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Bose–Einstein Condensation Beyond the Gross–Pitaevskii Regime

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Abstract. We consider N bosons in a box with volume one, interacting through a two-body potential with scattering length of the order $N^{-1+\kappa}$, for $\kappa > 0$. Assuming that $\kappa \in (0;1/43)$, we show that low-energy states exhibit Bose–Einstein condensation and we provide bounds on the expectation and on higher moments of the number of excitations.

1. Introduction

We consider systems of $N \in \mathbb{N}$ bosons trapped in the box $\Lambda = [0;1]^3$ with periodic boundary conditions (the three-dimensional torus with volume one) and interacting through a repulsive potential with scattering length of the order $N^{-1+\kappa}$, for $\kappa \in (0;1/43)$. We are interested in the limit of large N. The Hamilton operator has the form

$$H_N = \sum_{i=1}^{N} -\Delta_{x_i} + \sum_{1\leq i<j\leq N} N^{2-2\kappa} V(N^{1-\kappa}(x_i - x_j)) \tag{1.1}$$

and acts on a dense subspace of $L^2_s(\Lambda^N)$, the Hilbert space consisting of functions in $L^2(\Lambda^N)$ that are invariant with respect to permutations of the $N \in \mathbb{N}$ particles. Here, we assume the interaction potential $V \in L^3(\mathbb{R}^3)$ to have compact support and to be nonnegative, ie. $V(x) \geq 0$ for almost all $x \in \mathbb{R}^3$.

For $\kappa = 0$, the Hamilton operator (1.1) describes bosons in the so-called Gross–Pitaevskii limit. This regime is frequently used to model trapped Bose gases observed in recent experiments. Another important regime is the thermodynamic limit, where N bosons interacting through a fixed potential V (independent of N) are trapped in the box $\Lambda_L = [0;L]^3$ and where the limits $N, L \to \infty$ are taken, keeping the density $\rho = N/L^3$ fixed. After rescaling lengths (introducing new coordinates $x' = x/L$), the Hamilton operator of the Bose gas in the thermodynamic limit is given (up to a multiplicative constant) by (1.1), with $\kappa = 2/3$. Choosing $0 < \kappa < 2/3$, we are interpolating therefore between the Gross–Pitaevskii and the thermodynamic limits.

The goal of this paper is to show that low-energy states of (1.1) exhibit Bose–Einstein condensation in the zero-momentum mode $\varphi_0 \in L^2(\Lambda)$ defined by $\varphi_0(x) = 1$ for all $x \in \Lambda$ and to give bounds on the number of excitations of the condensate. To achieve this goal, it is convenient to switch to an equivalent representation of the bosonic system, removing the condensate and focusing instead on its orthogonal excitations. To this end, we notice that every $\psi_N \in L^2_s(\Lambda^N)$ can be uniquely decomposed as

$$\psi_N = \alpha_0 \varphi_0^{\otimes N} + \alpha_1 \otimes_s \varphi_0^{\otimes (N-1)} + \alpha_2 \otimes_s \varphi_0^{\otimes (N-2)} + \cdots + \alpha_N$$

where $\otimes_s$ denotes the symmetric tensor product and $\alpha_j \in L^2_\perp(\Lambda)^{\otimes_s j}$ for all $j = 0, \dots, N$, with $L^2_\perp(\Lambda)$ the orthogonal complement in $L^2(\Lambda)$ of φ_0. This observation allows us to define a unitary map $U_N : L^2_s(\Lambda^N) \to \mathcal{F}_+^{\leq N} = \bigoplus_{j=0}^N L^2_\perp(\Lambda)^{\otimes_s j}$ by setting

$$U_N \psi_N = \{\alpha_0, \alpha_1, \dots, \alpha_N\}. \tag{1.2}$$

The truncated Fock space $\mathcal{F}_+^{\leq N} = \bigoplus_{j=0}^N L^2_\perp(\Lambda)^{\otimes_s j}$ is used to describe orthogonal excitations of the condensate (some properties of the map U_N will be discussed in Sect. 2 below). On $\mathcal{F}_+^{\leq N}$, we introduce the number of particles operator, defining $(\mathcal{N}_+ \xi)^{(n)} = n \xi^{(n)}$ for every $\xi = \{\xi^{(0)}, \dots \xi^{(N)}\} \in \mathcal{F}_+^{\leq N}$.

We are now ready to state our main theorem, which provides estimates of the expectation and on higher moments of the number of orthogonal excitations of the Bose–Einstein condensate for low-energy states of (1.1).

Theorem 1.1. *Let $V \in L^3(\mathbb{R}^3)$ be pointwise nonnegative and spherically symmetric. Let $\mathfrak{a}_0 > 0$ denote the scattering length of V. Let H_N be defined as in (1.1) with $0 < \kappa < 1/43$. Then, for every $\varepsilon > 0$, there exists a constant $C > 0$ such that*

$$\left| E_N - 4\pi \mathfrak{a}_0 N^{1+\kappa} \right| \leq C N^{43\kappa + \varepsilon}. \tag{1.3}$$

for all $N \in \mathbb{N}$ large enough.

Let $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and

$$\langle \psi_N, (H_N - E_N)^2 \psi_N \rangle \leq \zeta^2, \tag{1.4}$$

for a $\zeta > 0$. Then, for every $\varepsilon > 0$ there exists a constant $C > 0$ such that

$$\langle U_N \psi_N, \mathcal{N}_+ U_N \psi_N \rangle \leq C \left[\zeta + \zeta^2 N^{13\kappa + \varepsilon - 1} + N^{43\kappa + 4\varepsilon} \right] \tag{1.5}$$

for all $N \in \mathbb{N}$ large enough. If moreover $\psi_N = \chi(H_N \leq E_N + \zeta)\psi_N$, then for all $k \in \mathbb{N}$ and all $\varepsilon > 0$ there exists $C > 0$ such that

$$\langle U_N \psi_N, \mathcal{N}_+^k U_N \psi_N \rangle \leq C \left[N^{20\kappa + \varepsilon} \zeta^2 + N^{44\kappa + 2\varepsilon} \right]^k \tag{1.6}$$

for all $N \in \mathbb{N}$ large enough.

The convergence $E_N / 4\pi \mathfrak{a}_0 N^{1+\kappa} \to 1$, as $N \to \infty$, has been first established, for Bose gases trapped by an external potential, in [19] (the choice $\kappa > 0$ corresponds, in the terminology of [19], to the Thomas–Fermi limit).

It follows from (1.5) that the one-particle density matrix $\gamma_N = \mathrm{tr}_{2,\dots,N}$

$|\psi_N\rangle\langle\psi_N|$ associated with a normalized $\psi_N \in L^2_s(\Lambda^N)$ satisfying (1.4) is such that

$$\begin{aligned} 1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle &= \frac{1}{N} \left[N - \langle \psi_N, a^*(\varphi_0) a(\varphi_0) \psi_N \rangle \right] \\ &= \frac{1}{N} \langle U_N \psi_N, \mathcal{N}_+ U_N \psi_N \rangle \\ &\leq C \left[\zeta N^{-1} + \zeta^2 N^{13\kappa+\varepsilon-2} + N^{43\kappa+4\varepsilon-1} \right] \end{aligned} \tag{1.7}$$

as $N \to \infty$. Here, we used the formula $U_N a^*(\varphi_0) a(\varphi_0) U_N = N - \mathcal{N}_+$; see (2.5). Equation (1.7) implies that low-energy states of (1.1) exhibit complete Bose–Einstein condensation, **if** $\kappa < 1/43$.

We remark that the estimate (1.6) follows, in our analysis, from a stronger bound controlling not only the number but also the energy of the excitations of the condensate. As we will explain in Sect. 3, in order to estimate the energy of excitations in low-energy states, we first need to remove (at least part of) their correlations. If we choose, as we do in (1.6), $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and $\psi_N = \chi(H_N \leq E_N + \zeta)\psi_N$, we can introduce the corresponding renormalized excitation vector $\xi_N = e^B U_N \psi_N \in \mathcal{F}_+^{\leq N}$, with the antisymmetric operator B defined as in (3.21) (the unitary operator e^B will be referred to as a generalized Bogoliubov transformation). We will show in Sect. 6 that for every $k \in \mathbb{N}$, there exists $C > 0$ such that

$$\langle \xi_N, (\mathcal{H}_N + 1)(\mathcal{N}_+ + 1)^{2k} \xi_N \rangle \leq C \left[N^{20\kappa+\varepsilon} \zeta^2 + N^{44\kappa+2\varepsilon} \right]^{2k+1} \tag{1.8}$$

for all N large enough. Here $\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N$, where

$$\mathcal{K} = \sum_{p \in \Lambda_+^*} p^2 a_p^* a_p, \quad \text{and} \quad \mathcal{V}_N = \frac{1}{2N} \sum_{\substack{p,q \in \Lambda_+^*, r \in \Lambda^*: \\ r \neq -p,-q}} N^\kappa \widehat{V}(r/N^{1-\kappa}) a_{p+r}^* a_q^* a_{q+r} a_p \tag{1.9}$$

are the kinetic and potential energy operators, restricted to $\mathcal{F}_+^{\leq N}$. (Here, $\widehat{V}$ is the Fourier transform of the potential V, defined as in (2.4).) Equation (1.6) follows then from (1.8), because $\mathcal{N}_+$ commutes with $\mathcal{H}_N$, $\mathcal{N}_+ \leq \mathcal{K} \leq \mathcal{H}_N$ and because conjugation with the generalized Bogoliubov transformation e^B does not change the number of particles substantially; see Lemma 3.2 (for $k \in \mathbb{N}$ even, we also use simple interpolation).

In the Gross–Pitaevskii regime corresponding to $\kappa = 0$ the convergence $\gamma_N \to |\varphi_0\rangle\langle\varphi_0|$ has been first established in [16–18] and later, using a different approach, in [21].[1] In this case (ie. $\kappa = 0$), the bounds (1.3), (1.5) and (1.6) with $\varepsilon = 0$ (which are optimal in their N-dependence) have been shown in [4]. Previously, they have been established in [2], under the additional assumption of small potential. A simpler proof of the results of [2], extended also to systems of bosons trapped by an external potential, has been recently given in [20]. The result of [4] was used in [3] to determine the second order corrections to the ground state energy and the low-energy excitation spectrum of the Bose gas in the Gross–Pitaevskii regime. Note that our approach in the present

[1] Going through the proof of [18, Theorem 5.1], one can observe that the authors actually show that $1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle \leq C N^{-2/17}$.

paper could be easily extended to the case $\kappa = 0$, leading to the same bounds obtained in [4]. We exclude the case $\kappa = 0$ because we would have to modify certain definitions, making the notation more complicated (for example, the sets P_H in (3.14) and P_L in (4.2) would have to be defined in terms of cutoffs independent of N).

The methods of [16–18] can also be extended to show Bose–Einstein condensation for low-energy states of (1.1), for some $\kappa > 0$. In fact, following the proof of [18, Theorem 5.1], it is possible to show that for a normalized $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and such that $\langle \psi_N, H_N \psi_N \rangle \leq E_N + \zeta$, the expectation of the number of excitations is bounded by

$$\langle U_N \psi_N, \mathcal{N}_+ U_N \psi_N \rangle \leq C \left[N^{\frac{15+20\kappa}{17}} + \zeta \right] \tag{1.10}$$

which implies complete Bose–Einstein condensation for low-energy states, for all $\kappa < 1/10$. For sufficiently small $\kappa > 0$, Theorem 1.1 improves (1.10) because it gives a better rate[2] (if $\kappa < 15/711$) and because, through (1.6), it also provides (under stronger conditions on ψ_N) bounds for higher moments of the number of excitations $\mathcal{N}_+$.

In [10], in a slightly different setting, the authors obtain a bound of the form (1.6) for $k = 1$, for the choice $\kappa = 1/(55 + 1/3)$ (for normalized $\psi_N \in L^2_s(\Lambda^N)$ that satisfy $\langle \psi_N, H_N \psi_N \rangle \leq E_N + \zeta$). They use this result to show a lower bound on the ground state energy of the dilute Bose gas in the thermodynamic limit matching the prediction of Lee–Yang and Lee–Huang–Yang [13,14].

After completion of our work, two more papers have appeared whose results are related with Theorem 1.1. Based on localization arguments from [8, 10], a bound for the expectation of $\mathcal{N}_+$ in low-energy states has been shown in [9], establishing Bose–Einstein condensation for all $\kappa < 2/5$ (as pointed out there, using a refined analysis similar to that of [10], the range of κ can be slightly improved). On the other hand, following an approach similar to [2], but with substantial simplifications (partly due to the fact that the author works in the grand canonical, rather than the canonical, ensemble), a new proof of Bose–Einstein condensation was obtained in [11], in the Gross–Pitaevskii regime, under the assumption of small potential. There is hope that the approach of [11] can be extended beyond the Gross–Pitaevskii regime, providing a simplified proof of Theorem 1.1, potentially allowing for larger values of κ.

The derivation of the bounds (1.5), (1.6), (1.8) is crucial to resolve the low-energy spectrum of the Hamiltonian (1.1). The extension of estimates on the ground state energy and on the excitation spectrum obtained in [3] for the Gross–Pitaevskii limit, to regimes with $\kappa > 0$ small enough will be addressed in a separate paper [6], using the results of Theorem 1.1. With our techniques, it does not seem possible to obtain such precise information on the spectrum of (1.1) using only previously available bounds like (1.10).

[2] For $\kappa > 0$, the rate (1.6) is not expected to be optimal. Bogoliubov theory predicts that the number of excitations of the Bose–Einstein condensate in a Bose gas with density ρ is of the order $N\rho^{1/2}$; see [5]. In our regime, this corresponds to $N^{3\kappa/2}$ excitations.

Let us now briefly explain the strategy we use to prove Theorem 1.1. The first part of our analysis follows closely [4]. We start in Sect. 2 by introducing the excitation Hamiltonian $\mathcal{L}_N = U_N H_N U_N^*$, acting on the truncated Fock space $\mathcal{F}_+^{\leq N}$; the result is given in (2.6), (2.7). The vacuum expectation $\langle \Omega, \mathcal{L}_N \Omega\rangle = N^{1+\kappa}\widehat{V}(0)/2$ is still very far from the correct ground state energy of $\mathcal{L}_N$ (and thus of H_N); the difference is of order $N^{1+\kappa}$. This is a consequence of the definition (1.2) of the unitary map U_N, whose action removes products of the condensate wave function φ_0, leaving however all correlations among particles in the wave functions $\alpha_j \in L^2_\perp(\Lambda)^{\otimes_s j}$, $j = 1, \dots, N$.

To factor out correlations, we introduce in Sect. 3 a renormalized excitation Hamiltonian $\mathcal{G}_N = e^{-B}\mathcal{L}_N e^B$, defined through unitary conjugation of $\mathcal{L}_N$ with a generalized Bogoliubov transformation e^B. The antisymmetric operator $B : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$ is quadratic in the modified creation and annihilation operators b_p, b_p^* defined, for every momentum $p \in \Lambda_+^* = 2\pi\mathbb{Z}^3\backslash\{0\}$, in (2.8) ($b_p^*$ creates a particle with momentum p annihilating, at the same time, a particle with momentum zero; in other words, b_p^* creates an excitation, moving a particle out of the condensate). The properties of $\mathcal{G}_N$ are listed in Prop. 3.3. In particular, Proposition 3.3 implies that to leading order, $\langle \Omega, \mathcal{G}_N \Omega\rangle \simeq 4\pi\mathfrak{a}_0 N^{1+\kappa}$, if κ is small enough.

Unfortunately, $\mathcal{G}_N$ is not coercive enough to prove directly that low-energy states exhibit condensation (in the sense that it is not clear how to estimate the difference between $\mathcal{G}_N$ and its vacuum expectation from below by the number of particle operator $\mathcal{N}_+$). For this reason, in Sect. 4, we define yet another renormalized excitation Hamiltonian $\mathcal{J}_N = e^{-A}\mathcal{G}_N e^A$, where now A is the antisymmetric operator (4.1), cubic in (modified) creation and annihilation operators (to be more precise, we only conjugate the main part of $\mathcal{G}_N$ with e^A; see (4.3)). Important properties of $\mathcal{J}_N$ are stated in Proposition 4.1. Up to negligible errors, the conjugation with e^A completes the renormalization of quadratic and cubic terms; in (4.5), these terms have the same form they would have for particles interacting through a mean-field potential with Fourier transform $8\pi\mathfrak{a}_0 N^\kappa \mathbf{1}(|p| < N^\alpha)$, with a parameter $\alpha > 0$ that will be chosen small enough, depending on κ (in other words, the renormalization procedure allows us to replace, in all quadratic and cubic terms, the original interaction with Fourier transform $N^{-1+\kappa}\widehat{V}(p/N^{1-\kappa})$ decaying only for momenta $|p| > N^{1-\kappa}$, with a potential whose Fourier transform already decays on scales $N^\alpha \ll N^{1-\kappa}$).

The main problem with $\mathcal{J}_N$ is that its quartic terms (the restriction of the initial potential energy on the orthogonal complement of the condensate wave function) are still proportional to the local interaction with Fourier transform $N^{-1+\kappa}\widehat{V}(p/N^{1-\kappa})$.

One possibility to solve this problem is to neglect the original quartic terms (they are positive) and insert instead quartic terms proportional to the renormalized mean-field potential $8\pi\mathfrak{a}_0 N^\kappa \mathbf{1}(|p| < N^\alpha)$, so that Bose–Einstein condensation follows as it does for mean-field systems (see [22]). Since (with the notation $\check{\chi}$ for the inverse Fourier transform of the characteristic function on the ball of radius one)

$$\frac{8\pi\mathfrak{a}_0 N^\kappa}{N}\sum_{|r|<N^\alpha} a^*_{p+r}a^*_q a_{q+r}a_p = 8\pi\mathfrak{a}_0 N^{3\alpha+\kappa-1}\int \check{\chi}(N^\alpha(x-y))\check{a}^*_x\check{a}^*_y\check{a}_y\check{a}_x\,dxdy \leq CN^{3\alpha+\kappa-1}\mathcal{N}_+^2$$

and since we know from (1.10) that $\mathcal{N}_+ \lesssim N^{\frac{15+20\kappa}{17}}$ in low-energy states, the insertion of the renormalized quartic terms produces an error that can be controlled by localization in the number of particles, if

$$3\alpha+\kappa-1+\frac{15+20\kappa}{17} = 3\alpha+\frac{37}{17}\kappa-\frac{2}{17} < 0$$

This strategy was used in [4] to prove Bose–Einstein condensation with optimal rate in the Gross–Pitaevskii regime $\kappa = 0$ (in this case, one can choose $\alpha = 0$).

Here, we follow a different approach. We perform a last renormalization step, conjugating $\mathcal{J}_N$ through a unitary operator e^D, with D quartic in creation and annihilation operators. This leads to a new Hamiltonian $\mathcal{M}_N = e^{-D}\mathcal{J}_N e^D$ (in fact, it is more convenient to conjugate only the main part of $\mathcal{J}_N$, ignoring small contributions that can be controlled by other means; see (5.5)), where the original interaction $N^{-1+\kappa}\widehat{V}(p/N^{1-\kappa})$ is replaced by the mean-field potential $8\pi\mathfrak{a}_0 N^\kappa \mathbf{1}(|p| < N^\alpha)$ in all relevant terms.[3] Condensation can then be shown as it is done for mean-field systems, with no need for localization. This is the main novelty of our analysis, compared with [4]. In Sect. 5, we define the final Hamiltonian $\mathcal{M}_N$ and in Proposition 5.1 we bound it from below. The proof of Proposition 5.1, which is technically the main part of our paper, is deferred to Sect. 7. In Sect. 6, we combine the results of the previous sections to conclude the proof of Theorem 1.1.

The results we prove with our new technique are stronger than what we would obtain using the approach of [4] in the sense that they allow for larger values of κ and better rates. More importantly, we believe that the approach we propose here is more natural and that it leaves more space for extensions. In particular, with the final quartic renormalization step, we map the original Hamilton operator (1.1), with an interaction varying on momenta of order $N^{1-\kappa}$, into a new Hamiltonian having the same form, but now with an interaction restricted to momenta smaller than N^α. If $\alpha < 1-\kappa$, this leads to an effective regularization of the potential and it suggests that further improvements may be achieved by iteration; we plan to follow this strategy, which bears some similarities to the renormalization group analysis developed in [1], in future work.

In order to control errors arising from the quartic conjugation, it is important to use observables that were not employed in [4]. In particular, the expectation of the number of excitations with large momenta

[3]Observe that the renormalized potential with Fourier transform $8\pi\mathfrak{a}_0 N^{-1+\kappa}\mathbf{1}(|p| < N^\alpha)$ that emerges in our rigorous analysis after a series of unitary transformations is reminiscent of the interaction that appears through an ad hoc substitution in the pseudo-potential method of [12,13].

$$\mathcal{N}_{\geq N^\gamma} = \sum_{p\in\Lambda_+^*:|p|\geq N^\gamma} a_p^* a_p$$

and of its powers $\mathcal{N}_{\geq N^\gamma}^2, \mathcal{N}_{\geq N^\gamma}^3$, as well as the expectation of products of the form $\mathcal{K}_L\mathcal{N}_{\geq N^\gamma}$ and $\mathcal{K}_L\mathcal{N}_{\geq N^\gamma}^2$, involving the kinetic energy operator restricted to low momenta $\mathcal{K}_L$, will play a crucial role in our analysis. It will therefore be important to establish bounds for the growth of these observables through all steps of the renormalization procedure (Lemmas 4.2, 4.3, 7.1, 7.2). In Sect. 6, an important step in the proof of Theorem 1.1 will consist in controlling the expectation of these observables on low-energy states of the renormalized Hamiltonian $\mathcal{G}_N$.

2. The Excitation Hamiltonian

We denote by $\mathcal{F} = \bigoplus_{n\geq 0} L^2(\Lambda)^{\otimes_s n}$ the bosonic Fock space over the one-particle space $L^2(\Lambda)$ and by $\Omega = \{1, 0, \ldots\}$ the vacuum vector. We can define the number of particles operator $\mathcal{N}$ by setting $(\mathcal{N}\psi)^{(n)} = n\psi^{(n)}$ for all $\psi = \{\psi^{(0)}, \psi^{(1)}, \ldots\}$ in a dense subspace of $\mathcal{F}$. For every one-particle wave function $g \in L^2(\Lambda)$, we define the creation operator $a^*(g)$ and its hermitian conjugate, the annihilation operator $a(g)$, through

$$(a^*(g)\Psi)^{(n)}(x_1,\ldots,x_n) = \frac{1}{\sqrt{n}}\sum_{j=1}^n g(x_j)\Psi^{(n-1)}(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_n)$$

$$(a(g)\Psi)^{(n)}(x_1,\ldots,x_n) = \sqrt{n+1}\int_\Lambda \bar{g}(x)\Psi^{(n+1)}(x,x_1,\ldots,x_n)\,dx$$

Creation and annihilation operators are defined on the domain of $\mathcal{N}^{1/2}$, where they satisfy the bounds

$$\|a(f)\psi\| \leq \|f\|\|\mathcal{N}^{1/2}\psi\|, \qquad \|a^*(f)\psi\| \leq \|f\|\|(\mathcal{N}_+ + 1)^{1/2}\psi\|$$

and the canonical commutation relations

$$[a(g), a^*(h)] = \langle g, h\rangle, \quad [a(g), a(h)] = [a^*(g), a^*(h)] = 0 \tag{2.1}$$

for all $g, h \in L^2(\Lambda)$ ($\langle .,.\rangle$ denotes here the inner product on $L^2(\Lambda)$). For $p \in \Lambda^* = 2\pi\mathbb{Z}^3$, we define the plane wave $\varphi_p \in L^2(\Lambda)$ through $\varphi_p(x) = e^{-ip\cdot x}$ for all $x \in \Lambda$, and the operators $a_p^* = a(\varphi_p)$ and $a_p = a(\varphi_p)$ creating and, respectively, annihilating a particle with momentum p. It is sometimes convenient to switch to position space, introducing operator valued distributions $\check{a}_x, \check{a}_x^*$ such that

$$a(f) = \int_\Lambda \bar{f}(x)\,\check{a}_x\,dx, \quad a^*(f) = \int_\Lambda f(x)\,\check{a}_x^*\,dx$$

In terms of creation and annihilation operators, the number of particles operator can be written as

$$\mathcal{N} = \sum_{p\in\Lambda^*} a_p^* a_p = \int a_x^* a_x\,dx$$

We will describe excitations of the Bose–Einstein condensate on the truncated Fock space

$$\mathcal{F}_+^{\leq N} = \bigoplus_{j=0}^{N} L^2_\perp(\Lambda)^{\otimes_s j}$$

constructed over the orthogonal complement $L^2_\perp(\Lambda)$ of the condensate wave function φ_0. On $\mathcal{F}_+^{\leq N}$, we denote the number of particles operator by $\mathcal{N}_+$. It is given by $\mathcal{N}_+ = \sum_{p\in\Lambda_+^*} a_p^* a_p$, where $\Lambda_+^* = \Lambda^*\backslash\{0\} = 2\pi\mathbb{Z}^3\backslash\{0\}$ is the momentum space for excitations. Given $\Theta \geq 0$, we also introduce the restricted number of particles operators

$$\mathcal{N}_{\geq\Theta} = \sum_{p\in\Lambda_+^* : |p|\geq\Theta} a_p^* a_p, \tag{2.2}$$

measuring the number of excitations with momentum larger or equal to Θ, and $\mathcal{N}_{<\Theta} = \mathcal{N}_+ - \mathcal{N}_{\geq\Theta}$.

Consider the operator $U_N : L^2_s(\Lambda^N) \to \mathcal{F}_+^{\leq N}$ defined in (1.2). Identifying $\psi_N \in L^2_s(\Lambda^N)$ with the Fock space vector $\{0, \dots, 0, \psi_N, 0, \dots\}$, we can also express U_N in terms of creation and annihilation operators; we obtain

$$U_N = \bigoplus_{n=0}^{N} (1 - |\varphi_0\rangle\langle\varphi_0|)^{\otimes n} \frac{a(\varphi_0)^{N-n}}{\sqrt{(N-n)!}}$$

It is then easy to check that $U_N^* : \mathcal{F}_+^{\leq N} \to L^2_s(\Lambda^N)$ is given by

$$U_N^* \{\alpha^{(0)}, \dots, \alpha^{(N)}\} = \sum_{n=0}^{N} \frac{a^*(\varphi_0)^{N-n}}{\sqrt{(N-n)!}} \alpha^{(n)}$$

and that $U_N^* U_N = 1$, ie. U_N is unitary.

