

Boris Venkov's Theory of Lattices and Spherical Designs

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1. Introduction

Boris Venkov passed away on November 10, 2011, just 5 days before his 77th birthday. His death overshadowed the conference “Diophantine methods, lattices, and arithmetic theory of quadratic forms” November 13-18, 2011, at the BIRS in Banff (Canada), where his important contributions to the theory of lattices, modular forms and spherical designs played a central role. This article gives a short survey of the mathematical work of Boris Venkov in this direction.

Boris Venkov's first work on lattices was a new proof [20] of the classification of even unimodular lattices in dimension 24, in 1978, which was reprinted as Chapter 18 of the book “Sphere packings, lattices and groups” [24]. This was the first application of the theory of spherical designs to lattices shortly after their definition in the fundamental work [25] by Delsarte, Goethals, and Seidel. In the same spirit a combination of the theory of spherical designs with modular forms allowed Venkov to prove that all layers of extremal even unimodular lattices form spherical designs of strength 11, respectively 7, if $n \equiv 0, 8 \pmod{24}$. Since then, lattices became an important tool for the construction and investigation of spherical designs (see for instance [6] or Section 5). Boris Venkov's work on the connection between lattices and spherical designs finally led him to the definition of strongly perfect lattices. His lecture series in Bordeaux and Aachen on this topic (see [8]) initiated many fruitful applications of this theory, some of which are collected in [30]. Strongly perfect lattices provide interesting examples of locally densest lattices, so called extreme lattices. The definition of strongly perfect lattices was very successful, for instance it allows us to apply the theory of modular forms to show that all extremal even unimodular lattices of dimension 32 are extreme lattices. It also permits us to apply representation theory of the automorphism group to show that a lattice is extreme. The notion of strong perfection has been generalized to other metric spaces such as Grassmanians or Hermitian spaces and also to coding theory.

Boris Venkov spent a great part of his mathematical life visiting other universities. When I asked him whether he has a complete list of his visits for proposing him for his Humboldt Research Award in 2007, he answered “A complete list of my visits would be too long. It contains also exotic visits like Tata Institute in Bombay, Universidad Autonoma in Mexico or Universidad de Habana, Kuba.” He

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had visiting professorships in Bonn (1989), Cambridge (1989), Geneva (1992, and 3 months every year since 1997), Lyon (1993), Paris (1994), Aachen (1994, 1996, ff.), Berlin (1995/96), Grenoble (1995, 1998), Bordeaux (1997), Fukuoka (2002), Kyoto (2006). All groups enjoyed interesting and fruitful discussions and productive collaborations with Boris Venkov. Some of the resulting articles are given in the references; for a complete list of Boris Venkov's papers I refer to the Zentralblatt or MathSciNet.

2. Lattices, designs and modular forms

2.1. Extreme lattices. A classical problem asks for the densest packing of equal spheres in Euclidean space. Already in dimension 3 this turned out to be very hard; in this generality it was solved by T. Hales, 1998, with a computer based proof of the Kepler conjecture. The sphere packing problem becomes easier if one restricts to lattice sphere packings, where the centers of the spheres form a group. The density function has only finitely many local maxima on the space of similarity classes of n -dimensional lattices, the so called extreme lattices. Korkine and Zolotareff and later Voronoi developed methods to compute all extreme lattices of a given dimension. The necessary definitions are given in this section. Details and proofs may be found in the textbook [31].

We always work in Euclidean n -space $(\mathbb{R}^n, (\cdot, \cdot))$ where $(x, y) = \sum_{i=1}^n x_i y_i$ is the standard inner product with associated quadratic form

$$Q : \mathbb{R}^n \rightarrow \mathbb{R}, Q(x) := \frac{1}{2}(x, x) = \frac{1}{2} \sum_{i=1}^n x_i^2.$$

- DEFINITION 2.1. (a) A **lattice** is the \mathbb{Z} -span $L = \langle b_1, \dots, b_n \rangle_{\mathbb{Z}} = \{ \sum_{i=1}^n a_i b_i \mid a_i \in \mathbb{Z} \}$ of a basis $B = (b_1, \dots, b_n)$ of \mathbb{R}^n .
- (b) The **determinant** of L is the square of the covolume of L in \mathbb{R}^n and can be computed as the determinant of a **Gram matrix** $\det(L) = \det((b_i, b_j))_{i,j=1}^n$.
- (c) L is called **integral** if $(\ell, m) \in \mathbb{Z}$ for all $\ell, m \in L$.
- (d) L is called **even** if $(\ell, \ell) \in 2\mathbb{Z}$ and hence $Q(\ell) \in \mathbb{Z}$ for all $\ell \in L$.
- (e) The **minimum** of L is $\min(L) = \min\{(x, x) \mid 0 \neq x \in L\}$. We denote the set of minimal vectors by $\text{Min}(L) := \{x \in L \mid (x, x) = \min(L)\}$.
- (f) The sphere packing density of L is then proportional to the $n/2$ -th power of the **Hermite invariant** of L , $\gamma(L) := \frac{\min(L)}{\det(L)^{1/n}}$.
- (g) A **similarity** of norm $\alpha \in \mathbb{R}_{>0}$ is an element $\sigma \in \text{GL}_n(\mathbb{R})$ with $(\sigma(x), \sigma(y)) = \alpha(x, y)$ for all $x, y \in \mathbb{R}^n$. Two lattices L and M are called **similar** if there is a similarity σ with $\sigma(L) = M$.
- (h) The Hermite function γ is well defined on similarity classes of lattices. A lattice L is called **extreme** if its similarity class realises a local maximum of γ .

The definition of extreme lattices goes back to Korkine and Zolotareff in the 1870s. They showed that extreme lattices are perfect, where a lattice L is **perfect** if the projections onto the minimal vectors span the space of all symmetric endomorphisms, i.e.

$$\langle x x^{tr} \mid x \in \text{Min}(L) \rangle = \mathbb{R}_{sym}^{n \times n}.$$

30 years later Voronoi gave an algorithm to enumerate all finitely many similarity classes of perfect lattices in a given dimension. In the study of perfect lattices we may always restrict to integral lattices by the following result:

THEOREM 2.2. *Any perfect lattice L is similar to some integral lattice.*

PROOF. Let L be some perfect lattice of minimum 1. Choose some basis B of L and let $F := ((b_i, b_j))_{i,j=1}^n \in \mathbb{R}^{n \times n}$ be the Gram matrix. Then $\text{Min}(L) = \{\sum_{i=1}^n x_i b_i \mid x \in \mathbb{Z}^n, x^{tr} F x = 1\}$. View this as a system of linear equations with rational coefficients on the entries of the Gram matrix F :

$$(2.1) \quad x^{tr} F x = 1 \text{ for all } \sum_{i=1}^n x_i b_i \in \text{Min}(L)$$

We show that F is the unique solution of this system: Let G be a second solution of (2.1). Then

$$0 = x^{tr}(F - G)x = \text{trace}(x^{tr}(F - G)x) = \text{trace}(xx^{tr}(F - G)) \text{ for all } x \in \text{Min}(L)$$

so $F - G \in \mathbb{R}_{sym}^{n \times n}$ is perpendicular to $\langle xx^{tr} \mid x \in \text{Min}(L) \rangle$ with respect to the positive definite symmetric bilinear form $(A, B) \mapsto \text{trace}(AB)$. Since L is perfect this yields $F - G = 0$, so F is uniquely determined by (2.1). All coefficients of (2.1) are integers, so the solution is rational, $F \in \mathbb{Q}^{n \times n}$. Multiplying by the common denominator yields an integral lattice that is similar to L . \square

All perfect lattices are known up to dimension 8. Due to the existence of the famous **Leech lattice** Λ_{24} we also know the absolutely densest lattice of dimension 24 by work of Elkies, Cohn and Kumar.

