series $\Delta_{2\pi\mathrm{i}N_\ell}\circ\cdots\circ\Delta_{2\pi\mathrm{i}N_1}\tilde{u}_n$ lie in $\mathrm{RES}^{(1)}$, and we obtain that the series $\tilde{u}_n$ belong to $\mathrm{RES}^{(N+1)}$ by checking the shape of the singularities at $\pm 2\pi\mathrm{i}(N+1)$ of the lateral continuations of the minors.

5.5. Further remarks.

a) *Analytic classification of a class of symplectic mappings*

All the previous results can be generalized to the case of the equation

$$u(z+1) - 2u(z) + u(z-1) = F\big(u(z)\big) \tag{30}$$

associated to an analytic function $F(X) = -X^2 + \mathcal{O}(X^3)$.

Any such function F determines a symplectic mapping of the plane:

$$\mathcal{F} : (X, Y) \longmapsto (X_1, Y_1),$$
$$X_1 = X + Y + F(X), \quad Y_1 = Y + F(X).$$

A particular solution $u(z)$ of (30) yields an invariant curve

$$\big(x(z) = u(z),\, y(z) = u(z) - u(z-1)\big)$$

for $\mathcal{F}$, in the sense that $\mathcal{F}$ maps $(x(z), y(z))$ onto $(x(z+1), y(z+1))$.

The general normalized solution $\tilde{u}(z, b)$ that one can construct in this case provides a formal conjugation

$$\begin{pmatrix} X = \tilde{u}(z, b) \\ Y = \tilde{u}(z, b) - \tilde{u}(z-1, b) \end{pmatrix}$$

between $\mathcal{F}$ and the "normal form at infinity"

$$\mathcal{T} : \ (z, b) \longmapsto (z+1, b).$$

The coefficients $A_{\omega||n}$ and $B_{\omega||n}$ which appear in the Bridge equation can now be interpreted as *analytic invariants*, *i.e.* they allow one to describe the analytic classification of the mappings $\mathcal{F}$.

b) *Connection formulas*

We can use the Laplace transform to define two families of entire functions $u_0^+, u_1^+, \ldots$ and $u_0^-, u_1^-, \ldots$ in the following way:

$$\forall n \geq 0, \quad u_n^\pm(z) = P_n(z) + \int_0^{\pm\infty} \hat{u}_n(\zeta) \mathrm{e}^{-z\zeta}\, \mathrm{d}\zeta,$$

where P_n is the polynomial part of $\tilde{u}_n$:

$$\tilde{u}_n(z) = P_n(z) + \mathcal{O}(z^{-1}).$$

Due to the general properties of Borel and Laplace transforms, $u_n^\pm$ admits $\tilde{u}_n$ as Gevrey-1 asymptotic expansion in any sectorial neighborhood of $\pm\infty$ of aperture strictly less than 2π.

The formal sums

$$u^{\pm}(b,z)=\sum_{n\geq 0} b^n u_n^{\pm}(z)$$

satisfy the equation (3), and we conjecture their convergence with respect to b.

The operators in the right-hand side of the Bridge equation give rise to two formal automorphisms

$$\Phi_{\text{down}}=\exp\Big(\sum_{N\geq 1}\big(A_{2\pi \mathrm{i}N}(b)\mathrm{e}^{-2\pi \mathrm{i}Nz}\partial_b+B_{2\pi \mathrm{i}N}(b)\mathrm{e}^{-2\pi \mathrm{i}Nz}\partial_z\big)\Big),$$

$$\Phi_{\text{down}}(b,z)=\Big(b+\sum_{N\geq 1}\mathcal{A}_{2\pi \mathrm{i}N}(b)\mathrm{e}^{-2\pi \mathrm{i}Nz},z+\sum_{N\geq 1}\mathcal{B}_{2\pi \mathrm{i}N}(b)\mathrm{e}^{-2\pi \mathrm{i}Nz}\Big),$$

$$\Phi_{\text{up}}=\exp\Big(\sum_{N\geq 1}\big(A_{-2\pi \mathrm{i}N}(b)\mathrm{e}^{2\pi \mathrm{i}Nz}\partial_b+B_{-2\pi \mathrm{i}N}(b)\mathrm{e}^{2\pi \mathrm{i}Nz}\partial_z\big)\Big),$$

$$\Phi_{\text{up}}(b,z)=\Big(b+\sum_{N\geq 1}\mathcal{A}_{-2\pi \mathrm{i}N}(b)\mathrm{e}^{2\pi \mathrm{i}Nz},z+\sum_{N\geq 1}\mathcal{B}_{-2\pi \mathrm{i}N}(b)\mathrm{e}^{2\pi \mathrm{i}Nz}\Big),$$

which should allow one to describe the passage from u^+ to u^-. We conjecture that they are convergent (at least with respect to z) and mutually inverse, and that

$$u^-(b,z)=u^+\big(\Phi_{\text{down}}(b,z)\big),\quad u^+(b,z)=u^-\big(\Phi_{\text{up}}(b,z)\big).$$