Using U_N, we can define the excitation Hamiltonian $\mathcal{L}_N := U_N H_N U_N^*$, acting on a dense subspace of $\mathcal{F}_+^{\leq N}$. To compute $\mathcal{L}_N$, we first write the Hamiltonian (1.1) in momentum space, in terms of creation and annihilation operators. We find

$$H_N = \sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{1}{2N^{1-\kappa}} \sum_{p,q,r\in\Lambda^*} \widehat{V}(r/N^{1-\kappa}) a_{p+r}^* a_q^* a_p a_{q+r} \tag{2.3}$$

where

$$\widehat{V}(k) = \int_{\mathbb{R}^3} V(x) e^{-ik\cdot x} dx \tag{2.4}$$

is the Fourier transform of V, defined for all $k \in \mathbb{R}^3$ (in fact, (1.1) is the restriction of (2.3) to the $N \in \mathbb{N}$-particle sector of the Fock space $\mathcal{F}$). We can now determine the excitation Hamiltonian $\mathcal{L}_N$ using the following rules, describing the action of the unitary operator U_N on products of a creation and an annihilation operator (products of the form $a_p^* a_q$ can be thought of as operators mapping $L^2_s(\Lambda^N)$ to itself). For any $p, q \in \Lambda_+^* = 2\pi\mathbb{Z}^3\backslash\{0\}$, we find (see [15]):

$$\begin{aligned} U_N\, a_0^* a_0\, U_N^* &= N - \mathcal{N}_+ \\ U_N\, a_p^* a_0\, U_N^* &= a_p^* \sqrt{N-\mathcal{N}_+} \\ U_N\, a_0^* a_p\, U_N^* &= \sqrt{N-\mathcal{N}_+}\, a_p \\ U_N\, a_p^* a_q\, U_N^* &= a_p^* a_q \end{aligned} \tag{2.5}$$

We conclude that

$$\mathcal{L}_N = \mathcal{L}_N^{(0)} + \mathcal{L}_N^{(2)} + \mathcal{L}_N^{(3)} + \mathcal{L}_N^{(4)} \tag{2.6}$$

with

$$\begin{aligned} \mathcal{L}_N^{(0)} &= \frac{N-1}{2N} N^\kappa \widehat{V}(0)(N-\mathcal{N}_+) + \frac{N^\kappa \widehat{V}(0)}{2N} \mathcal{N}_+(N-\mathcal{N}_+) \\ \mathcal{L}_N^{(2)} &= \sum_{p\in\Lambda_+^*} p^2 a_p^* a_p + \sum_{p\in\Lambda_+^*} N^\kappa \widehat{V}(p/N^{1-\kappa}) \left[b_p^* b_p - \frac{1}{N} a_p^* a_p \right] \\ &\quad + \frac{1}{2} \sum_{p\in\Lambda_+^*} N^\kappa \widehat{V}(p/N^{1-\kappa}) \left[b_p^* b_{-p}^* + b_p b_{-p} \right] \\ \mathcal{L}_N^{(3)} &= \frac{1}{\sqrt{N}} \sum_{p,q\in\Lambda_+^*: p+q\neq 0} N^\kappa \widehat{V}(p/N^{1-\kappa}) \left[b_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} b_{p+q} \right] \\ \mathcal{L}_N^{(4)} &= \frac{1}{2N} \sum_{\substack{p,q\in\Lambda_+^*, r\in\Lambda^*: \\ r\neq -p,-q}} N^\kappa \widehat{V}(r/N^{1-\kappa}) a_{p+r}^* a_q^* a_p a_{q+r} \end{aligned} \tag{2.7}$$

where we introduced generalized creation and annihilation operators

$$b_p^* = a_p^* \sqrt{\frac{N-\mathcal{N}_+}{N}}, \qquad \text{and} \qquad b_p = \sqrt{\frac{N-\mathcal{N}_+}{N}}\, a_p \tag{2.8}$$

for all $p \in \Lambda_+^*$. Observe that by (2.5),

$$U_N^* b_p^* U_N = a_p^* \frac{a_0}{\sqrt{N}}, \qquad U_N^* b_p U_N = \frac{a_0^*}{\sqrt{N}} a_p$$

In other words, b_p^* creates a particle with momentum $p \in \Lambda_+^*$ but, at the same time, it annihilates a particle from the condensate; it creates an excitation, preserving the total number of particles in the system. On states exhibiting complete Bose–Einstein condensation in the zero-momentum mode φ_0, we have $a_0, a_0^* \simeq \sqrt{N}$ and we can therefore expect that $b_p^* \simeq a_p^*$ and that $b_p \simeq a_p$. Modified creation and annihilation operators satisfy the commutation relations

$$\begin{aligned} [b_p, b_q^*] &= \left(1 - \frac{\mathcal{N}_+}{N}\right) \delta_{p,q} - \frac{1}{N} a_q^* a_p \\ [b_p, b_q] &= [b_p^*, b_q^*] = 0 \end{aligned} \tag{2.9}$$

Furthermore, we find

$$[b_p, a_q^* a_r] = \delta_{pq} b_r, \qquad [b_p^*, a_q^* a_r] = -\delta_{pr} b_q^* \tag{2.10}$$

for all $p, q, r \in \Lambda_+^*$; this implies in particular that $[b_p, \mathcal{N}_+] = b_p$, $[b_p^*, \mathcal{N}_+] = -b_p^*$. It is also useful to notice that the operators b_p^*, b_p, like the standard creation and annihilation operators a_p^*, a_p, can be bounded by the square root of the number of particles operators; we find

$$\begin{aligned} \|b_p \xi\| &\leq \left\| \mathcal{N}_+^{1/2} \Big(\frac{N+1-\mathcal{N}_+}{N} \Big)^{1/2} \xi \right\| \leq \|\mathcal{N}_+^{1/2} \xi\| \\ \|b_p^* \xi\| &\leq \left\| (\mathcal{N}_+ + 1)^{1/2} \Big(\frac{N-\mathcal{N}_+}{N} \Big)^{1/2} \xi \right\| \leq \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \end{aligned}$$

for all $\xi \in \mathcal{F}_+^{\leq N}$. Since $\mathcal{N}_+ \leq N$ on $\mathcal{F}_+^{\leq N}$, the operators b_p^*, b_p are bounded, with $\|b_p\|, \|b_p^*\| \leq (N+1)^{1/2}$.

We can also define modified operator valued distributions

$$\check{b}_x = \sqrt{\frac{N-\mathcal{N}_+}{N}}\, \check{a}_x, \qquad \text{and} \qquad \check{b}_x^* = \check{a}_x^* \sqrt{\frac{N-\mathcal{N}_+}{N}}$$

in position space, for $x \in \Lambda$. The commutation relations (2.9) take the form

$$\begin{aligned} [\check{b}_x, \check{b}_y^*] &= \left(1 - \frac{\mathcal{N}_+}{N} \right) \delta(x-y) - \frac{1}{N} \check{a}_y^* \check{a}_x \\ [\check{b}_x, \check{b}_y] &= [\check{b}_x^*, \check{b}_y^*] = 0 \end{aligned}$$

Moreover, (2.10) translates to

$$[\check{b}_x, \check{a}_y^* \check{a}_z] = \delta(x-y) \check{b}_z, \qquad [\check{b}_x^*, \check{a}_y^* \check{a}_z] = -\delta(x-z) \check{b}_y^*$$

which also implies that $[\check{b}_x, \mathcal{N}_+] = \check{b}_x$, $[\check{b}_x^*, \mathcal{N}_+] = -\check{b}_x^*$.

3. Renormalized Excitation Hamiltonian

Conjugation with U_N extracts, from the original quartic interaction in (2.3), some constant and some quadratic contributions, collected in $\mathcal{L}_N^{(0)}$ and $\mathcal{L}_N^{(2)}$ in (2.7). For bosons described by the Hamiltonian (1.1), this is not enough; there are still large contributions to the energy that are hidden in $\mathcal{L}_N^{(3)}$ and $\mathcal{L}_N^{(4)}$.

To extract the missing energy, we have to take into account correlations. To this end, we consider the ground state solution f_ℓ of the Neumann problem

$$\left[-\Delta + \frac{1}{2} V \right] f_\ell = \lambda_\ell f_\ell \tag{3.1}$$

on the ball $|x| \leq N^{1-\kappa}\ell$ (we omit the $N \in \mathbb{N}$-dependence in the notation for f_ℓ and for λ_ℓ; notice that λ_ℓ scales as $N^{3\kappa-3}$), with the normalization $f_\ell(x) = 1$ if $|x| = N^{1-\kappa}\ell$. By scaling, we observe that $f_\ell(N^{1-\kappa}.)$ satisfies the equation

$$\left[-\Delta + \frac{1}{2} N^{2-2\kappa} V(N^{1-\kappa} x) \right] f_\ell(N^{1-\kappa} x) = N^{2-2\kappa} \lambda_\ell f_\ell(N^{1-\kappa} x)$$

on the ball $|x| \leq \ell$. From now on, we fix some $0 < \ell < 1/2$, so that the ball of radius ℓ is contained in the box $\Lambda = [-1/2; 1/2]^3$. We then extend $f_\ell(N^{1-\kappa}.)$

to Λ, by setting $f_N(x) = f_\ell(N^{1-\kappa}x)$, if $|x| \le \ell$ and $f_N(x) = 1$ for $x \in \Lambda$, with $|x| > \ell$. As a consequence,

$$\left[-\Delta + \frac{1}{2}N^{2-2\kappa}V(N^{1-\kappa}.)\right] f_N = N^{2-2\kappa}\lambda_\ell f_N \chi_\ell, \tag{3.2}$$

where χ_ℓ denotes the characteristic function of the ball of radius ℓ. The Fourier coefficients of the function f_N are given by

$$\widehat{f}_N(p) = \int_\Lambda f_\ell(N^{1-\kappa}x)e^{-ip\cdot x}dx \tag{3.3}$$

for all $p \in \Lambda^*$. Next, we define $w_\ell(x) = 1 - f_\ell(x)$ for $|x| \le N^{1-\kappa}\ell$ and $w_\ell(x) = 0$ for all $|x| > N^{1-\kappa}\ell$. Its rescaled version $w_N : \Lambda \to \mathbb{R}$ is defined through $w_N(x) = w_\ell(N^{1-\kappa}x)$ if $|x| \le \ell$ and $w_N(x) = 0$ if $x \in \Lambda$ with $|x| > \ell$. The Fourier coefficients of w_N are given by

$$\widehat{w}_N(p) = \int_\Lambda w_\ell(N^{1-\kappa}x)e^{-ip\cdot x}dx = \frac{1}{N^{3-3\kappa}}\widehat{w}_\ell(p/N^{1-\kappa}),$$

where

$$\widehat{w}_\ell(k) = \int_{\mathbb{R}^3} w_\ell(x)e^{-ik\cdot x}dx$$

denotes the Fourier transform of the (compactly supported) function w_ℓ. We find $\widehat{f}_N(p) = \delta_{p,0} - N^{3\kappa-3}\widehat{w}_\ell(p/N^{1-\kappa})$. From (3.2), we obtain

$$\begin{aligned} &- p^2\widehat{w}_\ell(p/N^{1-\kappa}) + \frac{N^{2-2\kappa}}{2}\sum_{q\in\Lambda^*}\widehat{V}((p-q)/N^{1-\kappa})\widehat{f}_N(q) \\ &= N^{5-5\kappa}\lambda_\ell\sum_{q\in\Lambda^*}\widehat{\chi}_\ell(p-q)\widehat{f}_N(q). \end{aligned} \tag{3.4}$$

The next lemma summarizes important properties of the functions w_ℓ and f_ℓ. Its proof can be found in [4, Appendix A] (replacing $N \in \mathbb{N}$ by $N^{1-\kappa}$ and noting that still $N^{1-\kappa}\ell \gg 1$ for $N \in \mathbb{N}$ sufficiently large and fixed $\ell \in (0; 1/2)$).

Lemma 3.1. *Let $V \in L^3(\mathbb{R}^3)$ be nonnegative, compactly supported and spherically symmetric. Fix $\ell > 0$ and let f_ℓ denote the solution of (3.1). For $N \in \mathbb{N}$ large enough, the following properties hold true.*

(i) *We have*

$$\lambda_\ell = \frac{3\mathfrak{a}_0}{N^{3-3\kappa}\ell^3}\left(1 + \mathcal{O}\big(\mathfrak{a}_0/\ell N^{1-\kappa}\big)\right). \tag{3.5}$$

(ii) *We have $0 \le f_\ell, w_\ell \le 1$. Moreover there exists a constant $C > 0$ such that*

$$\left|\int V(x)f_\ell(x)dx - 8\pi\mathfrak{a}_0\right| \le \frac{C\mathfrak{a}_0^2}{\ell N^{1-\kappa}}. \tag{3.6}$$

(iii) *There exists a constant $C > 0$ such that*

$$w_\ell(x) \le \frac{C}{|x|+1} \quad \text{and} \quad |\nabla w_\ell(x)| \le \frac{C}{x^2+1}. \tag{3.7}$$

for all $x \in \mathbb{R}^3$ and all $N \in \mathbb{N}$ large enough.

(iv) *There exists a constant* $C > 0$ *such that*

$$|\widehat{w}_N(p)| \leq \frac{C}{N^{1-\kappa} p^2}$$

for all $p \in \mathbb{R}^3$ *and all* $N \in \mathbb{N}$ *large enough (such that* $N^{1-\kappa} \geq \ell^{-1}$*).*

We define $\eta : \Lambda^* \to \mathbb{R}$ through

$$\eta_p = -N\widehat{w}_N(p) = -\frac{N^\kappa}{N^{2-2\kappa}} \widehat{\omega}_\ell(p/N^{1-\kappa}). \tag{3.8}$$

In position space, this means that for $x \in \Lambda$, we have

$$\check{\eta}(x) = -N w_\ell(N^{1-\kappa} x), \tag{3.9}$$

so that we have in particular the L^∞-bound

$$\|\check{\eta}\|_\infty \leq CN. \tag{3.10}$$

Lemma 3.1 also implies

$$|\eta_p| \leq \frac{CN^\kappa}{|p|^2} \tag{3.11}$$

for all $p \in \Lambda_+^* = 2\pi\mathbb{Z}^3 \backslash \{0\}$, and for some constant $C > 0$ independent of $N \in \mathbb{N}$ (for $N \in \mathbb{N}$ large enough). From (3.4), we find the relation

$$p^2 \eta_p + \frac{1}{2} N^\kappa (\widehat{V}(./N^{1-\kappa}) * \widehat{f}_N)(p) = N^{3-2\kappa} \lambda_\ell (\widehat{\chi}_\ell * \widehat{f}_N)(p) \tag{3.12}$$

or equivalently, expressing the r.h.s. through the coefficients η_p,

$$\begin{aligned} p^2 \eta_p &+ \frac{1}{2} N^\kappa \widehat{V}(p/N^{1-\kappa}) + \frac{1}{2N} \sum_{q \in \Lambda^*} N^\kappa \widehat{V}((p-q)/N^{1-\kappa}) \eta_q \\ &= N^{3-2\kappa} \lambda_\ell \widehat{\chi}_\ell(p) + N^{2-2\kappa} \lambda_\ell \sum_{q \in \Lambda^*} \widehat{\chi}_\ell(p-q) \eta_q. \end{aligned} \tag{3.13}$$

In our analysis, it is useful to restrict η to high momenta. To this end, let $\alpha > 0$ and

$$P_H = \{p \in \Lambda_+^* : |p| \geq N^\alpha\}. \tag{3.14}$$

We define $\eta_H \in \ell^2(\Lambda_+^*)$ by

$$\eta_H(p) = \eta_p \, \chi(p \in P_H) = \eta_p \chi(|p| \geq N^\alpha). \tag{3.15}$$

Equation (3.11) implies that

$$\|\eta_H\| \leq CN^{\kappa - \alpha/2} \tag{3.16}$$

and we assume from now on that $\alpha > 2\kappa$ such that in particular

$$\lim_{N \to \infty} \|\eta_H\| = 0. \tag{3.17}$$

Notice, on the other hand, that the H^1-norm of η and η_H diverge, as $N \to \infty$. From (3.9) and Lemma 3.1, part iii), we find

$$\sum_{p \in P_H} p^2 |\eta_p|^2 \leq \sum_{p \in \Lambda_+^*} p^2 |\eta_p|^2 = \int |\nabla \check{\eta}(x)|^2 dx \leq CN^{1+\kappa} \tag{3.18}$$

for all $N \in \mathbb{N}$ large enough. We will mostly use the coefficients η_p with $p \neq 0$. Sometimes, however, it will be useful to have an estimate on η_0 (because Eq. (3.13) involves η_0). From Lemma 3.1, part iii), we obtain

$$|\eta_0| \leq N^{3\kappa-2} \int_{\mathbb{R}^3} w_\ell(x) dx \leq C N^\kappa \ell^2 \tag{3.19}$$

It will also be useful to have bounds for the function $\check{\eta}_H : \Lambda \to \mathbb{R}$, having Fourier coefficients $\eta_H(p)$ as defined in (3.15). Writing $\eta_H(p) = \eta_p - \eta_p \chi(|p| \leq N^\alpha)$, we obtain

$$\check{\eta}_H(x) = \check{\eta}(x) - \sum_{\substack{p\in\Lambda^*: \\ |p|\leq N^\alpha}} \eta_p e^{ip\cdot x} = -N w_\ell(N^{1-\kappa} x) - \sum_{\substack{p\in\Lambda^*: \\ |p|\leq N^\alpha}} \eta_p e^{ip\cdot x}$$

so that

$$|\check{\eta}_H(x)| \leq CN + CN^\kappa \sum_{\substack{p\in\Lambda^*: \\ |p|\leq N^\alpha}} |p|^{-2} \leq C(N + N^{\alpha+\kappa}) \leq C(N + N^{\alpha+\kappa}) \tag{3.20}$$

for all $x \in \Lambda$, if $N \in \mathbb{N}$ is large enough.

With the coefficients (3.15), we define the antisymmetric operator

$$B = \frac{1}{2} \sum_{p\in P_H} \left(\eta_p b_p^* b_{-p}^* - \bar{\eta}_p b_{-p} b_p \right) \tag{3.21}$$

and the generalized Bogoliubov transformation $e^B : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$. A first important observation is that conjugation with this unitary operator does not change the number of particles by too much. The proof of the following Lemma can be found in [7, Lemma 3.1] (a similar result has been previously established in [22]).

Lemma 3.2. *Assume B is defined as in (3.21), with the coefficients η_p as in (3.8), satisfying (3.17). For every $n \in \mathbb{N}$, there exists a constant $C > 0$ such that*

$$e^{-B} (\mathcal{N}_+ + 1)^n e^B \leq C (\mathcal{N}_+ + 1)^n \tag{3.22}$$

as an operator inequality on $\mathcal{F}_+^{\leq N}$. (The constant depends only on $\|\eta_H\|$ and on $n \in \mathbb{N}$.)

With the generalized Bogoliubov transformation e^B, we can now define the renormalized excitation Hamiltonian $\mathcal{G}_N : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$ by setting

$$\mathcal{G}_N = e^{-B} \mathcal{L}_N e^B = e^{-B} U_N H_N U_N^* e^B. \tag{3.23}$$

In the next propositions, we collect important properties of $\mathcal{G}_N$. Recall the notation $\mathcal{H}_N = \mathcal{K} + \mathcal{V}_N$, introduced in (1.9).

Proposition 3.3. *Let $V \in L^3(\mathbb{R}^3)$ be compactly supported, pointwise nonnegative and spherically symmetric. Let $\mathcal{G}_N$ be defined as in (3.23). Assume that the exponent α introduced in (3.14) is such that*

$$\alpha > 6\kappa, \qquad 2\alpha + 3\kappa < 1 \tag{3.24}$$

Then,

$$\mathcal{G}_N = 4\pi\mathfrak{a}_0 N^{1+\kappa} + \mathcal{H}_N + \theta_{\mathcal{G}_N} \tag{3.25}$$

and there exists $C > 0$ such that, for all $\delta > 0$ and all $N \in \mathbb{N}$ large enough, we have

$$\pm\, \theta_{\mathcal{G}_N} \leq \delta\mathcal{H}_N + C\delta^{-1}N^{\alpha+2\kappa}\mathcal{N}_+ + CN^{\alpha+2\kappa} \tag{3.26}$$

and the improved lower bound

$$\theta_{\mathcal{G}_N} \geq -\delta\mathcal{H}_N - C\delta^{-1}N^{\kappa}\mathcal{N}_+ - CN^{\alpha+2\kappa}. \tag{3.27}$$

Furthermore, for $\beta > 0$, denote by $\mathcal{G}_N^{\mathit{eff}}$ the excitation Hamiltonian

$$\begin{aligned}\mathcal{G}_N^{\mathit{eff}} &= 4\pi\mathfrak{a}_0 N^{\kappa}(N-\mathcal{N}_+) + \big[\widehat{V}(0) - 4\pi\mathfrak{a}_0\big]N^{\kappa}\mathcal{N}_+\frac{(N-\mathcal{N}_+)}{N}\\ &\quad + N^{\kappa}\widehat{V}(0)\sum_{p\in P_H^c} a_p^* a_p(1-\mathcal{N}_+/N) + 4\pi\mathfrak{a}_0 N^{\kappa}\sum_{p\in P_H^c}\big[b_p^* b_{-p}^* + b_p b_{-p}\big]\\ &\quad + \frac{1}{\sqrt{N}}\sum_{\substack{p,q\in\Lambda_+^*:|q|\leq N^{\beta},\\ p+q\neq 0}} N^{\kappa}\widehat{V}(p/N^{1-\kappa})\big[b_{p+q}^* a_{-p}^* a_q + \text{h.c.}\big] + \mathcal{H}_N\end{aligned} \tag{3.28}$$

Then, there exists $C > 0$ such that $\mathcal{E}_{\mathcal{G}_N} = \mathcal{G}_N - \mathcal{G}_N^{\mathit{eff}}$ is bounded by

$$\pm\, \mathcal{E}_{\mathcal{G}_N} \leq C(N^{3\kappa-\alpha/2} + N^{\alpha+3\kappa/2-1/2} + N^{\kappa/2-\beta})\mathcal{H}_N + CN^{\alpha+2\kappa} \tag{3.29}$$

for all $N \in \mathbb{N}$ sufficiently large.

Furthermore, there exists a constant $C > 0$ such that

$$\pm i[\mathcal{N}_{\geq cN^{\gamma}}, \mathcal{G}_N],\ \pm i[\mathcal{N}_{<cN^{\gamma}}, \mathcal{G}_N] \leq C(N^{\kappa+\alpha/2-\gamma} + N^{\kappa+\gamma/2})(\mathcal{H}_N + 1) \tag{3.30}$$

for all $\alpha \geq \gamma > 0$, $c > 0$ fixed (independent of $N \in \mathbb{N}$) and $N \in \mathbb{N}$ large enough.

Finally, for every $k \in \mathbb{N}$, there exists a constant $C > 0$ such that

$$\pm\, \mathrm{ad}^{(k)}_{i\mathcal{N}_+}(\mathcal{G}_N) = \pm\big[i\mathcal{N}_+, \dots \big[i\mathcal{N}_+, \mathcal{G}_N\big]\dots\big] \leq CN^{\kappa+\alpha/6}(\mathcal{H}_N + 1). \tag{3.31}$$

The proof of Proposition 3.3 is similar to the proof of [4, Prop. 4.2] and [3, Prop. 3.2], with the appropriate modifications dictated by the different scaling of the interaction. The main novelty in Proposition 3.3 is the bound (3.30) involving commutators of the restricted number of particles operator $\mathcal{N}_{\geq cN^{\gamma}}$. This can be obtained similarly to the bounds for $\mathcal{E}_{\mathcal{G}_N}$ and for $i[\mathcal{N}_+, \mathcal{G}_N]$, because we have a full expansion of the operator $\mathcal{G}_N$ in a sum of terms whose commutators with $\mathcal{N}_+$ and with $\mathcal{N}_{\geq cN^{\gamma}}$ retains essentially the same form. In the version of this paper that is posted on the arXiv, we give a complete proof of Proposition 3.3 in "Appendix A", adapting the arguments of [4, Prop. 4.2], [3, Prop. 3.2].

4. Cubic Renormalization

From Eq. (3.28), we observe that the cubic terms in $\mathcal{G}_N^{\rm eff}$ still depend on the original interaction, which decays slowly in momentum (in contrast to the quadratic terms in the second line of (3.28), where the sum is now restricted to $P_H^c = \{p \in \Lambda_+^* : |p| < N^\alpha\}$).

To renormalize the cubic terms in (3.28), we are going to conjugate $\mathcal{G}_N^{\rm eff}$ with a unitary operator e^A, where the antisymmetric operator $A : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$ is defined by

$$A = A_1 - A_1^*, \qquad \text{with} \qquad A_1 = \frac{1}{\sqrt{N}} \sum_{r \in P_H, p \in P_L} \eta_r b^*_{r+p} a^*_{-r} a_p. \tag{4.1}$$

The high-momentum set $P_H = \{p \in \Lambda_+^* : |p| \geq N^\alpha\}$ is as in (3.14). The low-momentum set P_L is defined by

$$P_L = \{p \in \Lambda_+^* : |p| \leq N^\beta\} \tag{4.2}$$

with exponent $\beta > 0$, that will be chosen as in (3.28).

Using the unitary operator e^A, we define $\mathcal{J}_N : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$ by

$$\mathcal{J}_N = e^{-A} \mathcal{G}_N^{\rm eff} e^A. \tag{4.3}$$

Observe here that we only conjugate the main part $\mathcal{G}_N^{\rm eff}$ of the renormalized excitation Hamiltonian $\mathcal{G}_N$; this makes the analysis a bit simpler (the difference $\mathcal{G}_N - \mathcal{G}_N^{\rm eff}$ is small and can be estimated before applying the cubic conjugation).

The next proposition summarizes important properties of $\mathcal{J}_N$; it can be shown very similarly to [4, Prop. 5.2], of course with the appropriate changes of the scaling of the interaction. In the version of this paper that is posted on the arXiv, we give a complete proof of Proposition 4.1 in "Appendix B", adapting the arguments of [4, Prop. 5.2].

Proposition 4.1. *Suppose the exponents α and β are such that*

$$i)\ \alpha > 3\beta + 2\kappa, \quad ii)\ 3\alpha/2 + 2\kappa < 1, \quad iii)\ \alpha < 5\beta, \quad iv)\ \beta > 3\kappa/2, \quad v)\ \beta < 1/2 \tag{4.4}$$

Let $\mathcal{J}_N$ be defined as in (4.3), let

$$\begin{aligned} \mathcal{J}_N^{\rm eff} = {} & 4\pi \mathfrak{a}_0 N^{1+\kappa} - 4\pi \mathfrak{a}_0 N^\kappa \mathcal{N}_+^2/N + 8\pi \mathfrak{a}_0 N^\kappa \sum_{p \in P_H^c} \Big[b_p^* b_p + \frac{1}{2} b_p^* b_{-p}^* + \frac{1}{2} b_p b_{-p} \Big] \\ & + \frac{8\pi \mathfrak{a}_0 N^\kappa}{\sqrt{N}} \sum_{\substack{p \in P_H^c, q \in P_L: \\ p+q \neq 0}} \big[b^*_{p+q} a^*_{-p} a_q + \text{h.c.} \big] + \mathcal{H}_N, \end{aligned} \tag{4.5}$$

and set $\mu = \max(3\alpha/2 + 2\kappa - 1, 3\kappa/2 - \beta)$ ($\mu < 0$ follows from (4.4)). Then, there exists a constant $C > 0$ such that the self-adjoint operator $\mathcal{E}_{\mathcal{J}_N} = \mathcal{J}_N - \mathcal{J}_N^{\rm eff}$ satisfies the operator inequality

$$\pm e^A \mathcal{E}_{\mathcal{J}_N} e^{-A} \leq C(N^{-\beta/2} + N^\mu)\mathcal{K} + CN^\mu \mathcal{V}_N + CN^{\mu-\kappa}\mathcal{N}_+ + CN^{\alpha+2\kappa}(1 + N^{\alpha+\beta/2-1}) \tag{4.6}$$

in $\mathcal{F}_+^{\leq N}$ for all $N \in \mathbb{N}$ sufficiently large.

The bounds for $\mathcal{J}_N$ given in Proposition 4.1 are still not enough to show Theorem 1.1. As we will discuss in the next section, the main problem is the quartic interaction term, contained in $\mathcal{H}_N$, which still depends on the singular interaction potential (in all other terms on the r.h.s. of (4.5), the singular potential has been replaced by the regular mean-field type potential, with Fourier transform $8\pi\mathfrak{a}_0 N^\kappa \mathbf{1}_{P_H^c}(p)$, supported on momenta $|p| < N^\alpha$). To renormalize the quartic interaction, we will have to conjugate $\mathcal{J}_N^{\text{eff}}$ with yet another unitary operator, this time quartic in creation and annihilation operators. This last conjugation (which will be performed in the next section) will produce error terms. These errors will controlled in terms of the observables $\mathcal{N}_+$, $\mathcal{K}$ and $\mathcal{V}_N$ (as in (4.6)) but also, as we stressed at the end of Sect. 1, in terms of observables having the form $\mathcal{N}_{\geq N^\gamma}$ (the number of excitations having momentum larger or equal to N^γ), $\mathcal{N}^2_{\geq N^\gamma}$, $\mathcal{N}^3_{\geq N^\gamma}$, $\mathcal{K}_{\leq N^\gamma}$ (the kinetic energy of excitations with momentum below N^γ), $\mathcal{K}_L \mathcal{N}_{\geq N^\gamma}$. For this reason, we need to control the action of e^A on all these observables.

First of all, we bound the action of the cubic phase on the restricted number of particles operators $\mathcal{N}_{\geq\theta} = \sum_{p\in\Lambda_+^*:|p|\geq\theta} a_p^* a_p$. We will make use of the pull-through formula $a_p\mathcal{N}_{\geq\theta} = (\mathcal{N}_{\geq\theta} + \mathbf{1}_{[\theta,\infty)}(p))a_p$, which in particular implies that

$$\begin{aligned} \|(\mathcal{N}_{\geq\theta}+1)^{1/2}a_p\xi\| &\leq C\|a_p(\mathcal{N}_{\geq\theta}+1)^{1/2}\xi\|,\\ \|(\mathcal{N}_{\geq\theta}+1)^{-1/2}a_p\xi\| &\leq C\|a_p(\mathcal{N}_{\geq\theta}+1)^{-1/2}\xi\|\,. \end{aligned} \tag{4.7}$$

Lemma 4.2. *Assume the exponents α, β satisfy (4.4) (in fact, here it is enough to assume that $\alpha > 2\kappa$). Let $k \in \mathbb{N}_0$, $m = 0, 1, 2$, $0 < \gamma \leq \alpha$, $c \geq 0$ (and $c < 1$ if $\gamma = \alpha$). Then, there exists a constant $C > 0$ such that the operator inequalities*

$$e^{-sA}(\mathcal{N}_++1)^k(\mathcal{N}_{\geq cN^\gamma}+1)^m e^{sA} \leq C(\mathcal{N}_++1)^k(\mathcal{N}_{\geq cN^\gamma}+1)^m \tag{4.8}$$

for all $s \in [-1; 1]$ and all $N \in \mathbb{N}$.

