

# Born–Infeld–Einstein Actions?

S. Deser<sup>1</sup>

Department of Physics, Brandeis University  
Waltham, Massachusetts 02254, USA

and

G.W. Gibbons<sup>2</sup>

D.A.M.T.P., Cambridge University  
Silver Street, Cambridge CB3 9EW, UK

## Abstract

We present some obvious physical requirements on gravitational avatars of non-linear electrodynamics and illustrate them with explicit determinantal Born–Infeld–Einstein models. A related procedure, using compensating Weyl scalars, permits us to formulate conformally invariant versions of these systems as well.

Born–Infeld (BI) electrodynamics [1] has earned its longevity through its elegant, compact, determinantal form,

$$I_{BI} = -1/2\lambda^2 \int d^4x \{-|g_{\mu\nu} + \lambda F_{\mu\nu}|\}^{\frac{1}{2}} . \quad (1)$$

It reduces to Maxwell theory for small amplitudes and shares with it two special properties, duality invariance [2] and causal, physical propagation [3]. Its quartic terms reproduce the effective action of one-loop SUSY QED. Not surprisingly, it regularly surfaces in more general contexts, most recently in various aspects of strings, branes and M. A further asset of (1) is the absence of ghost photon modes that are associated with models involving explicit derivatives on quadratic terms. This means that its famous taming of the Coulomb self-energy is not obtained at the price of

---

<sup>1</sup>deser@binah.cc.brandeis.edu; <sup>2</sup>G.W.Gibbons@damtp.cam.ac.uk

ghost compensation, but really stems from its nonpolynomial nature<sup>2</sup> and concomitant dimensional constant  $\lambda$ .

Is there a gravitational analog of BI? This note is part of the problem rather than of the solution: we will merely present and illustrate, but without giving any compelling exemplar, some criteria that it would have to satisfy.

We begin with the purely gravitational sector. Determinant forms of gravity have a long history of their own, although in a spirit different from that of BI. The first such action was given by Eddington [4] in a remarkably “modern” spirit; the metric enters as an integration constant in solving the equations of an ostensibly purely affine action,

$$I_{EDD} = \int d^4x |R_{(\mu\nu)}(\Gamma)|^{\frac{1}{2}} \quad (2)$$

where the independent field is a symmetric affinity  $\Gamma_{\nu\mu}^\alpha = \Gamma_{\mu\nu}^\alpha$ , and  $R_{(\mu\nu)}$  the symmetric part of its (generically nonsymmetric) Ricci tensor  $R_{\mu\nu}$ . Its (purely “Palatini”) variation implies that the covariant gradient of the normalized minor of  $R_{(\mu\nu)}$  vanishes. Consequently,  $R_{(\mu\nu)}$  is a “metric” for the affinity,  $R_{(\mu\nu)} = \lambda g_{\mu\nu}$ ,  $D_\alpha(\Gamma)g_{\mu\nu} = 0$ . This model has given rise to large literature of its own, including such extensions as using the full Ricci tensor and nonsymmetric  $\Gamma$  to represent electromagnetism. Although we hope to return to these aspects, it is not the road we take here. Ours is closer to the spirit of [1], working with a metric manifold from the start, with the generic geometrical action

$$I_G = \int d^4x \{-|ag_{\mu\nu} + bR_{\mu\nu} + cX_{\mu\nu}|\}^{\frac{1}{2}}. \quad (3)$$

We have separated the linear Ricci term from terms  $X_{\mu\nu}(R)$  quadratic or higher in curvature. The major necessary condition here is that (3) describe gravitons but no ghosts. This simply means

<sup>2</sup>One must distinguish, however, between “non-singular”, Bionic, finite energy and “solitonic”: there is no singularity in the sense that the “true” Coulomb field is bounded, but it is still generated by a point charge in a normal Poisson equation.

