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BORNOLOGICAL LOCALLY CONVEX CONES

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In this paper we define bornological and b-bornological cones and investigate their properties. We give some characterizations for these cones. In the special case of locally convex topological vector spaces these both concepts reduce to the known concept of bornological spaces. We introduce and investigate the convex quasiuniform structures $\mathfrak{U}_\tau$, $\mathfrak{U}_\sigma(\mathcal{P},\mathcal{P}^*)$ and $\mathfrak{U}_\beta(\mathcal{P},\mathcal{P}^*)$ on locally convex cone $(\mathcal{P},\mathfrak{U})$.

1. Introduction

A *cone* is a set $\mathcal{P}$ endowed with an addition and a scalar multiplication for nonnegative real numbers. The addition is assumed to be associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold, that is $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha+\beta)a = \alpha a + \beta a$, $\alpha(a+b) = \alpha a + \alpha b$, $1a = a$ and $0a = 0$ for all $a,b \in \mathcal{P}$ and $\alpha,\beta \geq 0$.

The theory of locally convex cones as developed in [3] and [8] uses an order theoretical concept or a convex quasiuniform structure to introduce a topological structure on a cone. In this paper we use the latter. For recent researches see [1, 4, 7].

Let $\mathcal{P}$ be a cone. A collection $\mathfrak{U}$ of convex subsets $U \subseteq \mathcal{P}^2 = \mathcal{P} \times \mathcal{P}$ is called a convex quasiuniform structure on $\mathcal{P}$, if the following properties hold:

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(U_1) $\Delta \subseteq U$ for every $U \in \mathfrak{U}$ ($\Delta = \{(a,a) : a \in \mathcal{P}\}$);

(U_2) for all $U,V \in \mathfrak{U}$ there is a $W \in \mathfrak{U}$ such that $W \subseteq U \cap V$;

(U_3) $\lambda U \circ \mu U \subseteq (\lambda + \mu)U$ for all $U \in \mathfrak{U}$ and $\lambda, \mu > 0$;

(U_4) $\alpha U \in \mathfrak{U}$ for all $U \in \mathfrak{U}$ and $\alpha > 0$.

Here, for $U,V \subseteq \mathcal{P}^2$, by $U \circ V$ we mean the set of all $(a,b) \in \mathcal{P}^2$ such that there is some $c \in \mathcal{P}$ with $(a,c) \in U$ and $(c,b) \in V$.

Let $\mathcal{P}$ be a cone and $\mathfrak{U}$ be a convex quasiuniform structure on $\mathcal{P}$. We shall say $(\mathcal{P},\mathfrak{U})$ is a locally convex cone if

(U_5) for each $a \in \mathcal{P}$ and $U \in \mathfrak{U}$ there is some $\rho > 0$ such that $(0,a) \in \rho U$.

With every convex quasiuniform structure $\mathfrak{U}$ on $\mathcal{P}$ we associate two topologies: the neighborhood bases for an element a in the upper and lower topologies are given by the sets

$$U(a) = \{b \in \mathcal{P} : (b,a) \in U\}, \quad \text{resp.} \quad (a)U = \{b \in P : (a,b) \in U\}, \quad U \in \mathfrak{U}.$$

The common refinement of the upper and lower topologies is called symmetric topology. A neighborhood base for $a \in \mathcal{P}$ in this topology is given by the sets

$$U(a)U = U(a) \cap (a)U, \quad U \in \mathfrak{U}.$$

Let $\mathfrak{U}$ and $\mathcal{W}$ be convex quasiuniform structures on $\mathcal{P}$. We say that $\mathfrak{U}$ is finer than $\mathcal{W}$ whenever for every $W \in \mathcal{W}$ there is $U \in \mathfrak{U}$ such that $U \subseteq W$.

The extended real number system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a cone endowed with the usual algebraic operations, in particular $a + \infty = +\infty$ for all $a \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. We set $\tilde{\mathcal{V}} = \{\tilde{\varepsilon} : \varepsilon > 0\}$, where

$$\tilde{\varepsilon} = \{(a,b) \in \overline{\mathbb{R}}^2 : a \leq b + \varepsilon\}.$$

Then $\tilde{\mathcal{V}}$ is a convex quasiuniform structure on $\overline{\mathbb{R}}$ and $(\overline{\mathbb{R}}, \tilde{\mathcal{V}})$ is a locally convex cone. For $a \in \mathbb{R}$ the intervals $(-\infty, a+\varepsilon]$ are the upper and the intervals $[a - \varepsilon, +\infty]$ are the lower neighborhoods, while for $a = +\infty$ the entire cone $\overline{\mathbb{R}}$ is the only upper neighborhood, and $\{+\infty\}$ is open in the lower topology. The symmetric topology is the usual topology on $\mathbb{R}$ with $+\infty$ as an isolated point.

For cones $\mathcal{P}$ and $\mathcal{Q}$, a mapping $T : \mathcal{P} \to \mathcal{Q}$ is called a *linear operator* if $T(a+b) = T(a) + T(b)$ and $T(\alpha a) = \alpha T(a)$ hold for all $a,b \in \mathcal{P}$ and $\alpha \geq 0$. If both $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{W})$ are locally convex cones, the operator T is called *(uniformly) continuous* if for every $W \in \mathcal{W}$ one can find $U \in \mathfrak{U}$ such that $(T \times T)(U) \subseteq W$.

A *linear functional* on $\mathcal{P}$ is a linear operator $\mu : \mathcal{P} \to \overline{\mathbb{R}}$. We denote the set of all linear functional on $\mathcal{P}$ by $L(\mathcal{P})$ (the algebraic dual of $\mathcal{P}$). For a subset F of $\mathcal{P}^2$ we define its *polar* F° as follows

$$F^\circ = \{\mu \in L(\mathcal{P}) : \mu(a) \le \mu(b) + 1, \forall (a,b) \in F\}.$$

Clearly $(\{(0,0)\})^\circ = L(\mathcal{P})$. A linear functional μ on $(\mathcal{P},\mathfrak{U})$ is (uniformly) continuous if $\mu \in U^\circ$ for some $U \in \mathfrak{U}$. The *dual cone* $\mathcal{P}^*$ of a locally convex cone $(\mathcal{P},\mathfrak{U})$ consists of all continuous linear functionals on $\mathcal{P}$ and is the union of all polars U° of neighborhoods $U \in \mathfrak{U}$.

We shall say that a locally convex cone $(\mathcal{P},\mathfrak{U})$ has the *strict separation property* if the following holds:

(SP) For all $a,b \in \mathcal{P}$ and $U \in \mathfrak{U}$ such that $(a,b) \notin \rho U$ for some $\rho > 1$, there is a linear functional $\mu \in U^\circ$ such that $\mu(a) > \mu(b) + 1$ ([3], II, 2.12).

Let $\mathfrak{U}$ be a convex quasiuniform structure on $\mathcal{P}$. The subset $\mathcal{B}$ of $\mathfrak{U}$ is called a *base* for $\mathfrak{U}$, whenever for every $U \in \mathfrak{U}$ there are $n \in \mathbb{N}$, $U_1, \cdots, U_n \in \mathcal{B}$ and $\lambda_1, \cdots, \lambda_n > 0$ such that $\lambda_1 U_1 \cap \cdots \cap \lambda_n U_n \subseteq U$.

2. Uniformly convex sets and *uc*-cones

Definition 2.1. Let $\mathcal{P}$ be a cone. We say that the convex subset E of $\mathcal{P}^2$ is *uniformly convex* whenever E has properties (U_1) and (U_3).

Proposition 2.2. *Let $\mathcal{P}$ be a cone and $\mathcal{B}$ be a collection of uniformly convex subsets of $\mathcal{P}^2$. Then there exists coarsest convex quasiuniform structure $\mathfrak{U}$ on $\mathcal{P}$ that contains $\mathcal{B}$. If for every $a \in \mathcal{P}$ and $U \in \mathcal{B}$ there is $\lambda > 0$ such that $(0,a) \in \lambda U$, then $(\mathcal{P},\mathfrak{U})$ is a locally convex cone.*

Proof. We suppose that $\mathfrak{U}$ is the collection of all uniformly convex subsets U of $\mathcal{P}^2$ for which there are $n \in \mathbb{N}$, $\lambda_1, ..., \lambda_n > 0$ and $U_1, ...U_n \in \mathcal{B}$ such that

$$\lambda_1 U_1 \cap ... \cap \lambda_n U_n \subseteq U.$$

It is easy to see that $\mathfrak{U}$ satisfies the conditions (U_1) to (U_4). Also, if for every $a \in \mathcal{P}$ and $U \in \mathcal{B}$ there is $\lambda > 0$ such that $(0,a) \in \lambda U$, then $\mathfrak{U}$ satisfies (U_5) and $(\mathcal{P},\mathfrak{U})$ becomes a locally convex cone. □

Corollary 2.3. *Let $\mathcal{P}$ be a cone. There is the finest convex quasiuniform structure $\mathfrak{U}_\beta$ on $\mathcal{P}$ that makes $(\mathcal{P},\mathfrak{U}_\beta)$ into a locally convex cone.*

If $\mathcal{B}$ is the collection of all uniformly convex subsets of $\mathcal{P}^2$ such that for every $a \in \mathcal{P}$ and $U \in \mathcal{B}$ there is $\lambda > 0$ such that $(0,a) \in \lambda U$, then $\mathfrak{U}_\beta$ is the coarsest convex quasiuniform structure on $\mathcal{P}$ that contains $\mathcal{B}$.

Definition 2.4. Let $\mathcal{P}$ be a cone and $F \subseteq \mathcal{P}^2$. The smallest uniformly convex subset of $\mathcal{P}^2$ that contains F is called *uniformly convex hull* of F. We denote it by $uch(F)$.

Obviously, $uch(F)$ is the intersection of all uniformly convex subsets of $\mathcal{P}^2$ which contain F.