Proof. The case $m = 0$ follows from $m = 1$. We start therefore with the case $m = 1$. For $\xi \in \mathcal{F}_+^{\leq N}$, we define the function $\varphi_\xi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_\xi(s) = \langle \xi, e^{-sA}(\mathcal{N}_++1)^k(\mathcal{N}_{\geq cN^\gamma}+1)e^{sA}\xi\rangle$$

which has derivative

$$\begin{aligned} \partial_s\varphi_\xi(s) &= 2\text{Re}\,\langle e^{sA}\xi, (\mathcal{N}_++1)^k\big[\mathcal{N}_{\geq cN^\gamma}, A_1\big]e^{sA}\xi\rangle\\ &\quad + 2\text{Re}\,\langle e^{sA}\xi, \big[(\mathcal{N}_++1)^k, A_1\big](\mathcal{N}_{\geq cN^\gamma}+1)e^{sA}\xi\rangle, \end{aligned} \tag{4.9}$$

where A_1 as in (4.1). By the assumptions on γ and c, we have $N^\alpha \geq N^\alpha - N^\beta \geq cN^\gamma$ for $N \in \mathbb{N}$ large enough. This implies in particular that

$$[\mathcal{N}_{\geq cN^\gamma}, b^*_{p+r}] = b^*_{p+r}, \quad [\mathcal{N}_{\geq cN^\gamma}, a^*_{-r}] = a^*_{-r}, \quad [\mathcal{N}_{\geq cN^\gamma}, a_p] = \chi(|p| \geq cN^\gamma)a_p$$

for $r \in P_H$ and $p \in P_L$, by (2.1) and (2.10). We then obtain

$$\big[\mathcal{N}_{\geq cN^\gamma}, A_1\big] = \frac{2}{\sqrt{N}} \sum_{r\in P_H, p\in P_L} \eta_r b^*_{r+p} a^*_{-r} a_p - \frac{1}{\sqrt{N}} \sum_{\substack{r\in P_H, p\in P_L,\\ |p|\geq cN^\gamma}} \eta_r b^*_{r+p} a^*_{-r} a_p \tag{4.10}$$

as well as

$$\big[(\mathcal{N}_+ + 1)^k, A_1\big] = \frac{k}{\sqrt{N}} \sum_{r\in P_H, p\in P_L} \eta_r b^*_{r+p} a^*_{-r} a_p (\mathcal{N}_+ + \Theta(\mathcal{N}_+) + 1)^{k-1}, \tag{4.11}$$

for some function $\Theta : \mathbb{N} \to (0;1)$ by the mean value theorem. Using the pull-through formula $\mathcal{N}_+ a^*_p = a^*_p(\mathcal{N}_+ + 1)$ and Cauchy–Schwarz, we estimate

$$\begin{aligned}
&\Big|\frac{1}{\sqrt{N}} \sum_{r\in P_H, p\in P_L} \eta_r \langle e^{sA}\xi, (\mathcal{N}_+ + 1)^k b^*_{r+p} a^*_{-r} a_p e^{sA}\xi\rangle\Big| \\
&\quad \leq \frac{1}{\sqrt{N}} \Big(\sum_{r\in P_H, p\in P_L} \|(\mathcal{N}_{\geq cN^\gamma} + 1)^{-1/2} a_{r+p} a_{-r} (\mathcal{N}_+ + 1)^{k/2} e^{sA}\xi\|^2 \Big)^{1/2} \\
&\qquad \times \Big(\sum_{r\in P_H, p\in P_L} \eta_r^2 \|(\mathcal{N}_{\geq cN^\gamma} + 1)^{1/2} a_p (\mathcal{N}_+ + 1)^{k/2} e^{sA}\xi\|^2 \Big)^{1/2}
\end{aligned}$$

With the operator inequality $\mathcal{N}_{\geq cN^\gamma} \geq \mathcal{N}_{\geq N^\alpha}$ and with (4.7), we find that

$$\begin{aligned}
&\Big|\frac{1}{\sqrt{N}} \sum_{r\in P_H, p\in P_L} \eta_r \langle e^{sA}\xi, (\mathcal{N}_+ + 1)^k b^*_{r+p} a^*_{-r} a_p e^{sA}\xi\rangle\Big| \\
&\quad \leq \frac{C}{\sqrt{N}} \Big(\sum_{r\in P_H, p\in P_L : |p+r|\geq cN^\gamma} \|a_{p+r} (\mathcal{N}_{\geq cN^\gamma} + 1)^{-1/2} a_{-r} (\mathcal{N}_+ + 1)^{k/2} e^{sA}\xi\|^2 \Big)^{1/2} \\
&\quad \times \|\eta_H\| \Big(\sum_{p\in P_L} \|a_p (\mathcal{N}_{\geq cN^\gamma} + 1)^{1/2} (\mathcal{N}_+ + 1)^{k/2} e^{sA}\xi\|^2 \Big)^{1/2} \\
&\quad \leq \frac{CN^{\kappa-\alpha/2}}{\sqrt{N}} \|(\mathcal{N}_{\geq N^\alpha} + 1)^{1/2} (\mathcal{N}_+ + 1)^{k/2} e^{sA}\xi\| \|(\mathcal{N}_{\geq cN^\gamma} + 1)^{1/2} (\mathcal{N}_+ + 1)^{(k+1)/2} e^{sA}\xi\| \\
&\quad \leq CN^{\kappa-\alpha/2} \|(\mathcal{N}_{\geq cN^\gamma} + 1)^{1/2} (\mathcal{N}_+ + 1)^{k/2} e^{sA}\xi\|^2 = CN^{\kappa-\alpha/2} \varphi_\xi(s).
\end{aligned} \tag{4.12}$$

The same arguments show that

$$\begin{aligned}
&\Big|\frac{1}{\sqrt{N}} \sum_{\substack{r\in P_H, p\in P_L,\\ |p|\geq cN^\gamma}} \eta_r \langle e^{sA}\xi, (\mathcal{N}_+ + 1)^k b^*_{r+p} a^*_{-r} a_p e^{sA}\xi\rangle\Big| \\
&\quad \leq \frac{C}{\sqrt{N}} \Big(\sum_{r\in P_H, p\in P_L : |p+r|\geq cN^\gamma} \|a_{p+r} (\mathcal{N}_{\geq cN^\gamma} + 1)^{-1/2} a_{-r} (\mathcal{N}_+ + 1)^{k/2} e^{sA}\xi\|^2 \Big)^{1/2} \\
&\qquad \times \|\eta_H\| \Big(\sum_{p\in P_L} \|a_p (\mathcal{N}_{\geq cN^\gamma} + 1)^{1/2} (\mathcal{N}_+ + 1)^{k/2} e^{sA}\xi\|^2 \Big)^{1/2} \\
&\quad \leq CN^{\kappa-\alpha/2} \varphi_\xi(s).
\end{aligned} \tag{4.13}$$

Finally, we have that

$$\begin{aligned}&\Big|\frac{k}{\sqrt{N}}\sum_{r\in P_H,p\in P_L}\eta_r\langle e^{sA}\xi,b^*_{r+p}a^*_{-r}a_p(\mathcal{N}_++\Theta(\mathcal{N}_+)+1)^{k-1}(\mathcal{N}_{\geq cN^\gamma}+1)e^{sA}\xi\rangle\Big|\\&\leq\frac{C}{\sqrt{N}}\Big(\sum_{r\in P_H,p\in P_L:|p+r|\geq cN^\gamma}\|a_{r+p}a_{-r}(\mathcal{N}_++1)^{(k-1)/2}e^{sA}\xi\|^2\Big)^{1/2}\\&\quad\times\Big(\sum_{r\in P_H,p\in P_L}\eta_r^2\|a_p(\mathcal{N}_++1)^{(k-1)/2}(\mathcal{N}_{\geq cN^\gamma}+1)e^{sA}\xi\|^2\Big)^{1/2}\\&\leq CN^{\kappa-\alpha/2}\|(\mathcal{N}_{\geq cN^\gamma}+1)^{1/2}(\mathcal{N}_++1)^{k/2}e^{sA}\xi\|^2=CN^{\kappa-\alpha/2}\varphi_\xi(s).\end{aligned}\tag{4.14}$$

Recalling (4.9), (4.10) and that $\alpha\geq 2\kappa$, the bounds (4.12) to (4.14) show that

$$\partial_s\varphi_\xi(s)\leq CN^{\kappa-\alpha/2}\varphi_\xi(s)\leq C\varphi_\xi(s).$$

Since the bounds are independent of $\xi\in\mathcal{F}_+^{\leq N}$ and the same bounds hold true replacing A by $-A$ in the definition of φ_ξ, the first inequality in (4.8) follows by Gronwall's Lemma.

To prove (4.8) with $m=2$, we proceed similarly. Given $\xi\in\mathcal{F}_+^{\leq N}$, we define the function $\psi_\xi:\mathbb{R}\to\mathbb{R}$ by

$$\psi_\xi(s)=\langle\xi,e^{-sA}(\mathcal{N}_++1)^k(\mathcal{N}_{\geq cN^\gamma}+1)^2e^{sA}\xi\rangle.$$

Its derivative is equal to

$$\begin{aligned}\partial_s\psi_\xi(s)&=2\mathrm{Re}\,\langle e^{sA}\xi,(\mathcal{N}_++1)^k\big[(\mathcal{N}_{\geq cN^\gamma}+1)^2,A_1\big]e^{sA}\xi\rangle\\&\quad+2\mathrm{Re}\,\langle e^{sA}\xi,\big[(\mathcal{N}_++1)^k,A_1\big](\mathcal{N}_{\geq cN^\gamma}+1)^2e^{sA}\xi\rangle\\&=2\mathrm{Re}\,\langle e^{sA}\xi,(\mathcal{N}_++1)^k\big[\mathcal{N}_{\geq cN^\gamma},\big[\mathcal{N}_{\geq cN^\gamma},A_1\big]\big]e^{sA}\xi\rangle\\&\quad+4\mathrm{Re}\,\langle e^{sA}\xi,(\mathcal{N}_++1)^k\big[\mathcal{N}_{\geq cN^\gamma},A_1\big](\mathcal{N}_{\geq cN^\gamma}+1)e^{sA}\xi\rangle\\&\quad+2\mathrm{Re}\,\langle e^{sA}\xi,\big[(\mathcal{N}_++1)^k,A_1\big](\mathcal{N}_{\geq cN^\gamma}+1)^2e^{sA}\xi\rangle.\end{aligned}\tag{4.15}$$

Comparing the contribution containing the double commutator in the last line on the r.h.s. of the last equation with (4.10) and using once again that $N^\alpha\geq N^\alpha-N^\beta\geq cN^\gamma$ for $N\in\mathbb{N}$ large enough, we observe that

$$\big[\mathcal{N}_{\geq cN^\gamma},\big[\mathcal{N}_{\geq cN^\gamma},A_1\big]\big]=\frac{4}{\sqrt{N}}\sum_{r\in P_H,p\in P_L}\eta_rb^*_{r+p}a^*_{-r}a_p-\frac{3}{\sqrt{N}}\sum_{\substack{r\in P_H,p\in P_L,\\|p|\geq cN^\gamma}}\eta_rb^*_{r+p}a^*_{-r}a_p.\tag{4.16}$$

Hence, the bounds (4.12) and (4.13) prove that

$$\big|\langle e^{sA}\xi,(\mathcal{N}_++1)^k\big[\mathcal{N}_{\geq cN^\gamma},\big[\mathcal{N}_{\geq cN^\gamma},A_1\big]\big]e^{sA}\xi\rangle\big|\leq C\varphi_\xi(s)\leq C\psi_\xi(s).$$

To bound the second contribution on the r.h.s. in (4.15), we recall (4.10) and we estimate

$$\begin{aligned}
&\Big| \frac{1}{\sqrt{N}} \sum_{r \in P_H, p \in P_L} \eta_r \langle e^{sA}\xi, (\mathcal{N}_+ + 1)^k b^*_{r+p} a^*_{-r} a_p (\mathcal{N}_{\geq cN^\gamma} + 1) e^{sA}\xi\rangle \Big| \\
&\quad + \Big| \frac{1}{\sqrt{N}} \sum_{\substack{r \in P_H, p \in P_L, \\ |p| \geq cN^\gamma}} \eta_r \langle e^{sA}\xi, (\mathcal{N}_+ + 1)^k b^*_{r+p} a^*_{-r} a_p (\mathcal{N}_{\geq cN^\gamma} + 1) e^{sA}\xi\rangle \Big| \\
&\leq \frac{C}{\sqrt{N}} \Big(\sum_{r \in P_H, p \in P_L : |p+r| \geq cN^\gamma} \| a_{p+r} a_{-r} (\mathcal{N}_+ + 1)^{k/2} e^{sA}\xi\|^2 \Big)^{1/2} \\
&\quad \times \|\eta_H\| \Big(\sum_{p \in P_L} \| a_p (\mathcal{N}_+ + 1)^{k/2} (\mathcal{N}_{\geq cN^\gamma} + 1) e^{sA}\xi\|^2 \Big)^{1/2} \\
&\leq CN^{\kappa - \alpha/2} \|(\mathcal{N}_{\geq cN^\gamma} + 1)(\mathcal{N}_+ + 1)^{k/2} e^{sA}\xi\|^2 = CN^{\kappa-\alpha/2}\psi_\xi(s)
\end{aligned}$$

Finally, the last contribution in (4.15) can be bounded as in (4.14), using (4.11). We have

$$\begin{aligned}
&\Big| \frac{k}{\sqrt{N}} \sum_{r \in P_H, p \in P_L} \eta_r \langle e^{sA}\xi, b^*_{r+p} a^*_{-r} a_p (\mathcal{N}_+ + \Theta(\mathcal{N}_+) + 1)^{k-1} (\mathcal{N}_{\geq cN^\gamma} + 1)^2 e^{sA}\xi\rangle \Big| \\
&\leq \frac{C}{\sqrt{N}} \Big(\sum_{r \in P_H, p \in P_L : |p+r| \geq cN^\gamma} \| a_{r+p} a_{-r} (\mathcal{N}_+ + 1)^{k/2} e^{sA}\xi\|^2 \Big)^{1/2} \\
&\quad \times \Big(\sum_{r \in P_H, p \in P_L} \eta_r^2 \| a_p (\mathcal{N}_+ + 1)^{(k-2)/2} (\mathcal{N}_{\geq cN^\gamma} + 1)^2 e^{sA}\xi\|^2 \Big)^{1/2} \\
&\leq CN^{\kappa - \alpha/2} \|(\mathcal{N}_{\geq cN^\gamma} + 1)(\mathcal{N}_+ + 1)^{k/2} e^{sA}\xi\|^2 = CN^{\kappa-\alpha/2}\psi_\xi(s),
\end{aligned}$$

where, in the last step, we used that $\mathcal{N}_{\geq cN^\gamma} \leq \mathcal{N}_+$. In conclusion, we have proved that

$$\partial_s \psi_\xi(s) \leq CN^{\kappa-\alpha/2}\psi_\xi(s) \leq C\psi_\xi(s).$$

Since the bounds are independent of $\xi \in \mathcal{F}_+^{\leq N}$ and the same bounds hold true replacing $-A$ by A in the definition ψ_ξ, Gronwall's lemma implies the last inequality in (4.8). □

We denote the kinetic energy restricted to low momenta by

$$\mathcal{K}_{\leq cN^\gamma} = \sum_{p \in \Lambda_+^* : |p| \leq cN^\gamma} p^2 a_p^* a_p. \tag{4.17}$$

We will need the following estimates for the growth of the restricted kinetic energy.

Lemma 4.3. *Assume the exponents α, β satisfy (4.4) (here we only need $\alpha \geq 2\kappa$ and $\alpha > \beta$). Let $0 < \gamma_1, \gamma_2 \leq \alpha$, and $c_1, c_2 \geq 0$ (and also $c_j < 1$, if $\gamma_j = \alpha$,* *for $j = 1, 2$). Then, there exists a constant $C > 0$ such that the operator inequalities*

$$\begin{aligned} e^{-sA}\mathcal{K}_{\leq c_1 N^{\gamma_1}} e^{sA} &\leq \mathcal{K}_{\leq c_1 N^{\gamma_1}} + N^{2\beta+2\kappa-\alpha-1}(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)^2, \\ e^{-sA}\mathcal{K}_{\leq c_1 N^{\gamma_1}}(\mathcal{N}_{\geq c_2 N^{\gamma_2}}+1)e^{sA} &\leq \mathcal{K}_{\leq c_1 N^{\gamma_1}}(\mathcal{N}_{\geq c_2 N^{\gamma_2}}+1) \\ &\quad + N^{2\beta+2\kappa-\alpha-1}(\mathcal{N}_{\geq c_2 N^{\gamma_2}}+1)^2(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1) \end{aligned} \tag{4.18}$$

for all $s \in [-1; 1]$ and all $N \in \mathbb{N}$ sufficiently large.

Proof. Like the previous Lemma 4.2, this is an application of Gronwall's lemma. Let us start to prove the first inequality in (4.18). Fix $\xi \in \mathcal{F}_+^{\leq N}$ and define $\varphi_\xi : \mathbb{R} \to \mathbb{R}$ by $\varphi_\xi(s) = \langle \xi, e^{-sA}\mathcal{K}_{\leq c_1 N^{\gamma_1}} e^{sA}\xi\rangle$ such that

$$\partial_s \varphi_\xi(s) = 2\mathrm{Re}\,\langle \xi, e^{-sA}[\mathcal{K}_{\leq c_1 N^{\gamma_1}}, A_1]e^{sA}\xi\rangle.$$

We notice first that

$$\left[\mathcal{K}_{\leq c_1 N^{\gamma_1}}, b^*_{p+r}\right] = \left[\mathcal{K}_{\leq c_1 N^{\gamma_1}}, a^*_{-r}\right] = 0$$

if $r \in P_H$ and $p \in P_L$, because $|r|, |p+r| \geq N^\alpha - N^\beta > c_1 N^{\gamma_1}$ for all $N \in \mathbb{N}$.

Using the commutation relations (2.1), we then compute

$$[\mathcal{K}_{\leq c_1 N^{\gamma_1}}, A_1] = -\frac{1}{\sqrt{N}} \sum_{r\in P_H, p\in P_L : |p|\leq c_1 N^{\gamma_1}} p^2 \eta_r b^*_{r+p} a^*_{-r} a_p. \tag{4.19}$$

With (4.19) and $|p| \leq N^\beta$ for $p \in P_L$, we then find that

$$\begin{aligned} &\left|\langle \xi, e^{-sA}[\mathcal{K}_{\leq c_1 N^{\gamma_1}}, A_1]e^{sA}\xi\rangle\right| \\ &\quad \leq \frac{CN^\beta}{\sqrt{N}} \sum_{r\in P_H, p\in P_L : |p|\leq c_1 N^{\gamma_1}} |p||\eta_r|\|a_{r+p}a_{-r}e^{sA}\xi\|\|a_p e^{sA}\xi\| \\ &\quad \leq \frac{CN^{\beta+\kappa-\alpha/2}}{\sqrt{N}} \|(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)e^{sA}\xi\|\|\mathcal{K}^{1/2}_{\leq c_1 N^{\gamma_1}} e^{sA}\xi\|. \end{aligned} \tag{4.20}$$

Finally, using Lemma 4.2 (with $c = \frac{1}{2}$, $\gamma = \alpha$ and $N \in \mathbb{N}$ sufficiently large), we conclude

$$\begin{aligned} \partial_s \varphi_\xi(s) &\leq CN^{\beta+\kappa-\alpha/2-1/2}\|(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)e^{sA}\xi\|\|\mathcal{K}^{1/2}_{\leq c_1 N^{\gamma_1}} e^{sA}\xi\| \\ &\leq CN^{2\beta+2\kappa-\alpha-1}\langle \xi, (\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)^2\xi\rangle + C\varphi_\xi(s). \end{aligned}$$

This proves the first inequality in (4.18), by Gronwall's lemma.

Next, let us prove the second inequality in (4.18). We define $\psi_\xi : \mathbb{R} \to \mathbb{R}$ by

$$\psi_\xi(s) = \langle \xi, e^{-sA}\mathcal{K}_{\leq c_1 N^{\gamma_1}}(\mathcal{N}_{\geq c_2 N^{\gamma_2}}+1)e^{sA}\xi\rangle,$$

and we compute

$$\begin{aligned} \partial_s \psi_\xi(s) &= 2\mathrm{Re}\,\langle \xi, e^{-sA}\big[\mathcal{K}_{\leq c_1 N^{\gamma_1}}, A_1\big](\mathcal{N}_{\geq c_2 N^{\gamma_2}}+1)e^{sA}\xi\rangle \\ &\quad + 2\mathrm{Re}\,\langle \xi, e^{-sA}\mathcal{K}_{\leq c_1 N^{\gamma_1}}\big[\mathcal{N}_{\geq c_2 N^{\gamma_2}}, A_1\big]e^{sA}\xi\rangle. \end{aligned}$$

First, we proceed as in (4.20) and obtain with (4.7) that

$$\begin{aligned}&\big|\langle \xi, e^{-sA}[\mathcal{K}_{\leq c_1 N^{\gamma_1}}, A_1](\mathcal{N}_{\geq c_2 N^{\gamma_2}}+1)e^{sA}\xi\rangle\big|\\ &\quad\leq \frac{CN^{\beta}}{\sqrt{N}}\sum_{\substack{r\in P_H, p\in P_L:\\ |p|\leq c_1 N^{\gamma_1}}}|p||\eta_r|\|a_{r+p}(\mathcal{N}_{\geq c_2 N^{\gamma_2}}+1)^{1/2}a_{-r}e^{sA}\xi\|\|a_p(\mathcal{N}_{\geq c_2 N^{\gamma_2}}+1)^{1/2}e^{sA}\xi\|\\ &\quad\leq \frac{CN^{\beta+\kappa-\alpha/2}}{\sqrt{N}}\|(\mathcal{N}_{\geq c_2 N^{\gamma_2}}+1)(\mathcal{N}_{\geq \frac{1}{2}N^{\alpha}}+1)^{1/2}e^{sD}\xi\|\|\mathcal{K}^{1/2}_{\leq c_1 N^{\gamma_1}}(\mathcal{N}_{\geq c_2 N^{\gamma_2}}+1)^{1/2}e^{sA}\xi\|.\end{aligned} \tag{4.21}$$

Equation (4.21) and Lemma 4.2 then imply

$$\begin{aligned}&\big|\langle \xi, e^{-sA}[\mathcal{K}_{\leq c_1 N^{\gamma_1}}, A_1](\mathcal{N}_{\geq c_2 N^{\gamma_2}}+1)e^{sA}\xi\rangle\big|\\ &\quad\leq CN^{2\beta+2\kappa-\alpha-1}\langle \xi, (\mathcal{N}_{\geq c_2 N^{\gamma_2}}+1)^2(\mathcal{N}_{\geq \frac{1}{2}N^{\alpha}}+1)\xi\rangle + C\psi_\xi(s).\end{aligned} \tag{4.22}$$

Next, we recall the identity in (4.10) and that

$$\big[\mathcal{K}_{\leq c_1 N^{\gamma_1}}, b^*_{p+r}\big] = \big[\mathcal{K}_{\leq c_1 N^{\gamma_1}}, a^*_{-r}\big] = 0$$

whenever $r\in P_H, p\in P_L$ and $N\in\mathbb{N}$, by assumption on c_1 and γ_1. We then estimate

$$\begin{aligned}&\big|\langle \xi, e^{-sA}\mathcal{K}_{\leq c_1 N^{\gamma_1}}\big[\mathcal{N}_{\geq c_2 N^{\gamma_2}}, A_1\big]e^{sA}\xi\rangle\big|\\ &\quad\leq \frac{C}{\sqrt{N}}\sum_{\substack{r\in P_H, p\in P_L,\\ v\in\Lambda_+^*:|v|\leq c_1 N^{\gamma_1}}}|v|^2|\eta_r|\|a_{r+p}(\mathcal{N}_{\geq c_2 N^{\gamma_2}}+1)^{-1/2}a_{-r}a_v e^{sD}\xi\|\\ &\qquad\times \|a_p(\mathcal{N}_{\geq c_2 N^{\gamma_2}}+1)^{1/2}a_v e^{sD}\xi\|\\ &\quad\leq CN^{\kappa-\alpha/2}\langle e^{sA}\xi, \mathcal{K}_{\leq c_1 N^{\gamma_1}}(\mathcal{N}_{\geq c_2 N^{\gamma_2}}+1)e^{sA}\xi\rangle \leq C\psi_\xi(s).\end{aligned} \tag{4.23}$$

Hence, putting (4.22) and (4.23) together, we have proved that

$$\partial_s\psi_\xi(s)\leq CN^{2\beta+2\kappa-\alpha-1}\langle \xi, (\mathcal{N}_{\geq c_2 N^{\gamma_2}}+1)^2(\mathcal{N}_{\geq \frac{1}{2}N^{\alpha}}+1)\xi\rangle + C\psi_\xi(s).$$

This implies the second bound in (4.18), by Gronwall's lemma. □

Next, we seek a bound for the growth of the potential energy operator. To this end, we first compute the commutator of $\mathcal{V}_N$ with the antisymmetric operator A. We introduce here the shorthand notation for the low-momentum part of the kinetic energy

$$\mathcal{K}_L = \sum_{p\in\Lambda_+^*:|p|\leq N^{\beta}} p^2 a^*_p a_p = \sum_{p\in P_L} p^2 a^*_p a_p. \tag{4.24}$$

Proposition 4.4. *Assume the exponents α,β satisfy (4.4). There exists a constant $C>0$ such that*

$$[\mathcal{V}_N, A] = \frac{1}{\sqrt{N}}\sum_{\substack{u\in\Lambda_+^*, p\in P_L:\\ p+u\neq 0}} N^{\kappa}(\widehat{V}(./N^{1-\kappa})*\eta/N)(u)\big[b^*_{p+u}a^*_{-u}a_p+\text{h.c.}\big]+\mathcal{E}_{[\mathcal{V}_N, A]} \tag{4.25}$$

where the self-adjoint operator $\mathcal{E}_{[\mathcal{V}_N,A]}$ *satisfies*

$$\pm\mathcal{E}_{[\mathcal{V}_N,A]} \le \delta\mathcal{V}_N + \delta^{-1}CN^{\kappa-2\beta-1}\mathcal{K}_L(\mathcal{N}_{\ge\frac{1}{2}N^\alpha}+1) + \delta^{-1}CN^{2\alpha+3\kappa-2}\mathcal{N}_+ + \delta^{-1}CN^{\kappa-1}(\mathcal{N}_{\ge\frac{1}{2}N^\alpha}+1)^2 \tag{4.26}$$

for all $\delta > 0$ *and for all* $N \in \mathbb{N}$ *sufficiently large.*

Proof. From (4.1), we have

$$[\mathcal{V}_N, A] = [\mathcal{V}_N, A_1] + \text{h.c.}$$

Following [4, Prop. 8.1], we find

$$[\mathcal{V}_N, A_1] + \text{h.c.} = \frac{1}{\sqrt{N}}\sum^*_{u\in\Lambda_+^*, v\in P_L} N^\kappa(\widehat{V}(./N^{1-\kappa}) * \eta/N)(u) b^*_{u+v}a^*_{-u}a_v + \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \text{h.c.}, \tag{4.27}$$

where

$$\begin{aligned}
\Theta_1 &= -\frac{1}{N^{3/2}}\sum^*_{\substack{u\in\Lambda^*, v\in P_L,\\ r\in P_H^c\cup\{0\}}} N^\kappa\widehat{V}((u-r)/N^{1-\kappa})\eta_r b^*_{u+v}a^*_{-u}a_v,\\
\Theta_2 &= \frac{1}{N^{3/2}}\sum^*_{\substack{u\in\Lambda^*, p\in\Lambda_+^*,\\ r\in P_H, v\in P_L}} N^\kappa\widehat{V}(u/N^{1-\kappa})\eta_r b^*_{p+u}a^*_{v+r-u}a^*_{-r}a_pa_v,\\
\Theta_3 &= \frac{1}{N^{3/2}}\sum^*_{\substack{u\in\Lambda^*, p\in\Lambda_+^*,\\ r\in P_H, v\in P_L}} N^\kappa\widehat{V}(u/N^{1-\kappa})\eta_r b^*_{v+r}a^*_{p+u}a^*_{-r-u}a_pa_v,\\
\Theta_4 &= -\frac{1}{N^{3/2}}\sum^*_{\substack{u\in\Lambda^*, p\in\Lambda_+^*,\\ r\in P_H, v\in P_L}} N^\kappa\widehat{V}(u/N^{1-\kappa})\eta_r b^*_{v+r}a^*_{-r}a^*_{p+u}a_pa_{v+u}.
\end{aligned} \tag{4.28}$$

Here and in the following, the notation $\sum^*$ indicates that we only sum over those momenta for which the arguments of the creation and annihilation operators are nonzero. The first term on the r.h.s. of (4.27) appears explicitly in (4.25), so let us estimate next the size of the operators Θ_1 to Θ_4, defined in (4.28). The bounds can be obtained similarly as in the proof of [4, Prop. 8.1].