When expanded with respect to b, these relations provide exact connection formulas between the u_n^+'s and the u_n^-'s. At first order, for large negative $\operatorname{Im} z$, we find

$$\begin{aligned}u_0^+(z)&=u_0^-\big(z+B_{2\pi \mathrm{i}|0}\mathrm{e}^{-2\pi \mathrm{i}z}+\mathcal{O}(\mathrm{e}^{-4\pi \mathrm{i}z})\big)\\&\qquad\qquad+A_{2\pi \mathrm{i}|0}\mathrm{e}^{-2\pi \mathrm{i}z}u_1^-\big(z+\mathcal{O}(\mathrm{e}^{-2\pi \mathrm{i}z})\big)+\mathcal{O}(z^{10}\mathrm{e}^{-4\pi \mathrm{i}z})\\&=u_0^-(z)+\mathrm{e}^{-2\pi \mathrm{i}z}\big(A_{2\pi \mathrm{i}|0}u_1^-(z)+B_{2\pi \mathrm{i}|0}\partial_z u_0^-(z)\big)+\mathcal{O}(z^{10}\mathrm{e}^{-4\pi \mathrm{i}z}),\end{aligned}$$

which ties up with Formula (15) in Proposition 1.

6. Numerical evaluation of splitting constants.

The complete knowledge on the singularities structure of $\hat{u}$ leads to the construction of a very efficient numerical method for the evaluation of the splitting constants Θ and μ. The algorithm is extremely simple, and it is suitable for some similar problems. But we first need some analytic preliminaries.

Asymptotic behavior of Taylor coefficients

The information about the first singularities of $\hat{u}$ can be extracted from the asymptotic behavior of its Taylor coefficients at the origin. This function has singularities on the boundary of $D_{2\pi} = \{\, \zeta \in \mathbb{C} : |\zeta| < 2\pi \,\}$ at $\pm 2\pi\mathrm{i}$. Let g be the corresponding polar part:

$$\begin{aligned} g(\zeta) &= \frac{A_5}{(\zeta-2\pi\mathrm{i})^5} + \frac{A_3}{(\zeta-2\pi\mathrm{i})^3} + \frac{A_1}{\zeta-2\pi\mathrm{i}} \\ &\qquad + \frac{A_5}{(\zeta+2\pi\mathrm{i})^5} + \frac{A_3}{(\zeta+2\pi\mathrm{i})^3} + \frac{A_1}{\zeta+2\pi\mathrm{i}} \\ &= \Big(\frac{A_5}{4!}\partial_\zeta^4 + \frac{A_3}{2!}\partial_\zeta^2 + A_1\Big)\frac{2\zeta}{4\pi^2+\zeta^2}, \end{aligned}$$

where we used the symmetries of the singularities due to the fact that $\hat{u}$ is real-analytic and odd. According to Theorem 2,

$$A_5 = \frac{\Theta}{7\pi\mathrm{i}}, \quad A_3 = \frac{17\Theta}{840\pi\mathrm{i}}, \quad A_1 = -\frac{17\Theta}{4480\pi\mathrm{i}}.$$

Since

$$\frac{2\zeta}{4\pi^2+\zeta^2} = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\Big(\frac{\zeta}{2\pi}\Big)^{2k-1},$$

we obtain

$$\begin{aligned} \partial_\zeta^{2k-1} g(0) &= \frac{A_5}{4!\pi}\,\frac{(-1)^{k-1}(2k+3)!}{(2\pi)^{2k+3}} + \frac{A_3}{2!\pi}\,\frac{(-1)^{k-1}(2k+1)!}{(2\pi)^{2k+1}} \\ &\qquad + \frac{A_1}{\pi}\,\frac{(-1)^{k-1}(2k-1)!}{(2\pi)^{2k-1}}. \end{aligned} \tag{31}$$

All the derivatives of even order vanish at zero.