The densest lattices.

| | | | | | | | | | |
|-----------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| dimension | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 24 |
| # perfect | 1 | 1 | 1 | 2 | 3 | 7 | 33 | 10916 | |
| # extreme | 1 | 1 | 1 | 2 | 3 | 6 | 30 | 2408 | |
| densest | \mathbb{A}_1 | \mathbb{A}_2 | \mathbb{A}_3 | \mathbb{D}_4 | \mathbb{D}_5 | \mathbb{E}_6 | \mathbb{E}_7 | \mathbb{E}_8 | Λ_{24} |

For a perfect lattice to be a local maximum of the Hermite function an additional convexity condition is needed:

DEFINITION 2.3. A lattice L is called **eutactic** if there are $\lambda_x > 0$ such that the unit matrix I_n is the sum $I_n = \sum_{x \in \text{Min}(L)} \lambda_x (xx^{tr})$. It is called **strongly eutactic** if all λ_x can be chosen to be equal.

THEOREM 2.4 ([**31**, Theorem 3.4.6]). *A lattice L is extreme if and only if it is perfect and eutactic.*

Up to dimension 8, the densest lattices are similar to root lattices.

DEFINITION 2.5. A lattice L is called a **root lattice** if L is even and $L = \langle \ell \in L \mid Q(\ell) = 1 \rangle_{\mathbb{Z}}$.

Any root lattice is a unique orthogonal sum of indecomposable root lattices. The orthogonally indecomposable root lattices are classified. They form two infinite series \mathbb{A}_n ($n \geq 1$), \mathbb{D}_n ($n \geq 4$) with three exceptional lattices \mathbb{E}_6 , \mathbb{E}_7 , \mathbb{E}_8 . An important invariant attached to an indecomposable root lattice is its **Coxeter number** $h(L) := |\text{Min}(L)| / \dim(L)$.

| L | $ \text{Min}(L) $ | $h(L)$ | $\det(L)$ | n |
|----------------|-------------------|----------|-----------|----------|
| \mathbb{A}_n | $n(n+1)$ | $n+1$ | $n+1$ | ≥ 1 |
| \mathbb{D}_n | $2n(n-1)$ | $2(n-1)$ | 4 | ≥ 4 |
| \mathbb{E}_6 | 72 | 12 | 3 | 6 |
| \mathbb{E}_7 | 126 | 18 | 2 | 7 |
| \mathbb{E}_8 | 240 | 30 | 1 | 8 |

Venkov's study of root lattices described in Section 2.4 gave the first connection between Voronoi's characterisation of extreme lattices and spherical designs. A guiding observation comes from the fact that indecomposable root lattices are strongly eutactic.

2.2. Strongly eutactic lattices.

REMARK 2.6. A lattice L is strongly eutactic if and only if there is some constant c such that

$$(2.2) \quad \sum_{x \in \text{Min}(L)} (x, \alpha)^2 = c(\alpha, \alpha) \text{ for all } \alpha \in \mathbb{R}^n$$

Applying the Laplace operator $\Delta_\alpha := \sum_{i=1}^n \frac{\partial^2}{\partial \alpha_i^2}$ to both sides of Equation (2.2) one gets $c = \text{min}(L)|\text{Min}(L)|/n$.

PROOF. The equation (2.2) reads as

$$\sum_{x \in \text{Min}(L)} \alpha^{tr} x x^{tr} \alpha = c \alpha^{tr} \alpha \text{ for all } \alpha \in \mathbb{R}^n$$

and therefore is equivalent to the equation $\sum_{x \in \text{Min}(L)} x x^{tr} = c I_n$ from Definition 2.3. \square

DEFINITION 2.7. The space of **harmonic polynomials** of degree t in n variables is

$$\text{Harm}_t := \{p \in \mathbb{R}[x_1, \dots, x_n] \mid \deg(p) = t \text{ and } \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} p = 0\}$$

So Harm_t is the kernel of the Laplace operator.

REMARK 2.8. The harmonic polynomials of degree 2 are linear combinations of

$$(2.3) \quad p_\alpha : x \mapsto (x, \alpha)^2 - \frac{1}{n}(x, x)(\alpha, \alpha) \text{ for } \alpha \in \mathbb{R}^n.$$

So a lattice is strongly eutactic if and only if

$$\sum_{x \in \text{Min}(L)} p(x) = 0 \text{ for all } p \in \text{Harm}_2.$$

Root lattices are important in the classification of complex semisimple Lie algebras but also for the classification of finite reflection groups. Any root $\ell \in \text{Min}(L)$ of a root lattice L defines an automorphism, the **reflection** along ℓ

$$\sigma_\ell : L \rightarrow L : x \mapsto x - (x, \ell)\ell,$$

so $\sigma_\ell \in \text{Aut}(L) = \{\sigma \in O(\mathbb{R}^n) \mid \sigma(L) = L\}$. The **automorphism group** $\text{Aut}(L)$ is a finite subgroup of $\text{GL}_n(\mathbb{R})$.

PROPOSITION 2.9. *If $\text{Aut}(L)$ is irreducible, then L is strongly eutactic.*

PROOF. Let $G = \text{Aut}(L)$. Since G is real irreducible all G -invariant quadratic forms are scalar multiples of $Q = \frac{1}{2} \sum_{i=1}^n x_i^2$. Now G permutes the vectors in $\text{Min}(L)$ and so

$$Q' : \mathbb{R}^n \rightarrow \mathbb{R}, \alpha \mapsto \sum_{x \in \text{Min}(L)} (x, \alpha)^2$$

is a positive G -invariant quadratic form; so there is some $c \in \mathbb{R}_{>0}$ such that $Q' = cQ$. By Remark 2.6 this means that L is strongly eutactic. \square

COROLLARY 2.10. *Let $L \leq \mathbb{R}^n$ be an indecomposable root lattice. Then $\text{Aut}(L)$ is irreducible and hence L is strongly eutactic, so $\sum_{x \in \text{Min}(L)} (x, \alpha)^2 = 2h(L)(\alpha, \alpha)$ for all $\alpha \in \mathbb{R}^n$.*

2.3. Extremal lattices. The notion of (analytic) extremality has first been defined for even unimodular lattices and has then been generalized by Quebbemann [32] to modular lattices. Roughly speaking, extremal lattices are lattices in some arithmetically defined family of lattices for which the density is as big as the theory of modular forms allows it to be. The idea is to translate arithmetic properties of the lattice L into invariance conditions of its theta series $\theta_L(z)$ and to prove that $\theta_L(z)$ is some homogeneous element in a finitely generated graded ring (or module) of modular forms. The knowledge of explicit generators then allows us to derive a priori upper bounds on the minimum of L . Details on this section can be found in the books [26] and [34].

DEFINITION 2.11. Let $L = \langle B \rangle_{\mathbb{Z}} \leq \mathbb{R}^n$ be a lattice.

- (a) The **dual lattice** $L^\# := \{x \in \mathbb{R}^n \mid (x, \ell) \in \mathbb{Z} \text{ for all } \ell \in L\}$ is the lattice spanned by the dual basis B^* .
- (b) L is called **unimodular** if $L = L^\#$.
- (c) Let L be an even lattice. Then the **theta series** of L is

$$\theta_L := \sum_{\ell \in L} q^{Q(\ell)} = 1 + \sum_{j=\text{min}(L)}^{\infty} a_j q^j$$

with $a_j = |A_j(L)|$ and $A_j(L) = \{\ell \in L \mid Q(\ell) = j\}$. The substitution $q := \exp(2\pi iz)$ then defines a holomorphic function $\theta_L(z) = \sum_{j=0}^{\infty} a_j \exp(2\pi iz)^j$ on the upper half plane $\mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$.