Consider first Θ_1. For $\xi \in \mathcal{F}_+^{\le N}$, we switch to position space and find

$$|\langle\xi, \Theta_1\xi\rangle| \le \frac{1}{N^{1/2}}\sum_{r\in P_H^c}|\eta_r|\Big(\int_{\Lambda^2} dxdy\; N^{2-2\kappa}V(N^{1-\kappa}(x-y))\|\check{b}_x\check{a}_y\xi\|^2\Big)^{1/2} \times\Big(\int_{\Lambda^2} dxdy\; N^{2-2\kappa}V(N^{1-\kappa}(x-y))\Big\|\sum_{v\in P_L}e^{ivx}a_v\xi\Big\|^2\Big)^{1/2}$$

$$\begin{aligned}&\leq CN^{\alpha+3\kappa/2-1}\|\mathcal{V}_N^{1/2}\xi\|\Big(\int_\Lambda dx\, e^{i(v-v')x}\sum_{v,v'\in P_L}\langle\xi,a^*_{v'}a_v\xi\rangle\Big)^{1/2}\\&\leq CN^{\alpha+3\kappa/2-1}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{N}^{1/2}_{\leq N^\beta}\xi\|.\end{aligned} \tag{4.29}$$

The term Θ_2 on the r.h.s. of (4.28) can be controlled by

$$\begin{aligned}&|\langle\xi,\Theta_2\xi\rangle|\\&\quad=\Big|\frac{1}{N^{1/2}}\int_{\Lambda^2}dxdy\,N^{2-2\kappa}V(N^{1-\kappa}(x-y))\sum_{r\in P_H,v\in P_L}e^{ivy}e^{iry}\eta_r\langle\xi,\check{b}^*_x\check{a}^*_y a^*_{-r}\check{a}_x a_v\xi\rangle\Big|\\&\quad\leq\frac{\|\eta_H\|}{N^{1/2}}\Big[\int_{\Lambda^2}dxdy\,N^{2-2\kappa}V(N^{1-\kappa}(x-y))\sum_{v\in P_L}|v|^{-2}\|\check{b}_x\check{a}_y\xi\|^2\Big)^{1/2}\\&\qquad\times\Big(\int_{\Lambda^2}dxdy\,N^{2-2\kappa}V(N^{1-\kappa}(x-y))\sum_{v\in P_L}|v|^2\|(\mathcal{N}_{\geq\frac12N^\alpha}+1)^{1/2}\check{a}_x a_v\xi\|^2\Big)^{1/2}\\&\quad\leq CN^{\beta/2+3\kappa/2-\alpha/2-1/2}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{K}_L^{1/2}(\mathcal{N}_{\geq\frac12N^\alpha}+1)^{1/2}\xi\|.\end{aligned}$$

In the last step, we used (4.7) to estimate

$$\begin{aligned}\int_\Lambda dx\,\|(\mathcal{N}_{\geq\frac12N^\alpha}+1)^{1/2}\check{a}_x\xi\|^2&=\sum_{p\in\Lambda^*_+}\|(\mathcal{N}_{\geq\frac12N^\alpha}+1)^{1/2}a_p\xi\|^2\\&\leq C\sum_{p\in\Lambda^*_+}\|a_p(\mathcal{N}_{\geq\frac12N^\alpha}+1)^{1/2}\xi\|^2\\&=C\|\mathcal{N}_+^{1/2}(\mathcal{N}_{\geq\frac12N^\alpha}+1)^{1/2}\xi\|^2\end{aligned} \tag{4.30}$$

for any $\xi\in\mathcal{F}_+^{\leq N}$. The contributions Θ_3 and Θ_4 can be bounded similarly. We find

$$\begin{aligned}&|\langle\xi,\Theta_3\xi\rangle|\\&\quad=\Big|\frac{1}{N^{1/2}}\int_{\Lambda^2}dxdy\,N^{2-2\kappa}V(N^{1-\kappa}(x-y))\sum_{r\in P_H,v\in P_L}e^{-iry}\eta_r\langle\xi,b^*_{v+r}\check{a}^*_x\check{a}^*_y\check{a}_x a_v\xi\rangle\Big|\\&\quad\leq\frac{C\|\eta_H\|}{N^{1/2}}\Big(\int_{\Lambda^2}dxdy\,N^{2-2\kappa}V(N^{1-\kappa}(x-y))\sum_{v\in P_L}|v|^{-2}\|\check{a}_x\check{a}_y\xi\|^2\Big)^{1/2}\\&\qquad\times\Big(\int_{\Lambda^2}dxdy\,N^{2-2\kappa}V(N^{1-\kappa}(x-y))\sum_{v\in P_L}|v|^2\|(\mathcal{N}_{\geq\frac12N^\alpha}+1)^{1/2}\check{a}_x a_v\xi\|^2\Big)^{1/2}\\&\quad\leq CN^{\beta/2+3\kappa/2-\alpha/2-1/2}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{K}_L^{1/2}(\mathcal{N}_{\geq\frac12N^\alpha}+1)^{1/2}\xi\|\end{aligned}$$

as well as

$$\begin{aligned}
&|\langle \xi, \Theta_4 \xi\rangle| \\
&\quad = \left| \frac{1}{N^{1/2}} \int_{\Lambda^2} dxdy\; N^{2-2\kappa} V(N^{1-\kappa}(x-y)) \sum_{r\in P_H, v\in P_L} \eta_r e^{-ivy} \langle \xi, b^*_{v+r} a^*_{-r} \check{a}^*_x \check{a}_x \check{a}_y \xi\rangle \right| \\
&\quad \le \frac{C\|\eta_H\|}{N^{1/2}} \Big[\int_{\Lambda^2} dxdy\; N^{2-2\kappa} V(N^{1-\kappa}(x-y)) \sum_{v\in P_L} \|\check{a}_x \check{a}_y \xi\|^2 \Big)^{1/2} \\
&\qquad \times \Big[\int_{\Lambda^2} dxdy\; N^{2-2\kappa} V(N^{1-\kappa}(x-y)) \sum_{r\in P_H, v\in P_L} \|\check{a}_x a_{v+r} a_{-r} \xi\|^2 \Big)^{1/2} \\
&\quad \le CN^{3\beta/2+3\kappa/2-\alpha/2-1/2} \|\mathcal{V}_N^{1/2}\xi\| \|(\mathcal{N}_{\ge \frac{1}{2}N^\alpha}+1)\xi\|.
\end{aligned}$$

Summarizing (using $\alpha > 3\beta + 2\kappa$) we proved that

$$\begin{aligned}
\pm \sum_{i=1}^4 (\Theta_i + \text{h.c.}) &\le \delta \mathcal{V}_N + \delta^{-1} C N^{2\alpha+3\kappa-2} \mathcal{N}_+ + \delta^{-1} C N^{\kappa-2\beta-1} \mathcal{K}_L (\mathcal{N}_{\ge \frac{1}{2}N^\alpha}+1) \\
&\quad + \delta^{-1} C N^{\kappa-1} (\mathcal{N}_{\ge \frac{1}{2}N^\alpha}+1)^2
\end{aligned} \tag{4.31}$$

for any $\delta > 0$. Setting $\mathcal{E}_{[\mathcal{V}_N, A]} = \sum_{i=1}^4 (\Theta_i + \text{h.c.})$, this proves the claim. □

From Proposition 4.4, we immediately get a bound for the action of e^A on $\mathcal{V}_N$.

Corollary 4.5. *Assume the exponents α, β satisfy (4.4). Then, there exists a constant $C > 0$ such that*

$$\begin{aligned}
e^{-sA} \mathcal{V}_N e^{sA} &\le C\mathcal{V}_N + C(N^\kappa + N^{2\alpha+3\kappa-2})(\mathcal{N}_+ + 1) \\
&\quad + CN^{\kappa-2\beta-1} \mathcal{K}_L (\mathcal{N}_{\ge \frac{1}{2}N^\alpha}+1) + CN^{\kappa-3\beta-2}(\mathcal{N}_{\ge \frac{1}{2}N^\alpha}+1)^3.
\end{aligned} \tag{4.32}$$

for all $s \in [-1;1]$ and $N \in \mathbb{N}$ large enough.

Proof. We apply Gronwall's lemma. Given $\xi \in \mathcal{F}_+^{\le N}$, we define $\varphi_\xi(s) = \langle \xi, e^{-sA} \mathcal{V}_N e^{sA} \xi\rangle$ and compute its derivative s.t.

$$\partial_s \varphi_\xi(s) = \langle \xi, e^{-sA} [\mathcal{V}_N, A] e^{sA} \xi\rangle.$$

Hence, we can apply (4.25) and estimate

$$\begin{aligned}
&\left| \frac{1}{\sqrt{N}} \sum_{\substack{u\in\Lambda^*_+, v\in P_L: \\ v+u\neq 0}} N^\kappa \langle e^{sA}\xi, (\widehat{V}(./N^{1-\kappa}) * \eta/N)(u) b^*_{v+u} a^*_{-u} a_v e^{sA}\xi\rangle \right| \\
&\quad \le \frac{N^{\kappa/2}\|\check{\eta}\|_\infty}{N} \Big(\int_{\Lambda^2} dxdy\; N^{2-2\kappa} V(N^{1-\kappa}(x-y)) \|\check{a}_x \check{a}_y e^{sA}\xi\|^2 \Big)^{1/2} \\
&\qquad \times \Big(\int_{\Lambda^2} dxdy\; N^{3-3\kappa} V(N^{1-\kappa}(x-y)) \Big\| \sum_{v\in P_L} e^{ivx} a_v e^{sA}\xi\|^2 \Big)^{1/2} \\
&\quad \le CN^{\kappa/2} \|\mathcal{V}_N^{1/2} e^{sA}\xi\| \|\mathcal{N}_{\le N^\beta} e^{sA}\xi\| \le CN^\kappa \langle \xi, e^{-sA}\mathcal{N}_+ e^{sA}\xi\rangle + C\varphi_\xi(s).
\end{aligned}$$

Here, we used (3.10), which shows that $\|\check{\eta}\|_\infty \leq CN$. Using Lemma 4.2, this simplifies to

$$\begin{aligned}&\Big|\frac{1}{\sqrt{N}}\sum_{\substack{u\in\Lambda_+^*, v\in P_L:\\ v+u\neq 0}} N^\kappa \langle e^{sD}\xi, (\widehat{V}(./N^{1-\kappa}) * \eta/N)(u) b^*_{u+v} a^*_{-u} a_v e^{sD}\xi\rangle\Big| \\ &\quad\leq C\varphi_\xi(s) + CN^\kappa\langle\xi, (\mathcal{N}_+ + 1)\xi\rangle. \end{aligned} \tag{4.33}$$

Together with (4.25), the bound (4.26) (choosing $\delta = 1$) and an application of Lemma 4.2 as well as of Lemma 4.3, the claim follows from Gronwall's lemma. □

5. Quartic Renormalization

To explain why the bounds for $\mathcal{J}_N$ obtained in Prop. 4.1 are not enough to show Theorem 1.1, we introduce, for $r \in \Lambda_+^*$, the operators

$$c_r^* = \frac{1}{\sqrt{N}} \sum_{\substack{v\in\Lambda_+^*: v\neq -r,\\ v\in P_L, v+r\in P_L^c}} a^*_{v+r} a_v, \qquad e_r^* = \frac{1}{2\sqrt{N}} \sum_{\substack{v\in\Lambda_+^*: v\neq -r,\\ v\in P_L, v+r\in P_L}} a^*_{v+r} a_v. \tag{5.1}$$

We denote the adjoints of c_r^* and e_r^* by c_r and e_r, respectively. Notice in particular that $e_r^* = e_{-r}$ for all $r \in \Lambda_+^*$. A straightforward computation shows that

$$\begin{aligned}&\frac{8\pi\mathfrak{a}_0 N^\kappa}{\sqrt{N}} \sum_{\substack{p\in P_H^c, q\in P_L:\\ p+q\neq 0}} \big[b^*_{p+q} a^*_{-p} a_q + \text{h.c.}\big] \\ &\quad= 8\pi\mathfrak{a}_0 N^\kappa \sum_{p\in P_H^c} \Big[b^*_{-p} e_{-p} + e^*_{-p} b_{-p} + b^*_{-p} e^*_p + e_p b_{-p} + b^*_{-p} c^*_p + c_p b_{-p}\Big]. \end{aligned} \tag{5.2}$$

Together with (4.5), this suggests to bound the Hamiltonian $\mathcal{J}_N$ from below by completing the square in the operators $g_r^* := b_r^* + c_r^* + e_r^*$ and $g_r := b_r + c_r + e_r$, for $r \in P_H^c \subset \Lambda_+^*$. A better look at (4.5) reveals, however, that several terms that are needed to complete the square are still hidden in the energy $\mathcal{H}_N$. Since these terms are not small, we need to extract them from $\mathcal{H}_N$ by conjugation with a unitary operator e^D, with

$$D = D_1 - D_1^*, \qquad \text{where} \qquad D_1 = \frac{1}{2N} \sum_{r\in P_H, p,q\in P_L} \eta_r a^*_{p+r} a^*_{q-r} a_p a_q. \tag{5.3}$$

Since $[D, \mathcal{N}_+] = 0$, we have the identity

$$e^{-sD}(\mathcal{N}_+ + 1)^k e^{sD} = (\mathcal{N}_+ + 1)^k \tag{5.4}$$

for all $k \in \mathbb{N}$.

Using e^D, we define the final excitation Hamiltonian

$$\mathcal{M}_N = e^{-D}\mathcal{J}_N^{\text{eff}} e^D. \tag{5.5}$$

The next proposition provides an important lower bound for $\mathcal{M}_N$. Its proof is given in Sect. 7.

Proposition 5.1. *Suppose the exponents α (in the definition of the set P_H in (3.14)) and β (in the definition of the set P_L in (4.2)) are such that*

$$i)\ \alpha > 3\beta+2\kappa, \quad ii)\ 1 > \alpha+\beta+2\kappa, \quad iii)\ 5\beta > \alpha, \quad iv)\ \beta > 3\kappa, \quad v)\ 1/2 > \beta, \tag{5.6}$$

Set $\gamma = \min(\alpha, 1-\alpha-\kappa)$ ($\gamma > 0$ from (5.6)) and let $m_0 \in \mathbb{R}$ be s.t. $m_0\beta = \alpha$. Let $V \in L^3(\mathbb{R}^3)$ be compactly supported, pointwise nonnegative and spherically symmetric. Then, $\mathcal{M}_N$, as defined as in (5.5), is bounded from below by

$$\mathcal{M}_N \geq 4\pi\mathfrak{a}_0 N^{1+\kappa} + \frac{1}{4}\mathcal{K} + \mathcal{E}_{\mathcal{M}_N} \tag{5.7}$$

for a self-adjoint operator $\mathcal{E}_{\mathcal{M}_N}$ satisfying

$$\begin{aligned} e^A e^D \mathcal{E}_{\mathcal{M}_N} e^{-D} e^{-A} &\geq -CN^{-\beta}\mathcal{K} - CN^{-\beta-\kappa}\mathcal{V}_N - CN^{\beta+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq N^\beta} \\ &\quad - CN^{\alpha+\beta+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq N^{\lfloor m_0\rfloor\beta}} \\ &\quad - C\sum_{j=3}^{2\lfloor m_0\rfloor-1} N^{j\beta/2+\beta/2+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq \frac{1}{2}N^{j\beta/2}} - CN^{3\alpha+\kappa} \end{aligned} \tag{5.8}$$

for all $N \in \mathbb{N}$ sufficiently large.

6. Proof of Theorem 1.1

For $\varepsilon > 0$ sufficiently small, we define

$$\alpha = 14\kappa + 4\varepsilon, \qquad \beta = 4\kappa + \varepsilon\,. \tag{6.1}$$

The choice $\kappa < 1/43$ guarantees, if $\varepsilon > 0$ is small enough, that all conditions in (5.6) (and thus also in (3.24) and (4.4)) are satisfied.

From (3.25) and (3.26), we obtain the upper bound

$$E_N \leq 4\pi\mathfrak{a}_0 N^{1+\kappa} + CN^{16\kappa+4\varepsilon} \tag{6.2}$$

for the ground state energy of H_N. From (3.25) and (3.27), on the other hand, we obtain

$$\mathcal{H}_N \leq 2(\mathcal{G}_N - 4\pi\mathfrak{a}_0 N^{1+\kappa}) + CN^\kappa\mathcal{N}_+ + CN^{16\kappa+4\varepsilon}$$

With (6.2) and setting $\mathcal{G}'_N = \mathcal{G}_N - E_N$, we deduce that

$$\mathcal{H}_N \leq 2\mathcal{G}'_N + CN^\kappa\mathcal{N}_+ + CN^{16\kappa+4\varepsilon} \tag{6.3}$$

Next, we prove (1.5). From (3.29) and (6.3) we arrive at

$$\mathcal{G}_N = \mathcal{G}_N^{\text{eff}} + \mathcal{E}_{\mathcal{G}_N} \geq \mathcal{G}_N^{\text{eff}} - CN^{-(7\kappa+2\varepsilon)/2}\mathcal{G}'_N - CN^{-(5\kappa+2\varepsilon)/2}\mathcal{N}_+ - CN^{16\kappa+4\varepsilon}$$

Writing $\mathcal{G}_{\text{eff}} = e^A \mathcal{J}_N e^{-A}$ and recalling that $\kappa < 1/43$ (and that $\varepsilon > 0$ is small enough), Prop. 4.1 and (6.3) imply that

$$\begin{aligned}\mathcal{G}_N &\geq e^A \mathcal{J}_N^{\text{eff}} e^{-A} + e^A \mathcal{E}_{\mathcal{J}_N} e^{-A} - CN^{-(7\kappa+2\varepsilon)/2} \mathcal{G}'_N - CN^{-(5\kappa+2\varepsilon)/2} \mathcal{N}_+ - CN^{16\kappa+4\varepsilon} \\ &\geq e^A \mathcal{J}_N^{\text{eff}} e^{-A} - CN^{-(5\kappa+2\varepsilon)/2} \mathcal{G}'_N - CN^{-(3\kappa+2\varepsilon)/2} \mathcal{N}_+ - CN^{16\kappa+4\varepsilon}\end{aligned}$$

Inserting $\mathcal{J}_{\text{eff}} = e^D \mathcal{M}_N e^{-D}$ and applying Prop. 5.1, we obtain

$$\begin{aligned}\mathcal{G}_N \geq\ & 4\pi \mathfrak{a}_0 N^{1+\kappa} + \frac{1}{4} e^A e^D \mathcal{K} e^{-D} e^{-A} + e^A e^D \mathcal{E}_{\mathcal{M}_N} e^{-D} e^{-A} \\ & - CN^{-(5\kappa+2\varepsilon)/2} \mathcal{G}'_N - CN^{-(3\kappa+2\varepsilon)/2} \mathcal{N}_+ - CN^{16\kappa+4\varepsilon}\end{aligned} \tag{6.4}$$

With $\mathcal{K} \geq (2\pi)^2 \mathcal{N}_+$ and Lemma 4.2 (with $m = 0$ and $k = 1$) we have

$$e^A e^D \mathcal{K} e^{-D} e^{-A} \geq (2\pi)^2 e^A e^D \mathcal{N}_+ e^{-D} e^{-A} = (2\pi)^2 e^A \mathcal{N}_+ e^{-A} \geq c\mathcal{N}_+ \tag{6.5}$$

for a constant $c > 0$ small enough (but independent of N). If N is large enough, we conclude (using also the upper bound (6.2)), that

$$\mathcal{N}_+ \leq C\mathcal{G}'_N - Ce^A e^D \mathcal{E}_{\mathcal{M}_N} e^{-D} e^{-A} + CN^{16\kappa+4\varepsilon} \tag{6.6}$$

To bound the error term $e^A e^D \mathcal{E}_{\mathcal{M}_N} e^{-D} e^{-A}$, we need (according to (5.8)) to control observables of the form $N^{-1} \mathcal{K} \mathcal{N}_{\geq cN^\gamma}$. To this end, we observe, first of all, that, by Cauchy–Schwarz and by (6.3),

$$\begin{aligned}N^{-1} \mathcal{K} \mathcal{N}_{\geq cN^\gamma} &\leq \delta^{-1} N^{\kappa-2\gamma} \mathcal{K} + \delta N^{2\gamma-\kappa-2} \mathcal{K} \mathcal{N}^2_{\geq cN^\gamma} \\ &\leq \delta^{-1} N^{\kappa-2\gamma} \mathcal{K} + 2\delta N^{2\gamma-\kappa-2} \mathcal{N}_{\geq cN^\gamma} \mathcal{G}'_N \mathcal{N}_{\geq cN^\gamma} + C\delta N^{-1} \mathcal{K} \mathcal{N}_{\geq cN^\gamma}.\end{aligned} \tag{6.7}$$

Choosing $\delta > 0$ sufficiently small, we thus have

$$N^{-1} \mathcal{K} \mathcal{N}_{\geq cN^\gamma} \leq CN^{\kappa-2\gamma} \mathcal{K} + CN^{2\gamma-\kappa-2} \mathcal{N}_{\geq cN^\gamma} \mathcal{G}'_N \mathcal{N}_{\geq cN^\gamma}. \tag{6.8}$$

We write

$$\mathcal{N}_{\geq cN^\gamma} \mathcal{G}'_N \mathcal{N}_{\geq cN^\gamma} = \mathcal{N}^2_{\geq cN^\gamma} \mathcal{G}'_N + \mathcal{N}_{\geq cN^\gamma} [\mathcal{G}'_N, \mathcal{N}_{\geq cN^\gamma}]. \tag{6.9}$$

Using (6.3) (similarly as we did in (6.7)) and $\mathcal{N}_{\geq cN^\gamma} \leq N$, $\mathcal{N}_{\geq cN^\gamma} \leq CN^{-2\gamma} \mathcal{K}$, we can bound the expectation of the first term on the r.h.s. of the last equation, for an arbitrary $\xi \in \mathcal{F}_+^{\leq N}$, by

$$\begin{aligned}&|\langle \xi, \mathcal{N}^2_{\geq cN^\gamma} \mathcal{G}'_N \xi \rangle| \\ &\quad \leq \langle \xi, \mathcal{N}^3_{\geq cN^\gamma} \xi \rangle^{1/2} \langle \xi, \mathcal{G}'_N \mathcal{N}_{\geq cN^\gamma} \mathcal{G}'_N \xi \rangle^{1/2} \\ &\quad \leq CN^{1/2-\gamma} \langle \xi, \mathcal{K} \mathcal{N}^2_{\geq cN^\gamma} \xi \rangle^{1/2} \langle \xi, \mathcal{G}'^2_N \xi \rangle^{1/2} \\ &\quad \leq CN^{1/2-\gamma} \langle \xi, \mathcal{G}'^2_N \xi \rangle^{1/2} \langle \xi, \mathcal{N}_{\geq cN^\gamma} \mathcal{G}'_N \mathcal{N}_{\geq cN^\gamma} \xi \rangle^{1/2} \\ &\qquad + CN^{1+\kappa/2-2\gamma} \langle \xi, \mathcal{G}'^2_N \xi \rangle^{1/2} \langle \xi, \mathcal{K} \mathcal{N}_{\geq cN^\gamma} \xi \rangle^{1/2} \\ &\quad \leq \delta \langle \xi, \mathcal{N}_{\geq cN^\gamma} \mathcal{G}'_N \mathcal{N}_{\geq cN^\gamma} \xi \rangle + C\delta^{-1} N^{1-2\gamma} \langle \xi, \mathcal{G}'^2_N \xi \rangle \\ &\qquad + C\delta N^{1+\kappa-2\gamma} \langle \xi, \mathcal{K} \mathcal{N}_{\geq cN^\gamma} \xi \rangle^{1/2}.\end{aligned} \tag{6.10}$$

On the other hand, to estimate the commutator term in Eq. (6.9), we notice that $\mathcal{A} := (\mathcal{H}_N+1)^{-1/2} i[\mathcal{G}'_N, \mathcal{N}_{\geq cN^\gamma}](\mathcal{H}_N+1)^{-1/2}$ is a bounded, self-adjoint operator with $\|\mathcal{A}\| \leq CN^{\kappa+\alpha/2-\gamma} + CN^{\kappa+\gamma/2}$, by (3.30). Setting $\mu = \max(\alpha, 3\gamma)$, this implies, with (6.3),

$$\begin{aligned} &|\langle \xi, \mathcal{N}_{\geq cN^\gamma}[\mathcal{G}'_N, \mathcal{N}_{\geq cN^\gamma}]\xi\rangle| \\ &\quad \leq \delta \langle \xi, \mathcal{N}_{\geq cN^\gamma}(\mathcal{H}_N+1)\mathcal{N}_{\geq cN^\gamma}\xi\rangle + C\delta^{-1} N^{2\kappa-2\gamma+\mu}\langle \xi, (\mathcal{H}_N+1)\xi\rangle \\ &\quad \leq 2\delta\langle \xi, \mathcal{N}_{\geq cN^\gamma}\mathcal{G}'_N\mathcal{N}_{\geq cN^\gamma}\xi\rangle + C\delta N^{1+\kappa-2\gamma}\langle \xi, \mathcal{K}\mathcal{N}_{\geq cN^\gamma}\xi\rangle \\ &\qquad + C\delta^{-1}N^{3\kappa-2\gamma+\mu}\langle \xi, \mathcal{N}_+\xi\rangle + C\delta^{-1}N^{3\kappa+\alpha-2\gamma+\mu}\|\xi\|^2 \end{aligned} \tag{6.11}$$

for all $\xi \in \mathcal{F}_+^{\leq N}$. Plugging (6.10) and (6.11) into (6.9), we find that, for sufficiently small $\delta > 0$,

$$\begin{aligned} \mathcal{N}_{\geq cN^\gamma}\mathcal{G}'_N\mathcal{N}_{\geq cN^\gamma} \leq C\delta N^{1+\kappa-2\gamma}\mathcal{K}\mathcal{N}_{\geq cN^\gamma} + C\delta^{-1}N^{1-2\gamma}\mathcal{G}'^2_N \\ + C\delta^{-1}N^{3\kappa-2\gamma+\mu}\mathcal{N}_+ + C\delta^{-1}N^{3\kappa-2\gamma+\mu+\alpha} \end{aligned} \tag{6.12}$$

Inserting into (6.8) and choosing $\delta > 0$ small enough, we obtain

$$N^{-1}\mathcal{K}\mathcal{N}_{\geq cN^\gamma} \leq CN^{\kappa-2\gamma}\mathcal{K} + CN^{-\kappa-1}\mathcal{G}'^2_N + CN^{2\kappa+\mu-2}\mathcal{N}_+ + CN^{2\kappa+\mu+\alpha-2} \tag{6.13}$$

Applying (6.13) to the r.h.s. of (5.8) we find, using also (6.3), (6.1), and the choice $\kappa < 1/43$,

$$e^A e^D \mathcal{E}_{\mathcal{M}_N} e^{-D} e^{-A} \geq -CN^{-\varepsilon}\mathcal{N}_+ - CN^{-(\kappa+\varepsilon)}\mathcal{G}'_N - CN^{13\kappa+3\varepsilon-1}\mathcal{G}'^2_N - CN^{43\kappa+12\varepsilon} \tag{6.14}$$

Inserting the last equation into (6.6) and using (6.2), we conclude that for N large enough,

$$\mathcal{N}_+ \leq C\mathcal{G}'_N + CN^{13\kappa+3\varepsilon-1}\mathcal{G}'^2_N + CN^{43\kappa+12\varepsilon}$$

For $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and $\langle \psi_N, (H_N - E_N)^2\psi_N\rangle \leq \zeta^2$, the corresponding excitation vector $\xi_N = e^B U_N \psi_N$ is such that $\langle \xi_N, \mathcal{G}'^2_N \xi_N\rangle \leq \zeta^2$ and thus

$$\langle \xi_N, \mathcal{N}_+\xi_N\rangle \leq C\left[\zeta + \zeta^2 N^{13\kappa+3\varepsilon-1} + N^{43\kappa+12\varepsilon}\right]$$

which proves (1.5), using Lemma 3.2. From (6.3), we obtain also

$$\langle \xi_N, \mathcal{H}_N\xi_N\rangle \leq C\left[\zeta N^\kappa + \zeta^2 N^{14\kappa+3\varepsilon-1} + N^{44\kappa+12\varepsilon}\right], \tag{6.15}$$

an estimate that will be needed to arrive at (1.6).

Evaluating (6.14) on a normalized ground state ξ_N of $\mathcal{G}_N$ and inserting the result in (6.4) we also deduce that

$$E_N \geq 4\pi\mathfrak{a}_0 N^{1+\kappa} - CN^{43\kappa+12\varepsilon}$$

Together with the upper bound (6.2), this concludes the proof of (1.3).