A subset A of $\mathcal{P}$ is called *balanced* if $b \in A$ whenever $b = \lambda a$ or $b + \lambda a = 0$ for some $a \in A$ and $0 \leq \lambda \leq 1$ (see [8], I, 1).

Proposition 2.5. *Let $\mathcal{P}$ be a cone and U be a uniformly convex subset of $\mathcal{P}^2$. Then $B = U(0)U = \{b \in \mathcal{P} | (0,b) \in U, (b,0) \in U\}$ is balanced and convex.*

Proof. The convexity of B is obvious. Let $a \in B$. For $0 \leq \lambda \leq 1$ we have $(0, \lambda a) \in \lambda U \subseteq U$ and $(\lambda a, 0) \in \lambda U \subseteq U$, then $b = \lambda a \in B$. On other hand if $b + \lambda a = 0$, then

$$(0,b) = (b+\lambda a, b) = (\lambda a, 0) + (b,b) \in \lambda U + (1-\lambda)\triangle \subseteq \lambda U + (1-\lambda)U \subseteq U.$$

Similarly $(b,0) \in U$. Thus $b \in B$ and B is balanced. □

Definition 2.6. Suppose that $(\mathcal{P}, \mathfrak{U})$ is a locally convex cone. We shall say that $F \subseteq \mathcal{P}^2$ is *u-bounded* if it is absorbed by each $U \in \mathfrak{U}$.

A subset A of $\mathcal{P}$ is called *bounded above (below)* whenever $A \times \{0\}$ (res. $\{0\} \times A$) is u-bounded.

Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone and F be a u-bounded subset of $\mathcal{P}^2$. Then $uch(F)$ is u-bounded. Indeed, for every $U \in \mathfrak{U}$ there is $\lambda > 0$ such that $F \subseteq \lambda U$. This shows that $uch(F) \subseteq uch(\lambda U) = \lambda U$, since each $U \in \mathfrak{U}$ is uniformly convex.

Definition 2.7. We say that the locally convex cone $(\mathcal{P}, \mathfrak{U})$ is a *uc-cone*, whenever $\mathfrak{U} = \{\alpha U : \alpha > 0\}$ for some $U \in \mathfrak{U}$ (the subset U of $\mathcal{P}^2$ is uniformly convex and $\mathfrak{U}$ is created by U).

Every normed space is a *uc*-cone as a locally convex cone. Indeed, if $(E, \|.\|)$ is a normed space, $B = \{(a,b) \in E^2 : \|a-b\| \leq 1\}$ and $\mathfrak{U} = \{\alpha B : \alpha > 0\}$, then $(E, \mathfrak{U})$ is a *uc*-cone.

Example 2.8. Let E be a normed space with unit ball B and let $Conv(E)$ be the collection of all non-empty convex subsets of E. Then $Conv(E)$ is a cone endowed with the usual addition and multiplication. The convexity of $A \in Conv(E)$ implies that $(\alpha + \beta)A = \alpha A + \beta A$. We set

$$\tilde{B} = \{(A,C) : A, C \in Conv(E),\ A \subseteq C + B\}.$$

The subset $\tilde{B}$ of $Conv(E) \times Conv(E)$ is uniformly convex and for $A \in Conv(E)$ there is $\lambda > 0$ such that $(\{0\}, A) \in \lambda \tilde{B}$. Therefore $\mathfrak{B} = \{\alpha \tilde{B} : \alpha > 0\}$ is a convex quasiuniform structure on $Conv(E)$ and $(Conv(E), \mathfrak{B})$ is a *uc*-cone.

Remark 2.9. If $(\mathcal{P},\mathfrak{U})$ is a *uc*-cone and $\mathcal{P}$ is a vector space over $\mathbb{R}$, then $\mathcal{P}$ is a seminormed space endowed with the symmetric topology of $(\mathcal{P},\mathfrak{U})$. If $\mathfrak{U}=\{\alpha U:\alpha>0\}$, then this seminorm is given by $P(a)=\inf\{\lambda>0:a\in\lambda U(0)U\}$ for $a\in\mathcal{P}$. If the symmetric topology on $\mathcal{P}$ is Hausdorff, then p is a norm on $\mathcal{P}$.

If a (Hausdorff) locally convex space has a bounded neighborhood, then it is a (normed) seminormed space. In locally convex cones we have:

Proposition 2.10. *Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone. If $\mathfrak{U}$ has a u-bounded element, then $(\mathcal{P},\mathfrak{U})$ is a uc-cone.*

Proof. Suppose that $U\in\mathfrak{U}$ is u-bounded. We claim that $\mathfrak{U}=\{\alpha U:\alpha>0\}$. Obviously, $\mathfrak{U}$ is finer than $\{\alpha U:\alpha>0\}$. On the other hand, if $W\in\mathfrak{U}$, then there is $\alpha>0$ such that $U\subseteq\alpha W$ or $\frac{1}{\alpha}U\subseteq W$. This shows that $\{\alpha U:\alpha>0\}$ is finer than $\mathfrak{U}$. □

The inductive and projective limit of locally convex convex cones is defined in [5].

Proposition 2.11. *Every locally convex cone is a projective limit of uc-cones.*

Proof. Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone. For every $U\in\mathfrak{U}$ we set $\mathfrak{U}_U=\{\alpha U:\alpha>0\}$. Then $\mathfrak{U}_U$ is a convex quasiuniform structure and $(\mathcal{P},\mathfrak{U}_U)$ is a *uc*-cone. We claim that $(\mathcal{P},\mathfrak{U})$ is the projective limit of *uc*-cones $(\mathcal{P},\mathfrak{U}_U)_{U\in\mathfrak{U}}$ with identity mappings. Indeed, the identity mappings $I_U:(\mathcal{P},\mathfrak{U})\to(\mathcal{P},\mathfrak{U}_U)$ are continuous, since $\mathfrak{U}$ is finer than $\mathfrak{U}_U$ for all $U\in\mathfrak{U}$. Let $\mathcal{W}$ be a convex quasiuniform structure on $\mathcal{P}$ that makes all $I_U:(\mathcal{P},\mathcal{W})\to(\mathcal{P}_U,\mathfrak{U}_U)$ continuous. If $U\in\mathfrak{U}$, then $U\in\mathfrak{U}_U$. Now the continuity of I_U shows that there is $W\in\mathcal{W}$ such that $W=I_U(W)\subseteq U$. This yields that $\mathcal{W}$ is finer than $\mathfrak{U}$. Therefore $(\mathcal{P},\mathfrak{U})$ is the projective limit of *uc*-cones $(\mathcal{P},\mathfrak{U}_U)$ by the identity mappings $I_U:(\mathcal{P},\mathfrak{U})\to(\mathcal{P},\mathfrak{U}_U)$. □

3. Bornological and *b*-Bornological Cones

In the following we define bornological and *b*-bornological cones. We extend some classical results from locally convex vector spaces to locally convex cones (see for example [2] and [6]). We obtain new results and present some examples of bornological cones. Also, we introduce extreme convex quasiuniform structures.

Suppose that $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{W})$ are locally convex cones and $T:\mathcal{P}\to\mathcal{Q}$ is a linear operator. We shall say T is *u-bounded* if $(T\times T)(F)$ is u-bounded in $\mathcal{Q}^2$ for every u-bounded subset F of $\mathcal{P}^2$.

Obviously, every continuous linear operator is u-bounded. The converse is not true in general. If $T:(\mathcal{P},\mathfrak{U})\to(\mathcal{Q},\mathcal{W})$ is a *u-bounded* linear operator and A is a bounded subset of $\mathcal{P}$, then $T(A)$ is a bounded subset of $\mathcal{Q}$.

Definition 3.1. Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone. We shall say $(\mathcal{P},\mathfrak{U})$ is a *bornological cone* if every u-bounded linear operator from $(\mathcal{P},\mathfrak{U})$ into any locally convex cone is continuous.

Example 3.2. Let E be a bornological locally convex vector space over $\mathbb{R}$ and $\mathcal{V}$ be a base of convex and balanced neighborhoods of the origin. If we set $\tilde{\mathcal{V}} = \{\tilde{V} : V \in \mathcal{V}\}$, where $\tilde{V} = \{(a,b) : a-b \in V\}$, then $(E,\tilde{\mathcal{V}})$ is a locally convex cone. We prove that $(E,\tilde{\mathcal{V}})$ is a bornological cone. Let T be a u-bounded linear operator from $(E,\tilde{\mathcal{V}})$ to another locally convex cone $(\mathcal{Q},\mathcal{W})$. Then $T(E)$ is a locally convex space endowed with the symmetric topology induced by $\mathcal{W}$. Let $W \in \mathcal{W}$. The operator T maps bounded subsets of locally convex space E into the locally convex space $T(E)$ (endowed with the symmetric topology). Since E is bornological as a locally convex space, T is continuous. Then there is $V \in \mathcal{V}$ such that $T(V) \subseteq (\frac{1}{2}W)(0)(\frac{1}{2}W)$. If we set $B = \{(T(a),T(b)) : T(a)-T(b) \in (\frac{1}{2}W)(0)(\frac{1}{2}W)\}$, then we have $B \subseteq W$. Now, we have $(T\times T)(\tilde{V}) \subseteq B \subseteq W$. Then $T : (E,\tilde{\mathcal{V}}) \to (\mathcal{Q},\mathcal{W})$ is continuous and $(E,\tilde{\mathcal{V}})$ is a bornological cone.