In the following we will study **even unimodular lattices**. They correspond to positive definite regular integral quadratic forms $Q : L \rightarrow \mathbb{Z}$. The theory of quadratic forms shows that even unimodular lattices only exist if the dimension n is a multiple of 8.

By the periodicity of the exponential function, the theta function of an even lattice is invariant under the substitution $z \mapsto z + 1$. The so called **theta transformation formula** ([26, Proposition 2.1], [34, Proposition 16]) relates the theta series of the dual lattice $L^\#$ to θ_L . In particular the theta function of an even unimodular lattice L is a modular form of weight $n/2$ for the full modular group $\text{SL}_2(\mathbb{Z})$. For details on modular forms (including their definition) I refer to [26] or [34]. The main result we need here is the following theorem describing the structure of the graded ring of modular forms.

THEOREM 2.12. Let E_4 and E_6 denote the normalized Eisenstein series of weight 4 and 6,

$$E_4 = 1 + 240 \sum_{j=1}^{\infty} \sigma_3(j)q^j = \theta_{\mathbb{E}_8}, \quad E_6 = 1 - 504 \sum_{j=1}^{\infty} \sigma_5(j)q^j$$

where $\sigma_r(j)$ is the sum over the r -th powers of all divisors of j . Then the ring of modular forms for the full modular group is

$$\mathcal{M}(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$$

the polynomial ring in E_4 and E_6 .

So any modular form f of weight k has a unique expression as

$$f = \sum_{4a+6b=k} c(a, b)E_4^a E_6^b \text{ with } c(a, b) \in \mathbb{C}.$$

The vanishing order of f at $z = i\infty$ (so $q = 0$) defines a valuation on $\mathcal{M}(\mathrm{SL}_2(\mathbb{Z}))$ with associated maximal ideal $\mathcal{S}(\mathrm{SL}_2(\mathbb{Z}))$, the space of **cusp forms**. This is a principal ideal generated by Δ with

$$(2.4) \quad \Delta = \frac{1}{1728}(E_4^3 - E_6^2) = q - 24q^2 + 252q^3 - 1472q^4 + \dots$$

THEOREM 2.13. Let L be an even unimodular lattice of dimension n . Then n is a multiple of 8 and $\theta_L \in \mathbb{C}[E_4, \Delta]_{n/2}$.

For $4k = n/2$ the space $\mathcal{M}_{4k}(\mathrm{SL}_2(\mathbb{Z}))$ has a very nice basis.

$$\begin{array}{llll} E_4^k = & 1 + & 240kq + & *q^2 + \dots \\ E_4^{k-3} \Delta = & & q + & *q^2 + \dots \\ E_4^{k-6} \Delta^2 = & & & q^2 + \dots \\ \vdots & & & \\ E_4^{k-3m_k} \Delta^{m_k} = & \dots & & q^{m_k} + \dots \end{array}$$

where $m_k = \lfloor \frac{n}{24} \rfloor = \lfloor \frac{k}{3} \rfloor$.

In particular $\mathcal{M}_{4k}(\mathrm{SL}_2(\mathbb{Z}))$ contains a unique form

$$f^{(k)} := 1 + 0q + 0q^2 + \dots + 0q^{m_k} + a(f^{(k)})q^{m_k+1} + b(f^{(k)})q^{m_k+2} + \dots$$

the **extremal modular form** of weight $4k$. If the minimum of an even unimodular lattice L of dimension $n = 8k$ is greater than or equal to $2 + 2\lfloor \frac{n}{24} \rfloor$, then $\theta_L = f^{(k)}$ is equal to the extremal modular form. Already Siegel [35, end of proof of Satz 2] has shown that the first nontrivial coefficient $a(f^{(k)})$ is always a positive integer. In particular

COROLLARY 2.14. Let L be an even unimodular lattice of dimension n . Then

$$\min(L) \leq 2 + 2 \left\lfloor \frac{n}{24} \right\rfloor.$$

The lattice L is called **extremal** if $\min(L) = 2 + 2\lfloor \frac{n}{24} \rfloor$.

Since the second nontrivial coefficient $b(f^{(k)})$ of the extremal modular becomes negative for all $k \geq 20,408$ there are no extremal even unimodular lattices in dimension $n \geq 163,264$.

The densest lattices in dimension 8 and 24 and the densest known lattices in dimension 48 and 72 are extremal even unimodular lattices. As we will see

in Section 4.1 Venkov's theory of strongly perfect lattices allows us to show that extremal even unimodular lattices of dimension $n \equiv 0, 8 \pmod{24}$ are extreme, i.e. realise a local maximum of the density function.

Extremal even unimodular lattices $L \leq \mathbb{R}^{n1}$

| | | | | | | | | | |
|-----------------------------------|---|----|----|-------------|----------------|----------|----------|----------|-----------------|
| n | 8 | 16 | 24 | 32 | 40 | 48 | 72 | 80 | $\geq 163, 264$ |
| $\min(L)$ | 2 | 2 | 4 | 4 | 4 | 6 | 8 | 8 | |
| number of extremal lattices | 1 | 2 | 1 | $\geq 10^7$ | $\geq 10^{51}$ | ≥ 3 | ≥ 1 | ≥ 4 | 0 |

2.4. Venkov's classification of Niemeier lattices. In 1968 Niemeier classified the even unimodular lattices of dimension 24. Up to isometry there are 24 such lattices L and they are distinguished by their root sublattice

$$R(L) := \langle \ell \in L \mid Q(\ell) = 1 \rangle_{\mathbb{Z}}.$$

In 1978 Boris Venkov [20] gave a more structural approach to Niemeier's list by showing the following theorem.

THEOREM 2.15. *Let L be an even unimodular lattice of dimension 24. Then*

- (a) *The root sublattice $R(L)$ is either 0 or has full rank.*
- (b) *The indecomposable components of $R(L)$ have the same Coxeter number.*

The possible root systems are then found combinatorially from the classification of indecomposable root systems and their Coxeter numbers:

$$\begin{aligned} & \emptyset, 24\mathbb{A}_1, 12\mathbb{A}_2, 8\mathbb{A}_3, 6\mathbb{A}_4, 4\mathbb{A}_6, 3\mathbb{A}_8, 2\mathbb{A}_{12}, \mathbb{A}_{24}, 6\mathbb{D}_4, 4\mathbb{D}_6, \\ & 3\mathbb{D}_8, 2\mathbb{D}_{12}, \mathbb{D}_{24}, 4\mathbb{E}_6, 3\mathbb{E}_8, 4\mathbb{A}_5 \perp \mathbb{D}_4, 2\mathbb{A}_7 \perp 2\mathbb{D}_5, 2\mathbb{A}_9 \perp \mathbb{D}_6, \\ & \mathbb{A}_{15} \perp \mathbb{D}_9, \mathbb{E}_8 \perp \mathbb{D}_{16}, 2\mathbb{E}_7 \perp \mathbb{D}_{10}, \mathbb{E}_7 \perp \mathbb{A}_{17}, \mathbb{E}_6 \perp \mathbb{D}_7 \perp \mathbb{A}_{11} \end{aligned}$$

If $R(L) \neq 0$, then $R(L) \leq L = L^\# \leq R(L)^\#$ so L corresponds to a maximal isotropic subgroup of the discriminant group $R(L)^\# / R(L)$. For each root system of Venkov's list, a case-by-case investigation of the possible glue-codes, generating the lattice over its root sublattice, leads to a unique possibility for a unimodular overlattice.