We still have to show (1.6) for $k > 0$. To this end, we will prove the stronger bound (1.8); Eq. (1.6) follows then immediately from $\mathcal{N}_+ \leq \mathcal{H}_N$ and by Lemma 3.2. We denote by Q_ζ the spectral subspace of $\mathcal{G}_N$ associated with energies below $E_N + \zeta$. We use induction to show that for all $k \in \mathbb{N}$, there exists a constant $C > 0$ (depending on k) such that

$$\sup_{\xi \in Q_\zeta} \frac{\langle \xi, (\mathcal{H}_N + 1)(\mathcal{N}_+ + 1)^{2k} \xi \rangle}{\|\xi\|^2} \leq C \left[N^{44\kappa + 12\varepsilon} + \zeta^2 N^{20\kappa + 5\varepsilon} \right]^{2k+1} \tag{6.16}$$

for all $k \in \mathbb{N}$. This proves (1.8) and thus, with the bound $\mathcal{N}_+ \leq \mathcal{H}_N$ and with Lemma 3.2, also (1.6). The case $k = 0$ follows from (6.15). From now on, we assume (6.16) to hold true, and we prove the same bound, with k replaced by $(k + 1)$ (and with a new constant C). To this end, we start by observing that combining (6.3) and (6.6),

$$\mathcal{H}_N + 1 \leq CN^\kappa \mathcal{G}'_N - CN^\kappa e^A e^D \mathcal{E}_{\mathcal{M}_N} e^{-D} e^{-A} + CN^{17\kappa + 4\varepsilon}$$

Hence,

$$\begin{aligned} (\mathcal{N}_+ + 1)^{2(k+1)} (\mathcal{H}_N + 1) &= (\mathcal{N}_+ + 1)^{k+1} (\mathcal{H}_N + 1)(\mathcal{N}_+ + 1)^{k+1} \\ &\leq CN^\kappa (\mathcal{N}_+ + 1)^{k+1} \mathcal{G}'_N (\mathcal{N}_+ + 1)^{k+1} \\ &\quad - CN^\kappa (\mathcal{N}_+ + 1)^{k+1} e^A e^D \mathcal{E}_{\mathcal{M}_N} e^{-D} e^{-A} (\mathcal{N}_+ + 1)^{k+1} \\ &\quad + CN^{17\kappa + 4\varepsilon} (\mathcal{N}_+ + 1)^{2(k+1)} \end{aligned} \tag{6.17}$$

We estimate the first term on the r.h.s. by

$$\begin{aligned} &N^\kappa (\mathcal{N}_+ + 1)^{k+1} \mathcal{G}'_N (\mathcal{N}_+ + 1)^{k+1} \\ &\quad \leq N^\kappa (\mathcal{N}_+ + 1)^{2(k+1)} \mathcal{G}'_N + N^\kappa (\mathcal{N}_+ + 1)^{k+1} [\mathcal{G}'_N, (\mathcal{N}_+ + 1)^{k+1}] \\ &\quad = N^\kappa (\mathcal{N}_+ + 1)^{2(k+1)} \mathcal{G}'_N \\ &\qquad + N^\kappa \sum_{j=1}^{k+1} \binom{k+1}{j} (\mathcal{N}_+ + 1)^{k+1} \mathrm{ad}^{(j)}_{\mathcal{N}_+}(\mathcal{G}_N)(\mathcal{N}_+ + 1)^{k+1-j} \end{aligned}$$

By Cauchy–Schwarz, we find

$$\begin{aligned} &N^\kappa (\mathcal{N}_+ + 1)^{k+1} \mathcal{G}'_N (\mathcal{N}_+ + 1)^{k+1} \\ &\quad \leq N^\kappa (\mathcal{N}_+ + 1)^{2(k+1)} + N^\kappa \mathcal{G}'_N (\mathcal{N}_+ + 1)^{2(k+1)} \mathcal{G}'_N \\ &\qquad + N^\kappa \sum_{j=1}^{k+1} \binom{k+1}{j} (\mathcal{N}_+ + 1)^{k+1} \mathrm{ad}^{(j)}_{\mathcal{N}_+}(\mathcal{G}_N)(\mathcal{N}_+ + 1)^{k+1-j} \end{aligned}$$

With $(\mathcal{N}_+ + 1)^{2(k+1)} \leq (\mathcal{N}_+ + 1)^{2k+1}(\mathcal{H}_N + 1)$ and with the estimate

$$\|(\mathcal{H}_N + 1)^{-1/2} \mathrm{ad}^{(j)}_{\mathcal{N}_+}(\mathcal{G}_N)(\mathcal{H}_N + 1)^{-1/2}\| \leq CN^{7\kappa/3 + 2\varepsilon/3} \tag{6.18}$$

from (3.31) we obtain, using again Cauchy–Schwarz,

$$\begin{aligned}
&N^\kappa \langle \xi, (\mathcal{N}_+ + 1)^{k+1} \mathcal{G}'_N (\mathcal{N}_+ + 1)^{k+1} \xi \rangle \\
&\quad \le \; C \left[N^\kappa \zeta^2 + N^{7\kappa/3 + 2\varepsilon/3} \right] \|\xi\|^2 \\
&\qquad \times \left[\sup_{\xi \in Q_\zeta} \frac{\langle \xi, (\mathcal{N}_+ + 1)^{2(k+1)} (\mathcal{H}_N + 1) \xi \rangle}{\|\xi\|^2} \right]^{1/2} \\
&\qquad \left[\sup_{\xi \in Q_\zeta} \frac{\langle \xi, (\mathcal{N}_+ + 1)^{2k} (\mathcal{H}_N + 1) \xi \rangle}{\|\xi\|^2} \right]^{1/2}
\end{aligned}$$

for every $\xi \in Q_\zeta$. Hence, for any $\delta > 0$, we have

$$\begin{aligned}
&N^\kappa \frac{\langle \xi, (\mathcal{N}_+ + 1)^{k+1} \mathcal{G}'_N (\mathcal{N}_+ + 1)^{k+1} \xi \rangle}{\|\xi\|^2} \\
&\quad \le \delta \sup_{\xi \in Q_\zeta} \frac{\langle \xi, (\mathcal{N}_+ + 1)^{2(k+1)} (\mathcal{H}_N + 1) \xi \rangle}{\|\xi\|^2} \\
&\qquad + C\delta^{-1} \left[N^\kappa \zeta^2 + N^{7\kappa/3 + 2\varepsilon/3} \right]^2 \sup_{\xi \in Q_\zeta} \frac{\langle \xi, (\mathcal{N}_+ + 1)^{2k} (\mathcal{H}_N + 1) \xi \rangle}{\|\xi\|^2}
\end{aligned} \tag{6.19}$$

To bound the contribution proportional to $e^A e^D \mathcal{E}_{\mathcal{M}_N} e^{-D} e^{-A}$ on the r.h.s. of (6.17), we have to control, according to (6.8), terms of the form

$$\begin{aligned}
&(\mathcal{N}_+ + 1)^{k+1} \mathcal{N}_{\ge cN^\gamma} \mathcal{G}'_N \mathcal{N}_{\ge cN^\gamma} (\mathcal{N}_+ + 1)^{k+1} \\
&\quad = ((\mathcal{N}_+ + 1)^{k+1} \mathcal{N}_{\ge cN^\gamma})^2 \mathcal{G}'_N + (\mathcal{N}_+ + 1)^{k+1} \mathcal{N}_{\ge cN^\gamma} \left[\mathcal{G}'_N, (\mathcal{N}_+ + 1)^{k+1} \mathcal{N}_{\ge cN^\gamma} \right] \\
&\quad =: \mathrm{A} + \mathrm{B}
\end{aligned}$$

For an arbitrary $\xi \in Q_\zeta$, we can bound the expectation of A by Cauchy–Schwarz as

$$\begin{aligned}
\frac{\langle \xi, \mathrm{A} \xi \rangle}{\|\xi\|^2} &\le \frac{\langle \xi, ((\mathcal{N}_+ + 1)^{k+1} \mathcal{N}_{\ge cN^\gamma})^2 \xi \rangle}{\|\xi\|^2} + \frac{\langle \mathcal{G}'_N \xi, ((\mathcal{N}_+ + 1)^{k+1} \mathcal{N}_{\ge cN^\gamma})^2 \mathcal{G}'_N \xi \rangle}{\|\xi\|^2} \\
&\le N^2 (1 + \zeta^2) \sup_{\xi \in Q_\zeta} \frac{\langle \xi, (\mathcal{N}_+ + 1)^{2k} \mathcal{N}^2_{\ge cN^\gamma} \xi \rangle}{\|\xi\|^2} \\
&\le N^{2-2\gamma} (1 + \zeta^2) \sup_{\xi \in Q_\zeta} \frac{\langle \xi, (\mathcal{N}_+ + 1)^{2k+1} \mathcal{K} \xi \rangle}{\|\xi\|^2} \\
&\le N^{2-2\gamma} (1 + \zeta^2) \left[\sup_{\xi \in Q_\zeta} \frac{\langle \xi, (\mathcal{N}_+ + 1)^{2k} \mathcal{K} \xi \rangle}{\|\xi\|^2} \right]^{1/2} \\
&\quad \times \left[\sup_{\xi \in Q_\zeta} \frac{\langle \xi, (\mathcal{N}_+ + 1)^{2(k+1)} \mathcal{K} \xi \rangle}{\|\xi\|^2} \right]^{1/2}
\end{aligned} \tag{6.20}$$

As for the term B, we can write

$$\begin{aligned}\mathrm{B} &= (\mathcal{N}_+ + 1)^{k+1}\mathcal{N}^2_{\geq cN^\gamma}\left[\mathcal{G}'_N, (\mathcal{N}_+ + 1)^{k+1}\right] \\ &\quad + (\mathcal{N}_+ + 1)^{k+1}\mathcal{N}_{\geq cN^\gamma}\left[\mathcal{G}'_N, \mathcal{N}_{\geq cN^\gamma}\right](\mathcal{N}_+ + 1)^{k+1} \\ &= \sum_{j=1}^{k+1}\binom{k+1}{j}(\mathcal{N}_+ + 1)^{k+1}\mathcal{N}^2_{\geq cN^\gamma}\mathrm{ad}^{(j)}_{\mathcal{N}_+}(\mathcal{G}'_N)(\mathcal{N}_+ + 1)^{k+1-j} \\ &\quad + (\mathcal{N}_+ + 1)^{k+1}\mathcal{N}_{\geq cN^\gamma}\left[\mathcal{G}'_N, \mathcal{N}_{\geq cN^\gamma}\right](\mathcal{N}_+ + 1)^{k+1}\end{aligned}$$

From (6.18) and using (3.30) to estimate

$$\|(\mathcal{H}_N + 1)^{-1/2}[\mathcal{N}_{\geq cN^\gamma}, \mathcal{G}_N](\mathcal{H}_N + 1)^{-1/2}\| \leq CN^{8\kappa+2\varepsilon-\gamma} + CN^{\kappa+\gamma/2},$$

we obtain for every $\xi \in Q_\zeta$ that

$$\begin{aligned}&|\langle\xi, \mathrm{B}\xi\rangle| \\ &\quad\leq CN^{7\kappa/3+2\varepsilon/3}\|(\mathcal{H}_N + 1)^{1/2}\mathcal{N}^2_{\geq cN^\gamma}(\mathcal{N}_+ + 1)^{k+1}\xi\|\|(\mathcal{H}_N + 1)^{1/2}(\mathcal{N}_+ + 1)^k\xi\| \\ &\qquad + CN^{8\kappa+2\varepsilon-\gamma}\|(\mathcal{H}_N + 1)^{1/2}\mathcal{N}_{\geq cN^\gamma}(\mathcal{N}_+ + 1)^{k+1}\xi\|\|(\mathcal{H}_N + 1)^{1/2}(\mathcal{N}_+ + 1)^{k+1}\xi\| \\ &\qquad + CN^{\kappa+\gamma/2}\|(\mathcal{H}_N + 1)^{1/2}\mathcal{N}_{\geq cN^\gamma}(\mathcal{N}_+ + 1)^{k+1}\xi\|\|(\mathcal{H}_N + 1)^{1/2}(\mathcal{N}_+ + 1)^{k+1}\xi\|.\end{aligned}$$

Applying the bounds $\mathcal{N}_+ \leq N$, $\mathcal{N}_{\geq cN^\gamma} \leq CN^{-2\gamma}\mathcal{K}$ and (6.3) yields on the one hand

$$\begin{aligned}&\|(\mathcal{H}_N + 1)^{1/2}\mathcal{N}_{\geq cN^\gamma}(\mathcal{N}_+ + 1)^{k+1}\xi\|\|(\mathcal{H}_N + 1)^{1/2}(\mathcal{N}_+ + 1)^{k+1}\xi\| \\ &\quad\leq C\|\mathcal{G}'_N\mathcal{N}_{\geq cN^\gamma}(\mathcal{N}_+ + 1)^{k+1}\xi\|\|(\mathcal{H}_N + 1)^{1/2}(\mathcal{N}_+ + 1)^{k+1}\xi\| \\ &\qquad + CN^{1+\kappa/2-\gamma}\|(\mathcal{H}_N + 1)^{1/2}(\mathcal{N}_+ + 1)^{k+1}\xi\|^2 \\ &\quad\leq \delta\langle\xi, (\mathcal{N}_+ + 1)^{k+1}\mathcal{N}_{\geq cN^\gamma}\mathcal{G}'_N\mathcal{N}_{\geq cN^\gamma}(\mathcal{N}_+ + 1)^{k+1}\xi\rangle \\ &\qquad + C(\delta^{-1} + N^{1+\kappa/2-\gamma})\|(\mathcal{H}_N + 1)^{1/2}(\mathcal{N}_+ + 1)^{k+1}\xi\|^2\end{aligned}$$

for any $\delta > 0$. Since $8\kappa + 2\varepsilon - \gamma \leq 1 + \kappa/2 - \gamma$ and $\kappa + \gamma/2 \leq 1 + \kappa/2 - \gamma$ for all $\gamma \leq \alpha$ if $\kappa < 1/43$, this implies with the choice $\delta = \frac{1}{4}(N^{8\kappa+2\varepsilon-\gamma} + N^{\kappa+\gamma/2})^{-1}$ that

$$\begin{aligned}|\langle\xi, \mathrm{B}\xi\rangle| &\leq CN^{7\kappa/3+2\varepsilon/3}\|(\mathcal{H}_N + 1)^{1/2}\mathcal{N}^2_{\geq cN^\gamma}(\mathcal{N}_+ + 1)^{k+1}\xi\|\|(\mathcal{H}_N + 1)^{1/2}(\mathcal{N}_+ + 1)^k\xi\| \\ &\quad + C(N^{1+17\kappa/2+2\varepsilon-\gamma} + N^{1+3\kappa/2-\gamma/2})\|(\mathcal{H}_N + 1)^{1/2}(\mathcal{N}_+ + 1)^{k+1}\xi\|^2 \\ &\quad + \frac{1}{4}\langle\xi, (\mathcal{N}_+ + 1)^{k+1}\mathcal{N}_{\geq cN^\gamma}\mathcal{G}'_N\mathcal{N}_{\geq cN^\gamma}(\mathcal{N}_+ + 1)^{k+1}\xi\rangle.\end{aligned} \tag{6.21}$$

On the other hand, we can estimate

$$\begin{aligned}&\|(\mathcal{H}_N + 1)^{1/2}\mathcal{N}^2_{\geq cN^\gamma}(\mathcal{N}_+ + 1)^{k+1}\xi\| \\ &\quad\leq N\|(\mathcal{K} + 1)^{1/2}\mathcal{N}_{\geq cN^\gamma}(\mathcal{N}_+ + 1)^{k+1}\xi\| + \|\mathcal{V}_N^{1/2}\mathcal{N}^2_{\geq cN^\gamma}(\mathcal{N}_+ + 1)^{k+1}\xi\|.\end{aligned} \tag{6.22}$$

Expressing $\mathcal{V}_N$ in position space, we find, with $\phi = \mathcal{N}_{\geq cN^\gamma}(\mathcal{N}_+ + 1)^{k+1}\xi$,

$$\|\mathcal{V}_N^{1/2}\mathcal{N}_{\geq cN^\gamma}\phi\|^2 = \int dxdy\, N^{2-2\kappa}V(N^{1-\kappa}(x-y))\|\check{a}_x\check{a}_y\mathcal{N}_{\geq cN^\gamma}\phi\|^2 \tag{6.23}$$

We have

$$\check{a}_x \mathcal{N}_{\geq cN^\gamma} = (\mathcal{N}_{\geq cN^\gamma} + 1)\check{a}_x - a(\check{\chi}_x)$$

where

$$\check{\chi}_x(y) = \check{\chi}(y-x) = \sum_{p\in\Lambda_+^*:|p|\leq cN^\gamma} e^{ip\cdot(x-y)}$$

is such that $\|\check{\chi}_x\| = \|\chi\| \leq CN^{3\gamma/2}$. Hence, we find

$$\|\check{a}_x\check{a}_y\mathcal{N}_{\geq cN^\gamma}\phi\| \leq N\|\check{a}_x\check{a}_y\phi\| + N^{1/2}\|\check{\chi}_x\|\|\check{a}_y\phi\| + N^{1/2}\|\check{\chi}_y\|\|\check{a}_x\phi\|.$$

Inserting in (6.23), we find

$$\|\mathcal{V}_N^{1/2}\mathcal{N}_{\geq cN^\gamma}\phi\|^2 \leq CN^2\|\mathcal{V}_N^{1/2}\phi\|^2 + CN^{3\gamma+\kappa}\|\mathcal{N}_+^{1/2}\phi\|^2.$$

From (6.22), we conclude that

$$\|(\mathcal{H}_N+1)^{1/2}\mathcal{N}_{\geq cN^\gamma}^2\mathcal{N}_+^{k+1}\xi\| \leq N\|(\mathcal{H}_N+1)^{1/2}\mathcal{N}_{\geq cN^\gamma}(\mathcal{N}_++1)^{k+1}\xi\|$$

for all $\gamma \leq \alpha = 14\kappa + 4\varepsilon$, if $\kappa < 1/43$. Using now similar arguments as before (6.21), we conclude that together with (6.21), we have

$$\begin{aligned}
&|\langle\xi, \mathrm{B}\xi\rangle| \\
&\quad\leq \frac{1}{2}\langle\xi, (\mathcal{N}_++1)^{k+1}\mathcal{N}_{\geq cN^\gamma}\mathcal{G}_N'\mathcal{N}_{\geq cN^\gamma}(\mathcal{N}_++1)^{k+1}\xi\rangle \\
&\qquad + CN^{2+10\kappa/3+2\varepsilon/3-\gamma}\|(\mathcal{H}_N+1)^{1/2}(\mathcal{N}_++1)^{k+1}\xi\|\|(\mathcal{H}_N+1)^{1/2}(\mathcal{N}_++1)^k\xi\| \\
&\qquad + CN^{2+14\kappa/3+4\varepsilon/3}\|(\mathcal{H}_N+1)^{1/2}(\mathcal{N}_++1)^k\xi\|^2 \\
&\qquad + C(N^{1+17\kappa/2+2\varepsilon-2\gamma} + N^{1+3\kappa/2-\gamma/2})\|(\mathcal{H}_N+1)^{1/2}(\mathcal{N}_++1)^{k+1}\xi\|^2
\end{aligned}$$

Combining this with (6.20), we arrive at

$$\begin{aligned}
&\frac{\langle\xi, (\mathcal{N}_++1)^{k+1}\mathcal{N}_{\geq cN^\gamma}\mathcal{G}_N'\mathcal{N}_{\geq cN^\gamma}(\mathcal{N}_++1)^{k+1}\xi\rangle}{\|\xi\|^2} \\
&\quad\leq \left[N^{2-2\gamma}\zeta^2 + N^{2+10\kappa/3+2\varepsilon/3-\gamma}\right]\left[\sup_{\xi\in Q_\zeta}\frac{\langle\xi, (\mathcal{N}_++1)^{2k}(\mathcal{H}_N+1)\xi\rangle}{\|\xi\|^2}\right]^{1/2} \\
&\qquad\times\left[\sup_{\xi\in Q_\zeta}\frac{\langle\xi, (\mathcal{N}_++1)^{2(k+1)}(\mathcal{H}_N+1)\xi\rangle}{\|\xi\|^2}\right]^{1/2} \\
&\qquad + CN^{2+14\kappa/3+4\varepsilon/3}\left[\sup_{\xi\in Q_\zeta}\frac{\langle\xi, (\mathcal{N}_++1)^{2k}(\mathcal{H}_N+1)\xi\rangle}{\|\xi\|^2}\right] \\
&\qquad + C(N^{1+17\kappa/2+2\varepsilon-2\gamma} + N^{1+3\kappa/2-\gamma/2})\left[\sup_{\xi\in Q_\zeta}\frac{\langle\xi, (\mathcal{N}_++1)^{2(k+1)}(\mathcal{H}_N+1)\xi\rangle}{\|\xi\|^2}\right]
\end{aligned}$$

for all $\xi \in Q_z$. With (6.8), we obtain

$$
\begin{aligned}
&\frac{N^{-1}\langle \xi, (\mathcal{N}_+ +1)^{k+1} \mathcal{K} \mathcal{N}_{\geq cN^\gamma} (\mathcal{N}_+ +1)^{k+1} \xi\rangle}{\|\xi\|^2} \\
&\quad \leq\; C N^{\kappa - 2\gamma} \frac{\langle \xi, (\mathcal{N}_+ +1)^{k+1} \mathcal{K} (\mathcal{N}_+ +1)^{k+1} \xi\rangle}{\|\xi\|^2} \\
&\qquad + C \left[N^{-\kappa} \zeta^2 + N^{\gamma + 7\kappa/3 + 2\varepsilon/3} \right] \left[\sup_{\xi \in Q_\zeta} \frac{\langle \xi, (\mathcal{N}_+ +1)^{2k} (\mathcal{H}_N +1) \xi\rangle}{\|\xi\|^2} \right]^{1/2} \\
&\qquad \times \left[\sup_{\xi \in Q_\zeta} \frac{\langle \xi, (\mathcal{N}_+ +1)^{2(k+1)} (\mathcal{H}_N +1) \xi\rangle}{\|\xi\|^2} \right]^{1/2} \\
&\qquad + C N^{2\gamma + 11\kappa/3 + 4\varepsilon/3} \left[\sup_{\xi \in Q_\zeta} \frac{\langle \xi, (\mathcal{N}_+ +1)^{2k} (\mathcal{H}_N +1) \xi\rangle}{\|\xi\|^2} \right] \\
&\qquad + C (N^{15\kappa/2 + 2\varepsilon - 1} + N^{\kappa/2 + 3\gamma/2 - 1}) \left[\sup_{\xi \in Q_\zeta} \frac{\langle \xi, (\mathcal{N}_+ +1)^{2(k+1)} (\mathcal{H}_N +1) \xi\rangle}{\|\xi\|^2} \right].
\end{aligned}
$$

Applying this bound to (5.8) and recalling that $\kappa < 1/43$, we conclude that

$$
\begin{aligned}
&\frac{N^{\kappa}\langle \xi, (\mathcal{N}_+ +1)^{k+1} e^A e^D \mathcal{E}_{\mathcal{M}_N} e^{-D} e^{-A} (\mathcal{N}_+ +1)^{k+1} \xi\rangle}{\|\xi\|^2} \\
&\quad \geq\; -C N^{-\varepsilon} \left[\sup_{\xi \in Q_\zeta} \frac{\langle \xi, (\mathcal{H}_N +1)(\mathcal{N}_+ +1)^{2(k+1)} \xi\rangle}{\|\xi\|^2} \right] \\
&\qquad - C \left[N^{20\kappa + 5\varepsilon} \zeta^2 + N^{44\kappa + 12\varepsilon} \right] \left[\sup_{\xi \in Q_\zeta} \frac{\langle \xi, (\mathcal{N}_+ +1)^{2k} (\mathcal{H}_N +1) \xi\rangle}{\|\xi\|^2} \right]^{1/2} \\
&\qquad \times \left[\sup_{\xi \in Q_\zeta} \frac{\langle \xi, (\mathcal{N}_+ +1)^{2(k+1)} (\mathcal{H}_N +1) \xi\rangle}{\|\xi\|^2} \right]^{1/2}.
\end{aligned}
$$

Therefore, for any $\delta > 0$, we find (if N is large enough)

$$
\begin{aligned}
&\frac{N^{\kappa}\langle \xi, (\mathcal{N}_+ +1)^{k+1} e^A e^D \mathcal{E}_{\mathcal{M}_N} e^{-D} e^{-A} (\mathcal{N}_+ +1)^{k+1} \xi\rangle}{\|\xi\|^2} \\
&\quad \geq -\delta \sup_{\xi \in Q_\zeta} \frac{\langle \xi, (\mathcal{H}_N +1)(\mathcal{N}_+ +1)^{2(k+1)} \xi\rangle}{\|\xi\|^2} \\
&\qquad - C\delta^{-1} \left[N^{20\kappa + 5\varepsilon} \zeta^2 + N^{44\kappa + 12\varepsilon} \right]^2 \sup_{\xi \in Q_\zeta} \frac{\langle \xi, (\mathcal{H}_N +1)(\mathcal{N}_+ +1)^{2k} \xi\rangle}{\|\xi\|^2}.
\end{aligned}
$$

From the last bound, (6.19) and (6.17), we obtain

$$\begin{aligned}
&\frac{\langle \xi, (\mathcal{N}_+ + 1)^{2(k+1)}(\mathcal{H}_N+1)\xi\rangle}{\|\xi\|^2}\\
&\quad\le \delta \sup_{\xi\in Q_\zeta} \frac{\langle \xi, (\mathcal{N}_+ + 1)^{2(k+1)}(\mathcal{H}_N+1)\xi\rangle}{\|\xi\|^2}\\
&\qquad + C\delta^{-1}\left[N^{20\kappa+5\varepsilon}\zeta^2 + N^{44\kappa+12\varepsilon}\right]^2 \sup_{\xi\in Q_\zeta} \frac{\langle \xi., (\mathcal{N}_+ + 1)^{2k}(\mathcal{H}_N+1)\xi\rangle}{\|\xi\|^2}
\end{aligned}$$

for any $\xi \in Q_\zeta$. Taking the supremum over all $\xi \in Q_\zeta$, and choosing $\delta > 0$ small enough, we arrive at

$$\begin{aligned}
&\sup_{\xi\in Q_\zeta} \frac{\langle \xi, (\mathcal{N}_+ + 1)^{2(k+1)}(\mathcal{H}_N+1)\xi\rangle}{\|\xi\|^2}\\
&\quad\le C\left[N^{20\kappa+5\varepsilon}\zeta^2 + N^{44\kappa+12\varepsilon}\right]^2 \sup_{\xi\in Q_\zeta} \frac{\langle \xi, (\mathcal{N}_+ + 1)^{2k}(\mathcal{H}_N+1)\xi\rangle}{\|\xi\|^2}\\
&\quad\le C\left[N^{20\kappa+5\varepsilon}\zeta^2 + N^{44\kappa+12\varepsilon}\right]^{2k+1}
\end{aligned}$$

by the induction assumption. □

7. Analysis of $\mathcal{M}_N$

This section is devoted to the proof of Proposition 5.1. In Sect. 7.1 we establish bounds on the growth of the number of excitations and of their energy with respect to the action of e^D, with the quartic operator $D = D_1 - D_1^*$ with

$$D_1 = \frac{1}{2N} \sum_{r\in P_H, p,q\in P_L} \eta_r a^*_{p+r} a^*_{q-r} a_p a_q \tag{7.1}$$

as defined in (5.3). In Sect. 7.2, we compute the different parts of the excitation Hamiltonian $\mathcal{M}_N$, introduced in (5.5). Finally, in Sect. 7.3, we conclude the proof of Proposition 5.1.

7.1. Growth of Number and Energy of Excitations

The first lemma of this section controls the growth of the number of excitations with high momentum.

Lemma 7.1. *Assume the exponents α, β satisfy (5.6). Let $k \in \mathbb{N}_0$, $m = 1, 2, 3$, $0 < \gamma \le \alpha$ and $c > 0$ ($c < 1$ if $\gamma = \alpha$). Then, there exists a constant $C > 0$ such that*

$$e^{-sD}(\mathcal{N}_+ + 1)^k(\mathcal{N}_{\ge cN^\gamma} + 1)^m e^{sD} \le C(\mathcal{N}_+ + 1)^k(\mathcal{N}_{\ge cN^\gamma} + 1)^m, \tag{7.2}$$

for all $s \in [-1; 1]$ and all $N \in \mathbb{N}$ large enough.

Proof. Since $[\mathcal{N}_+, \mathcal{N}_{\ge cN^\gamma}] = 0$ and $[\mathcal{N}_+, D] = 0$, it is enough to prove the lemma for $k = 0$. We consider first $m = 1$. For $\xi \in \mathcal{F}_+^{\le N}$, we define the function $\varphi_\xi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_\xi(s) = \langle \xi, e^{-sD}(\mathcal{N}_{\ge cN^\gamma} + 1)e^{sD}\xi\rangle$$

so that differentiating yields

$$\partial_s \varphi_\xi(s) = 2\mathrm{Re}\,\langle e^{sD}\xi, \big[\mathcal{N}_{\geq cN^\gamma}, D_1\big] e^{sD}\xi\rangle \tag{7.3}$$

with D_1 as in (7.1). By assumption, $N^\alpha \geq N^\alpha - N^\beta \geq cN^\gamma$ for sufficiently large $N \in \mathbb{N}$. This implies that

$$[\mathcal{N}_{\geq cN^\gamma}, a^*_{p+r}] = a^*_{p+r}, \quad [\mathcal{N}_{\geq cN^\gamma}, a^*_{q-r}] = a^*_{q-r}$$

for $r \in P_H$ and $p, q \in P_L$, by (2.1) and (2.10). We then compute

$$\big[\mathcal{N}_{\geq cN^\gamma}, D_1\big] = \frac{1}{N}\sum_{r\in P_H, p,q\in P_L} \eta_r a^*_{p+r}a^*_{q-r}a_p a_q - \frac{1}{N}\sum_{\substack{r\in P_H, p,q\in P_L,\\ |p|\geq cN^\gamma}} \eta_r a^*_{p+r}a^*_{q-r}a_p a_q. \tag{7.4}$$

and apply Cauchy–Schwarz to obtain

$$\begin{aligned}|\partial_s\varphi_\xi(s)| &\leq \frac{C}{N}\Bigg(\sum_{\substack{r\in P_H, p,q\in P_L,\\ |p+r|\geq cN^\gamma, |q-r|\geq cN^\gamma}} \|a_{p+r}(\mathcal{N}_{\geq cN^\gamma}+1)^{-1/2}a_{q-r}e^{sD}\xi\|^2\Bigg)^{1/2}\\ &\qquad\times \|\eta_H\|\Bigg(\sum_{p,q\in P_L}\|a_p(\mathcal{N}_{\geq cN^\gamma}+1)^{1/2}a_q e^{sD}\xi\|^2\Bigg)^{1/2}\\ &\leq CN^{\kappa+3\beta/2-\alpha/2}\varphi_\xi(s) \leq C\varphi_\xi(s).\end{aligned} \tag{7.5}$$

Since the bound is independent of $\xi \in \mathcal{F}_+^{\leq N}$ and it also holds true if we replace D by $-D$ in the definition of φ_ξ, this proves (7.2), for $m = 1$.