Example 3.3. The locally convex cone $(\mathcal{P},\mathfrak{U}_\beta)$ from Corollary 2.3 is a bornological cone. For, if T is a linear operator from $\mathcal{P}$ into other locally convex cone $(Q,\mathcal{W})$, $t = T\times T$, and $W \in \mathcal{W}$, then $t^{-1}(W) = (T\times T)^{-1}(W)$ is uniformly convex. Also for every $a \in \mathcal{P}$ we have $(T(0),T(a)) = (0,T(a)) \in \lambda W$ for some $\lambda > 0$. Then $(0,a) \in \lambda t^{-1}(W)$. This shows that $t^{-1}(W) \in \mathfrak{U}_\beta$. Therefore every linear operator on $\mathcal{P}$ is continuous. This implies that $(\mathcal{P},\mathfrak{U}_\beta)$ is a bornological cone.

Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone. If the convex quasiuniform structure $\mathfrak{U}$ has a countable base, then it has a base $(U_n)_{n\in\mathbb{N}}$ such that $U_{n+1} \subseteq U_n$. Indeed, let $(\omega_n)_{n\in\mathbb{N}}$ be a base of $\mathfrak{U}$. We set $U_1 = \omega_1$ and $U_n = U_{n-1}\cap\omega_n$ for $n \geq 2$. Now, $(U_n)_{n\in\mathbb{N}}$ is a base of $\mathfrak{U}$ and we have $U_{n+1} \subseteq U_n$ for all $n \in \mathbb{N}$.

Theorem 3.4. *Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone. If the convex quasiuniform structure $\mathfrak{U}$ has a countable base, then $(\mathcal{P},\mathfrak{U})$ is a bornological cone.*

Proof. We consider the base $(U_n)_{n\in\mathbb{N}}$ for $\mathfrak{U}$ such that $U_{n+1} \subseteq U_n$ for all $n \in \mathbb{N}$. Let $(\mathcal{Q},\mathcal{W})$ be a locally convex cone and $T : (\mathcal{P},\mathfrak{U}) \to (\mathcal{Q},\mathcal{W})$ be a u-bounded linear operator. If T is not continuous, then there exists $W \in \mathcal{W}$ such that $(T\times T)(U_n) \nsubseteq nW$ for all $n \in \mathbb{N}$. We set

$$F = \bigcap_{i=1}^{\infty} U_n.$$

The subset F of $\mathcal{P}^2$ is nonempty and u-bounded because $\triangle \subseteq F \subseteq U_n$ for all $n \in \mathbb{N}$. On the other hand $(T\times T)(F)$ is not u-bounded in $\mathcal{Q}^2$. In fact it is not absorbed by $W \in \mathcal{W}$. This contradiction proves our claim. $\square$

Corollary 3.5. *Every uc-cone is bornological.*

In the case of locally convex spaces, Theorem 3.4 shows that every metrizable convex space is bornological as a locally convex cone.

Proposition 3.6. *An inductive limit of bornological cones is a bornological cone.*

Proof. Let the locally convex cone $(\mathcal{P},\mathfrak{U})$ be the inductive limit of bornological cones $(\mathcal{P}_\gamma,\mathfrak{U}_\gamma)$ by linear mappings $f_\gamma:\mathcal{P}_\gamma\to\mathcal{P}$, for $\gamma\in\Gamma$. If T is a u-bounded linear mapping from $\mathcal{P}$ into other locally convex cone $(\mathcal{Q},\mathcal{W})$, then Tof_γ is a u-bounded linear mapping from $\mathcal{P}_\gamma$ into $\mathcal{Q}$, for each $\gamma\in\Gamma$, by the continuity of f_γ. Now, Tof_γ is continuous, since $(\mathcal{P}_\gamma,\mathfrak{U}_\gamma)$ is bornological for each $\gamma\in\Gamma$. Thus T is continuous by Proposition 3.3 from [5]. □

Corollary 3.7. *Every inductive limit of locally convex cones with a countable base is a bornological cone.*

Theorem 3.8. *Let $(\mathcal{P},\mathcal{W})$ be a locally convex cone. Then there is the finest convex quasiuniform structure $\mathfrak{U}$ on $\mathcal{P}$ under which $\mathcal{P}^2$ has the same u-bounded sets as under $\mathcal{W}$. Under the convex quasiuniform structure $\mathfrak{U}$, $\mathcal{P}$ is a bornological cone, the inductive limit of a family of uc-subcones of $\mathcal{P}$. The locally convex cone $(\mathcal{P},\mathcal{W})$ is bornological if and if $\mathfrak{U}$ and $\mathcal{W}$ are equivalent.*

Proof. Let $\mathfrak{B}$ be the collection of all u-bounded uniformly convex subsets of $\mathcal{P}^2$. For $U\in\mathfrak{B}$ we set

$$\mathcal{P}_U=\{a\in\mathcal{P}:\exists\lambda>0\,\text{s. t.}\,(0,a)\in\lambda U\}\quad\text{and}\quad\mathfrak{U}_U=\{\alpha U:\alpha>0\}.$$

It is easy to see that $(\mathcal{P}_U,\mathfrak{U}_U)$ is a uc-cone. We claim that $\mathcal{P}=\bigcup_{U\in\mathfrak{B}}\mathcal{P}_U$. For, if $a\in\mathcal{P}$, then $\{(0,a)\}$ is a u-bounded subset of $\mathcal{P}^2$. We set $U'=uch(\{(0,a)\})$. Then U' is a u-bounded uniformly convex subset of $\mathcal{P}^2$ and we have $a\in\mathcal{P}_{U'}$. We shall demonstrate that $(\mathcal{P},\mathfrak{U})$ is the inductive limit of locally convex cones $(\mathcal{P}_U,\mathfrak{U}_U)_{U\in\mathfrak{B}}$ by the inclusion mappings: $i_U:(\mathcal{P}_U,\mathfrak{U}_U)\to(\mathcal{P},\mathcal{W})$, $U\in\mathfrak{B}$. The u-boundedness of U implies that the inclusion mapping $i_U:(\mathcal{P}_U,\mathfrak{U}_U)\to(\mathcal{P},\mathcal{W})$ is continuous. This shows that $\mathfrak{U}$ is finer than $\mathcal{W}$, by the definition of inductive limit. Then u-boundedness in $\mathfrak{U}$ implies u-boundedness in $\mathcal{W}$. For the converse, suppose that $F\subset\mathcal{P}^2$ is u-bounded under $\mathcal{W}$. If $\tilde{F}=uch(F)$, then $\tilde{F}$ is u-bounded and $\tilde{F}\in\mathfrak{B}$. Also F is u-bounded in $(\mathcal{P}_{\tilde{F}},\mathfrak{U}_{\tilde{F}})$. Now, the continuity of the inclusion mapping $i_{\tilde{F}}:(\mathcal{P}_{\tilde{F}},\mathfrak{U}_{\tilde{F}})\to(\mathcal{P},\mathfrak{U})$ implies the u-boundedness of F in $(\mathcal{P},\mathfrak{U})$. The locally convex cone $(\mathcal{P},\mathfrak{U})$ is bornological by Proposition 3.6 and Corollary 3.5. Now, if $(\mathcal{P},\mathcal{W})$ is bornological, then the inclusion mapping $i:(\mathcal{P},\mathcal{W})\to(\mathcal{P},\mathfrak{U})$ is u-bounded and then it is continuous. This shows that $\mathfrak{U}$ and $\mathcal{W}$ are equivalent. □

Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone. In the following we denote the finest convex quasiuniform structure under which $\mathcal{P}^2$ has the same u-bounded sets as under $\mathfrak{U}$, by $\mathfrak{U}_\tau$. If $(\mathcal{P},\mathfrak{U})$ is bornological, then $\mathfrak{U}=\mathfrak{U}_\tau$. If $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{W})$ are locally convex cones, we shall call the linear operator $T:(\mathcal{P},\mathfrak{U})\to(\mathcal{Q},\mathcal{W})$, τ-continuous whenever $T:(\mathcal{P},\mathfrak{U}_\tau)\to(\mathcal{Q},\mathcal{W}_\tau)$ is continuous.

Theorem 3.9. *Let $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{W})$ be locally convex cones and $T:\mathcal{P}\to\mathcal{Q}$ be a linear operator. Then $T:(\mathcal{P},\mathfrak{U})\to(\mathcal{Q},\mathcal{W})$ is u-bounded if and only if $T:(\mathcal{P},\mathfrak{U}_\tau)\to(\mathcal{Q},\mathcal{W})$ is continuous.*

Proof. We suppose that $T:(\mathcal{P},\mathfrak{U})\to(\mathcal{Q},\mathcal{W})$ is u-bounded. Then $T:(\mathcal{P},\mathfrak{U}_\tau)\to(\mathcal{Q},\mathcal{W})$ is u-bounded, since $\mathcal{P}^2$ has the same u-bounded sets under $\mathfrak{U}$ and $\mathfrak{U}_\tau$. Now, $T:(\mathcal{P},\mathfrak{U}_\tau)\to(\mathcal{Q},\mathcal{W})$ is continuous because $(\mathcal{P},\mathfrak{U}_\tau)$ is bornological.

Conversely, if $T:(\mathcal{P},\mathfrak{U}_\tau)\to(\mathcal{Q},\mathcal{W})$ is continuous then it is u-bounded. This implies that $T:(\mathcal{P},\mathfrak{U})\to(\mathcal{Q},\mathcal{W})$ is u-bounded, since $\mathcal{P}^2$ has the same u-bounded sets under $\mathfrak{U}$ and $\mathfrak{U}_\tau$. □

We characterize bornological cones in the following theorem.

Theorem 3.10. *Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone. The following are equivalent:*

(a) *$(\mathcal{P},\mathfrak{U})$ is bornological.*

(b) *Every uniformly convex subset of $\mathcal{P}^2$ that absorbs all u-bounded subsets of $\mathcal{P}^2$ contains an element of $\mathfrak{U}$.*

(c) *Every u-bounded linear mapping of $(\mathcal{P},\mathfrak{U})$ into each uc-cone is continuous.*

(d) *$(\mathcal{P},\mathfrak{U})$ is an inductive limit of a family of uc- subcones of $\mathcal{P}$.*

Proof. The statements (a) and (d) are equivalent by Corollary 3.5 and Theorem 3.8.