THEOREM 2.16. *For each of the 23 nonzero root lattices listed above there is a unique even unimodular lattice in dimension 24 having this root sublattice.*

The uniqueness of the Leech lattice, the unique even unimodular lattice of dimension 24 with no roots is proved differently. It follows for instance from the uniqueness of the Golay code, but also by applying the mass formula.

$$\sum_{i=1}^h |\text{Aut}(L_i)|^{-1} = m_{2k} = \frac{|B_k|}{2k} \prod_{j=1}^{k-1} \frac{B_{2j}}{4j}$$

where L_1, \dots, L_h represent the isometry classes of even unimodular lattices in \mathbb{R}^{2k} .

For the proof of Theorem 2.15 we need the following result by Hecke.

¹The known extremal lattices in the jump dimensions $24k$ are found in the online database of lattices, <http://www.math.rwth-aachen.de/~Gabriele.Nebe/LATTICES/>

THEOREM 2.17 (see [26, Corollary 3.3]). *Let L be an even unimodular lattice of dimension n and let $p \in \text{Harm}_t$ be a harmonic polynomial of degree t (see Definition 2.7). Then*

$$\theta_{L,p} := \sum_{\ell \in L} p(\ell) q^{Q(\ell)} \in \mathcal{M}_{n/2+t}(\text{SL}_2(\mathbb{Z}))$$

is a modular form of weight $n/2 + t$ for the full modular group.

If $p = 1$, then $\theta_{L,p} = \theta_L$. For non constant homogeneous polynomials p one has $p(0) = 0$ and therefore $\theta_{L,p} \in \mathcal{S}_{n/2+t}(\text{SL}_2(\mathbb{Z}))$ is a cusp form and hence divisible by the form Δ from Equation (2.4).

THEOREM 2.15. Let L be an even unimodular lattice of dimension 24 with $R(L) \neq 0$. For $\alpha \in \mathbb{R}^n$ let $p_\alpha \in \text{Harm}_2$ be the harmonic polynomial defined in Equation (2.3). Then by Theorem 2.17 the theta series

$$\theta_{L,p_\alpha} \in \Delta \mathcal{M}_{12+2-12}(\text{SL}_2(\mathbb{Z})) = \Delta \mathcal{M}_2(\text{SL}_2(\mathbb{Z})) = 0$$

since there are no nonzero modular forms of weight 2. But this implies that

$$\sum_{x \in \text{Min}(L)} p_\alpha(x) = 0 \text{ so } \sum_{x \in \text{Min}(L)} (x, \alpha)^2 = \frac{2|\text{Min}(L)|}{24}(\alpha, \alpha).$$

In particular if $(x, \alpha) = 0$ for all $x \in \text{Min}(L)$ then $\alpha = 0$ and hence $R(L)^\perp = 0$.

Now write $R(L) = R_1 \perp \dots \perp R_s$ with indecomposable root lattices R_i of dimension $n_i = \dim(R_i)$. For $\alpha \in \langle R_i \rangle_{\mathbb{R}}$ we obtain

$$\sum_{x \in \text{Min}(L)} (x, \alpha)^2 = \sum_{x \in \text{Min}(R_i)} (x, \alpha)^2 = \frac{2|\text{Min}(R_i)|}{n_i}(\alpha, \alpha)$$

by Corollary 2.10 and Remark 2.6. Hence $h(R_i) = \frac{|\text{Min}(R_i)|}{n_i} = \frac{|\text{Min}(L)|}{24}$ for all i . \square

2.5. The Koch-Venkov invariant. Even unimodular lattices are fully classified up to dimension 24. In dimension 32 the mass formula shows that there are more than 80 million such lattices, more than 10 million of which are extremal ([28]). Nevertheless Koch (1988) and Venkov ([19], [18], [17]) started to investigate 32-dimensional even unimodular lattices. At this time it was not possible to algorithmically decide equivalence of 32-dimensional extremal lattices.

To understand the motivation of Koch and Venkov one should recall the well known correspondence between framed unimodular lattices and self-dual codes.

REMARK 2.18. Let $L = L^\# \leq \mathbb{R}^n$ be a unimodular lattice and $F := \{v_1, \dots, v_n\} \subset L$ be a **p-frame**, so

$$(v_i, v_i) = p, (v_i, v_j) = 0 \text{ for all } i \neq j.$$

Then any $\ell \in L$ is a unique sum $\ell = \sum_{i=1}^n a_i v_i$ with $a_i \in \frac{1}{p}\mathbb{Z}$ and

$$C(L, F) := \{(\bar{a}_1, \dots, \bar{a}_n) \mid \sum a_i v_i \in L\} \leq \mathbb{F}_p^n$$

is a self-dual code. Here $\bar{a} = a + \mathbb{Z} \in \frac{1}{p}\mathbb{Z}/\mathbb{Z} \cong \mathbb{F}_p$. Conversely, given some $C = C^\perp \leq \mathbb{F}_p^n$ and a p -frame $\{v_1, \dots, v_n\}$ the lattice

$$L(C) := \left\{ \sum_{i=1}^n a_i v_i \mid (\bar{a}_1, \dots, \bar{a}_n) \in C, a_i \in \frac{1}{p}\mathbb{Z} \right\}$$

is a unimodular lattice. $L(C)$ is even if and only if $p = 2$ and C is **doubly-even**, i.e. all weights of code words are multiples of 4.

Koch and Venkov define the **defect** of an integral n -dimensional lattice L as $\delta(L) := n - s$, where s is the maximal cardinality of a set of pairwise orthogonal roots in L . So the lattices of defect 0 are exactly the lattices $L(C)$ for self-dual binary codes C . Koch and Venkov show the following

THEOREM 2.19. *Let L be an integral unimodular lattice of even dimension n . If $\delta(L) \leq 13$, then $\delta(L)$ is one of 0, 8, or 12.*

PROOF. For the proof they use their notion of perestroika of a lattice. Let $m = n - \delta(L)$ and $v_1, \dots, v_m \in L$ be pairwise orthogonal roots. Then $\langle v_1, \dots, v_m, 2L \rangle / 2L \leq L/2L$ is an isotropic space and hence contained in some maximal isotropic space $M/2L$. The sublattice M of L is called a **perestroika** of L . Since n is even, the dimension of $M/2L$ is $\frac{n}{2}$ and hence $\sqrt{2}^{-1}M$ is a unimodular lattice containing the sublattice $\perp_{i=1}^m \frac{1}{\sqrt{2}}v_i\mathbb{Z} \cong \mathbb{Z}^m$. So $M \cong \mathbb{Z}^m \perp N$ for some unimodular lattice N of dimension $\delta(L)$ of minimum ≥ 2 . By [29] there is no such lattice N of dimension 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13. If $\delta(L) = 8$ then $N \cong \mathbb{E}_8$ and for $\delta(L) = 12$ the lattice N is \mathbb{D}_{12}^+ . \square

Of course extremal lattices of dimension 32 have minimum 4, so they do not contain any roots. Nevertheless the definition of defect is helpful here by considering neighbors of the lattice.

DEFINITION 2.20 ([29]). Two unimodular lattices L and M are called **neighbors** if $L \cap M$ has index 2 in L (and hence also in M).

All neighbors M of the unimodular lattice L are of the form

$$M = L^{(v)} := L_v + \mathbb{Z}\frac{v}{2} \text{ with } L_v := \{\ell \in L \mid (\ell, v) \in 2\mathbb{Z}\}$$

for some $v \in L$ such that $(v, v) \in 4\mathbb{Z}$. If L is an even lattice then M is even if and only if $(v, v) \in 8\mathbb{Z}$.