For $m = 3$, we define

$$\psi_\xi(s) = \langle \xi, e^{-sD}(\mathcal{N}_{\geq cN^\gamma}+1)^3 e^{sD}\xi\rangle$$

with derivative

$$\partial_s\psi_\xi(s) = 2\mathrm{Re}\,\langle e^{sD}\xi, [(\mathcal{N}_{\geq cN^\gamma}+1)^3, D_1]e^{sD}\xi\rangle$$

We have

$$\begin{aligned}[(\mathcal{N}_{\geq cN^\gamma}+1)^3, D_1] &= 3(\mathcal{N}_{\geq cN^\gamma}+1)[\mathcal{N}_{\geq cN^\gamma}, D_1](\mathcal{N}_{\geq cN^\gamma}+1)\\ &\quad + [\mathcal{N}_{\geq cN^\gamma},[\mathcal{N}_{\geq cN^\gamma},[\mathcal{N}_{\geq cN^\gamma}, D_1]]].\end{aligned} \tag{7.6}$$

The contribution of the first term on the r.h.s. of (7.6) can be controlled as in (7.5) (replacing $e^{sD}\xi$ with $(\mathcal{N}_{\geq cN^\gamma}+1)e^{sD}\xi$). With (7.4) and using again that $N^\alpha \geq N^\alpha - N^\beta \geq cN^\gamma$, we obtain that

$$\begin{aligned}&[\mathcal{N}_{\geq cN^\gamma},[\mathcal{N}_{\geq cN^\gamma},[\mathcal{N}_{\geq cN^\gamma}, D_1]]]\\ &\quad= \frac{4}{N}\sum_{r\in P_H, p,q\in P_L}\eta_r a^*_{p+r}a^*_{q-r}a_p a_q - \frac{7}{N}\sum_{\substack{r\in P_H, p,q\in P_L,\\ |p|\geq cN^\gamma}}\eta_r a^*_{p+r}a^*_{q-r}a_p a_q\\ &\qquad + \frac{3}{N}\sum_{\substack{r\in P_H, p,q\in P_L,\\ |p|,|q|\geq cN^\gamma}}\eta_r a^*_{p+r}a^*_{q-r}a_p a_q.\end{aligned}$$

All these contributions can be controlled like those in (7.4). We conclude that

$$|\partial_s \psi_\xi(s)| \leq C\psi_\xi(s)$$

This proves (7.2) with $m = 3$. The case $m = 2$ follows by operator monotonicity of the function $x \mapsto x^{2/3}$. □

Next, we prove bounds for the growth of the low-momentum part of the kinetic energy, defined as in (4.17).

Lemma 7.2. *Assume the exponents α, β satisfy (5.6). Let $0 < \gamma_1, \gamma_2 \leq \alpha$, $c_1, c_2 \geq 0$ (and $c_j \leq 1$ if $\gamma_j = \alpha$, for $j = 1, 2$). Then, there exists a constant $C > 0$ such that*

$$\begin{aligned} e^{-sD}\mathcal{K}_{\leq c_1 N^{\gamma_1}} e^{sD} &\leq \mathcal{K}_{\leq c_1 N^{\gamma_1}} + N^{2\beta-1}(\mathcal{N}_{\geq \frac{1}{2}N^\alpha} + 1)^2, \\ e^{-sD}\mathcal{K}_{\leq c_1 N^{\gamma_1}}(\mathcal{N}_{\geq c_2 N^{\gamma_2}} + 1)e^{sD} &\leq \mathcal{K}_{\leq c_1 N^{\gamma_1}}(\mathcal{N}_{\geq c_2 N^{\gamma_2}} + 1) \\ &\quad + N^{2\beta-1}(\mathcal{N}_{\geq c_2 N^{\gamma_2}} + 1)^2(\mathcal{N}_{\geq \frac{1}{2}N^\alpha} + 1) \end{aligned} \tag{7.7}$$

for all $s \in [-1; 1]$ and all $N \in \mathbb{N}$ sufficiently large.

Proof. Fix $\xi \in \mathcal{F}_+^{\leq N}$ and define $\varphi_\xi : \mathbb{R} \to \mathbb{R}$ by $\varphi_\xi(s) = \langle \xi, e^{-sD}\mathcal{K}_{\leq c_1 N^{\gamma_1}} e^{sD}\xi\rangle$ such that

$$\partial_s \varphi_\xi(s) = 2\mathrm{Re}\,\langle \xi, e^{-sD}[\mathcal{K}_{\leq c_1 N^{\gamma_1}}, D_1]e^{sD}\xi\rangle.$$

We notice that

$$\left[\mathcal{K}_{\leq c_1 N^{\gamma_1}}, a^*_{p+r}\right] = \left[\mathcal{K}_{\leq c_1 N^{\gamma_1}}, a^*_{q-r}\right] = 0$$

if $r \in P_H$ and $p, q \in P_L$, because $|r|, |p+r|, |q-r| \geq N^\alpha - N^\beta > c_1 N^{\gamma_1}$ for $N \in \mathbb{N}$ large enough.

Using (2.1), we then compute

$$[\mathcal{K}_{\leq c_1 N^{\gamma_1}}, D_1] = -\frac{1}{N}\sum_{r\in P_H, p,q\in P_L : |p|\leq c_1 N^{\gamma_1}} p^2 \eta_r a^*_{p+r}a^*_{q-r}a_p a_q. \tag{7.8}$$

and, using that $|p| \leq N^\beta$ for $p \in P_L$, we obtain with Cauchy–Schwarz

$$\begin{aligned} &\left|\langle \xi, e^{-sD}[\mathcal{K}_{\leq c_1 N^{\gamma_1}}, D_1]e^{sD}\xi\rangle\right| \\ &\quad\leq \frac{CN^\beta}{N}\sum_{r\in P_H, p,q\in P_L : |p|\leq c_1 N^{\gamma_1}} |p||\eta_r|\|a_{r+p}a_{q-r}e^{sD}\xi\|\|a_p a_q e^{sD}\xi\| \\ &\quad\leq CN^{5\beta/2+\kappa-\alpha/2-1/2}\|(\mathcal{N}_{\geq \frac{1}{2}N^\alpha} + 1)e^{sD}\xi\|\|\mathcal{K}^{1/2}_{\leq c_1 N^{\gamma_1}} e^{sD}\xi\|. \end{aligned} \tag{7.9}$$

With Lemma 7.1 choosing $c = \frac{1}{2}$ and $\gamma = \alpha$, this implies for $N \in \mathbb{N}$ large enough that

$$\begin{aligned} \partial_s \varphi_\xi(s) &\leq CN^{5\beta/2+\kappa-\alpha/2-1/2}\|(\mathcal{N}_{\geq \frac{1}{2}N^\alpha} + 1)e^{sD}\xi\|\|\mathcal{K}^{1/2}_{\leq c_1 N^{\gamma_1}} e^{sD}\xi\| \\ &\leq CN^{2\beta-1}\langle \xi, (\mathcal{N}_{\geq \frac{1}{2}N^\alpha} + 1)^2\xi\rangle + C\varphi_\xi(s). \end{aligned}$$

This proves the first inequality in (7.7), by Gronwall's lemma and $\alpha > 3\beta + 2\kappa \geq 0$.

Next, let us prove the second inequality in (7.7). We define $\psi_\xi : \mathbb{R} \to \mathbb{R}$ by

$$\psi_\xi(s) = \langle \xi, e^{-sD} \mathcal{K}_{\leq c_1 N^{\gamma_1}} (\mathcal{N}_{\geq c_2 N^{\gamma_2}} + 1) e^{sD} \xi \rangle,$$

and we compute

$$\begin{aligned}\partial_s \psi_\xi(s) &= 2\mathrm{Re}\, \langle \xi, e^{-sD} \big[\mathcal{K}_{\leq c_1 N^{\gamma_1}}, D_1 \big] (\mathcal{N}_{\geq c_2 N^{\gamma_2}} + 1) e^{sD} \xi \rangle \\ &\quad + 2\mathrm{Re}\, \langle \xi, e^{-sD} \mathcal{K}_{\leq c_1 N^{\gamma_1}} \big[\mathcal{N}_{\geq c_2 N^{\gamma_2}}, D_1 \big] e^{sD} \xi \rangle.\end{aligned}$$

First, we proceed as in (7.9) and obtain with (4.7) that

$$\begin{aligned}&\big| \langle \xi, e^{-sD} [\mathcal{K}_{\leq c_1 N^{\gamma_1}}, D_1] (\mathcal{N}_{\geq c_2 N^{\gamma_2}} + 1) e^{sD} \xi \rangle \big| \\ &\quad \leq \frac{CN^\beta}{N} \sum_{\substack{r \in P_H, p, q \in P_L: \\ |p| \leq c_1 N^{\gamma_1}}} |p| |\eta_r| \| a_{r+p} a_{q-r} (\mathcal{N}_{\geq c_2 N^{\gamma_2}} + 1)^{1/2} e^{sD} \xi \| \\ &\qquad \times \| a_q a_p (\mathcal{N}_{\geq c_2 N^{\gamma_2}} + 1)^{1/2} e^{sD} \xi \| \\ &\quad \leq CN^{5\beta/2 + \kappa - \alpha/2 - 1/2} \| (\mathcal{N}_{\geq c_2 N^{\gamma_2}} + 1) (\mathcal{N}_{\geq \frac{1}{2} N^\alpha} + 1)^{1/2} e^{sD} \xi \| \\ &\qquad \times \| \mathcal{K}^{1/2}_{\leq c_1 N^{\gamma_1}} (\mathcal{N}_{\geq c_2 N^{\gamma_2}} + 1)^{1/2} e^{sD} \xi \|.\end{aligned}$$

Here, we used in the last step that $[a_{q-r}, \mathcal{N}_{\geq c_2 N^{\gamma_2}}] = a_{q-r}$ for $r \in P_H$, $q \in P_L$ and that $\mathcal{N}_{c_2 N^{\gamma_2}} \geq \mathcal{N}_{N^\alpha - N^\beta}$ for $N \in \mathbb{N}$ large enough. The last bound and Lemma 7.1 imply that

$$\begin{aligned}&\big| \langle \xi, e^{-sD} [\mathcal{K}_{\leq c_1 N^{\gamma_1}}, D_1] (\mathcal{N}_{\geq c_2 N^{\gamma_2}} + 1) e^{sD} \xi \rangle \big| \\ &\quad \leq CN^{2\beta - 1} \langle \xi, (\mathcal{N}_{\geq c_2 N^{\gamma_2}} + 1)^2 (\mathcal{N}_{\geq \frac{1}{2} N^\alpha} + 1) \xi \rangle + C \psi_\xi(s).\end{aligned} \tag{7.10}$$

Next, we recall the identity (7.4) and that

$$\big[\mathcal{K}_{\leq c_1 N^{\gamma_1}}, a^*_{p+r} \big] = \big[\mathcal{K}_{\leq c_1 N^{\gamma_1}}, a^*_{q-r} \big] = 0$$

whenever $r \in P_H, p, q \in P_L$ and $N \in \mathbb{N}$ is sufficiently large. We then obtain

$$\begin{aligned}&\big| \langle \xi, e^{-sD} \mathcal{K}_{\leq c_1 N^{\gamma_1}} \big[\mathcal{N}_{\geq c_2 N^{\gamma_2}}, D_1 \big] e^{sD} \xi \rangle \big| \\ &\quad \leq \frac{C}{N} \sum_{\substack{r \in P_H, p, q \in P_L, \\ v \in \Lambda^*_+ : |v| \leq c_1 N^{\gamma_1}}} |v|^2 |\eta_r| \| a_{r+p} (\mathcal{N}_{\geq c_2 N^{\gamma_2}} + 1)^{-1/2} a_{q-r} a_v e^{sD} \xi \| \\ &\qquad \times \| a_p a_q (\mathcal{N}_{\geq c_2 N^{\gamma_2}} + 1)^{1/2} a_v e^{sD} \xi \| \\ &\quad \leq CN^{3\beta/2 + \kappa - \alpha/2} \langle e^{sD} \xi, \mathcal{K}_{\leq c_1 N^{\gamma_1}} (\mathcal{N}_{\geq c_2 N^{\gamma_2}} + 1) e^{sD} \xi \rangle \leq C \psi_\xi(s).\end{aligned} \tag{7.11}$$

Hence, putting (7.10) and (7.11) together, we have proved that

$$\partial_s \psi_\xi(s) \leq CN^{2\beta - 1} \langle \xi, (\mathcal{N}_{\geq c_2 N^{\gamma_2}} + 1)^2 (\mathcal{N}_{\geq \frac{1}{2} N^\alpha} + 1) \xi \rangle + C \psi_\xi(s),$$

which implies the second bound in (7.7), by Gronwall's lemma. □

It will also be important to control the potential energy operator, restricted to low momenta. We define

$$\mathcal{V}_{N,L} = \frac{1}{2N} \sum_{\substack{u \in \Lambda^*, p, q \in \Lambda^*_+: \\ p+u, q+u, p, q \in P_L}} N^\kappa \widehat{V}(u/N^{1-\kappa}) a^*_{p+u} a^*_q a_p a_{q+u}. \tag{7.12}$$

Notice that $\mathcal{V}_{N,L} = \mathcal{V}_{N,L}^*$ by symmetry of the momentum restrictions. To calculate $e^D \mathcal{V}_{N,L} e^{-D}$, we will use the next lemma, which will also be useful in the next subsections.

Lemma 7.3. *Assume the exponents α, β satisfy (5.6). Let $F = (F_p)_{p\in\Lambda_+^*} \in \ell^\infty(\Lambda_+^*)$ and define*

$$\mathrm{Z} = \frac{1}{2N} \sum_{\substack{u\in\Lambda^*, p,q\in\Lambda_+^*:\\ p+u,q+u,p,q\in P_L}} F_u a^*_{p+u} a^*_q a_p a_{q+u} \tag{7.13}$$

Then, there exists a constant $C > 0$ such that

$$\pm\big(e^{-sD}\mathrm{Z}e^{sD} - Z\big) \le C\|F\|_\infty N^{\beta-1}\mathcal{K}_L(\mathcal{N}_{\ge\frac{1}{2}N^\alpha}+1) + C\|F\|_\infty N^{3\beta-2}(\mathcal{N}_{\ge\frac{1}{2}N^\alpha}+1)^3 \tag{7.14}$$

for all $s \in [-1;1]$, and for all $N \in \mathbb{N}$ sufficiently large.

Proof. Given $\xi \in \mathcal{F}_+^{\le N}$, we define $\varphi_\xi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_\xi(s) = \langle \xi, e^{-sD}\mathrm{Z}e^{sD}\xi\rangle,$$

which has derivative

$$\partial_s\varphi_\xi(s) = 2\mathrm{Re}\,\langle \xi, e^{-sD}[\mathrm{Z}, D_1]e^{sD}\xi\rangle.$$

By assumption, we have $\alpha > 3\beta+2\kappa$ so that $|r|, |v+r|, |w-r| \ge N^\alpha - N^\beta > N^\beta$ if $r \in P_H$ and $v, w \in P_L$, for sufficiently large $N \in \mathbb{N}$. This implies in particular that

$$[a_p a_{q+u}, a^*_{v+r}a^*_{w-r}] = 0$$

whenever $q+u, p \in P_L$ and $r \in P_H$, $v, w \in P_L$. As a consequence, we find

$$\begin{aligned} [\mathrm{Z}, D_1] = &-\frac{1}{2N^2} \sum_{\substack{u\in\Lambda^*, r\in P_H, v,w\in P_L:\\ w-u,v+u\in P_L}} F_u \eta_r a^*_{v+r}a^*_{w-r}a_{w-u}a_{v+u} \\ &- \frac{1}{N^2} \sum_{\substack{u\in\Lambda^*, r\in P_H, v,w,p\in P_L:\\ p+u,v+u\in P_L}} F_u \eta_r a^*_{v+r}a^*_{w-r}a^*_{p+u}a_w a_{v+u}a_p. \end{aligned} \tag{7.15}$$

With (4.7) and $N^\alpha - N^\beta > \frac{1}{2}N^\alpha$ for $N \in \mathbb{N}$ large enough, we can bound

$$\begin{aligned} &\Bigg|\frac{1}{N^2} \sum_{\substack{u\in\Lambda^*, r\in P_H, v,w\in P_L:\\ w-u,v+u\in P_L}} F_u\eta_r\langle e^{sD}\xi, a^*_{v+r}a^*_{w-r}a_{w-u}a_{v+u}e^{sD}\xi\rangle\Bigg| \\ &\le \frac{C\|F\|_\infty}{N^2}\Bigg(\sum_{\substack{u\in\Lambda^*, r\in P_H, v,w\in P_L:\\ w-u,v+u\in P_L}} |v+u|^{-2}\|a_{v+r}(\mathcal{N}_{\ge\frac{1}{2}N^\alpha}+1)^{-1/2}a_{w-r}e^{sD}\xi\|^2\Bigg)^{1/2} \\ &\quad\times\Bigg(\sum_{\substack{u\in\Lambda^*, r\in P_H, v,w\in P_L:\\ w-u,v+u\in P_L}} \eta_r^2|v+u|^2\|a_{w-u}(\mathcal{N}_{\ge\frac{1}{2}N^\alpha}+1)^{1/2}a_{v+u}e^{sD}\xi\|^2\Bigg)^{1/2} \\ &\le C\|F\|_\infty N^{7\beta/2+\kappa-\alpha/2-3/2}\|(\mathcal{N}_{\ge\frac{1}{2}N^\alpha}+1)^{1/2}e^{sD}\xi\|\|\mathcal{K}_L^{1/2}(\mathcal{N}_{\ge\frac{1}{2}N^\alpha}+1)^{1/2}e^{sD}\xi\|. \end{aligned}$$

and

$$\begin{aligned}
&\Bigg| \frac{1}{N^2} \sum_{\substack{u\in\Lambda^*, r\in P_H, v,w,p\in P_L:\\ p+u, v+u\in P_L}} F_u \eta_r \langle e^{sD}\xi, a^*_{v+r} a^*_{w-r} a^*_{p+u} a_w a_{v+u} a_p e^{sD}\xi\rangle \Bigg| \\
&\quad \le \frac{C\|F\|_\infty}{N^2} \Bigg(\sum_{\substack{u\in\Lambda^*, r\in P_H, v,w,p\in P_L:\\ p+u, v+u\in P_L}} |p+u|^2 |p|^{-2} \\
&\qquad \times \|a_{v+r} (\mathcal{N}_{\ge \frac{1}{2}N^\alpha}+1)^{-1/2} a_{w-r} a_{p+u} e^{sD}\xi\|^2 \Bigg)^{1/2} \\
&\qquad \times \Bigg(\sum_{\substack{u\in\Lambda^*, r\in P_H, v,w,p\in P_L:\\ p+u, v+u\in P_L}} \eta_r^2 |p|^2 |p+u|^{-2} \\
&\qquad \times \|a_w (\mathcal{N}_{\ge \frac{1}{2}N^\alpha}+1)^{1/2} a_{v+u} a_p e^{sD}\xi\|^2 \Bigg)^{1/2} \\
&\quad \le C\|F\|_\infty N^{5\beta/2+\kappa-\alpha/2-1} \langle \xi, e^{-sD} \mathcal{K}_L (\mathcal{N}_{\ge \frac{1}{2}N^\alpha}+1) e^{sD}\xi\rangle.
\end{aligned}$$

Lemmas 7.1, 7.2 and the assumption $\alpha > 3\beta + 2\kappa \ge 0$ implies

$$\begin{aligned}
\pm \partial_s \varphi_s(\xi) &\le C\|F\|_\infty N^{\beta-1} \langle \xi, \mathcal{K}_L (\mathcal{N}_{\ge \frac{1}{2}N^\alpha}+1)\xi\rangle \\
&\quad + C\|F\|_\infty N^{3\beta-2} \langle \xi, (\mathcal{N}_{\ge \frac{1}{2}N^\alpha}+1)^3 \xi\rangle.
\end{aligned}$$

Hence, integrating the last equation from zero to $s \in [-1; 1]$ proves the lemma. □

With $\sup_{p\in\Lambda^*} |N^\kappa \widehat{V}(p/N^{1-\kappa})| \le CN^\kappa$, we obtain immediately the following result.

Corollary 7.4. *Assume the exponents α, β satisfy (5.6). Then, there exists a constant $C > 0$ such that*

$$\begin{aligned}
\pm\big(e^{-sD} \mathcal{V}_{N,L} e^{sD} - \mathcal{V}_{N,L}\big) &\le C N^{\beta+\kappa-1} \mathcal{K}_L (\mathcal{N}_{\ge \frac{1}{2}N^\alpha}+1) \\
&\quad + CN^{3\beta+\kappa-2} (\mathcal{N}_{\ge \frac{1}{2}N^\alpha}+1)^3
\end{aligned}$$

for all $s \in [-1; 1]$, and for all $N \in \mathbb{N}$ sufficiently large.

We also need rough bounds for the conjugation of the full potential energy operator $\mathcal{V}_N$. To this end, we will make use of the following estimate for the commutator of $\mathcal{V}_N$ with $D = D_1 - D_1^*$, with D_1 defined in (7.1).

Proposition 7.5. *Assume the exponents α, β satisfy (5.6). Then,*

$$\begin{aligned}
[\mathcal{V}_N, D] &= \frac{1}{2N} \sum_{\substack{u\in\Lambda^*_+, p,q\in P_L:\\ p+u, q-u\neq 0}} N^\kappa (\widehat{V}(./N^{1-\kappa}) * \eta/N)(u) \big(a^*_{p+u} a^*_{q-u} a_p a_q + \text{h.c.}\big) \\
&\quad + \mathcal{E}_{[\mathcal{V}_N, D]}
\end{aligned} \tag{7.16}$$

and there exists a constant $C > 0$ such that

$$\begin{aligned} \pm\mathcal{E}_{[\mathcal{V}_N,D]} &\leq \delta\mathcal{V}_N + CN^{\alpha+\kappa-1}\mathcal{V}_N + CN^{\alpha+\kappa-1}\mathcal{V}_{N,L} \\ &\quad + \delta^{-1}CN^{\beta+\kappa-1}\mathcal{K}_L(\mathcal{N}_{\geq\frac{1}{2}N^\alpha}+1) + \delta^{-1}CN^{3\beta+\kappa-1}(\mathcal{N}_{\geq\frac{1}{2}N^\alpha}+1)^2 \end{aligned} \tag{7.17}$$

for all $\delta > 0$ and for all $N \in \mathbb{N}$ sufficiently large.

Proof. We have

$$[\mathcal{V}_N, D] = [\mathcal{V}_N, D_1] + \text{h.c.}$$

To compute the commutator $[\mathcal{V}_N, D_1]$, we compute first of all that

$$\begin{aligned} &[a^*_{p+u}a^*_q a_p a_{q+u}, a^*_{v+r}a^*_{w-r}a_v a_w] \\ &\quad = a^*_{p+u}a^*_q a_{q+u}a^*_{w-r}a_v a_w\delta_{p,v+r} + a^*_{p+u}a^*_q a_p a^*_{w-r}a_v a_w\delta_{q+u,v+r} \\ &\qquad + a^*_{p+u}a^*_q a^*_{v+r}a_{q+u}a_v a_w\delta_{p,w-r} + a^*_{p+u}a^*_q a^*_{v+r}a_p a_v a_w\delta_{q+u,w-r} \\ &\qquad - a^*_{v+r}a^*_{w-r}a^*_q a_w a_p a_{q+u}\delta_{p+u,v} - a^*_{v+r}a^*_{w-r}a^*_{p+u}a_w a_p a_{q+u}\delta_{q,v} \\ &\qquad - a^*_{v+r}a^*_{w-r}a_v a^*_q a_p a_{q+u}\delta_{p+u,w} - a^*_{v+r}a^*_{w-r}a_v a^*_{p+u}a_p a_{q+u}\delta_{q,w}. \end{aligned}$$

Putting the terms in the first and last line on the r.h.s. into normal order, we obtain

$$\begin{aligned} [\mathcal{V}_N, D_1] + \text{h.c.} &= \frac{1}{2N}\sum^*_{u\in\Lambda^*,v,w\in P_L} N^\kappa(\widehat{V}(./N^{1-\kappa}) * \eta/N)(u)a^*_{v+u}a^*_{w-u}a_v a_w \\ &\quad + \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 + \text{h.c.}, \end{aligned} \tag{7.18}$$

where

$$\begin{aligned} \Phi_1 &= -\frac{1}{2N^2}\sum^*_{\substack{u\in\Lambda^*,v,w\in P_L,\\ r\in P_H^c\cup\{0\}}} N^\kappa\widehat{V}((u-r)/N^{1-\kappa})\eta_r a^*_{v+u}a^*_{w-u}a_v a_w, \\ \Phi_2 &= -\frac{1}{2N^2}\sum^*_{\substack{u\in\Lambda^*,r\in P_H,\\ v,w\in P_L}} N^\kappa\widehat{V}(u/N^{1-\kappa})\eta_r a^*_{v+r}a^*_{w-r}a_{w-u}a_{v+u}, \\ \Phi_3 &= \frac{1}{N^2}\sum^*_{\substack{u\in\Lambda^*,q\in\Lambda^*_+,\\ r\in P_H,v,w\in P_L}} N^\kappa\widehat{V}(u/N^{1-\kappa})\eta_r a^*_{w-r+u}a^*_{v+r}a^*_q a_{q+u}a_v a_w, \\ \Phi_4 &= -\frac{1}{N^2}\sum^*_{\substack{u\in\Lambda^*,q\in\Lambda^*_+,\\ r\in P_H,v,w\in P_L}} N^\kappa\widehat{V}(u/N^{1-\kappa})\eta_r a^*_{v+r}a^*_{w-r}a^*_q a_w a_{v-u}a_{q+u}. \end{aligned} \tag{7.19}$$

The first term on the r.h.s. in (7.18) appears explicitly in (7.16). Hence, let us estimate the size of the operators Φ_1 to Φ_4, defined in (7.19).