$(a)\to(b)$ Suppose that (a) holds and $\tilde{v}$ is a uniformly convex subset of $\mathcal{P}^2$ that absorbs all u-bounded subsets of $\mathcal{P}^2$. We set $\mathcal{V}=\{\alpha\tilde{v}:\alpha>0\}$. It is easy to see that $(\mathcal{P},\mathcal{V})$ is a locally convex cone. Since $\tilde{v}$ absorbs all u-bounded subsets of $\mathcal{P}^2$, the identity mapping $i:(\mathcal{P},\mathfrak{U})\to(\mathcal{P},\mathcal{V})$ is a u-bounded linear operator. Now, (a) implies that i is continuous and therefore $\mathfrak{U}$ is finer than $\mathcal{V}$. This shows that $\tilde{v}\in\mathfrak{U}$. Thus (a) implies (b).

$(b)\to(a)$ Suppose that (b) holds and $T:(\mathcal{P},\mathfrak{U})\to(\mathcal{Q},\mathcal{W})$ is a u-bounded linear operator. We set $t=T\times T$. For every $W\in\mathcal{W}$, $t^{-1}(W)$ is a uniformly convex and absorbs all u-bounded subsets of $\mathcal{P}^2$. Indeed, $(T(a),T(a))\in W$ implies $(a,a)\in t^{-1}(W)$. If $(a,c)\in\lambda t^{-1}(W)$ and $(c,d)\in\mu t^{-1}(W)$, then $(T(a),T(c))\in$

λW and $(T(c),T(d)) \in \mu W$. This shows that $(T(a),T(d)) \in (\lambda+\mu)W$ and then $(a,c) \in (\lambda+\mu)t^{-1}(W)$. Also the u-boundedness of T implies that $t^{-1}(W)$ absorbs all u-bounded subsets of $\mathcal{P}^2$. Now (b) implies that $t^{-1}(W) \in \mathfrak{U}$. Thus T is continuous and $(\mathcal{P},\mathfrak{U})$ is bornological.

Clearly (a) implies (c).

$(c) \to (a)$ Suppose that (c) holds and $T : (\mathcal{P},\mathfrak{U}) \to (\mathcal{Q},\mathcal{W})$ is a u-bounded linear operator. For $W \in \mathcal{W}$, we set $\mathcal{W}_W = \{\alpha W : \alpha > 0\}$. Clearly for every $W \in \mathcal{W}$, $T : (\mathcal{P},\mathfrak{U}) \to (\mathcal{Q},\mathcal{W}_W)$ is u-bounded and then it is continuous by (c). If $W \in \mathcal{W}$, then $W \in \mathcal{W}_W$. Therefore, there is $U \in \mathfrak{U}$ such that $(T\times T)(U) \subseteq W$. □

As a special case in locally convex spaces, Theorem 3.10, (c) yields that a locally convex space E is bornological if and only if every u-bounded linear operator from E into a seminormed space is continuous.

Remark 3.11. Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone with (SP) and $\mathcal{P}^*$ be its dual cone. We set

$$X = \{U^\circ : U \in \mathfrak{U}\} \text{ and } V_{U^\circ} = \{(a,b) \in \mathcal{P}^2 : \mu(a) \le \mu(b)+1 \quad for\,all\ \mu \in U^\circ\}.$$

It is proved in [3], chapter II, that $\mathfrak{U}_X = \{V_{U^\circ} : U \in \mathfrak{U}\}$ is a convex quasiuniform structure on $\mathcal{P}$ and $(\mathcal{P},\mathfrak{U}_X)$ is a locally convex cone. If $(\mathcal{P},\mathfrak{U})$ has (SP), then the convex quasiuniform structures $\mathfrak{U}$ and $\mathfrak{U}_X$ are equivalent (see [3], chapter II).

A net $(x_i)_{i\in\mathcal{I}}$ in locally convex cone $(\mathcal{P},\mathfrak{U})$ is called symmetric Cauchy if for each $U \in \mathfrak{U}$ there is some $\gamma_U \in \mathcal{I}$ such that $(x_\beta,x_\alpha) \in U$ for all $\alpha,\beta \in \mathcal{I}$ with $\beta,\alpha \ge \gamma_U$. The locally convex cone $(\mathcal{P},\mathfrak{U})$ is called symmetric complete if every symmetric Cauchy net converges in the symmetric topology.

Proposition 3.12. *Let $(\mathcal{P},\mathcal{W})$ be a locally convex cone with (SP) and U be a u-bounded uniformly convex subset of $\mathcal{P}^2$ such that*

$$(CP) \quad \text{if } (a,b) \notin U, \text{ then there is } \mu \in \mathcal{P}^* \text{ such that } \mu(a) > \mu(b)+1 \text{ and } \mu(c) \le \mu(d)+1 \text{ for all } (c,d) \in U.$$

Then the completeness of $(\mathcal{P},\mathcal{W})$ with respect to the symmetric topology implies the completeness of $(\mathcal{P}_U,\mathfrak{U}_U)$ with respect to the symmetric topology.

Proof. Let $(a_i)_{i\in\mathcal{I}}$ be a Cauchy net in $(\mathcal{P}_U,\mathfrak{U}_U)$. Then $(a_i)_{i\in\mathcal{I}}$ is a Cauchy net in $(\mathcal{P},\mathcal{W})$, since the topology induced on $\mathcal{P}_U$ by the symmetric topology of $(\mathcal{P},\mathcal{W})$ is coarser than the symmetric topology of $(\mathcal{P}_U,\mathfrak{U}_U)$. Now, the completeness of $(\mathcal{P},\mathcal{W})$ yields that there is $a \in \mathcal{P}$ such that $a_i \to a$ with respect to the symmetric topology of $(\mathcal{P},\mathcal{W})$. We show that $a \in \mathcal{P}_U$ and $a_i \to a$ with respect to the symmetric topology of $(\mathcal{P}_U,\mathfrak{U}_U)$. For $\varepsilon > 0$ there exists $i_0 \in \mathcal{I}$ such that $(a_i,a_j) \in \varepsilon U$ for all $i,j \geq i_0$. We claim that $(a_i,a) \in \varepsilon U$ and $(a,a_i) \in \varepsilon U$ for all $i \geq i_0$. Otherwise, there is $\mu \in \mathcal{P}^*$ such that $\mu(a_i) > \mu(a)+\varepsilon$ or there is $\mu_1 \in \mathcal{P}^*$ such that $\mu_1(a) > \mu_1(a_i)+\varepsilon$. This is a contradiction by Remark 3.11. Indeed, there is $W_1 \in \mathcal{W}$ such that $\mu,\mu_1 \in W_1^\circ$. Let $X = \{W^\circ : W \in \mathcal{W}\}$. Then the convex quasiuniform structures $\mathcal{W}$ and $\mathcal{W}_X$ are equivalent, since $(\mathcal{P},\mathcal{W})$ has (SP). This shows that $a_i \to a$ with respect to the symmetric topology of $(\mathcal{P},\mathcal{W}_X)$. Then there is i_1 such that for $i \geq i_1$, $(a_i,a) \in \varepsilon U_{W_1^\circ}$ and $(a,a_i) \in \varepsilon U_{W_1^\circ}$. This implies that $\mu(a_i) \leq \mu(a)+\varepsilon$ and $\mu_1(a) \leq \mu_1(a_i)+\varepsilon$ for $i \geq \max\{i_1,i_0\}$.

Also, there is $\lambda > 0$ such that $(0,a_{i_0}) \in \lambda U$, since $a_{i_0} \in \mathcal{P}_U$. On the other hand $(a_{i_0},a) \in \varepsilon U$. Since U is uniformly convex, we have $(0,a) \in (\lambda+\varepsilon)U$ and then $a \in \mathcal{P}_U$. □

Lemma 3.13. *In every locally convex cone with (SP), every uniformly convex u-bounded subset of $\mathcal{P}^2$ is contained in a uniformly convex u-bounded subset of $\mathcal{P}^2$ which has property (CP).*

Proof. Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone with (SP) and B be a uniformly convex u-bounded subset of $\mathcal{P}^2$. We set $\tilde{B} = \{(a,b) \in \mathcal{P}^2 : \mu(a) \leq \mu(b)+1,\ \forall \mu \in B^\circ\}$. Obviously, $\tilde{B}$ is uniformly convex and we have $B \subseteq \tilde{B}$. We prove that $\tilde{B}$ is u-bounded. Let $U \in \mathfrak{U}$. There is $\lambda > 1$ such that $B \subseteq \lambda U$, since B is u-bounded. This shows that $U^\circ \subseteq \lambda B^\circ$. We claim that $\tilde{B} \subseteq \lambda U$. If $(a,b) \notin \lambda U$ then there is $\mu \in U^\circ \subseteq \lambda B^\circ$ such that $\mu(a) > \mu(b)+1$, since $(\mathcal{P},\mathfrak{U})$ has (SP). This shows that $(a,b) \notin \tilde{B}$. □

Every complete Hausdorff bornological space is the inductive limit of Banach spaces. In locally convex cones we have:

Theorem 3.14. *Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone such that:*
(a) $\mathcal{P}$ is complete with respect to the symmetric topology,
(b) $(\mathcal{P},\mathfrak{U})$ has (SP).
Then $(\mathcal{P},\mathfrak{U}_\tau)$ is the inductive limit of a family of uc- subcones which are complete with respect to their symmetric topologies.