REMARK 2.21. Let L be an even unimodular lattice with no roots and $v \in L$ such that $(v, v) \in 8\mathbb{Z}$. If $x, y \in L^{(v)} \setminus L$ then $x \pm y \in L$. So any two roots $x \neq \pm y \in L^{(v)}$ are orthogonal to each other and therefore $L^{(v)}$ is an even unimodular lattice with root system $(n - \delta(L^{(v)}))\mathbb{A}_1$.

DEFINITION 2.22 (Koch and Venkov). Let L be an extremal even unimodular lattice of dimension 32. For $1 \leq i \leq 32$ let

$$f_L(i) := |\{v \in L \mid (v, v) = 8, \delta(L^{(v)}) = 32 - i\}| \text{ and } g_L(i) := \frac{f_L(i)}{2^i}.$$

Since $L^{(v)} = L^{(2w)}$ for any root $w \in L^{(v)}$ the function g_L takes only integer values. By Theorem 2.19, $g_L(i) = 0$ for $i = 19, 21, 22, 23, 25, \dots, 31$. Using modular forms Koch and Venkov [14] prove the following equations for g_L :

$$\sum_{i=1}^{32} i g_L(i) = 2^5 3^5 5^2 17 \cdot 733, \quad \sum_{i=1}^{32} i^2 g_L(i) = 2^{10} 3^6 5^2 17^2, \quad \sum_{i=1}^{32} i^3 g_L(i) = 2^{14} 3^7 5^2 17.$$

They also compute the function g_L for all lattices L with $g_L(32) \neq 0$, the neighbors $L = L(C)^{(v)}$ of the code-lattices $L(C)$ for one of the 5 doubly-even self-dual

extremal codes C . In my diploma thesis I computed the function g_L for those twelve lattices L with $g_L(24) \neq 0$. The function g_L seems to distinguish extremal 32-dimensional lattices.

3. Lattices and spherical designs

3.1. Strongly perfect lattices. Most of the material in this section can be found in Boris Venkov's fundamental lecture notes [8]. In 1977 Delsarte, Goethals, and Seidel [25] define the notion of spherical designs:

DEFINITION 3.1. Let $X \subset S^{n-1}(m) := \{x \in \mathbb{R}^n \mid (x, x) = m\}$ be some non-empty finite set. Then X is called a **spherical t -design** if for all polynomials $p \in \mathbb{R}[x_1, \dots, x_n]$ for degree $\leq t$

$$(3.1) \quad \frac{1}{|X|} \sum_{x \in X} p(x) = \int_{S^{n-1}(m)} p(x) dx.$$

Since the right hand side is the $O(\mathbb{R}^n)$ -invariant inner product of p with the constant function and the homogeneous polynomials of degree t are the orthogonal sum

$$\mathbb{R}[x_1, \dots, x_n]_t = \text{Harm}_t \perp Q \text{Harm}_{t-2} \perp Q^2 \text{Harm}_{t-4} \perp \dots$$

the condition (3.1) is equivalent to

$$(3.2) \quad \sum_{x \in X} p(x) = 0 \text{ for all non constant harmonic polynomials } p \text{ of degree } \leq t.$$

In particular Remark 2.8 says that a lattice L is strongly eutactic if and only if its minimal vectors form a spherical 2-design. Motivated by this observation Boris Venkov gave the following very fruitful definition.

DEFINITION 3.2. A lattice L is **strongly perfect** if its minimal vectors form a spherical 4-design.

Strongly perfect lattices provide interesting examples of locally densest lattices as shown in the following theorem. In contrast to arbitrary extreme lattices, they can be classified in small dimensions using the combinatorics of their minimal vectors.

THEOREM 3.3. *Strongly perfect lattices are strongly eutactic and perfect, so they are extreme.*

PROOF. Let L be a strongly perfect lattice. Then the minimal vectors of L form a spherical 2-design and hence L is strongly eutactic by Remark 2.8. We need to show that L is perfect, i.e. that

$$\langle xx^{tr} \mid x \in \text{Min}(L) \rangle = \mathbb{R}_{sym}^{n \times n}.$$

Note that any symmetric matrix $A \in \mathbb{R}_{sym}^{n \times n}$ defines a polynomial $p_A : \alpha \mapsto \alpha^{tr} A \alpha$. Then $p_{xx^{tr}}(\alpha) = (x, \alpha)^2$ and

$$\text{trace}(xx^{tr} A) = \text{trace}(x^{tr} A x) = p_A(x).$$

Assume that $A \in \langle xx^{tr} \mid x \in \text{Min}(L) \rangle^\perp$. Then $p_A(x) = 0$ for all $x \in \text{Min}(L)$. Since $\text{Min}(L)$ is a spherical 4-design we obtain

$$\int_{S^{n-1}(\text{min}(L))} p_A^2 = \frac{1}{|X|} \sum_{x \in X} p_A(x)^2 = 0$$

which implies that $p_A = 0$ and hence $A = 0$. \square

LEMMA 3.4. *A lattice L is strongly perfect if and only if there is some constant c such that*

$$\sum_{x \in \text{Min}(L)} (x, \alpha)^4 = c(\alpha, \alpha)^2 \text{ for all } \alpha \in \mathbb{R}^n.$$

As in Remark 2.6, the constant c is obtained by applying the Laplace operator Δ_α with respect to α twice, $c = \frac{3 \min(L)^2}{n(n+2)} |\text{Min}(L)|$. The lemma only gives a polynomial condition of degree 4. Applying Δ_α one obtains the condition of degree 2 from Remark 2.6 that characterises strongly eutactic lattices. Note that $\sum_{x \in \text{Min}(L)} p(x) = 0$ for all homogeneous polynomials p of odd degree since $\text{Min}(L)$ is **antipodal**, $\text{Min}(L) = -\text{Min}(L)$. Summarizing we obtain that L is strongly perfect if and only if for all $\alpha \in \mathbb{R}^n$

$$(3.3) \quad \begin{aligned} (D4) \quad \sum_{x \in \text{Min}(L)} (x, \alpha)^4 &= \frac{3|\text{Min}(L)|m^2}{n(n+2)} (\alpha, \alpha)^2 \\ (D2) \quad \sum_{x \in \text{Min}(L)} (x, \alpha)^2 &= \frac{|\text{Min}(L)|m}{n} (\alpha, \alpha) \end{aligned}$$

THEOREM 3.5. *Let L be a strongly perfect lattice of dimension n .*

Then $\min(L) \min(L^\#) \geq (n+2)/3$.

PROOF. Let $\alpha \in \text{Min}(L^\#)$ so $(\alpha, \alpha) = \min(L^\#)$. Then $(\alpha, x) \in \mathbb{Z}$ for all $x \in \text{Min}(L)$ and hence (D4)-(D2) =

$$\sum_{x \in \text{Min}(L)} \underbrace{(x, \alpha)^2 ((x, \alpha)^2 - 1)}_{\geq 0} = \frac{|\text{Min}(L)| \min(L)}{n} (\alpha, \alpha) \underbrace{\left(\frac{3 \min(L) (\alpha, \alpha)}{n+2} - 1 \right)}_{\Rightarrow \geq 0}$$

Therefore $\frac{3 \min(L) \min(L^\#)}{n+2} \geq 1$ and the theorem follows. \square

3.2. The classification of strongly perfect lattices. The formulas (D4) and (D2) from the last section allow us to classify strongly perfect lattices of small dimension as well as strongly perfect integral lattices of small minimum.