Starting with Φ_1, we switch to position space and find

$$\begin{aligned}|\langle\xi,\Phi_1\xi\rangle| &\leq \frac{1}{N}\sum_{r\in P_H^c\cup\{0\}}|\eta_r|\Big(\int_{\Lambda^2}dxdy\; N^{2-2\kappa}V(N^{1-\kappa}(x-y))\|\check{b}_x\check{a}_y\xi\|^2\Big)^{1/2}\\ &\quad\times\Big(\int_{\Lambda^2}dxdy\; N^{2-2\kappa}V(N^{1-\kappa}(x-y))\Big\|\sum_{w,v\in P_L}e^{ivx+iwy}a_va_w\xi\Big\|^2\Big)^{1/2}\\ &\leq CN^{\alpha+\kappa-1}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{V}_{N,L}^{1/2}\xi\|.\end{aligned} \tag{7.20}$$

The term Φ_2 on the r.h.s. of (7.19) can be controlled by

$$\begin{aligned}|\langle\xi,\Phi_2\xi\rangle| &= \Big|\frac{1}{N}\int_{\Lambda^2}dxdy\; N^{2-2\kappa}V(N^{1-\kappa}(x-y))\\ &\quad\times\sideset{}{^*}\sum_{\substack{r\in P_H,\\ v,w\in P_L}}e^{-iwx}e^{-ivy}\eta_r\langle\xi,a^*_{v+r}a^*_{w-r}\check{a}_x\check{a}_y\xi\rangle\Big|\\ &\leq \frac{CN^{3\beta}\|\eta_H\|}{N}\Big(\int_{\Lambda^2}dxdy\; N^{2-2\kappa}V(N^{1-\kappa}(x-y))\|\check{a}_x\check{a}_y\xi\|^2\Big)^{1/2}\\ &\quad\times\Big(\int_{\Lambda^2}dxdy\; N^{2-2\kappa}V(N^{1-\kappa}(x-y))\sum_{r\in P_H,v,w\in P_L}\|a_{v+r}a_{w-r}\xi\|^2\Big)^{1/2}\\ &\leq CN^{9\beta/2+3\kappa/2-\alpha/2-3/2}\|\mathcal{V}_N^{1/2}\xi\|\|(\mathcal{N}_{\geq\frac{1}{2}N^\alpha}+1)\xi\|.\end{aligned}$$

Finally, the contributions Φ_3 and Φ_4 can be bounded as follows. We obtain

$$\begin{aligned}|\langle\xi,\Phi_3\xi\rangle| &\leq \frac{1}{N}\int_{\Lambda^2}dxdy\; N^{2-2\kappa}V(N^{1-\kappa}(x-y))\sum_{\substack{r\in P_H,\\ v,w\in P_L}}|\eta_r||\langle\xi,a^*_{v+r}\check{a}^*_x\check{a}^*_y\check{a}_ya_va_w\xi\rangle|\\ &\leq \frac{CN^{3\beta/2}\|\eta_H\|}{N}\Big(\int_{\Lambda^2}dxdy\; N^{2-2\kappa}V(N^{1-\kappa}(x-y))\sum_{v\in P_L}|v|^{-2}\|\check{a}_x\check{a}_y\xi\|^2\Big)^{1/2}\\ &\quad\times\Big(N^{\kappa-1}\int_\Lambda dx\sum_{v,w\in P_L}|v|^2\|(\mathcal{N}_{\geq\frac{1}{2}N^\alpha}+1)^{1/2}\check{a}_xa_wa_v\xi\|^2\Big)^{1/2}\\ &\leq CN^{2\beta+3\kappa/2-\alpha/2-1/2}\|\mathcal{V}_N^{1/2}\xi\|\|\mathcal{K}_L^{1/2}(\mathcal{N}_{\geq\frac{1}{2}N^\alpha}+1)^{1/2}\xi\|\end{aligned}$$

as well as

$$\begin{aligned}|\langle\xi,\Phi_4\xi\rangle| &\leq \frac{1}{N}\int_{\Lambda^2}dxdy\; N^{2-2\kappa}V(N^{1-\kappa}(x-y))\\ &\quad\times\sum_{r\in P_H,v,w\in P_L}|\eta_r||\langle\xi,a^*_{v+r}a^*_{w-r}\check{a}^*_ya_w\check{a}_x\check{a}_y\xi\rangle|\\ &\leq \frac{CN^{3\beta/2}\|\eta_H\|}{N}\Big[\int_{\Lambda^2}dxdy\; N^{2-2\kappa}V(N^{1-\kappa}(x-y))\|\check{a}_x\check{a}_y\xi\|^2\Big)^{1/2}\end{aligned}$$

$$\times\left(N^{\kappa-1}\int_\Lambda dy\sum_{\substack{r\in P_H,\\ v,w\in P_L}}\|\check{a}_y a_{v+r}a_{w-r}(\mathcal{N}_++1)^{1/2}\xi\|^2\right)^{1/2}$$
$$\leq CN^{3\beta+3\kappa/2-\alpha/2-1/2}\|\mathcal{V}_N^{1/2}\xi\|\|(\mathcal{N}_{\geq\frac12 N^\alpha}+1)\xi\|.$$

In conclusion, the previous bounds imply with the assumption (5.6) (in particular, since $\alpha > 3\beta + 2\kappa$ and $3\beta - 2 < 0$) that

$$\begin{aligned}&\pm(\Phi_1+\Phi_2+\Phi_3+\Phi_4+\text{h.c.})\\&\quad\leq\delta\mathcal{V}_N+CN^{\alpha+\kappa-1}\mathcal{V}_N+CN^{\alpha+\kappa-1}\mathcal{V}_{N,L}+\delta^{-1}CN^{\beta+\kappa-1}\mathcal{K}_L(\mathcal{N}_{\geq\frac12 N^\alpha}+1)\\&\qquad+\delta^{-1}CN^{3\beta+\kappa-1}(\mathcal{N}_{\geq\frac12 N^\alpha}+1)^2\end{aligned}\tag{7.21}$$

holds true in $\mathcal{F}_+^{\leq N}$ for any $\delta > 0$. This concludes the proof. □

With Proposition 7.5, we obtain a bound for the growth of $\mathcal{V}_N$.

Corollary 7.6. *Assume the exponents α, β satisfy (5.6). Then, there exists a constant $C > 0$ such that the operator inequality*

$$\begin{aligned}e^{-sD}\mathcal{V}_Ne^{sD}\leq C\mathcal{V}_N+C\mathcal{V}_{N,L}+CN^{\beta+\kappa-1}\mathcal{K}_L(\mathcal{N}_{\geq\frac12 N^\alpha}+1)\\+CN^{3\beta+\kappa}(\mathcal{N}_{\geq\frac12 N^\alpha}+1).\end{aligned}$$

for all $s \in [-1; 1]$ and for all $N \in \mathbb{N}$ sufficiently large.

Proof. We apply Gronwall's lemma. Given a normalized vector $\xi \in \mathcal{F}_+^{\leq N}$, we define $\varphi_\xi(s) = \langle\xi, e^{-sD}\mathcal{V}_Ne^{sD}\xi\rangle$ and compute its derivative s.t.

$$\partial_s\varphi_\xi(s)=\langle\xi,e^{-sD}[\mathcal{V}_N,D]e^{sD}\xi\rangle.$$

Hence, we can apply (7.16) and estimate

$$\begin{aligned}&\left|\frac{1}{2N}\sum_{\substack{u\in\Lambda_+^*,v,w\in P_L:\\ v+u,w-u\neq0}}N^\kappa(\widehat{V}(./N^{1-\kappa})*\eta/N)(u)\langle e^{sD}\xi,a^*_{v+u}a^*_{w-u}a_va_we^{sD}\xi\rangle\right|\\&\quad\leq\frac{\|\check{\eta}\|_\infty}{N}\left(\int_{\Lambda^2}dxdy\,N^{2-2\kappa}V(N^{1-\kappa}(x-y))\|\check{a}_x\check{a}_ye^{sD}\xi\|^2\right)^{1/2}\\&\qquad\times\left(\int_{\Lambda^2}dxdy\,N^{2-2\kappa}V(N^{1-\kappa}(x-y))\Big\|\sum_{v,w\in P_L}e^{ivx+iwy}a_va_we^{sD}\xi\Big\|^2\right)^{1/2}\\&\quad\leq C\|\mathcal{V}_N^{1/2}e^{sD}\xi\|\|\mathcal{V}_{N,L}^{1/2}e^{sD}\xi\|\leq C\varphi_\xi(s)+C\langle\xi,e^{-sD}\mathcal{V}_{N,L}e^{sD}\xi\rangle.\end{aligned}\tag{7.22}$$

Here, we used (3.10), which shows that $\|\check{\eta}\|_\infty \leq CN$. Using Corollary 7.4 (recalling that $\alpha > 3\beta + 2\kappa$ and $2\beta \leq 1$) and $\mathcal{N}_{\geq \frac{1}{2}N^\alpha} \leq N$ in $\mathcal{F}_+^{\leq N}$, this simplifies to

$$\begin{aligned}&\Bigg|\frac{1}{2N}\sum_{\substack{u\in\Lambda_+^*,v,w\in P_L:\\ v+u,w-u\neq 0}} N^\kappa(\widehat{V}(./N^{1-\kappa})*\eta/N)(u)\langle e^{sD}\xi, a^*_{v+u}a^*_{w-u}a_va_we^{sD}\xi\rangle\Bigg|\\ &\quad\leq C\varphi_\xi(s) + C\langle\xi,\mathcal{V}_{N,L}\xi\rangle + CN^{\beta+\kappa-1}\langle\xi,\mathcal{K}_L(\mathcal{N}_{\geq\frac{1}{2}N^\alpha}+1)\xi\rangle\\ &\qquad + CN^{3\beta+\kappa}\langle\xi,(\mathcal{N}_{\geq\frac{1}{2}N^\alpha}+1)\xi\rangle.\end{aligned}$$

Together with (7.16), the bound (7.17) (choosing $\delta = 1$) and an application of Lemma 7.1 and of Lemma 7.2, the claim follows now from Gronwall's lemma. □

Finally, we need control for the growth of the full kinetic energy operator $\mathcal{K}$. To this end, we need to estimate its commutator with D.

Proposition 7.7. *Assume the exponents α,β satisfy (5.6). Let $m_0\in\mathbb{R}$ be such that $m_0\beta = \alpha$ (from (5.6) it follows that $3 < m_0 < 5$). Then,*

$$\begin{aligned}[\mathcal{K},D] = &-\frac{1}{2N}\sum_{\substack{u\in\Lambda^*,p,q\in P_L:\\ p+u,q-u\neq 0}} N^\kappa(\widehat{V}(./N^{1-\kappa})*\widehat{f}_N)(u)\big(a^*_{p+u}a^*_{q-u}a_pa_q + \text{h.c.}\big)\\ &+\mathcal{E}_{[\mathcal{K},D]},\end{aligned}\tag{7.23}$$

where the self-adjoint operator $\mathcal{E}_{[\mathcal{K},D]}$ satisfies

$$\begin{aligned}\pm\mathcal{E}_{[\mathcal{K},D]} \leq\ & CN^{5\beta/4+\kappa}\mathcal{K}_{\leq 2N^{3\beta/2}} + \delta\mathcal{K}\\ &+ C\delta^{-1}\sum_{j=3}^{2\lfloor m_0\rfloor-1} N^{j\beta/2+3\beta/2+2\kappa-1}\mathcal{K}_L(\mathcal{N}_{\geq\frac{1}{2}N^{j\beta/2}}+1)\\ &+ C\delta^{-1}N^{\alpha+\beta+2\kappa-1}\mathcal{K}_L(\mathcal{N}_{\geq\frac{1}{2}N^{\lfloor m_0\rfloor\beta}}+1) + C\end{aligned}\tag{7.24}$$

for all $\delta > 0$ and for all $N\in\mathbb{N}$ sufficiently large.

Proof. Using that $[\mathcal{K},D] = [\mathcal{K},D_1] + \text{h.c.}$, a straight forward computation shows that

$$\begin{aligned}[\mathcal{K},D_1] + \text{h.c.} = &-\frac{1}{2N}\sum_{\substack{r\in\Lambda^*,v,w\in P_L:\\ v+r,w-r\neq 0}} N^\kappa(\widehat{V}(./N^{1-\kappa})*\widehat{f}_N)(r)a^*_{v+r}a^*_{w-r}a_va_w\\ &+\Sigma_1+\Sigma_2+\Sigma_3+\text{h.c.},\end{aligned}\tag{7.25}$$

where

$$\begin{aligned}\Sigma_1 &= \frac{1}{2N} \sum_{\substack{r\in P_H^c\cup\{0\},v,w\in P_L:\\ v+r,w-r\neq 0}} N^\kappa(\widehat{V}(./N^{1-\kappa}) * \widehat{f}_N)(r) a^*_{v+r}a^*_{w-r}a_v a_w,\\ \Sigma_2 &= \frac{1}{2N} \sum_{\substack{r\in P_H,v,w\in P_L:\\ v+r,w-r\neq 0}} N^{3-2\kappa}\lambda_\ell(\widehat{\chi}_\ell * \widehat{f}_N)(r) a^*_{v+r}a^*_{w-r}a_v a_w,\\ \Sigma_3 &= \frac{2}{N} \sum_{\substack{r\in P_H,v,w\in P_L:\\ v+r,w-r\neq 0}} r\cdot v\, \eta_r a^*_{v+r}a^*_{w-r}a_v a_w.\end{aligned} \tag{7.26}$$

Let us estimate the size of the operators Σ_1, Σ_2 and Σ_3. Using $\big|(\widehat{V}(./N^{1-\kappa}) * \widehat{f}_N)(r)\big| \leq C$, we control the operator Σ_1 by

$$\begin{aligned}|\langle \xi, \Sigma_1\xi\rangle| &= \Bigg|\frac{1}{2N} \sum_{\substack{r\in P_H^c\cup\{0\},v,w\in P_L:\\ v+r,w-r\neq 0}} N^\kappa(\widehat{V}(./N^{1-\kappa}) * \widehat{f}_N)(r)\langle \xi, b^*_{v+r}a^*_{w-r}a_v a_w\xi\rangle\Bigg|\\ &\leq \frac{CN^\kappa}{N} \sum_{\substack{r\in\Lambda^*,v,w\in P_L:|r|\leq N^{3\beta/2},\\ v+r,w-r\neq 0}} \|a_{w-r}a_{v+r}\xi\|\|a_v a_w\xi\|\\ &\quad + \frac{CN^\kappa}{N}\sum_{j=3}^{2\lfloor m_0\rfloor-1}\\ &\quad \sum_{\substack{r\in P_H^c\cup\{0\},v,w\in P_L:\\ N^{j\beta/2}\leq |r|\leq N^{(j+1)\beta/2},\\ v+r,w-r\neq 0}} \|a_{w-r}(\mathcal{N}_{\geq \frac12 N^{j\beta/2}}+1)^{-1/2}a_{v+r}\xi\|\|a_v(\mathcal{N}_{\geq \frac12 N^{j\beta/2}}+1)^{1/2}a_w\xi\|\\ &\quad + \frac{CN^\kappa}{N}\\ &\quad \sum_{\substack{r\in P_H^c\cup\{0\},v,w\in P_L:\\ N^{\lfloor m_0\rfloor\beta}\leq |r|\leq N^{\alpha},\\ v+r,w-r\neq 0}} \|a_{w-r}(\mathcal{N}_{\geq \frac12 N^{\lfloor m_0\rfloor\beta}}+1)^{-1/2}a_{v+r}\xi\|\|a_v(\mathcal{N}_{\geq \frac12 N^{\lfloor m_0\rfloor\beta}}+1)^{1/2}a_w\xi\|.\end{aligned} \tag{7.27}$$

By Cauchy–Schwarz, the first term on the r.h.s. of (7.27) can be controlled by

$$\frac{CN^\kappa}{N} \sum_{\substack{r\in\Lambda^*,v,w\in P_L:|r|\leq N^{3\beta/2},\\ v+r,w-r\neq 0}} \|a_{w-r}a_{v+r}\xi\|\|a_v a_w\xi\| \leq CN^{5\beta/4+\kappa}\langle \xi, \mathcal{K}_{\leq 2N^{3\beta/2}}\xi\rangle.$$

The second contribution on the r.h.s. of (7.27) can be bounded by

$$\begin{aligned}&\frac{CN^\kappa}{N}\sum_{j=3}^{2\lfloor m_0\rfloor-1} \sum_{\substack{r\in P_H^c\cup\{0\},v,w\in P_L:\\ N^{j\beta/2}\leq |r|\leq N^{(j+1)\beta/2},\\ v+r,w-r\neq 0}} \|a_{w-r}(\mathcal{N}_{\geq \frac12 N^{j\beta/2}}+1)^{-1/2}a_{v+r}\xi\|\|a_w(\mathcal{N}_{\geq \frac12 N^{j\beta/2}}+1)^{1/2}a_v\xi\|\\ &\quad\leq C\sum_{j=3}^{2\lfloor m_0\rfloor-1} N^{j\beta/4+3\beta/4+\kappa-1/2}\|\mathcal{K}^{1/2}\xi\|\|\mathcal{K}_L^{1/2}(\mathcal{N}_{\geq\frac12 N^{j\beta/2}}+1)^{1/2}\xi\|.\end{aligned} \tag{7.28}$$

Similarly, we find that

$$\begin{aligned}\frac{CN^{\kappa}}{N} & \sum_{\substack{r\in P_H^c\cup\{0\}, v,w\in P_L:\\ N^{\lfloor m_0\rfloor\beta}\le |r|\le N^{\alpha},\\ v+r,w-r\neq 0}} \|a_{w-r}(\mathcal{N}_{\ge\frac12 N^{\lfloor m_0\rfloor\beta}}+1)^{-1/2}a_{v+r}\xi\|\|a_w(\mathcal{N}_{\ge\frac12 N^{\lfloor m_0\rfloor\beta}}+1)^{1/2}a_v\xi\| \\ &\le CN^{\alpha/2+\beta/2+\kappa-1/2}\|\mathcal{K}^{1/2}\xi\|\|\mathcal{K}_L^{1/2}(\mathcal{N}_{\ge\frac12 N^{\lfloor m_0\rfloor\beta}}+1)^{1/2}\xi\|.\end{aligned} \tag{7.29}$$

In summary, the previous three bounds imply that

$$\begin{aligned}\pm\Sigma_1 \le CN^{5\beta/4+\kappa}\mathcal{K}_{\le 2N^{3\beta/2}} + \delta\mathcal{K} + C\delta^{-1}N^{\alpha+\beta+2\kappa-1}\mathcal{K}_L(\mathcal{N}_{\ge\frac12 N^{\lfloor m_0\rfloor\beta}}+1) \\ + C\delta^{-1}\sum_{j=3}^{2\lfloor m_0\rfloor-1} N^{j\beta/2+3\beta/2+2\kappa-1}\mathcal{K}_L(\mathcal{N}_{\ge\frac12 N^{j\beta/2}}+1)\end{aligned} \tag{7.30}$$

for some constant $C>0$ and all $\delta>0$.

Next, let us switch to Σ_2 and Σ_3, defined in (7.26). Since $(\widehat{\chi}_\ell * \widehat{f}_N)(r) = \widehat{\chi}_\ell(r) + N^{-1}\eta_r$, with

$$\widehat{\chi}_\ell(r) = \frac{4\pi}{|r|^2}\left(\frac{\sin(\ell|r|)}{|r|} - \ell\cos(\ell|r|)\right)$$

we find

$$|(\widehat{\chi}_\ell * \widehat{f}_N)(r)| \le C|r|^{-2}$$

This, together with Lemma 3.1(i), Cauchy–Schwarz and $\alpha > 3\beta+2\kappa$, implies that

$$\begin{aligned}|\langle\xi,\Sigma_2\xi\rangle| \le \frac{CN^{\kappa}}{N} & \sum_{\substack{r\in P_H, v,w\in P_L:\\ v+r,w-r\neq 0}} |r|^{-2}\|a_{v+r}(\mathcal{N}_{\ge\frac12 N^{\alpha}}+1)^{-1/2}a_{w-r}\xi\| \\ &\times \|a_v(\mathcal{N}_{\ge\frac12 N^{\alpha}}+1)^{1/2}a_w\xi\| \\ &\le CN^{-\beta-1/2}\|(\mathcal{N}_{\ge\frac12 N^{\alpha}}+1)^{1/2}\xi\|\|\mathcal{K}_L^{1/2}(\mathcal{N}_{\ge\frac12 N^{\alpha}}+1)^{1/2}\xi\|.\end{aligned} \tag{7.31}$$

Similarly, we obtain

$$\begin{aligned}|\langle\xi,\Sigma_3\xi\rangle| \le \frac{C}{N} & \sum_{r\in P_H, v,w\in P_L} |r||v||\eta_r|\|a_{v+r}(\mathcal{N}_{\ge\frac12 N^{\alpha}}+1)^{-1/2}a_{w-r}\xi\| \\ &\times \|a_v a_w(\mathcal{N}_{\ge\frac12 N^{\alpha}}+1)^{1/2}\xi\| \\ &\le CN^{-1/2}\|\mathcal{K}^{1/2}\xi\|\|\mathcal{K}_L^{1/2}(\mathcal{N}_{\ge\frac12 N^{\alpha}}+1)^{1/2}\xi\|,\end{aligned} \tag{7.32}$$

where we used that $|r|/|v+r| \le 2$ for $r\in P_H$, $v\in P_L$ and $N\in\mathbb{N}$ large enough. Combining (7.30), (7.31) and (7.32) and defining $\mathcal{E}_{[\mathcal{K},D]} = \sum_{i=1}^3(\Sigma_i + \text{h.c.})$ proves the claim. □

Corollary 7.8. *Assume the exponents α,β satisfy (5.6). Let $m_0 \in \mathbb{R}$ be such that $m_0\beta = \alpha$ ($3 < m_0 < 5$ from (5.6)). Then, there exists a constant $C > 0$ such that*

$$\begin{aligned} &e^{-sD}\mathcal{K}e^{sD} \\ &\quad\le C\mathcal{K} + C\mathcal{V}_N + C\mathcal{V}_{N,L} + CN^{5\beta/4+\kappa}\mathcal{K}_{\le N^{3\beta/2}} \\ &\qquad + C\sum_{j=3}^{2\lfloor m_0\rfloor-1} N^{j\beta/2+3\beta/2+2\kappa-1}\Big[\mathcal{K}_L + N^{2\beta}(\mathcal{N}_{\ge\frac12 N^\alpha}+1)\Big](\mathcal{N}_{\ge\frac12 N^{j\beta/2}}+1) \\ &\qquad + CN^{\alpha+\beta+2\kappa-1}\Big[\mathcal{K}_L + N^{2\beta}(\mathcal{N}_{\ge\frac12 N^\alpha}+1)\Big](\mathcal{N}_{\ge\frac12 N^{\lfloor m_0\rfloor\beta}}+1) + CN^{13\beta/4+\kappa} \end{aligned} \tag{7.33}$$

for all $s \in [-1;1]$ and for all $N \in \mathbb{N}$ sufficiently large.

Proof. Given $\xi \in \mathcal{F}_+^{\le N}$, we define $\varphi_\xi(s) = \langle \xi, e^{-sD}\mathcal{K}e^{sD}\xi\rangle$. Differentiation yields

$$\partial_s\varphi_\xi(s) = \langle \xi, e^{-sD}[\mathcal{K},D]e^{sD}\xi\rangle,$$

s.t., to bound the derivative of φ_ξ, we can apply Proposition 7.7. Arguing exactly as in (7.22), we obtain with $\sup_{x\in\Lambda}|f_N(x)| \le 1$ the operator inequality

$$\pm\frac{1}{2N}\sum_{\substack{u\in\Lambda_+^*,v,w\in P_L:\\ v+u,w-u\neq0}} N^\kappa(\widehat{V}(./N^{1-\kappa}) * \widehat{f}_N)(u)a^*_{v+u}a^*_{w-u}a_va_w \le C\mathcal{V}_N + C\mathcal{V}_{N,L}.$$

Now, the claim follows from the bound (7.24) (choosing $\delta = 1$), the previous bound and an application of Corollaries 7.6, 7.4, Lemmas 7.1, 7.2 and the operator bound $\mathcal{N}_{\ge\frac12 N^\alpha} \le 4N^{-2\alpha}\mathcal{K}$, by Gronwall's Lemma. □

7.2. Action of Quartic Renormalization on Excitation Hamiltonian

We compute now the main contributions to $\mathcal{M}_N = e^{-D}\mathcal{J}_N^{\text{eff}}e^D$. From (4.5) and recalling that $[\mathcal{N}_+, D] = 0$, we can decompose

$$\mathcal{M}_N = 4\pi\mathfrak{a}_0N^{1+\kappa} - 4\pi\mathfrak{a}_0N^{\kappa-1}\mathcal{N}_+^2/N + \mathcal{M}_N^{(2)} + \mathcal{M}_N^{(3)} + \mathcal{M}_N^{(4)} \tag{7.34}$$

where the operators $\mathcal{M}_N^{(i)}, i = 2,3,4$, are defined by

$$\begin{aligned} \mathcal{M}_N^{(2)} &= 8\pi\mathfrak{a}_0N^\kappa\sum_{p\in P_H^c} e^{-D}b_p^*b_pe^D + 4\pi\mathfrak{a}_0N^\kappa\sum_{p\in P_H^c} e^{-D}\big[b_p^*b_{-p}^* + b_pb_{-p}\big]e^D \\ \mathcal{M}_N^{(3)} &= \frac{8\pi\mathfrak{a}_0N^\kappa}{\sqrt{N}}\sum_{\substack{p\in P_H^c,q\in P_L:\\ p+q\neq0}} e^{-D}\big[b^*_{p+q}a^*_{-p}a_q + \text{h.c.}\big]e^D, \\ \mathcal{M}_N^{(4)} &= e^{-D}\mathcal{H}_Ne^D = e^{-D}\mathcal{K}e^D + e^{-D}\mathcal{V}_Ne^D. \end{aligned} \tag{7.35}$$

7.2.1. Analysis of $\mathcal{M}_N^{(2)}$. In this section, we determine the main contributions to $\mathcal{M}_N^{(2)}$, defined in (7.35) by

$$\mathcal{M}_N^{(2)} = 8\pi\mathfrak{a}_0 N^\kappa \sum_{p\in P_H^c} e^{-D} b_p^* b_p e^D + 4\pi\mathfrak{a}_0 N^\kappa \sum_{p\in P_H^c} e^{-D} \big[b_p^* b_{-p}^* + b_p b_{-p}\big] e^D \tag{7.36}$$

The main result of this section is the following proposition.

Proposition 7.9. *Assume the exponents α,β satisfy (5.6). Then*

$$\mathcal{M}_N^{(2)} = 8\pi\mathfrak{a}_0 N^\kappa \sum_{p\in P_H^c} \Big[b_p^* b_p + \frac{1}{2} b_p^* b_{-p}^* + \frac{1}{2} b_p b_{-p}\Big] + \mathcal{E}_{\mathcal{M}_N}^{(2)} \tag{7.37}$$

and there exists a constant $C>0$ such that

$$\pm e^A e^D \mathcal{E}_{\mathcal{M}_N}^{(2)} e^{-D} e^{-A} \leq CN^{-\beta-2\kappa}\mathcal{K} + CN^\kappa \tag{7.38}$$

for all $N\in\mathbb{N}$ sufficiently large.