Proof. Let $\mathfrak{B}$ be the collection of all u-bounded uniformly convex subsets of $\mathcal{P}^2$ which have (CP) (the collection $\mathfrak{B}$ is not empty by Lemma 3.13). We can prove that $(\mathcal{P},\mathfrak{U}_\tau)$ is the inductive limit of uc-cones $(\mathcal{P}_U,\mathfrak{U}_U)_{U\in\mathfrak{B}}$ in a similar way to the proof of Theorem 3.8. Since $(\mathcal{P},\mathfrak{U})$ is complete with respect to the symmetric topology, $(\mathcal{P}_U,\mathfrak{U}_U)$ is complete with respect to the symmetric topology for all $U \in \mathfrak{B}$, by Proposition 3.12. □

Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone and $\mathcal{P}^*$ be whose dual cone. We suppose that $\mathfrak{U}_\sigma(\mathcal{P},\mathcal{P}^*)$ is the coarsest convex quasiuniform structure on $\mathcal{P}$ that makes all $\mu \in \mathcal{P}^*$ continuous. The finite intersections of the sets $(\mu\times\mu)^{-1}(\tilde{\varepsilon})$ where

$\mu \in \mathcal{P}^*$, $\varepsilon > 0$ and $\tilde{\varepsilon} = \{(a,b) \in \overline{\mathbb{R}}^2 : a \leq b + \varepsilon\}$, form a base for $\mathfrak{U}_\sigma(\mathcal{P},\mathcal{P}^*)$. We call $\mathfrak{U}_\sigma(\mathcal{P},\mathcal{P}^*)$ the *weak convex quasiuniform structure* on $\mathcal{P}$. It is easy to see that $(\mathcal{P},\mathfrak{U}_\sigma(\mathcal{P},\mathcal{P}^*))$ is a locally convex cone. We shall say that $F \subset \mathcal{P}^2$ is *weakly u-bounded*, if it is u-bounded in locally convex cone $(\mathcal{P},\mathfrak{U}_\sigma(\mathcal{P},\mathcal{P}^*))$. The operator $T : (\mathcal{P},\mathfrak{U}) \to (\mathcal{Q},\mathcal{W})$ is called *weakly u-bounded* if $(T \times T)(F)$ is weakly u-bounded for every weakly u-bounded subset F of $\mathcal{P}^2$ and it is called *weakly continuous or (σ-continuous)* whenever it is continuous with respect the weak convex quasiuniform structures on $\mathcal{P}$ and $\mathcal{Q}$. The linear operator T : $(\mathcal{P},\mathfrak{U}) \to (\mathcal{Q},\mathcal{W})$ is called *τ-continuous* whenever $T : (\mathcal{P},\mathfrak{U}_\tau) \to (\mathcal{Q},\mathcal{W}_\tau)$ is continuous.

Proposition 3.15. *Let $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{W})$ be locally convex cones and let T be a continuous linear operator from $(\mathcal{P},\mathfrak{U})$ into $(\mathcal{Q},\mathcal{W})$. Then T is σ and τ-continuous.*

Proof. Let $W_\sigma \in \mathcal{W}_\sigma$. There are $n \in \mathbb{N}$ and $\mu_1,\dots,\mu_n \in \mathcal{Q}^*$ such that

$$\bigcap_{i=1}^{n} \Lambda_i^{-1}(\tilde{1}) \subseteq W_\sigma,$$

where $\Lambda_i = \mu_i \times \mu_i$. We have $\mu_i \circ T \in \mathcal{P}^*$ for all $i \in \{1,\dots,n\}$, since T is continuous. We set $\Gamma_i = \mu_i \circ T \times \mu_i \circ T$. Then $U_\sigma = \bigcap_{i=1}^{n} \Gamma_i^{-1}(\tilde{1}) \in \mathfrak{U}_\sigma$ and $(T \times T)(U_\sigma) \subseteq \bigcap_{i=1}^{n} \Lambda_i^{-1}(\tilde{1}) \subseteq W_\sigma$. Then T is σ-continuous.

The linear operator $T : (\mathcal{P},\mathfrak{U}) \to (\mathcal{Q},\mathcal{W})$ is u-bounded, since it is continuous. Then $T : (\mathcal{P},\mathfrak{U}_\tau) \to (\mathcal{Q},\mathcal{W}_\tau)$ is u-bounded, since $\mathcal{P}^2$ has the same u-bounded sets under $\mathfrak{U}$ and $\mathfrak{U}_\tau$, also $\mathcal{Q}^2$ has the same u-bounded sets under $\mathcal{W}$ and $\mathcal{W}_\tau$ by Theorem 3.8. Now, since $(\mathcal{P},\mathfrak{U}_\tau)$ is a bornological cone, $T : (\mathcal{P},\mathfrak{U}_\tau) \to (\mathcal{Q},\mathcal{W}_\tau)$ is continuous. □

Proposition 3.16. *Let $(\mathcal{Q},\mathcal{W})$ be a locally convex cone such that $\mathcal{Q}^2$ has the same u-bounded subsets under $\mathcal{W}$ and $\mathcal{W}_\sigma(\mathcal{Q},\mathcal{Q}^*)$. If $(\mathcal{P},\mathfrak{U})$ is a bornological cone and $T : \mathcal{P} \to \mathcal{W}$ is weakly continuous, then it is continuous.*

Proof. Since $(\mathcal{P},\mathfrak{U})$ is a bornological cone, it is enough to show that T is a u-bounded operator. Let $F \subseteq \mathcal{P}^2$ be u-bounded. Then F is weakly u-bounded. The weak continuity of T yields that $(T \times T)(F)$ is weakly u-bounded and then u-bounded by the hypothesis. This shows that T is u-bounded. □

Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone and $\mathfrak{B}$ be a collection of u-bounded subsets of $\mathcal{P}^2$. For a u-bounded subset B of $\mathcal{P}^2$, we set $B^\circ = \{\mu \in \mathcal{P}^* : \mu(a) \leq \mu(b) + 1, \forall (a,b) \in B\}$ and $\tilde{B^\circ} = \{(\mu,\nu) \in \mathcal{P}^* \times \mathcal{P}^* : \nu \in \mu + B^\circ\}$. We show that $\tilde{B^\circ}$ is uniformly convex. The convexity of $\tilde{B^\circ}$ is obvious. Clearly we have $0 \in B^\circ$. This yields $\mu \in \mu + B^\circ$ for all $\mu \in \mathcal{P}^*$. Thus $\triangle \subseteq \tilde{B^\circ}$. Let $(\mu_1,\mu_3) \in (\lambda\tilde{B^\circ}) \circ$

$(\rho\tilde{B^\circ})$. Then there is $\mu_2 \in \mathcal{P}^*$ such that $(\mu_1,\mu_2) \in \lambda\tilde{B^\circ}$ and $(\mu_2,\mu_3) \in \rho\tilde{B^\circ}$. This implies $\mu_2 \in \mu_1 + \lambda B^\circ$ and $\mu_3 \in \mu_2 + \rho B^\circ$. Since $\tilde{B^\circ}$ is convex, this implies $\mu_3 \in \mu_1 + (\lambda+\rho)B^\circ$. Thus $(\mu_1,\mu_3) \in (\lambda+\rho)\tilde{B^\circ}$.

Suppose $\tilde{\mathfrak{B}} = \{\tilde{B^\circ} : B \in \mathfrak{B}\}$. Then there is the coarsest convex quasiuniform structure $\mathfrak{U}_{\mathfrak{B}}$ on $\mathcal{P}^*$ such that $\tilde{\mathfrak{B}} \subset \mathfrak{U}_{\mathfrak{B}}$ by Proposition 2.2. Let $\mu \in \mathcal{P}^*$ and $B \in \mathfrak{B}$ be arbitrary. There is $U \in \mathfrak{U}$ such that $\mu \in U^\circ$. Since B is u-bounded, there is $\lambda > 0$ such that $B \subseteq \lambda U$. This implies $U^\circ \subseteq \lambda B^\circ$. Thus $\mu \in \lambda B^\circ$ and then $(0,\mu) \in \lambda\tilde{B^\circ}$. Now Proposition 2.2 yields that $(\mathcal{P}^*,\mathfrak{U}_{\mathfrak{B}})$ is a locally convex cone. If $\mathfrak{B}$ is the collection of all u-bounded subsets of $\mathcal{P}^2$, then we denote the corresponding convex quasiuniform structure by $\mathfrak{U}_\beta(\mathcal{P}^*,\mathcal{P})$.

Proposition 3.17. *If $(\mathcal{P},\mathfrak{U})$ is a uc-cone, then $(\mathcal{P}^*,\mathfrak{U}_\beta(\mathcal{P}^*,\mathcal{P}))$ is a uc-cone*

Proof. Let $\mathfrak{U} = \{\alpha U : \alpha > 0\}$. We prove that $\mathfrak{U}_\beta(\mathcal{P}^*,\mathcal{P}) = \{\alpha\tilde{U}^\circ : \alpha > 0\}$. If B is a u-bounded subset of $\mathcal{P}^2$, then there is $\lambda > 0$ such that $B \subseteq \lambda U$. Thus $\frac{1}{\lambda}\tilde{U}^\circ \subseteq \tilde{B}^\circ$. This implies that $\{\alpha\tilde{U}^\circ : \alpha > 0\}$ is finer than $\mathfrak{U}_\beta(\mathcal{P}^*,\mathcal{P})$. Therefore $\mathfrak{U}_\beta(\mathcal{P}^*,\mathcal{P}) = \{\alpha\tilde{U}^\circ : \alpha > 0\}$. □

Let $\mathcal{P}$ be a (Hausdorff) locally convex space and $\mathcal{V}$ be a base of convex, balanced and closed neighborhoods. For $V \in \mathcal{V}$, suppose $\tilde{V} = \{(a,b) \in \mathcal{P}^2 : a-b \in V\}$. We set $\tilde{\mathcal{V}} = \{\tilde{V} : V \in \mathcal{V}\}$. Then $(\mathcal{P},\tilde{\mathcal{V}})$ is a locally convex cone. If $\mathcal{V} = \{\alpha V : \alpha > 0\}$ for some $V \in \mathcal{V}$, then the symmetric topology induced on $\mathcal{P}^*$ by $\mathfrak{U}_\beta(\mathcal{P}^*,\mathcal{P})$ is (normable) seminormable by Proposition 3.17 and Remark 2.9.