that the curvature expansion of (3) should begin with the Einstein  $R$  (of proper sign of course) but not contain any quadratic terms, since the latter are always responsible for ghost modes in an expansion about Minkowski (or de Sitter) backgrounds. [The cosmological term implicit in (3) (and indeed in (2)) can always be removed by subtraction or by suitable parameter limits.] To understand the effect of this constraint, we recall that in  $D=4$ , the Gauss–Bonnet combination

$$E_4 \equiv \sqrt{-g} [R_{\mu\nu\alpha\beta}^2 - 4R_{\mu\nu}^2 + R^2] \quad (4)$$

is a total divergence, so that effectively the generic quadratic Lagrangian is  $\alpha R_{\mu\nu}^2 + \beta R^2$ ; since these terms (in any combination) always generate ghosts or tachyons, they must be absent.<sup>3</sup> In fact, the quadratic part of  $E_4$  in  $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$  is a total divergence in any  $D$ , so the above remarks remain valid there. In (3) we could also have used the more general combination  $\tilde{R}_{\mu\nu} = R_{\mu\nu} - a g_{\mu\nu} R$ ,  $a \neq 1/4$ , including even  $\tilde{R}_{\mu\nu} \equiv g_{\mu\nu} R$ ; this trivializes the BI procedure, resembling a choice  $\sim |g_{\mu\nu}(1 - \lambda^2 F^2)|^{\frac{1}{2}}$  there. The expansion of a determinant,

$$|1 + A|^{\frac{1}{2}} = 1 + \frac{1}{2} \text{tr} A + \frac{1}{8} (\text{tr} A)^2 - \frac{1}{4} \text{tr}(A^2) + \mathcal{O}(A^3) \quad (5)$$

tells us that we must cancel the terms  $\sim \frac{b^2}{8} (\frac{1}{2} R^2 - R_{\mu\nu}^2)$  due to the quadratic expansion in  $R_{\mu\nu}$  by using the leading parts of  $\frac{1}{2} X_\alpha^\alpha$ . This leaves a wide latitude in the choice of  $X_{\mu\nu}$ : firstly, we can use any  $X_{\mu\nu}$  whose trace  $\sim -\frac{b^2}{8} (\frac{1}{2} R^2 - R_{\mu\nu}^2) + f E_4$  for arbitrary constant  $f$ . We can then obviously arrange for this  $X_\alpha^\alpha$  value by having  $X_{\mu\nu}$  be a pure trace term,  $X_{\mu\nu} = \frac{1}{4} g_{\mu\nu} X_\alpha^\alpha$ , or with more exotic choices such as  $X_{\mu\nu} \sim (R_{\mu\nu} R_\nu^\alpha - \frac{1}{2} R R_\alpha^\alpha)$  or any (suitably normalized) linear combinations of these. Their differences will only show up in cubic,  $\mathcal{O}(XR)$ , and higher contributions. There is no immediate criterion, obtainable from ghost-freedom, to further constrain  $X$ , although one may

---

<sup>3</sup>Also in the BI spirit, we exclude explicit derivatives in  $X_{\mu\nu}$ , although ghost-freedom alone does not exclude them in cubic or higher terms.

imagine adding higher and higher powers of  $R_{\mu\nu}$  in  $X$  to cancel particular unwanted contributions from expanding the  $R_{\mu\nu}$  and mixed contributions order by order.

In this connection, let us note that a “fudge tensor”  $X_{\mu\nu}$  actually permits one to write almost any action in BI form, so that there must be some *a priori* criterion for it as well. For example, any electromagnetic Lagrangian  $L_0(\alpha, \beta)$ , where  $\alpha \equiv \frac{1}{2}F_{\mu\nu}^2$ ,  $\beta \equiv \frac{1}{4}F^*F$  are the two invariants, can be so expressed: Simply write  $f \equiv L_{BI}/L_0$  and factorize  $f^{-\frac{1}{2}}$  into each element,  $(f^{-\frac{1}{2}}g_{\mu\nu} + \lambda f^{-\frac{1}{2}}F_{\mu\nu})$  of the new determinant that now represents  $L_0$ , then expand  $f$  about unity and call the rest  $X_{\mu\nu}$ .