THEOREM 3.6 ([8, Théorème 6.11]). *The strongly perfect root lattices are \mathbb{A}_1 , \mathbb{A}_2 , \mathbb{D}_4 , \mathbb{E}_6 , \mathbb{E}_7 , and \mathbb{E}_8 .*

THEOREM 3.7 ([8, Théorème 7.4]). *The strongly perfect integral lattices of minimum 3 are O_1 , O_7 , O_{16} , O_{22} , O_{23} .*

The lattices O_n of dimension n are as follows $O_7 = \sqrt{2}\mathbb{E}_7^\#$, $O_{16} = \langle \Lambda_{16}, x \rangle$, where Λ_{16} is the Barnes-Wall lattice in dimension 16 and $x \in \Lambda_{16}^\#$ satisfies $(x, x) = 3$. O_{23} is the unique unimodular lattice of minimum 3 and dimension 23, $O_{22} = x^\perp$ for any minimal vector $x \in O_{23}$.

These classifications have been extended by J. Martinet [30, pp 135-146] to integral lattices of higher minimum by imposing stronger design conditions on the minimal vectors.

The strongly perfect lattices up to dimension 12 are all classified ([8], [12], [5]). It is believed that the lattices given in [8, Tableau 19.1 and 19.2] are the only strongly perfect lattices up to dimension 24. In higher dimensions, the classifications get more and more involved. To simplify them one might either impose stronger design conditions (see for instance [4]) on the lattice or extra conditions on the dual lattice. Motivated by the fact that for most of the known strongly perfect lattices also the dual lattice is strongly perfect, we gave the following definition.

DEFINITION 3.8. A lattice L is called **dual strongly perfect** if L and $L^\#$ are both strongly perfect lattices.

One method to show that a lattice L is strongly perfect is to use its automorphism group $G = \text{Aut}(L)$. If this group has no harmonic invariant of degree ≤ 4 , then all G -orbits are spherical 4-designs (see Section 3.3) and hence the lattice is strongly perfect. Since $\text{Aut}(L) = \text{Aut}(L^\#)$ such lattices are also dual strongly perfect. In [2] we showed that there is a unique dual strongly perfect lattice of dimension 14. The general method to classify all strongly perfect lattices in a given dimension usually starts with a finite list of possible pairs (s, γ) , where $s = s(L) = \frac{1}{2}|\text{Min}(L)|$ is half of the kissing number of L and

$$\gamma = \gamma'(L)^2 = \min(L) \min(L^\#)$$

the Bergé-Martinet invariant of L . For both quantities there are good upper bounds known ([23]). Note that γ is just the product of the values of the Hermite function on L and $L^\#$. Using the general equations (3.3) of Section 3.1 a case by case analysis allows us either to exclude certain of the possibilities (s, γ) or to factor $\gamma = m \cdot r$ such that rescaled to minimum $\min(L^\#) = m$, the lattice $L^\#$ is integral (or even) and in particular contained in its dual lattice L (which is then of minimum r). For dual strongly perfect lattices we can use a similar argumentation to obtain a finite list of possibilities (s', γ) for $s' = s(L^\#)$ and in each case a factorization $\gamma = m' \cdot r'$ such that L is integral (or even) if rescaled to $\min(L) = m'$. This gives the exponent (in the latter scaling) $\exp(L^\#/L) = \frac{m}{r'}$. We proceed either by a direct classification of all such lattices L or use modular forms to exclude the existence of a modular form θ_L of level $\frac{m}{r'}$ and weight $\frac{n}{2}$ starting with $1 + 2sq^{m'/2} + \dots$, such that its image under the Fricke involution starts with $1 + 2s'q^{m/2} + \dots$ and both q -expansions have non-negative integral coefficients.

3.3. Application of group representations. Besides providing combinatorial tools for the classification of certain locally densest lattices, the notion of strongly perfect lattices opens up the possibility to apply representation theory of finite groups but also the theory of modular forms (Section 4.1) to prove that certain lattices are extreme.

Similar to Proposition 2.9 one shows the following Lemma.

LEMMA 3.9. *Let $G \leq \text{Aut}(L)$ and assume that all homogeneous G -invariant polynomials of degree 4 are multiples of Q^2 . Then L is dual strongly perfect.*

Together with Venkov we tried to apply this to obtain the minimum of the Thompson-Smith lattice of dimension 248: Let $G = \text{Th}$ be the sporadic simple Thompson group. Then G has a 248-dimensional rational representation $\rho : G \rightarrow O(\mathbb{R}^{248})$. Since G is finite, $\rho(G)$ fixes a lattice $L \leq \mathbb{Q}^{248}$. Modular representation theory tells us that for all primes p the $\mathbb{F}_p G$ -module L/pL is simple. Therefore $L = L^\#$ and L is even (otherwise the even sublattice L_0 of L provides an $\mathbb{F}_2 G$ -submodule $L_0/2L < L/2L$). From the character table of G one obtains that the space of G -invariant homogeneous polynomials of degree $2d$ is spanned by Q^d for $d = 1, 2, 3$. So all layers of L form spherical 6-designs and in particular L is strongly perfect. Theorem 3.5 implies that $\min(L) \min(L^\#) = \min(L)^2 \geq \frac{248+2}{3} > 83$, so $\min(L) \geq 10$. Constructing the lattice L one finds a vector $v \in L$ with $Q(v) = 6$, so $\min(L) \in \{10, 12\}$.

COROLLARY 3.10. *The minimum of the Thompson-Smith lattice is either 10 or 12.*

4. Unimodular lattices

4.1. Extremal even unimodular lattices are extreme. Boris Venkov was the first to use the theory of modular forms to study designs supported by extremal even unimodular lattices (see [16], [24, Chapter 7, Theorem 23]).

THEOREM 4.1. *Let L be an extremal even unimodular lattice of dimension $n = 24a + 8b$ with $b = 0, 1, 2$. Then all nonempty layers $A_j(L)$ are $(11 - 4b)$ -designs.*

PROOF. Since L is extremal, its minimum is $2a + 2$. Let $p \in \text{Harm}_t$ be a harmonic polynomial of degree $t \geq 1$. Then

$$\theta_{L,p} = \sum_{j=a+1}^{\infty} \left(\sum_{\ell \in A_j(L)} p(\ell) \right) q^j \in \Delta^{a+1} \mathcal{M}_{4b-12+t}$$

Therefore $\theta_{L,p} = 0$ whenever $4b+t < 12$, hence all layers $A_j(L) = \{\ell \in L \mid Q(\ell) = j\}$ form spherical $(11 - 4b)$ -designs. \square

COROLLARY 4.2. *If $b = 0$ or $b = 1$ then L is strongly perfect and hence extreme.*

In particular all extremal even unimodular lattices of dimension 32 are extreme. O. King [28] has shown that there are more than 10 million such lattices. A complete classification is unknown and the theory of strongly perfect lattices is the only known method to prove that all these lattices provide local maxima of the density function.

Theorem 4.1 has been generalized to extremal modular lattices (in the sense of Quebbemann [32]) in [10]. In this important paper Christine Bachoc and Boris Venkov develop very sophisticated methods to investigate extremal modular lattices using their harmonic theta series. Without classifying the complete genus they show that there are exactly three extremal 2-modular lattices of dimension 20, there is no extremal 7-modular lattice of dimension 18 and that there is a unique 5-modular lattice of dimension 16.

4.2. Odd unimodular lattices and their shadow. The theta series of an odd unimodular lattice is only a modular form for a subgroup of index 3 of the full modular group. The upper bound on the minimum of an odd unimodular lattice $L \leq \mathbb{R}^n$ obtained by the theory of modular forms in the same way as for even lattices in Corollary 2.14 is

$$\min(L) \leq \left\lfloor \frac{n}{8} \right\rfloor + 1.$$

The only unimodular lattices where equality is achieved are \mathbf{Z}^n ($n = 1, \dots, 7$), \mathbb{E}_8 , \mathbb{D}_{12}^+ , $(\mathbb{E}_7 \perp \mathbb{E}_7)^+$, \mathbb{A}_{15}^+ , O_{23} and Λ_{24} (see [24, Chapter 19]).