Proposition 3.18. *Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone. Then for every $U \in \mathfrak{U}$, U° is bounded below in $(\mathcal{P}^*,\mathfrak{U}_\beta(\mathcal{P}^*,\mathcal{P}))$.*

Proof. Let B be a u-bounded subset of $\mathcal{P}^2$ and $U \in \mathfrak{U}$. Then $B \subseteq \lambda U$ for some $\lambda > 0$. This implies that $U^\circ \subseteq \lambda B^\circ$, hence $\mu \in \lambda B^\circ$ for all $\mu \in U^\circ$. This yields $(0,\mu) \in \lambda\tilde{B}^\circ$ for all $\mu \in U^\circ$. Thus U is bounded below in $(\mathcal{P}^*,\mathfrak{U}_\beta(\mathcal{P}^*,\mathcal{P}))$. □

Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone and $\mathfrak{B}_*$ be the collection of all u-bounded subsets of $\mathcal{P}^* \times \mathcal{P}^*$ under $\mathfrak{U}_\beta(\mathcal{P}^*,\mathcal{P})$. For $B_* \in \mathcal{B}_*$, we set

$$B_*^\circ = \{a \in \mathcal{P} : \mu(a) \leq \nu(a) + 1, \forall (\mu,\nu) \in B_*\}.$$

Then $\tilde{\mathfrak{B}}_* = \{\tilde{B_*^\circ} : B_* \in \mathfrak{B}_*\}$ is a collection of uniformly convex subsets of $\mathcal{P}^2$, where $\tilde{B_*^\circ} = \{(a,b) \in \mathcal{P}^2 : b \in a + B_*^\circ\}$.

Lemma 3.19. *Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone and A be a subset of $\mathcal{P}$. If $B = \{0\} \times A$, then*

(i) $B \subseteq \tilde{E}^\circ$, *where* $E = \tilde{B}^\circ$ ($B \subseteq \tilde{\tilde{B}}^{\circ\circ}$).
(ii) $\tilde{\tilde{B}}^{\circ\circ}$ *is uniformly convex,*
(iii) $\tilde{\tilde{B}}^{\circ\circ} \subseteq U$ *for every subset* U *of* $\mathcal{P}^2$ *that has* (CP) *and contains* B.

Proof. For (i), let $(0,a) \in B$ and $(\mu,\nu) \in E$ be arbitrary. Then $\nu \in \mu + B^\circ$. This shows that there is $\psi \in B^\circ$ such that $\nu = \mu + \psi$. Obviously, we have $\psi(a) \geq -1$. This implies that $\mu(a) \leq \nu(a) + 1$. Therefore $a \in E^\circ$. This yields that $(0,a) \in \tilde{E}^\circ$.

(ii) is obvious.

For (iii), let U be a subset of $\mathcal{P}^2$ that has (CP) and contains B and let $(a,b) \notin U$. Then there is $\mu \in \mathcal{P}^*$ such that $\mu(a) > \mu(b) + 1$ and $\mu(c) \leq \mu(d) + 1$ for all $(c,d) \in U$. Clearly we have $\mu \in U^\circ \subset B^\circ$. This implies $(0,\mu) \in \tilde{B}^\circ$. If $(a,b) \in \tilde{\tilde{B}}^{\circ\circ}$, then $b = a + t$ for some $t \in E^\circ$, where $E = \tilde{B}^\circ$. Now, since $(0,\mu) \in \tilde{B}^\circ$ we have $\mu(t) \geq -1$. This implies that $\mu(a) \leq \mu(b) + 1$. This contradiction proves our claim. □

Corollary 3.20. *Let* $(\mathcal{P},\mathfrak{U})$ *be a locally convex cone. If* $U = uch(\{0\} \times A)$ *for some subset* A *of* $\mathcal{P}$ *and* U *has* (CP), *then* $U = \tilde{\tilde{U}}^{\circ\circ}$.

Let $a \in \mathcal{P}$, $B_* \in \mathfrak{B}_*$ and $B = \{0\} \times \{a\}$. We have $\tilde{B}^\circ \in \mathfrak{U}_\beta(\mathcal{P}^*,\mathcal{P})$, since B is u-bounded in $\mathcal{P}^2$. Now, there is $\lambda > 0$ such that $B_* \subseteq \lambda \tilde{B}^\circ$. This implies that $B \subseteq \tilde{\tilde{B}}^{\circ\circ} \subseteq \lambda \tilde{B}_*^\circ$ by Lemma 3.19. Therefore, there is the coarsest convex quasiuniform structure $\mathfrak{U}_\beta(\mathcal{P},\mathcal{P}^*)$ on $\mathcal{P}$ such that $\tilde{\mathfrak{B}}_* \subseteq \mathfrak{U}_\beta(\mathcal{P},\mathcal{P}^*)$ and $(\mathcal{P},\mathfrak{U}_\beta(\mathcal{P},\mathcal{P}^*))$ is a locally convex cone by Proposition 2.2.

Example 3.21. We consider the locally convex cone $(\overline{\mathbb{R}},\tilde{\mathcal{V}})$, where $\tilde{\mathcal{V}} = \{\varepsilon\tilde{1} : \varepsilon > 0\}$ and $\tilde{1} = \{(a,b) \in \overline{\mathbb{R}}^2 : a \leq b+1\}$. The dual cone of $(\overline{\mathbb{R}},\tilde{\mathcal{V}})$ consists of all nonnegative reals and the functional $\bar{0}$ acting as

$$\bar{0}(a) = \begin{cases} +\infty, & \text{if } a = +\infty \\ 0 & \text{else.} \end{cases}$$

We have $\tilde{\mathcal{V}}_\beta(\overline{\mathbb{R}}^*,\overline{\mathbb{R}}) = \{\varepsilon\tilde{\tilde{1}}^\circ : \varepsilon > 0\}$, where $\tilde{\tilde{1}}^\circ = \{(\alpha,\beta) \in (\overline{\mathbb{R}}^*)^2 : \beta \in \alpha + (\tilde{1})^\circ\}$. The subset $U = \tilde{1}$ of $\overline{\mathbb{R}}^2$ is uniformly convex and it has (CP). Then we have $U = \tilde{\tilde{U}}^{\circ\circ}$ by Corollary 3.20. This shows that the convex quasiuniform structures $\tilde{\mathcal{V}}_\beta(\overline{\mathbb{R}},\overline{\mathbb{R}}^*)$ and $\tilde{\mathcal{V}}$ are equivalent on $\overline{\mathbb{R}}$.

Proposition 3.22. *Let* $(\mathcal{P},\mathfrak{U})$ *be a locally convex cone with* (SP). *Then the convex quasiuniform structure* $\mathfrak{U}_\beta(\mathcal{P},\mathcal{P}^*)$ *is finer than* $\mathfrak{U}$ *on* $\mathcal{P}$. *A subset* A *of* $\mathcal{P}$ *is bounded below in* $(\mathcal{P},\mathfrak{U})$ *if and only if it is bounded below in* $(\mathcal{P},\mathfrak{U}_\beta(\mathcal{P},\mathcal{P}^*))$.

Proof. Let $U \in \mathfrak{U}$. Then $\{0\} \times U^\circ \in \mathfrak{B}_*$ by Proposition 3.18. Let $B_* = \{0\} \times U^\circ$ and $(a,b) \in \tilde{B}_*^\circ$. Then $b \in a + B_*^\circ$. This shows that $b = a + c$ for some $c \in B_*^\circ$. Since $c \in B_*^\circ$, then $0 \le \mu(c) + 1$ for all $\mu \in U^\circ$, hence $\mu(a) \le \mu(b) + 1$ for all $\mu \in U^\circ$. Now, if $(a,b) \notin 2U$ then there is $\mu \in U^\circ$ such that $\mu(a) > \mu(b) + 1$ by (SP). This shows that $(a,b) \notin \tilde{B}_*^\circ$. Thus $\tilde{B}_*^\circ \subseteq 2U$.

Let A be bounded below in $(\mathcal{P}, \mathfrak{U})$. Then $B = \{0\} \times A$ is u-bounded. If B_* is a u-bounded subset of $\mathcal{P}^* \times \mathcal{P}^*$ under $\mathfrak{U}_\beta(\mathcal{P}^*, \mathcal{P})$, then there is $\lambda > 0$ such that $B_* \subseteq \lambda \tilde{B}^\circ$. This shows that $\tilde{\tilde{B}}^{\circ\circ} \subseteq \lambda B_*^\circ$. Now we have $B \subseteq \tilde{\tilde{B}}^{\circ\circ} \subseteq \lambda \tilde{B}_*^\circ$ by Lemma 3.19. Thus B is u-bounded in $(\mathcal{P}, \mathfrak{U}_\beta(\mathcal{P}, \mathcal{P}^*))$ (the sets $\tilde{B}_*^\circ$ form a base for $\mathfrak{U}_\beta(\mathcal{P}, \mathcal{P}^*)$). Therefore A is bounded below in $(\mathcal{P}, \mathfrak{U}_\beta(\mathcal{P}, \mathcal{P}^*)$. The converse is obvious. □

Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones. The linear operator $T : \mathcal{P} \to \mathcal{Q}$ is called *bounded below* whenever T maps bounded below subsets of $\mathcal{P}$ into bounded below subsets of $\mathcal{Q}$. The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is called *b-bornological* whenever every bounded below linear operator from $(\mathcal{P}, \mathfrak{U})$ into other locally convex cone is continuous.