The difference $f(\zeta) = \hat{u}(\zeta) - g(\zeta)$ is analytic in $D_{2\pi}$ and continuous in its closure. Applying the Cauchy estimates

$$\big|\partial_\zeta^k f(0)\big| \le \frac{k!}{(2\pi)^k}\sup_{\zeta\in D_{2\pi}}\big|f(\zeta)\big|$$

and taking into account that $\partial_\zeta^{2k-1}\hat{u}(0) = a_k$, we see that

$$a_k - \partial_\zeta^{2k-1} g(0) = \mathcal{O}\Big(\frac{(2k-1)!}{(2\pi)^{2k-1}}\Big). \tag{32}$$

If we keep only the first term in the expression for $\partial_\zeta^{2k-1}g(0)$ and solve the equation with respect to A_5, we obtain

$$A_5 = \frac{4!}{2}\,\frac{(-1)^{k-1}a_k(2\pi)^{2k+4}}{(2k+3)!} + \mathcal{O}(k^{-2}).$$

In particular,

$$|\Theta| = 42 \lim_{k\to\infty} \frac{(-1)^k a_k (2\pi)^{2k+5}}{(2k+3)!}. \tag{33}$$

It is not too difficult to compute several hundreds of a_k, but the convergence in the above formula is rather slow (the relative error $\sim k^{-2}$). We can substantially improve the method using more detailed knowledge of the singularity structure. As a first improvement we note that $A_3 = 17A_5/120$. Then we substitute the first two terms of (31) into (32) and solve the equation with respect to A_5:

$$A_5 = \frac{(-1)^{k-1}a_k(2\pi)^{2k+4}}{(2k+3)!}\left(\frac{2}{4!} + \frac{17}{120}\,\frac{(2\pi)^2}{(2k+2)(2k+3)}\right)^{-1} + \mathcal{O}(k^{-4}).$$

In this way we constructed a sequence which converges to A_5 much faster.

We can repeat the same reasoning adding to g the first N terms of the logarithmic part of the singularities at $\pm 2\pi\mathrm{i}$ for some $N \geq 1$. Let us denote by $\sum_{m\geq 1} h_m\zeta^m$ the Taylor series at 0 of the function h in (12). The odd coefficients are proportional to Θ, the even ones to μ, and if instead of g we consider

$$g_N(\zeta) = s_N(\zeta - 2\pi\mathrm{i}) + \overline{s_N(\bar{\zeta} - 2\pi\mathrm{i})},$$

where

$$s_N(\xi) = \frac{A_5}{\xi^5} + \frac{A_3}{\xi^3} + \frac{A_1}{\xi} + \sum_{m=1}^{N} h_m \xi^m \frac{\log\xi}{2\pi\mathrm{i}},$$

we obtain an approximation for a_k with relative error $\mathcal{O}(k^{-N-7})$.

Of course, the constant in the $\mathcal{O}$ estimates depends on N. In the next section we numerically observe that choosing $N = \frac{1}{2}k$ leads to exponential convergence (relative error is $\mathcal{O}(\mathrm{e}^{-ck})$). From the theoretical point of view this is quite natural and can be rigorously proved by applying the techniques of the present paper. This is due to the fact that φ_1 and φ_2 are resurgent functions too, which provides Gevrey-1 estimates on the growth of the sequences b_k, d_k and the constants in the $\mathcal{O}$-terms.

Formula (33) was established by semi-empirical reasoning and used by several authors [Che98], [TTJ98] for numerical evaluation of the splitting constant Θ. The computation of V. Chernov [Che98] (14 correct decimals) are in excellent agreement with the present paper as well as with [Gel91], [GLT91], where an independent method was used. On the other hand we are not able to explain the discrepancy with the numerical experiments of Tovbis *et al.* [TTJ98], where for the constant $K = |\Theta|/168\pi$ it was obtained $K \approx 7374$, which is some 36 % larger than we expect.

Numerical algorithm

We first compute the coefficients of u and of φ_1, the first formal solution of the variational equation. We use the following recurrent formulas: $a_1 = -6$, $a_2 = 15/2$ and

$$(34) \quad a_m = -\frac{1}{2m^2+m-6}\Big(\sum_{k=1}^{m-1}\binom{2m+1}{2k-1}a_k + \sum_{k=2}^{m-1}\frac{a_{m+1-k}a_k}{2}\Big) \qquad \text{for } m \geq 3,$$

$$(35) \quad b_m = -2m\, a_m \qquad \text{for } m \geq 1.$$

Then we compute the coefficients of φ_2, the second solution of the variational equation. We let $d_{-2} = 1/84$, $d_{-1} = 17/84$, $d_0 = -17/2240$, and for $m \geq 1$,

$$(36) \qquad d_m = -\frac{1}{2m^2+m-6}\Big(\sum_{k=1}^{m-1}\binom{2m+1}{2k-1}d_k + \sum_{k=-2}^{m-1}a_{m+1-k}d_k\Big).$$