Any odd unimodular lattice L contains its **even sublattice**

$$(4.1) \quad L_0 := \{\ell \in L \mid (\ell, \ell) \in 2\mathbb{Z}\}$$

of index 2. The theta series of L_0 is $\theta_{L_0} = \frac{1}{2}(\theta_L(z) + \theta_L(z+1))$ and also $\theta_{L_0^\#}$ is obtained from θ_L using the theta transformation formula.

DEFINITION 4.3. Let L be an odd unimodular lattice. Then the **shadow** of L is $S(L) := L_0^\# \setminus L$.

Note that $S(L)$ is not a lattice but the union of the two cosets $\neq L$ of L_0 in $L_0^\#$. The theta series of $S(L)$ is obtained from the theta series of L as

$$\theta_{S(L)}(z) = S(\theta_L(z)) = \left(\frac{i}{z}\right)^{n/2} \theta_L\left(-\frac{1}{z} + 1\right).$$

Using the fact that $S(\theta_L)$ also has non-negative integer coefficients Rains and Sloane [33] prove the following theorem.

THEOREM 4.4. *Let $L \leq \mathbb{R}^n$ be an odd unimodular lattice. Then $\min(L) \leq 2 + 2\lfloor \frac{n}{24} \rfloor$ except for $n = 23$, where this bound is 3.*

A similar result holds for odd modular lattices.

Any $v \in S(L)$ satisfies $(v, \ell) \equiv Q(\ell) \pmod{\mathbb{Z}}$, so $2v$ is a **characteristic vector** of L . By the theory of quadratic forms, the norm $(2v, 2v) \equiv n \pmod{8}$. Define $\sigma(L) := 4 \min(S(L))$ to be the minimal norm of a characteristic vector in L .

Elkies [27] proved that \mathbb{Z}^n is the only unimodular lattice L with $\sigma(L) = n$. Any unimodular lattice L can be written uniquely as $L = M \perp \mathbb{Z}^k$ with $M = M^\#$ of minimum ≥ 2 . Then $\sigma(L) = \sigma(M) + k$, so one may assume that $\min(L) \geq 2$. Then Elkies found the short list of lattices L of minimum ≥ 2 with $\sigma(L) = n - 8$. The largest possible dimension here is $n = 23$ where the lattice L is the shorter Leech lattice O_{23} . In [7] we adapt the theory of theta series with spherical coefficients to the shadow theory of unimodular lattices to study lattices L with $\sigma(L) = n - 16$. If $\min(L) \geq 3$ then $n \leq 46$. This bound is the best possible, because $L = O_{23} \perp O_{23}$ satisfies $\dim(L) = 46$ and $\sigma(L) = 46 - 16$ and this is the only such lattice of dimension 46. In dimension 45 and 44 there are no such lattices of minimum ≥ 3 .

The combination of the minimum of the lattice and its shadow motivated Bachoc and Gaborit to define *s-extremal* lattices and codes. Gaborit showed that a unimodular lattice $L \leq \mathbb{R}^n$ always satisfies $8 \min(L) + \sigma(L) \leq 8 + n$ unless $n = 23$ and $L = O_{23}$. Lattices achieving this bound are called **s-extremal**. This notion has been generalized to modular lattices.

4.3. Classification of odd unimodular lattices. All unimodular lattices are classified up to dimension 25 [24, Chapter 16,17]. Borcherds also showed that there is a unique unimodular lattice in dimension 26 without roots. In higher dimensions the mass formula shows that there are too many unimodular lattices to classify them all. Roland Bacher and Boris Venkov [9] developed tools to classify only those odd unimodular lattices of dimension 27 and 28 that have minimum ≥ 3 . They show that there are 3 such lattices in dimension 27 and 38 such lattices in dimension 28. The correctness of their classification has later been also verified by the mass formulas in [28].

Bacher's and Venkov's method of classification is as follows: Let L be a unimodular lattice of dimension 28 with minimum 3. By the work by Elkies on lattices with long shadows, mentioned above, $\sigma(L)$ is either 4 or 12. This determines the two possibilities for the theta series of L . In particular L contains vectors v of norm $(v, v) = 4$. Each such vector $v \in L$ defines a neighbor $L^{(v)} = \mathbb{Z} \perp M$ for some unimodular lattice M of dimension 27 and of minimum ≥ 2 . It can be shown that the root system of M is $k\mathbb{A}_1$ and that there exists $v \in L$ such that $k \leq 4$. So it is enough to classify the 27-dimensional lattices M with root system $k\mathbb{A}_1$ for

$k = 0, 1, 2, 3, 4$ and then construct L as a neighbor of $M \perp \mathbb{Z}$. A clever counting argument using the symmetry of the neighboring graph makes the computation feasible.

4.4. An application to coding theory. The Bacher Venkov classification of the unimodular lattices in dimension 28 without roots has a very nice application to the classification of extremal self-dual ternary codes of length 28 using Remark 2.18. The paper [3] shows that there are exactly 6931 such codes. They correspond to pairs (L, F) of 28-dimensional unimodular lattices of minimum 3 and a 3-frame $F = \{v_1, \dots, v_{28}\} \subset L$. Since the minimum of the codes is 9 one obtains a bijection between the set of equivalence classes of extremal self-dual ternary codes of length 28 and the set of pairs (L, F) of isometry classes of unimodular lattices L of minimum 3 and representatives of the $\text{Aut}(L)$ -orbits of 3-frames $F \subset L$. Again theoretical arguments are needed to enable the enumeration of all frames with the computer.

5. Tight spherical designs

The most interesting t -designs are those of minimal cardinality. They have been studied by Bannai shortly after their definition in [25]: If $t = 2m$ is even, then any spherical t -design $X \subset S^{n-1}$ satisfies

$$|X| \geq \binom{n-1+m}{m} + \binom{n-2+m}{m-1}$$

and if $t = 2m + 1$ is odd then

$$|X| \geq 2 \binom{n-1+m}{m}.$$

A t -design X for which equality holds is called a **tight** t -design.

Tight t -designs in \mathbb{R}^n with $n \geq 3$ are very rare, they only exist if $t \leq 5$ and for $t = 7, 11$. The unique tight 11-design is supported by the minimal vectors $\text{Min}(\Lambda_{24})$ of the Leech lattice. The tight t -designs with $t = 1, 2, 3$ are also completely classified whereas their classification for $t = 4, 5, 7$ is still an open problem. It is known that the existence of a tight 4-design in dimension $n - 1$ is equivalent to the existence of a tight 5-design in dimension n , so the open cases are $t = 5$ and $t = 7$. It is also well known that tight spherical t -designs X for odd values of t are antipodal, i.e. $X = -X$ (see [25]).

There are certain numerical conditions on the dimension of such tight designs. A tight 5-design $X \subset S^{n-1}$ can only exist if either $n = 3$ and X is the set of 12 vertices of a regular icosahedron or $n = (2m + 1)^2 - 2$ for an integer m . Existence is only known for $m = 1, 2$ and these designs are unique and given by the minimal vectors of $\mathbb{E}_7^\#$ resp. $M_{23}^\#[2]$ from [8, Tableau 19.2]. Using lattices, Bannai, Mune-masa and Venkov exclude the next two open cases $m = 3, 4$ as well as an infinity of other values of m in the paper [6].