We prove that every b-bornological cone is bornological. Let $(\mathcal{P}, \mathfrak{U})$ be a b-bornological cone and T be a u-bounded linear operator from $(\mathcal{P}, \mathfrak{U})$ into another locally convex cone $(\mathcal{Q}, \mathcal{W})$. If B is a bounded below subset of $(\mathcal{P}, \mathfrak{U})$ then $\{0\} \times B$ is u-bounded and therefore $\{0\} \times T(B) = (T \times T)(\{0\} \times B)$ is u-bounded. This shows that $T(B)$ is bounded below in $(\mathcal{Q}, \mathcal{W})$. Thus T is bounded below. Now, T is continuous, since $(\mathcal{P}, \mathfrak{U})$ is b-bornological. We note that every locally convex bornological real vector space is both b-bornological and bornological as a locally convex cone. It is easy to see that an inductive limit of b-bornological cones is b-bornological.

Theorem 3.23. *Let $(\mathcal{P}, \mathfrak{U})$ be a b-bornological locally convex cone with (SP). Then $\mathfrak{U}$ and $\mathfrak{U}_\beta(\mathcal{P}, \mathcal{P}^*)$ are equivalent.*

Proof. The identity mapping $i : (\mathcal{P}, \mathfrak{U}) \to (\mathcal{P}, \mathfrak{U}_\beta(\mathcal{P}, \mathcal{P}^*))$ is bounded below. Since $(\mathcal{P}, \mathfrak{U})$ is a b-bornological, i is continuous. This implies that $\mathfrak{U}$ and $\mathfrak{U}_\beta(\mathcal{P}, \mathcal{P}^*)$ are equivalent by Proposition 3.22. □

Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone. We say that the subset A^* of $\mathcal{P}^*$ is equicontinuous if there is $U \in \mathfrak{U}$ such that $A^* \subset U^\circ$. Proposition 3.18 yields that every equicontinuous subset of $\mathcal{P}^*$ is bounded below in $(\mathcal{P}^*, \mathfrak{U}_\beta(\mathcal{P}^*, \mathcal{P}))$. The converse is not true in general. However, we have:

Theorem 3.24. *Let $(\mathcal{P}, \mathfrak{U})$ be a b-bornological locally convex cone with (SP). Then every bounded below subset of $(\mathcal{P}^*, \mathfrak{U}_\beta(\mathcal{P}^*, \mathcal{P}))$ is equicontinuous*

Proof. Let A^* be a bounded below subset of $(\mathcal{P}^*, \mathfrak{U}_\beta(\mathcal{P}^*, \mathcal{P}))$ and $B_* = \{0\} \times A^*$. Then $U = \tilde{B}_*^\circ \in \mathfrak{U}$ by Theorem 3.23. Now, we have $B_* \subseteq U^\circ$ by Lemma 3.19. □

Example 3.25. Let $\mathcal{P}$ be a cone and A be a subset of $\mathcal{P}$ such that for every $a \in \mathcal{P}$ there is $\lambda \geq 0$ such that $a \in \lambda A$. We set $U = uch(\{0\} \times A)$ and $\mathfrak{U} = \{\alpha U : \alpha > 0\}$. Then $(\mathcal{P}, \mathfrak{U})$ is a *uc*-cone. We claim that the *uc*-cone $(\mathcal{P}, \mathfrak{U})$ is *b*-bornological. Indeed, let T be a bounded below linear operator from $(\mathcal{P}, \mathfrak{U})$ into $(\mathcal{Q}, \mathcal{W})$. If T is not continuous, then there is $W \in \mathcal{W}$ such that $T(U) \nsubseteq \alpha W$ for all $\alpha > 0$. Thus $T(\{0\} \times A) \nsubseteq \alpha W$ for all $\alpha > 0$. This shows that $T(A)$ is not bounded below. Then T is not bounded below. This contradiction proves our claim. Let $\mathcal{P} = \overline{\mathbb{R}}$, $U = uch(\{0\} \times \{-1, 1, +\infty\})$ and $\mathfrak{U} = \{\alpha U : \alpha > 0\}$. The *uc*-cone $(\overline{\mathbb{R}}, \mathfrak{U})$ is *b*-bornological.

Theorem 3.26. *Let $(\mathcal{P}, \mathcal{W})$ be a locally convex cone. Then there is a finest convex quasiuniform structure $\mathfrak{U}$ on $\mathcal{P}$ under which $\mathcal{P}$ has the same bounded below subsets as under $\mathcal{W}$. Under the convex quasiuniform structure $\mathfrak{U}$, $\mathcal{P}$ is a b-bornological cone, the inductive limit of a family of uc-subcones of $\mathcal{P}$. The locally convex cone $(\mathcal{P}, \mathcal{W})$ is b-bornological if and if $\mathfrak{U}$ and $\mathcal{W}$ are equivalent.*

Proof. Let $\mathfrak{B}$ be the collection of all bounded below subsets of $(\mathcal{P}, \mathcal{W})$. For $B \in \mathfrak{B}$ we set $\tilde{B} = uch(\{0\} \times B)$ and

$$\mathcal{P}_{\tilde{B}} = \{a \in \mathcal{P} : \exists \lambda > 0 \,\text{such that}\, (0, a) \in \lambda \tilde{B}\} \quad \text{and} \quad \mathfrak{U}_{\tilde{B}} = \{\alpha \tilde{B} : \alpha > 0\}.$$

It is easy to see that $(\mathcal{P}_{\tilde{B}}, \mathfrak{U}_{\tilde{B}})$ is a *uc*-cone for each $B \in \mathfrak{B}$. The *uc*-cone $(\mathcal{P}_{\tilde{B}}, \mathfrak{U}_{\tilde{B}})$ is *b*-bornological by Example 3.25. We claim that $\mathcal{P} = \bigcup_{B \in \mathfrak{B}} \mathcal{P}_{\tilde{B}}$. For, if $a \in \mathcal{P}$, then $A' = \{a\}$ is a bounded below subset of $\mathcal{P}$ and we have $a \in \mathcal{P}_{\tilde{A'}}$. We suppose that $(\mathcal{P}, \mathfrak{U})$ is the inductive limit of locally convex cones $(\mathcal{P}_{\tilde{B}}, \mathfrak{U}_{\tilde{B}})_{B \in \mathfrak{B}}$ by the inclusion mappings: $i_B : (\mathcal{P}_{\tilde{B}}, \mathfrak{U}_{\tilde{B}}) \to (\mathcal{P}, \mathcal{W})$, $B \in \mathfrak{B}$. The *u*-boundedness of $\tilde{B}$ implies that the inclusion mapping $i_B : (\mathcal{P}_{\tilde{B}}, \mathfrak{U}_{\tilde{B}}) \to (\mathcal{P}, \mathcal{W})$ is continuous. This shows that $\mathfrak{U}$ is finer than $\mathcal{W}$, by the definition of inductive limit. Then the bounded below subsets of $\mathcal{P}$ under $\mathfrak{U}$ are bounded below under $\mathcal{W}$. For the converse, suppose that $A \subset \mathcal{P}$ is bounded below under $\mathcal{W}$. Then A is bounded below in $(\mathcal{P}_{\tilde{A}}, \mathfrak{U}_{\tilde{A}})$. Now, the continuity of the inclusion mapping $i_A : (\mathcal{P}_{\tilde{A}}, \mathfrak{U}_{\tilde{A}}) \to (\mathcal{P}, \mathfrak{U})$ implies that A is bounded below in $(\mathcal{P}, \mathfrak{U})$.

The locally convex cone $(\mathcal{P}, \mathfrak{U})$ is *b*-bornological, since it is the inductive limit of *b*-bornological cones. Now if $(\mathcal{P}, \mathcal{W})$ is *b*-bornological, then the inclusion mapping $i : (\mathcal{P}, \mathcal{W}) \to (\mathcal{P}, \mathfrak{U})$ is bounded below and then it is continuous. This shows that $\mathfrak{U}$ and $\mathcal{W}$ are equivalent. □

Let $(\mathcal{P}, \mathfrak{U})$ be a locally convex cone. In the following we denote the finest convex quasiuniform structure under which $\mathcal{P}$ has the same bounded below subsets as under $\mathfrak{U}$, by $\mathfrak{U}_{b\tau}$. If $(\mathcal{P},\mathfrak{U})$ is b-bornological, then $\mathfrak{U}$ and $\mathfrak{U}_{b\tau}$ are equivalent. If $(\mathcal{P},\mathfrak{U})$ and $(\mathcal{Q},\mathcal{W})$ are locally convex cones, we shall call the linear operator $T:(\mathcal{P},\mathfrak{U})\to(\mathcal{Q},\mathcal{W})$, $b\tau$-continuous whenever $T:(\mathcal{P},\mathfrak{U}_{b\tau})\to(\mathcal{Q},\mathcal{W}_{b\tau})$ is continuous. In a similar way to Theorem 3.9, we can prove that $T:(\mathcal{P},\mathfrak{U})\to(\mathcal{Q},\mathcal{W})$ is bounded below if and only if $T:(\mathcal{P},\mathfrak{U}_{b\tau})\to(\mathcal{Q},\mathcal{W})$ is continuous.

Now, we reformulate Theorem 3.10 for b-bornological cones.

Theorem 3.27. *Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone. The following are equivalent:*

- (a) *$(\mathcal{P},\mathfrak{U})$ is b-bornological.*
- (b) *If V is a uniformly convex subset of $\mathcal{P}^2$ such that for every bounded below subset A of $\mathcal{P}$ there is $\lambda>0$ such that $\{0\}\times A\subseteq\lambda V$, then there is $U\in\mathfrak{U}$ such that $U\subseteq V$.*
- (c) *Every bounded below linear mapping of $(\mathcal{P},\mathfrak{U})$ into each uc-cone is continuous.*
- (d) *$(\mathcal{P},\mathfrak{U})$ is an inductive limit of uc- subcones $(\mathcal{P}_{\tilde{B}},\mathfrak{U}_{\tilde{B}})_{B\in\mathfrak{B}}$ of $\mathcal{P}$, where $\mathfrak{B}$ is the collection of all bounded below subsets of $(\mathcal{P},\mathfrak{U})$, $\tilde{B}=uch(\{0\}\times B)$, $\mathcal{P}_{\tilde{B}}=\{a\in\mathcal{P}:\exists\lambda>0\, s.t.\,(0,a)\in\lambda\tilde{B}\}$ and $\mathfrak{U}_{\tilde{B}}=\{\alpha\tilde{B}:\alpha>0\}$.*

Proof. The statements (a) and (d) are equivalent by Theorem 3.26.