Then we evaluate two auxiliary sums:

$$s^1_{k,N} = \sum_{m=-2}^{N-1}\frac{2(-1)^\ell(2\ell+1)!\,d_m}{(2\pi)^{2\ell+3}}, \qquad (\ell = k-m-1)$$

$$s^2_{k,N} = \sum_{m=1}^{N-1}\frac{2(-1)^\ell(2\ell)!\,b_m}{(2\pi)^{2\ell+2}}, \qquad (\ell = k-m-1).$$

Finally, we evaluate the splitting constants Θ and μ by comparing a_k with

$$(37) \qquad \tilde{a}_k = s^1_{k,N}\Theta + s^2_{k,N}\mu,$$

the k^{th} derivative of $g_N(\zeta)$ at origin (provided $k \geq 2N$). According to the previous section $a_k = \tilde{a}_k(1+\mathcal{O}(k^{-N-7}))$. In our experiments we used

$N = \frac{1}{2}k$, which seems to minimize the error due to the replacement of $\tilde{a}_k$ by a_k. In fact our numerical method is not sensitive to this choice. The method of the present paper can be used to analyze this error analytically.

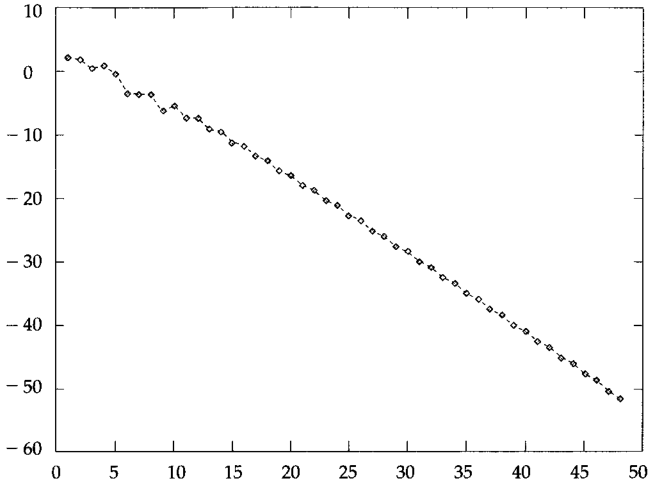

Figure 5. The plot of $\log|\delta_k|$ *versus* k

In the numerical experiments we computed $n = 50$ terms for each of the sequences $(a_k, b_k, d_k, s^1_{k,N}, s^2_{k,N})$. Then we found the constants,

$$\Theta^* = 2.474\,425\,593\,553\,251\,053\,840 \cdot 10^6,$$
$$\mu^* = 4\,908.934\,252\,164,$$

by replacing $\tilde{a}_k$ by a_k and solving (37) by the method of least squares using the last six values of $k = 45, \ldots, 50$.

In order to estimate the error due to the replacement of $\tilde{a}_k$ by a_k we computed the relative errors

$$\delta_k = \frac{\tilde{a}^*_k - a_k}{a_k},$$

where $\tilde{a}^*_k = s^1_{k,N}\Theta^* + s^2_{k,N}\mu^*$ is the "experimental" value of $\tilde{a}_k$. Some particular values are

$$\delta_{10} \approx 4 \cdot 10^{-3}, \quad \delta_{20} \approx -7 \cdot 10^{-7}, \quad \delta_{30} \approx -4 \cdot 10^{-13}.$$

Figure 5 gives a numerical evidence of $\delta_k \sim \mathrm{e}^{-ck}$. From the analytical viewpoint this error is due to a contribution from the other singularities of $\hat{u}$.

Finally, we repeated the computations for larger values of n. We used the coefficients with $k = 90, \dots, 100$ to determine the values of Θ and μ. This test confirmed that the previously computed decimals are all correct.

We also compared this method with [GLS94], where it was shown that the splitting constants can be found as the following limits:

$$\Theta = \lim_{\operatorname{Im} z \to -\infty} \mathrm{e}^{2\pi \mathrm{i} z} \mathcal{W}_{u^+ - u^-, \varphi_2^-}(z),$$

$$\mu = \lim_{\operatorname{Im} z \to -\infty} \mathrm{e}^{2\pi \mathrm{i} z} \mathcal{W}_{\varphi_1^-, u^+ - u^-}(z),$$

where $\mathcal{W}$ denotes finite-difference Wronskian like in (10) and φ_1^-, φ_2^- are the Borel-Laplace transforms along $\mathbb{R}^-$ of φ_1, φ_2. The limit is reached exponentially fast. The computations, based on this definition, afford to compute only between 6 and 8 correct decimals of Θ using the double precision complex arithmetic.

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