There are further possible criteria: one may require that there be no singular “Coulomb” – here Schwarzschild (or Schwarzschild–de Sitter ones if the cosmological constant is kept) – solutions. This in turn has the necessary consequence that Ricci-flat solutions are to be excluded, meaning that at some order in the field equations there must appear terms depending only on the Weyl tensor, as can be accomplished simply by endowing  $X_{\mu\nu}$  with Weyl tensor dependence.<sup>4</sup> Presumably the space of such theories will be further constrained by the requirement that *their* “Coulomb” solutions will be milder than the black holes they replace! Perhaps the strongest “physical” constraint on theories of this type, however, would come from demanding that (like BI) they be supersymmetrizable. We do not know even know if this is at all possible, since the SUSY would have to be a local one, a very stringent (and dimension-dependent) requirement. The positive-energy issue might constitute one major barrier. Our other conditions are only mildly dimension-dependent: Although for  $D > 4$ , the Gauss–Bonnet identity is replaced by higher curvature ones (in even  $D$ ) that are irrelevant to the ghost problem, we have seen that the linearized  $D=4$  identity is preserved in any  $D$ , so the  $D=4$

---

<sup>4</sup>This condition is not sufficient, as one could imagine combinations of such terms in the field equations that vanish for the simple Schwarzschild (or similar) form but not for generic Ricci-flat spaces. Of course certain Einstein spaces will remain solutions of any action, namely the (unbounded) *pp*-waves, all of whose scalar invariants vanish [5]; electromagnetic plane waves are likewise solutions of any nonlinear model, since their  $\alpha$  and  $\beta$  vanish.

discussion effectively stands. For  $D=3$ ,  $E_4 \equiv 0$ , Riemann and Ricci tensors coincide so the exercise would reduce to some variant of Einstein theory. For  $D=2$  only the scalar Euler density  $R$  remains and no “genuine” BI structure is possible, although one can still write [6] a form  $\sim |g_{\mu\nu} f(R)|^{\frac{1}{2}}$ . The original BI action (1) is of course insensitive to  $D$ .

Let us now turn to possible “BI-E” actions involving both photons and gravitons. Reinstating the linear  $\lambda F_{\mu\nu}$  term inside our determinant in (3) will now give rise to nonminimal cross-terms at least bilinear in  $F$  (because it is antisymmetric) times powers of curvatures. While such terms do not affect the excitation content, they do alter the propagation properties of both types of particles [7]. Indeed their “light cones” become governed by effective metrics of the schematic form  $(g_{\mu\nu} + \mathcal{O}_{\mu\nu})$  where  $\mathcal{O}_{\mu\nu}$  represents the nonminimal contribution, with attendant propagation complications that may in principle be used to narrow the ambiguity in  $X$ . Clearly, one would also want a suitably tamed Reissner–Nordstrom solution here, along with a bounded Coulomb field. Scalar fields (or multiplets) can be incorporated in a ghost-free way by adding terms of the form  $\sim (\partial_\mu \phi \partial_\nu \phi + m^2 g_{\mu\nu} \phi^2)$  under the determinant; their trace will reproduce the usual scalar action, but now there will be non-minimal cross-terms as well.

## 1 Weyl-invariance

In four spacetime dimensions, the Maxwell action is invariant under Weyl rescalings of the metric:

$$g_{\mu\nu} \rightarrow \Omega^2(x) g_{\mu\nu} \tag{6}$$

without transforming the  $A_\mu$ . The Born-Infeld action is of course not Weyl-invariant. However, it may readily be made so by introducing a scalar compensating field  $\phi$  of Weyl weight  $-1$ ,

$$\phi \rightarrow \Omega^{-1} \phi. \tag{7}$$

The modified action is

$$I_{BIW} = -1/2 \lambda^2 \int d^4x \left\{ -|\phi^2 g_{\mu\nu} + \phi^{-2} \mathcal{D}_\mu \phi \mathcal{D}_\nu \phi + \lambda F_{\mu\nu}| \right\}^{\frac{1}{2}}, \quad (8)$$

where the (real!) Weyl-covariant derivative is  $\mathcal{D}_\mu = \partial_\mu + A_\mu$  and the vector potential now undergoes an  $R^*$  gauge transformation:

$$A_\mu \rightarrow A_\mu + \partial_\mu \ln \Omega. \quad (9)$$

An appropriate choice of sign of the scalar term ensures its ghost-freedom.