$(a)\to(b)$ Suppose that (a) holds and V is a uniformly convex subset of $\mathcal{P}^2$ such that for every bounded below subset A of $\mathcal{P}$ there is $\lambda>0$ such that $\{0\}\times A\subseteq\lambda V$. We set $\mathcal{V}=\{\alpha V:\alpha>0\}$. It is easy to see that $(\mathcal{P},\mathcal{V})$ is a locally convex cone. Since for every bounded below subset A of $(\mathcal{P},\mathfrak{U})$, $\{0\}\times A$ is absorbed by V, the identity mapping $i:(\mathcal{P},\mathfrak{U})\to(\mathcal{P},\mathcal{V})$ is a bounded below linear operator. Now, (a) implies that i is continuous and then $\mathfrak{U}$ is finer than $\mathcal{V}$. This shows that $V\in\mathfrak{U}$. Thus (a) implies (b).

$(b)\to(a)$ Suppose that (b) holds and $T:(\mathcal{P},\mathfrak{U})\to(\mathcal{Q},\mathcal{W})$ is a bounded below linear operator. We set $t=T\times T$. For every $W\in\mathcal{W}$, $t^{-1}(W)$ is uniformly convex and absorbs all subsets $\{0\}\times A$ of $\mathcal{P}^2$, where A is bounded below in $(\mathcal{P},\mathfrak{U})$. Then $t^{-1}(W)\in\mathfrak{U}$ by (b). Now (b) implies that $t^{-1}(W)\in\mathfrak{U}$. Thus T is continuous and $(\mathcal{P},\mathfrak{U})$ is b-bornological.

Clearly (a) implies (c).

$(c)\to(a)$ Suppose that (c) holds and $T:(\mathcal{P},\mathfrak{U})\to(\mathcal{Q},\mathcal{W})$ is a bounded below linear operator. For $W\in\mathcal{W}$, we set $\mathcal{W}_W=\{\alpha W:\alpha>0\}$. Clearly for every $W\in\mathcal{W}$, $T:(\mathcal{P},\mathfrak{U})\to(\mathcal{Q},\mathcal{W}_W)$ is bounded below and then it is continuous by (c). If $W\in\mathcal{W}$, then $W\in\mathcal{W}_W$. Therefore, there is $U\in\mathfrak{U}$ such that $(T\times T)(U)\subseteq W$. □

Let $(\mathcal{P},\mathfrak{U})$ be a locally convex cone. The cone $\mathcal{P}$ has the same bounded below subsets under $\mathfrak{U}$ and $\mathfrak{U}_\tau$. This shows that $\mathfrak{U}_\tau \subseteq \mathfrak{U}_{b\tau}$, since $\mathfrak{U}_{b\tau}$ is the finest convex quasiuniform structure under which $\mathcal{P}$ has the same bounded below subsets as under $\mathfrak{U}$.

Suppose that $(\mathcal{P},\mathfrak{U})$ is a locally convex cone and $\mathcal{P}^*$ is its dual. We investigate the behavior of the convex quasiuniform structures $\mathfrak{U}_\sigma(\mathcal{P},\mathcal{P}^*)$ and $\mathfrak{U}_\tau$ under inductive and projective limits.

Theorem 3.28. *Let $(\mathcal{P},\mathfrak{U})$ be the inductive limit of locally convex cones $(\mathcal{P}_\gamma,\mathfrak{U}_\gamma)$ by the mappings f_γ, $\gamma\in\Gamma$. Then $(\mathcal{P},\mathfrak{U}_\tau)$ (or $(\mathcal{P},\mathfrak{U}_{b\tau})$) is the inductive limit of locally convex cones $(\mathcal{P}_\gamma,\mathfrak{U}_{\gamma\tau})$(or $(\mathcal{P}_\gamma,\mathfrak{U}_{\gamma b\tau})$) by the linear mappings f_γ, $\gamma\in\Gamma$.*

Proof. The linear mapping $f_\gamma : (\mathcal{P}_\gamma,\mathfrak{U}_{\gamma\tau}) \to (\mathcal{P},\mathfrak{U}_\tau)$ is continuous by Proposition 3.15 for each $\gamma\in\Gamma$. Let $(\mathcal{P},\mathcal{W})$ be the inductive limit of locally convex cones $(\mathcal{P}_\gamma,\mathfrak{U}_{\gamma\tau})$ by the mappings f_γ, $\gamma\in\Gamma$. Then we have $\mathfrak{U}_\tau\subseteq\mathcal{W}$ by the definition of an inductive limit. We claim that $\mathcal{P}^2$ has the same u-bounded sets under $\mathcal{W}$ and $\mathfrak{U}_\tau$. Indeed, if $B\subseteq\mathcal{P}^2$ is u-bounded under $\mathfrak{U}_\tau$, then it is u-bounded under $\mathfrak{U}$. This shows that $(f_\gamma\times f_\gamma)^{-1}(B)$ is u-bounded under $\mathfrak{U}_\gamma$ and then under $\mathfrak{U}_{\gamma\tau}$, for $\gamma\in\Gamma$. Thus B is u-bounded under $\mathcal{W}$ in $\mathcal{P}^2$. If $B\subseteq\mathcal{P}^2$ is u-bounded under $\mathcal{W}$, then it is u-bounded in $\mathfrak{U}_\tau$, since $\mathfrak{U}_\tau\subseteq W$. Since the identity mapping $i : (\mathcal{P},\mathfrak{U}_\tau)\to(\mathcal{P},\mathcal{W})$ is u-bounded and $(\mathcal{P},\mathfrak{U}_\tau)$ is bornological, $\mathcal{W}$ and $\mathfrak{U}_\tau$ are equivalent. □

Theorem 3.29. *Let $(\mathcal{P},\mathfrak{U})$ be the projective limit of locally convex cones $(\mathcal{P}_\gamma,\mathfrak{U}_\gamma)$ by the mappings g_γ, $\gamma\in\Gamma$. Then $(\mathcal{P},\mathfrak{U}_\sigma(\mathcal{P},\mathcal{P}^*))$ is the projective limit of locally convex cones $(\mathcal{P}_\gamma,\mathfrak{U}_{\gamma\sigma}(\mathcal{P}_\gamma,\mathcal{P}_\gamma^*))$ by the mappings g_γ, $\gamma\in\Gamma$.*

Proof. Obviously, $g_\gamma : (\mathcal{P},\mathfrak{U}_\sigma(\mathcal{P},\mathcal{P}^*))\to(\mathcal{P}_\gamma,\mathfrak{U}_{\gamma\sigma}(\mathcal{P}_\gamma,\mathcal{P}_\gamma^*))$ is continuous for each $\gamma\in\Gamma$. Let $(\mathcal{P},\mathcal{W})$ be the projective limit of locally convex cones $(\mathcal{P}_\gamma,\mathfrak{U}_{\gamma\sigma}(\mathcal{P}_\gamma,\mathcal{P}_\gamma^*))$ by the mappings g_γ, $\gamma\in\Gamma$. Let $U_\sigma\in\mathfrak{U}_\sigma(\mathcal{P},\mathcal{P}^*)$. Then there is $n\in\mathbb{N}$ and $\mu_1,...,\mu_n\in\mathcal{P}^*$ such that

$$\textstyle\bigcap_{i=1}^n \Lambda_i^{-1}(\tilde{1})\subseteq U_\sigma,\ \textit{where}\ \Lambda_i=\mu_i\times\mu_i.$$

There is $U\in\mathfrak{U}$ such that $\mu_1,...,\mu_n\in U^\circ$. Since $(\mathcal{P},\mathfrak{U})$ is the projective limit of locally convex cones $(\mathcal{P}_\gamma,\mathfrak{U}_\gamma)$ by the mappings g_γ, there is $m\in\mathbb{N}$ such that

$$\textstyle\bigcap_{j=1}^m G_{\gamma_j}^{-1}(U_{\gamma_j})\subseteq U,\qquad U_{\gamma_j}\in\mathfrak{U}_{\gamma_j},\gamma_j\in\Gamma,G_{\gamma_j}=g_{\gamma_j}\times g_{\gamma_j}.$$

This shows that $\mu_i\in(G_{\gamma_j}^{-1}(U_{\gamma_j}))^\circ$ for each $i\in\{1,...,n\}$ and some $j\in\{1,...,m\}$. Therefore $\theta_{ij}=\mu_i o g_{\gamma_j}^{-1}\in\mathcal{P}_{\gamma_j}^*$ for each $i\in\{1,...,n\}$. Now, we have

$$\bigcap_{i=1}^n \Theta_{ij}^{-1}(\tilde{1})\in\mathfrak{U}_{\gamma_j\sigma}(\mathcal{P}_{\gamma_j},\mathcal{P}_{\gamma_j}^*),$$

where $\Theta_{ij} = \theta_{ij} \times \theta_{ij}$. Then

$$\bigcap_{i=1}^{n} \Lambda_i^{-1}(\tilde{1}) = G_{\gamma_j}^{-1}(\bigcap_{i=1}^{n} \Theta_{ij}^{-1}(\tilde{1})) \in \mathcal{W}.$$

Thus $\mathcal{W}$ and $\mathfrak{U}_\sigma(\mathcal{P}, \mathcal{P}^*)$ are identical. □

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