There are similar results for tight 7-designs. Such designs only exist if $n = 3d^2 - 4$. The only known cases are $d = 2, 3$ and the corresponding designs are unique; they are given by the minimal vectors of the unimodular lattices \mathbb{E}_8 and O_{23} . The paper [6] excludes the cases $d = 4, 5$ and also gives partial results on the interesting case $d = 6$ which still remains open. The study of such designs in dimension 104 was part of our joint projects during Boris Venkov's last weeks, in October and November 2011, in Aachen. Just to illustrate the connection with

lattices a few arguments from [6] are recalled. So let $D = X \dot{\cup} -X \subset S^{n-1}(d)$ be a tight spherical 7-design where $n = 3d^2 - 4$, $d \in \mathbb{N}$. Then

$$|X| = \frac{n(n+1)(n+2)}{6} \text{ and } (x, y) \in 0, \pm 1 \text{ for all } x \neq y \in X.$$

Let $L := \langle X \rangle$. Then L is an integral lattice of dimension n . The design conditions (equation (D2) and (D4) from Section 3.1 and the analogous equation (D6)) yield linear equations on the cardinalities

$$\begin{aligned} n_k(\alpha) &:= |\{x \in X \mid (x, \alpha) = \pm k\}|, (\alpha \in \mathbb{R}^n, k \geq 0) \\ \sum n_k(\alpha) &= |X| = (1/2)(3d^2 - 4)(3d^2 - 2)(d^2 - 1) \\ \sum k^2 n_k(\alpha) &= (1/2)(3d^2 - 2)(d^2 - 1)d(\alpha, \alpha) \\ \sum k^4 n_k(\alpha) &= (3/2)(d^2 - 1)d^2(\alpha, \alpha)^2 \\ \sum k^6 n_k(\alpha) &= (5/2)(d^2 - 1)d(\alpha, \alpha)^3. \end{aligned}$$

Assume that $\alpha \in \text{Min}(L) \setminus D$. Then $(\alpha, x) \in \{0, \pm 1, \dots, \pm \lfloor \frac{d}{2} \rfloor\}$ for all $x \in X$. In particular for $d \leq 7$ the $n_k(\alpha)$ are uniquely determined by the 4 equations. In all cases one obtains $n_2(\alpha) < 0$ which is absurd. So $\text{min}(L) = d$ and $D = \text{Min}(L)$. Now let $\alpha \in L^\#$ be minimal in its class modulo L . Again $(\alpha, x) \in \{0, \pm 1, \dots, \pm \lfloor \frac{d}{2} \rfloor\}$ for all $x \in X$. For $d = 4$ and 5 the system is overdetermined and one should find (α, α) as rational root of the polynomial that determines $n_3(\alpha)$. But this polynomial has no nonzero rational roots. Therefore $\alpha = 0$ and $L = L^\#$ is unimodular. For $d = 4$ this immediately yields a contradiction since then L is even unimodular of dimension 44, which is not a multiple of 8. The case $d = 5$ is more involved. Here Venkov takes $\alpha \in L$ to be a characteristic vector of minimal norm. Then $(x, \alpha) \in \{\pm 1, \pm 3, \pm 5\}$ again yields an overdetermined system on the $n_k(\alpha)$. One obtains a polynomial equation for (α, α) that has no rational solution. A contradiction.

If k is odd then $(k^2 - 1)(k^2 - 9)(k^2 - 25)$ is a multiple of $2^{10}3^25$. This yields divisibility conditions on the norm of a characteristic vector. In [1] we show that tight 7-designs with d odd may only exist if $d \equiv \pm 1 \pmod{16}$ or $d \equiv \pm 3 \pmod{32}$.

6. Hecke operators

In the previous section we have seen that one may apply modular forms, spherical designs and codes to construct and investigate interesting lattices. We also saw application of lattices to the classification of codes and tight designs. This final section reports on Venkov's ideas to apply the Kneser 2-neighbor graph of the Niemeier lattices to construct the action of certain Hecke operators on the space of Siegel modular forms spanned by theta series [11]. This has later been applied to other genera of modular lattices, including the genus of the Barnes-Wall lattice from [13], but also to genera of Hermitian lattices to construct Siegel cusp forms as linear combinations of Siegel theta series. The transfer of this method to codes allowed me to define Hecke operators in coding theory which was an old question by Broué.

Let L_1, \dots, L_{24} represent the isometry classes of even unimodular lattices in dimension 24. The Kneser 2-neighbor graph for these lattices has been computed by Borcherds for the purpose of classifying odd unimodular lattices in dimension 24 (see [24, Chapter 17]). The adjacency matrix

$$K \in \mathbb{Z}^{24 \times 24}, K_{ij} := |\{M \mid M \cong L_j, [L_i : M \cap L_i] = 2\}|$$

defines the action of a Hecke operator on the Siegel theta series [36]. The operator K acts on the complex vector space

$$V := \left\{ \sum_{i=1}^{24} a_i [L_i] \mid a_i \in \mathbb{C} \right\} \cong \mathbb{C}^{24}$$

of formal linear combinations of the Niemeier lattices (see [21]). Taking the **Siegel theta series** defines a linear mapping

$$\Theta^{(d)} : V \rightarrow \mathcal{M}_{12}(\mathrm{Sp}_{2d}(\mathbb{Z})), \sum_{i=1}^{24} a_i [L_i] \mapsto \sum_{i=1}^{24} a_i \Theta^{(d)}(L_i).$$

Let $V_d := \ker(\Theta^{(d)})$ be the kernel of $\Theta^{(d)}$, i.e. those linear combinations of lattices which have trivial degree- d Siegel theta series. Then we get the filtration

$$(6.1) \quad V =: V_{-1} \supseteq V_0 \supseteq V_1 \supseteq \dots \supseteq V_m = \{0\}.$$

The space V has a natural positive definite Hermitian inner product defined by

$$(6.2) \quad \langle [\Gamma], [\Lambda] \rangle := (\# \mathrm{Aut}(\Gamma)) \delta_{[\Gamma], [\Lambda]}.$$

Let $Y_d := V_{d-1} \cap V_d^\perp$. The space S_d of degree- d Siegel cusp forms that are linear combinations of Siegel theta series is then isomorphic to $Y_d \cong V_{d-1}/V_d$. This yields the orthogonal decomposition

$$(6.3) \quad V = \bigoplus_{d=0}^m Y_d.$$

The purpose of [11] is to compute this decomposition and therefore the spaces S_d . The Kneser neighbor operator K is self-adjoint with respect to the inner product (6.2). It respects the filtration (6.1) and hence also the decomposition (6.3) and therefore each space Y_d has a basis consisting of eigenvectors of K . It turns out that K has a simple spectrum, so it remains for each eigenvector e_1, \dots, e_{24} of K to compute the number $d = w(i)$ such that $e_i \in Y_d$. This is a difficult problem which could not be solved completely. By computing some nonzero coefficient of e_i one can always obtain upper bounds on $w(i)$. One important tool is the definition of an associative and commutative multiplication \circ on V for which the dual filtration of (6.1) behaves well, i.e. $V_n^\perp \circ V_d^\perp \subseteq V_{n+d}^\perp$ for all n, d . The starting point was the cusp form $e_{24} \in Y_{12}$ constructed by Borcherds, Freitag, and Weissauer [22]. We computed $e_i \circ e_j = A_{ij} e_{24} + \sum_{l=1}^{23} b_{ij}^l e_l$ with a nonzero coefficient A_{ij} for certain pairs i, j . This gave us the lower bound $w(i) + w(j) \geq 12$ which allowed us to obtain exact values for $w(i)$ and $w(j)$. We could determine all $w(i)$ apart from one open conjecture which involves to prove that a certain linear combination of degree 9 Siegel theta series of weight 12 vanishes.

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