This model can also accommodate Weyl invariant gravity, though of course that always involves ghosts. In particular one could add in (8) a (fourth derivative) combination of the form

$$a \phi^2 g_{\mu\nu} C_{\beta\gamma\delta}^\alpha C_{\alpha\lambda\sigma}^\beta g^{\gamma\lambda} g^{\delta\sigma} + b \phi^2 C_{\mu\beta\delta}^\alpha C_{\nu\alpha\sigma}^\beta g^{\delta\sigma} \quad (10)$$

(for D=4 the two terms in (10) are proportional). An alternate route is to “improve” the Einstein term using the compensator: the relevant (D=4) action is proportional to the famous combination  $\phi^{-2} \sqrt{-g} (\frac{1}{6} R + \phi^{-2} (\partial_\mu \phi \partial_\nu \phi) g^{\mu\nu})$  where either the Einstein or the gravity kinetic term is now necessarily of the wrong sign [8]. This model can obviously be adapted to BI form using the previously discussed extensions of  $R_{\mu\nu}$  and  $X_{\mu\nu}$ .

In conclusion, it should be obvious from the rather loose conditions we have stated that any real progress on adding “E” to BI will require either better hints from string expansions or from supersymmetry requirements. The elegant pure BI insight has as yet found no counterpart here.

After this work was completed, an interesting non-determinantal two-metric reformulation of BI has been suggested [9]. In the process, evenness of BI in  $F_{\mu\nu}$  is used to rewrite the determinant (1) in terms of the tensor  $(g_{\mu\nu} + \lambda^2 F_{\mu\alpha} F_\nu^\alpha)$ . While it is perhaps formally more natural to include  $R_{\mu\nu} + X_{\mu\nu}$  into this symmetric array, our considerations remain unaltered.

The research of S.D. was supported by the National Science Foundation, under grant #PHY-9315811. Our work was begun at the VI Conference on the Quantum Mechanics of Fundamental

Systems, during a session held at the Presidente Frei Antarctic base; we thank the organizers for this unique opportunity.

## References

- [1] M. Born and L. Infeld, Proc. Roy. Soc. **A144** (1934) 425.
- [2] See for example, I. Bialynicki-Birula in “Quantum Theory of Particles and Fields,” eds. B. B. Jancewicz and J. Lukierski (World Scientific: 1983); G.W. Gibbons and D.A. Rasheed, Nucl. Phys. **B454** (1995) 185; S. Deser, A. Gomberoff, M. Henneaux, and C. Teitelboim, Phys. Lett. **B400** (1997) 80 and Nucl. Phys. **B** (in press).
- [3] J. Plebanski, Nordita Lectures on Nonlinear Electrodynamics (1968); G. Boillat J. Math. Phys. **11** (1970) 941.
- [4] A.S. Eddington “The Mathematical Theory of Relativity” (CUP 1924).
- [5] G.W. Gibbons, Commun. Math. Phys. **45** (1975) 191; S. Deser, J. Phys. **A8** (1975) 1972.
- [6] J.A. Feigenbaum, G.G.O. Freund, and M. Pigli hep-th/9709196.
- [7] See *e.g.*, C. Aragone and S. Deser, Nuovo Cimento **57B** (1980) 33; S. Deser and R. Puzalowski, J. Phys. **A13** (1980) 2501.
- [8] S. Deser, Ann. Phys. **59** (1970) 248.
- [9] M. Abou Zeid and C.M. Hull, hep-th/9802179; see also C.M. Hull, hep-th/9708048.