Proof. We start with the identity

$$\begin{aligned}\mathcal{M}_N^{(2)} &- 8\pi\mathfrak{a}_0 N^\kappa \sum_{p\in P_H^c} \Big[b_p^* b_p + \frac{1}{2} b_p^* b_{-p}^* + \frac{1}{2} b_p b_{-p}\Big] \\ &= 8\pi\mathfrak{a}_0 N^\kappa \int_0^1 dt \sum_{p\in P_H^c} e^{-tD}\big[b_p^* b_p + \frac{1}{2} b_p^* b_{-p}^* + \frac{1}{2} b_p b_{-p}, D_1\big] e^{tD} + \text{h.c.}\end{aligned} \tag{7.39}$$

and a straight-forward computation shows that

$$\begin{aligned}&\big[b_p^* b_p + \frac{1}{2} b_p^* b_{-p}^* + \frac{1}{2} b_p b_{-p}, a_{v+r}^* a_{w-r}^* a_w a_v\big] \\ &\quad = b_{v+r}^* a_{w-r}^* a_v b_w \big(\delta_{p,v+r} + \delta_{p,w-r} - \delta_{p,v} - \delta_{p,w}\big) \\ &\qquad - \frac{1}{2} b_{v+r}^* b_{w-r}^* \big(\delta_{p,w}\delta_{-p,v} + \delta_{-p,w}\delta_{p,v}\big) + \frac{1}{2} b_v b_w \big(\delta_{p,w-r}\delta_{-p,v+r} + \delta_{-p,w-r}\delta_{p,v+r}\big) \\ &\qquad - \frac{1}{2} b_{v+r}^* b_{w-r}^* \big(a_{-p}^* a_w \delta_{p,v} + a_p^* a_w \delta_{-p,v} + a_{-p}^* a_v \delta_{p,w} + a_p^* a_v \delta_{-p,w}\big) \\ &\qquad + \frac{1}{2} \big(a_{w-r}^* a_{-p}\delta_{p,v+r} + a_{v+r}^* a_{-p}\delta_{p,w-r} + a_{w-r}^* a_p \delta_{-p,v+r} + a_{v+r}^* a_p \delta_{-p,w-r}\big) b_v b_w.\end{aligned}$$

As a consequence, we find that

$$\mathcal{M}_N^{(2)} - 8\pi\mathfrak{a}_0 N^\kappa \sum_{p\in P_H^c} \Big[b_p^* b_p + \frac{1}{2} b_p^* b_{-p}^* + \frac{1}{2} b_p b_{-p}\Big] = \int_0^1 dt\, e^{-tD} \sum_{j=1}^5 (\mathrm{V}_j + \text{h.c.}) e^{tD}, \tag{7.40}$$

where

$$\begin{aligned}
\mathrm{V}_1 &= -\frac{8\pi\mathfrak{a}_0 N^\kappa}{2N} \sum_{r\in P_H, v\in P_L} \eta_r b^*_{v+r} b^*_{-v-r},\\
\mathrm{V}_2 &= \frac{8\pi\mathfrak{a}_0 N^\kappa}{2N} \sum_{\substack{r\in P_H, v\in P_L:\\ v+r\in P_H^c, v+r\neq 0}} \eta_r b_v b_{-v},\\
\mathrm{V}_3 &= \frac{8\pi\mathfrak{a}_0 N^\kappa}{2N} \sum_{\substack{r\in P_H, v,w\in P_L:\\ v+r, w-r\neq 0}} \eta_r \big(-2 + \chi_{\{r+v\in P_H^c\}} + \chi_{\{w-r\in P_H^c\}}\big) b^*_{v+r} a^*_{w-r} a_v b_w,\\
\mathrm{V}_4 &= -\frac{8\pi\mathfrak{a}_0 N^\kappa}{N} \sum_{\substack{r\in P_H, v,w\in P_L:\\ v+r, w-r\neq 0}} \eta_r b^*_{v+r} b^*_{w-r} a^*_{-v} a_w,\\
\mathrm{V}_5 &= \frac{8\pi\mathfrak{a}_0 N^\kappa}{N} \sum_{\substack{r\in P_H, v,w\in P_L:\\ r-w\in P_H^c, v+r, w-r\neq 0}} \eta_r a^*_{v+r} a_{r-w} b_v b_w.
\end{aligned} \tag{7.41}$$

Here, $\chi_{\{p\in S\}}$ denotes as usual the characteristic function for the set $S \subset \Lambda_+^*$, evaluated at $p \in \Lambda_+^*$. Let us briefly explain how to bound the different contributions V_1 to V_5, defined in (7.41). Using Cauchy–Schwarz, the first two contributions are bounded by

$$\pm(\mathrm{V}_1 + \mathrm{V}_2) \leq CN^{2\kappa+3\beta-\alpha/2-1}(\mathcal{N}_{\geq \frac{1}{2}N^\alpha} + 1) + CN^{2\kappa+3\beta/2-1}(\mathcal{K}_L + 1)$$

where, for V_2, we used that $v + r \in P_H^c$ implies that $|r| \leq N^\alpha + N^\beta$ and furthermore that $\sum_{N^\alpha \leq |r| \leq N^\alpha + N^\beta} |\eta_r| \leq N^{\kappa+\beta}$. The contributions V_3 to V_5, on the other hand, can be controlled by

$$\begin{aligned}
&|\langle \xi, (\mathrm{V}_3 + \mathrm{V}_4 + \mathrm{V}_5)\xi\rangle|\\
&\leq \frac{CN^\kappa}{N} \sum_{\substack{r\in P_H, v,w\in P_L:\\ v+r, w-r\neq 0}} |\eta_r| \|a_{v+r}(\mathcal{N}_{\geq \frac{1}{2}N^\alpha} + 1)^{-1/2} a_{w-r}\xi\| \|a_v(\mathcal{N}_{\geq \frac{1}{2}N^\alpha} + 1)^{1/2} a_w \xi\|\\
&\quad + \frac{CN^\kappa}{N} \sum_{\substack{r\in P_H, v,w\in P_L:\\ v+r, w-r\neq 0}} |\eta_r| \|a_{v+r}(\mathcal{N}_{\geq \frac{1}{2}N^\alpha} + 1)^{-1/2} a_{w-r} a_w \xi\| \|a_v(\mathcal{N}_{\geq \frac{1}{2}N^\alpha} + 1)^{1/2} \xi\|\\
&\quad + \frac{CN^\kappa}{N} \sum_{\substack{r\in P_H, v,w\in P_L:\\ v+r, w-r\neq 0}} |\eta_r| \|a_{v+r}\xi\| \|a_v a_w a_{w-r}\xi\|\\
&\leq CN^{2\kappa+3\beta/2-\alpha/2}\langle \xi, (\mathcal{N}_{\geq \frac{1}{2}N^\alpha} + 1)\xi\rangle \leq CN^\kappa \langle \xi, (\mathcal{N}_{\geq \frac{1}{2}N^\alpha} + 1)\xi\rangle
\end{aligned}$$

for any $\xi \in \mathcal{F}_+^{\leq N}$. In conclusion (since $2\kappa + 3\beta - \alpha/2 - 1 < \kappa$ from (5.6)), we have proved that

$$\pm \sum_{j=1}^5 \big(\mathrm{V}_j + \mathrm{h.c.}\big) \leq CN^{2\kappa+3\beta/2-1}\mathcal{K}_L + CN^\kappa(\mathcal{N}_{\geq \frac{1}{2}N^\alpha} + 1).$$

Now, applying this bound together with (7.40), Lemmas 4.2, 4.3, 7.1, 7.2 and the operator inequality $\mathcal{N}_{\geq \frac{1}{2}N^\alpha} \leq 4N^{-2\alpha}\mathcal{K}$ proves the claim. ☐

7.2.2. Analysis of $\mathcal{M}_N^{(3)}$. In this section, we determine the main contributions to $\mathcal{M}_N^{(3)}$, defined in (7.35) by

$$\mathcal{M}_N^{(3)} = \frac{8\pi\mathfrak{a}_0 N^\kappa}{\sqrt{N}} \sum_{\substack{p\in P_H^c, q\in P_L:\\ p+q\neq 0}} e^{-D}\big(b^*_{p+q}a^*_{-p}a_q + \text{h.c.}\big)e^D. \tag{7.42}$$

Proposition 7.10. *Assume the exponents α,β satisfy (5.6). Then, we have that*

$$\mathcal{M}_N^{(3)} = \frac{8\pi\mathfrak{a}_0 N^\kappa}{\sqrt{N}} \sum_{\substack{p\in P_H^c, q\in P_L:\\ p+q\neq 0}} \big(b^*_{p+q}a^*_{-p}a_q + \text{h.c.}\big) + \mathcal{E}^{(3)}_{\mathcal{M}_N} \tag{7.43}$$

and there exists a constant $C>0$ such that

$$\begin{aligned} &\pm\, e^A e^D \mathcal{E}^{(3)}_{\mathcal{M}_N} e^{-D}e^{-A} \\ &\quad \leq CN^{-\beta}\mathcal{K} + CN^{\alpha+\beta/2+2\kappa-1}\mathcal{K}(\mathcal{N}_{\geq\frac{1}{2}N^\alpha}+1) + CN^{\alpha+\beta/2+2\kappa} \end{aligned} \tag{7.44}$$

for all $N\in\mathbb{N}$ sufficiently large.

Proof. Let us define the operator $\mathrm{Y}:\mathcal{F}_+^{\leq N}\to\mathcal{F}_+^{\leq N}$ by

$$\mathrm{Y} = \frac{8\pi\mathfrak{a}_0 N^\kappa}{\sqrt{N}} \sum_{\substack{p\in P_H^c, q\in P_L:\\ p+q\neq 0}} \big(b^*_{p+q}a^*_{-p}a_q + \text{h.c.}\big), \tag{7.45}$$

so that $\mathcal{M}_N^{(3)} = e^{-D}\mathrm{Y}e^D$. We recall the definition (7.1) and observe that

$$e^{-D}\mathrm{Y}e^D - \mathrm{Y} = \int_0^1 ds\; e^{-sD}[\mathrm{Y},D_1]e^{sD} + \text{h.c.}. \tag{7.46}$$

This implies that it is enough to control the commutator $[\mathrm{Y},D_1]$ after conjugation with e^{tD}, for any $t\in[-1;1]$. Note that, if $p\in P_H^c, q\in P_L, r\in P_H$ and $v,w\in P_L$, we have $|v+r|\geq N^\alpha - N^\beta > \frac{1}{2}N^\alpha > N^\beta$ s.t. $[a^*_{-p}a_q, a^*_{v+r}a^*_{w-r}]=0$, for $N\in\mathbb{N}$ large enough. Then, a lengthy but straightforward calculation shows that

$$\begin{aligned} [b^*_{p+q}a^*_{-p}a_q, a^*_{v+r}a^*_{w-r}a_va_w] &= -b^*_{v+r}a^*_{w-r}a_q(\delta_{-p,w}\delta_{p+q,v}+\delta_{-p,v}\delta_{p+q,w}) \\ &\quad - b^*_{p+q}a^*_{v+r}a^*_{w-r}a_q(a_w\delta_{-p,v}+a_v\delta_{-p,w}) \\ &\quad - b^*_{-p}a^*_{v+r}a^*_{w-r}a_q(a_w\delta_{p+q,v}+a_v\delta_{p+q,w}) \end{aligned}$$

and

$$\begin{aligned} [a^*_q a_{-p} b_{p+q}, a^*_{v+r}a^*_{w-r}a_va_w] &= a^*_q a_v b_w \delta_{-p,w-r}\delta_{p+q,v+r} + a^*_q a_v b_w \delta_{-p,v+r}\delta_{p+q,w-r} \\ &\quad + a^*_q a^*_{w-r}a_va_wb_{p+q}\delta_{-p,v+r} + a^*_qa^*_{v+r}a_va_wb_{p+q}\delta_{-p,w-r} \\ &\quad - a^*_{v+r}a^*_{w-r}a_wa_{-p}b_{p+q}\delta_{q,v} - a^*_{v+r}a^*_{w-r}a_va_{-p}b_{p+q}\delta_{q,w} \\ &\quad + a^*_qa^*_{w-r}a_{-p}a_vb_w\delta_{p+q,v+r} + a^*_qa^*_{v+r}a_{-p}a_vb_w\delta_{p+q,w-r}. \end{aligned}$$

As a consequence, we conclude that

$$[\mathrm{Y},D_1] + \text{h.c.} = \sum_{i=1}^6 (\Psi_i + \text{h.c.}), \tag{7.47}$$

where

$$\begin{aligned}
\Psi_1 &= -\frac{8\pi\mathfrak{a}_0 N^\kappa}{N^{3/2}} \sideset{}{^*}\sum_{\substack{r\in P_H,v,w\in P_L:\\ v+w\in P_L}} \eta_r b^*_{v+r} a^*_{w-r} a_{v+w},\\
\Psi_2 &= \frac{8\pi\mathfrak{a}_0 N^\kappa}{N^{3/2}} \sideset{}{^*}\sum_{\substack{r\in P_H,v,w\in P_L:\\ v+r,r-w\in P_H^c,v+w\in P_L}} \eta_r a^*_{v+w} a_v b_w,\\
\Psi_3 &= -\frac{16\pi\mathfrak{a}_0 N^\kappa}{N^{3/2}} \sideset{}{^*}\sum_{r\in P_H,q,v,w\in P_L} \eta_r b^*_{q-v} a^*_{v+r} a^*_{w-r} a_q a_w,\\
\Psi_4 &= \frac{8\pi\mathfrak{a}_0 N^\kappa}{N^{3/2}} \sideset{}{^*}\sum_{\substack{r\in P_H,q,v,w\in P_L:\\ v+r\in P_H^c}} \eta_r a^*_q a^*_{w-r} a_v a_w b_{q-v-r},\\
\Psi_5 &= \frac{8\pi\mathfrak{a}_0 N^\kappa}{N^{3/2}} \sideset{}{^*}\sum_{\substack{r\in P_H,q,v,w\in P_L:\\ v+r-q\in P_H^c}} \eta_r a^*_q a^*_{w-r} a_v a_w b_{q-v-r},\\
\Psi_6 &= -\frac{8\pi\mathfrak{a}_0 N^\kappa}{N^{3/2}} \sideset{}{^*}\sum_{p\in P_H^c,r\in P_H,v,w\in P_L} \eta_r a^*_{v+r} a^*_{w-r} a_w a_{-p} b_{p+v}.
\end{aligned} \tag{7.48}$$

Let us explain how to control the operators Ψ_1 to Ψ_6, defined in (7.48). We start with Ψ_1. Given $\xi \in \mathcal{F}_+^{\leq N}$, we find that

$$\begin{aligned}
&|\langle \xi, \Psi_1 \xi\rangle|\\
&\quad= \left| \frac{8\pi\mathfrak{a}_0 N^\kappa}{N^{3/2}} \sideset{}{^*}\sum_{r\in P_H,v,w\in P_L} \eta_r \langle \xi, b^*_{v+r} a^*_{w-r} a_{v+w} \xi\rangle \right|\\
&\quad\leq \frac{CN^\kappa}{N^{3/2}} \sideset{}{^*}\sum_{r\in P_H,v,w\in P_L} |\eta_r| \|(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)^{-1/2} a_{v+r} a_{w-r}\xi\| \|(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)^{1/2} a_{v+w}\xi\|\\
&\quad\leq CN^{3\beta+2\kappa-\alpha/2-1} \langle \xi, (\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)\xi\rangle \leq CN^{3\beta/2+\kappa-1} \langle \xi, (\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)\xi\rangle.
\end{aligned}$$

The contribution Ψ_2 can be bounded by

$$\begin{aligned}
|\langle \xi, \Psi_2\xi\rangle| &= \left| \frac{8\pi\mathfrak{a}_0 N^\kappa}{N^{3/2}} \sideset{}{^*}\sum_{\substack{r\in P_H,v,w\in P_L:\\ v+r\in P_H^c}} \eta_r \langle \xi a^*_{v+w} a_v b_w \xi\rangle \right|\\
&\leq CN^{\beta/2+\kappa-1} \langle \xi, \mathcal{K}_{\leq 2N^\beta}\xi\rangle \sum_{N^\alpha\leq |r|\leq N^\alpha+N^\beta} |\eta_r| \leq CN^{3\beta/2+2\kappa-1} \langle \xi, \mathcal{K}_{\leq 2N^\beta}\xi\rangle.
\end{aligned}$$

Notice here, that we used that $|r| \leq N^\alpha + N^\beta$ if $r+v \in P_H^c$ and $v \in P_L$. Next, we apply as usual Cauchy–Schwarz to estimate the terms Ψ_3 to Ψ_5 by

$$\begin{aligned}
&|\langle \xi, \Psi_3\xi\rangle + \langle \xi, \Psi_4\xi\rangle + \langle \xi, \Psi_5\xi\rangle|\\
&\quad\leq CN^{3\beta+2\kappa-\alpha/2} \langle \xi, (\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)\xi\rangle \leq CN^{3\beta/2+\kappa} \langle \xi, (\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)\xi\rangle
\end{aligned}$$

for all $\alpha > 3\beta + 2\kappa$. Finally, the term Ψ_6 can be controlled by

$$\begin{aligned}|\langle\xi, \Psi_6\xi\rangle| &= \left|\frac{8\pi\mathfrak{a}_0 N^\kappa}{N^{3/2}} \sum^*_{p\in P_H^c, r\in P_H, v,w\in P_L} \eta_r \langle\xi, a^*_{v+r}a^*_{w-r}a_w a_{-p} b_{p+v}\xi\rangle\right| \\ &\leq CN^{\kappa-3/2} \sum^*_{p\in P_H^c, r\in P_H, v,w\in P_L} |w|^{-1}\|(\mathcal{N}_{\geq N^\alpha/2}+1)^{-1/2}a_{v+r}a_{w-r}\xi\| \\ &\quad \times |w||\eta_r|\|a_w a_{-p} b_{p+v}(\mathcal{N}_{\geq N^\alpha/2}+1)^{1/2}\xi\| \\ &\leq CN^{\alpha+\beta/2+2\kappa-1}\langle\xi, \mathcal{K}_L(\mathcal{N}_{\geq \frac12 N^\alpha}+1)\xi\rangle + CN^{\alpha+\beta/2+2\kappa}\langle\xi, (\mathcal{N}_{\geq \frac12 N^\alpha}+1)\xi\rangle.\end{aligned}$$

In conclusion, the previous estimates show that

$$\begin{aligned}\pm\Big[\sum_{i=1}^6 (\Psi_i + \text{h.c.})\Big] &\leq CN^{3\beta/2+2\kappa-1}\mathcal{K}_{\leq 2N^\beta} + CN^{\alpha+\beta/2+2\kappa-1}\mathcal{K}_L(\mathcal{N}_{\geq\frac12 N^\alpha}+1) \\ &\quad + CN^{\alpha+\beta/2+2\kappa}(\mathcal{N}_{\geq \frac12 N^\alpha}+1),\end{aligned}$$

so that, together with (7.46) and (7.47), an application of the Lemmas 4.2, 4.3, 7.1, 7.2 and the operator bound $\mathcal{N}_{\geq\frac12 N^\alpha} \leq 4N^{-2\alpha}\mathcal{K}$ proves the claim. □

7.2.3. Analysis of $\mathcal{M}_N^{(4)}$. In this section, we determine the main contributions to $\mathcal{M}_N^{(4)} = e^{-D}\mathcal{H}_N e^D$, defined in (7.35). To this end, we start with the observation that

$$\mathcal{M}_N^{(4)} = \mathcal{H}_N + \int_0^1 ds\, e^{-sD}\Big([\mathcal{K}, D_1] + [\mathcal{V}_N, D_1]\Big)e^{sD} + \text{h.c.}, \tag{7.49}$$

with D_1 defined in (7.1). By Propositions 7.5 and 7.7, this implies that

$$\begin{aligned}\mathcal{M}_N^{(4)} &= \mathcal{H}_N - \frac{N^\kappa}{2N}\sum_{\substack{r\in\Lambda^*, v,w\in P_L:\\ v+r, w-r\neq 0}} \int_0^1 ds\, \widehat{V}(r/N^{1-\kappa})e^{-sD}(a^*_{v+r}a^*_{w-r}a_v a_w + \text{h.c.})e^{sD} \\ &\quad + \int_0^1 ds\, e^{-sD}\Big(\mathcal{E}_{[\mathcal{K},D]} + \mathcal{E}_{[\mathcal{V}_N,D]}\Big)e^{sD},\end{aligned} \tag{7.50}$$

where we used that $\widehat{V}(\cdot/N^{1-\kappa})*(\widehat{f}_N - \eta/N)(r) = \widehat{V}(\cdot/N^{1-\kappa})(r)$ for all $r\in\Lambda^*_+$. Moreover, the operators $\mathcal{E}_{[\mathcal{V}_N,D]}$ and $\mathcal{E}_{[\mathcal{K},D]}$ are explicitly given by

$$\mathcal{E}_{[\mathcal{V}_N,D]} = \sum_{i=1}^4\big(\Phi_i + \text{h.c.}\big), \qquad \mathcal{E}_{[\mathcal{K},D]} = \sum_{j=1}^3\big(\Sigma_j + \text{h.c.}\big) \tag{7.51}$$

where we recall the definitions (7.19) and (7.26). Let us analyze the different contributions in (7.50), separately. We start with the second term on the r.h.s. of (7.50).

Proposition 7.11. *Assume the exponents α, β satisfy (5.6). Then, we have*

$$\begin{aligned}
&\frac{1}{2N}\sum_{\substack{u\in\Lambda^*,p,q\in P_L:\\ p+u,q-u\neq 0}} N^\kappa \widehat{V}(r/N^{1-\kappa})e^{-sD}\big(a^*_{p+u}a^*_{q-u}a_pa_q+\text{h.c.}\big)e^{sD}\\
&\quad=\frac{1}{2N}\sum_{\substack{u\in\Lambda^*,p,q\in P_L:\\ p+u,q-u\neq 0}} N^\kappa \widehat{V}(r/N^{1-\kappa})\big(a^*_{p+u}a^*_{q-u}a_pa_q+\text{h.c.}\big)\\
&\qquad+\frac{s}{N}\sum^*_{\substack{u\in\Lambda^*,v,w\in P_L:\\ v+u,w-u\in P_L}} N^\kappa(\widehat{V}(./N^{1-\kappa})*\eta/N)(u)a^*_{v+u}a^*_{w-u}a_va_w+\mathcal{E}_1(s)+\mathcal{E}_2(s)
\end{aligned} \tag{7.52}$$

and there exists a constant $C>0$ s.t. $\mathcal{E}_1(s)$ and $\mathcal{E}_2(s)$ satisfy

$$\begin{aligned}
&\pm\mathcal{E}_1(s)\le C(N^{\alpha+\beta+2\kappa-1}+N^{-3\beta-3\kappa})\mathcal{K}+CN^{2\beta+\kappa},\\
&\pm\mathcal{E}_2(s)\le CN^{\beta+\kappa-1}\mathcal{K}_L(\mathcal{N}_{\ge\frac12N^\alpha}+1)+C(N^{-\beta-\kappa}+CN^{3\beta/2+\kappa/2-1})\int_0^s dt\; e^{-tD}\mathcal{V}_Ne^{tD}\\
&\qquad+CN^{2\beta+2\kappa-1}\int_0^s dt\; e^{-tD}\mathcal{K}_{\le 2N^\beta}(\mathcal{N}_{\ge\frac12N^\alpha}+1)e^{tD},
\end{aligned} \tag{7.53}$$

for all $\delta>0$, $s\in[-1;1]$ and for all $N\in\mathbb{N}$ sufficiently large.

Proof. For definiteness, let us denote by $\mathrm{W}:\mathcal{F}_+^{\le N}\to\mathcal{F}_+^{\le N}$ the operator

$$\mathrm{W}=\frac{1}{2N}\sum_{\substack{u\in\Lambda^*,p,q\in P_L:\\ p+u,q-u\neq 0}} N^\kappa\widehat{V}(u/N^{1-\kappa})\big(a^*_{p+u}a^*_{q-u}a_pa_q+\text{h.c.}\big) \tag{7.54}$$

and consider the identity

$$\begin{aligned}
&e^{-sD}\mathrm{W}e^{sD}-\mathrm{W}\\
&\quad=\int_0^s dt\; e^{-tD}[\mathrm{W},D_1]e^{tD}+\text{h.c.}\\
&\quad=\frac{1}{2N}\int_0^s dt\sum_{\substack{u\in\Lambda^*,p,q\in P_L:\\ p+u,q-u\neq 0}} N^\kappa\widehat{V}(r/N^{1-\kappa})e^{-tD}\Big[\big(a^*_{p+u}a^*_{q-u}a_pa_q+\text{h.c.}\big),D_1\Big]e^{tD}+\text{h.c.}
\end{aligned} \tag{7.55}$$

Now, observe that

$$[a_p,a^*_{v+r}]=[a_q,a^*_{v+r}]=[a_p,a^*_{w-r}]=[a_q,a^*_{w-r}]=0$$

for all $p,q\in P_L$ and $r\in P_H$, $v,w\in P_L$ and $N\in\mathbb{N}$ sufficiently large. Then, proceeding as in the proof of Proposition 7.5, we obtain

$$\begin{aligned}
&[a^*_{p+u}a^*_{q-u}a_pa_q,\,a^*_{v+r}a^*_{w-r}a_va_w]\\
&\quad=-a^*_{v+r}a^*_{w-r}a^*_{q-u}a_wa_pa_q\delta_{p+u,v}-a^*_{v+r}a^*_{w-r}a^*_{p+u}a_wa_pa_q\delta_{q-u,v}\\
&\qquad-a^*_{v+r}a^*_{w-r}a_va^*_{q-u}a_pa_q\delta_{p+u,w}-a^*_{v+r}a^*_{w-r}a_va^*_{p+u}a_pa_q\delta_{q-u,w}.
\end{aligned} \tag{7.56}$$

and

$$\begin{aligned} &[a_p^* a_q^* a_{p-u} a_{q+u}, a_{v+r}^* a_{w-r}^* a_v a_w] \\ &\quad = a_p^* a_q^* a_{q+u} a_{w-r}^* a_v a_w \delta_{p-u,v+r} + a_p^* a_q^* a_{p-u} a_{w-r}^* a_v a_w \delta_{q+u,v+r} \\ &\qquad + a_p^* a_q^* a_{v+r}^* a_{q+u} a_v a_w \delta_{p-u,w-r} + a_p^* a_q^* a_{v+r}^* a_{p-u} a_v a_w \delta_{q+u,w-r} \\ &\qquad - a_{v+r}^* a_{w-r}^* a_q^* a_w a_{p-u} a_{q+u} \delta_{p,v} - a_{v+r}^* a_{w-r}^* a_p^* a_w a_{p-u} a_{q+u} \delta_{q,v} \\ &\qquad - a_{v+r}^* a_{w-r}^* a_v a_q^* a_{p-u} a_{q+u} \delta_{p,w} - a_{v+r}^* a_{w-r}^* a_v a_p^* a_{p-u} a_{q+u} \delta_{q,w}. \end{aligned} \tag{7.57}$$

Combining the last two identities and putting non-normally ordered contributions into normal order, we find that

$$\begin{aligned} [\mathrm{W}, D_1] + \text{h.c.} = \frac{1}{N} &\sum^*_{\substack{u\in\Lambda^*, v,w\in P_L: \\ v+u, w-u\in P_L}} N^\kappa (\widehat{V}(./N^{1-\kappa}) * \eta/N)(u) a_{v+u}^* a_{w-u}^* a_v a_w \\ &+ \sum_{j=1}^{6} \big(\zeta_j + \text{h.c.}\big), \end{aligned} \tag{7.58}$$

where

$$\begin{aligned} \zeta_1 &= -\frac{1}{2N^2} \sum^*_{\substack{u\in\Lambda^*, v,w\in P_L: \\ v+u, w-u\in P_L, \\ r\in P_H^c\cup\{0\}}} N^\kappa \widehat{V}((u-r)/N^{1-\kappa}) \eta_r a_{v+u}^* a_{w-u}^* a_v a_w, \\ \zeta_2 &= -\frac{1}{2N^2} \sum^*_{\substack{u\in\Lambda^*, r\in P_H, \\ v,w\in P_L: \\ w-u, v+u\in P_L}} N^\kappa \widehat{V}(u/N^{1-\kappa}) \eta_r a_{v+r}^* a_{w-r}^* a_{w-u} a_{v+u}, \\ \zeta_3 &= -\frac{1}{2N^2} \sum^*_{\substack{u\in\Lambda^*, r\in P_H, \\ v,w\in P_L}} N^\kappa \widehat{V}(u/N^{1-\kappa}) \eta_r a_{v+r}^* a_{w-r}^* a_{w-u} a_{v+u}, \\ \zeta_4 &= -\frac{1}{N^2} \sum^*_{\substack{u\in\Lambda^*, r\in P_H, \\ v,w,q\in P_L: \\ v-u\in P_L}} N^\kappa \widehat{V}(u/N^{1-\kappa}) \eta_r a_{v+r}^* a_{w-r}^* a_{q-u}^* a_w a_{v-u} a_q, \\ \zeta_5 &= \frac{1}{N^2} \sum^*_{\substack{u\in\Lambda^*, r\in P_H, \\ v,w,q\in P_L: \\ v+r+u\in P_L}} N^\kappa \widehat{V}(u/N^{1-\kappa}) \eta_r a_{v+r+u}^* a_q^* a_{w-r}^* a_{q+u} a_v a_w, \\ \zeta_6 &= -\frac{1}{N^2} \sum^*_{\substack{u\in\Lambda^*, r\in P_H, \\ v,w,q\in P_L}} N^\kappa \widehat{V}(u/N^{1-\kappa}) \eta_r a_{v+r}^* a_{w-r}^* a_q^* a_w a_{v-u} a_{q+u}. \end{aligned} \tag{7.59}$$

Let us briefly explain how to control the operators ζ_1 to ζ_6, defined in (7.2.3).

Noting that $v+u \in P_L$ implies $|u| \leq 2N^\beta$ whenever $v \in P_L$, the first two contributions ζ_1 and ζ_2 in (7.2.3) can be controlled by

$$\begin{aligned}
&|\langle \xi, \zeta_1 \xi\rangle| + |\langle \xi, \zeta_2 \xi\rangle| \\
&\leq \frac{CN^\kappa}{2N^2} \sum^{*}_{\substack{u\in\Lambda^*, v,w\in P_L:\\ v+u, w-u \in P_L,\\ r\in P_H^c\cup\{0\}}} |\eta_r| \frac{|w-u|}{|v|} \|a_{v+u}a_{w-u}\xi\| \frac{|v|}{|w-u|} \|a_v a_w \xi\| \\
&\quad + \frac{CN^\kappa}{2N^2} \sum^{*}_{\substack{u\in\Lambda^*, r\in P_H,\\ v,w\in P_L:\\ w-u, v+u\in P_L}} |\eta_r| \|a_{v+r}(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)^{-1/2} a_{w-r}\xi\| \\
&\quad \times \|a_{w-u}(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)^{1/2} a_{v+u}\xi\| \\
&\leq CN^{\alpha+\beta+2\kappa-1}\langle \xi, \mathcal{K}_{\leq 2N^\beta}\xi\rangle + N^{7\beta/2+2\kappa-\alpha/2-1}\langle \xi, (\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)\xi\rangle \\
&\quad + N^{7\beta/2+2\kappa-\alpha/2-2}\langle \xi, \mathcal{K}_{\leq 2N^\beta}(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)\xi\rangle \\
&\leq CN^{\alpha+\beta+2\kappa-1}\langle \xi, \mathcal{K}_{\leq 2N^\beta}\xi\rangle + CN^{2\beta+\kappa-1}\langle \xi, (\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)\xi\rangle.
\end{aligned} \tag{7.60}$$

By switching to position space, the term ζ_3 can be bounded by

$$\begin{aligned}
&|\langle \xi, \zeta_3\xi\rangle| \\
&\leq CN^{3\beta/2+\kappa-\alpha/2-1}\Big(\int_{\Lambda^2} dxdy\; N^{2-2\kappa}V(N^{1-\kappa}(x-y))\|\check{a}_x\check{a}_y\xi\|^2\Big)^{1/2} \\
&\quad \times \Big(\int_{\Lambda^2} dxdy\; N^{2-2\kappa}V(N^{1-\kappa}(x-y)) \sum_{r\in P_H, w\in P_L} \Big\| \sum_{v\in P_L} e^{ivx} a_{v+r}a_{w-r}\xi\Big\|^2\Big)^{1/2} \\
&\leq CN^{3\beta/2+\kappa-\alpha/2-1}\|\mathcal{V}_N^{1/2}\xi\| \Big(N^{\kappa-1}\int_\Lambda dx \sum_{r\in P_H, w\in P_L} \Big\| \sum_{v\in P_L} e^{ivx} a_{v+r}a_{w-r}\xi\Big\|^2\Big)^{1/2} \\
&\leq CN^{3\beta/2+\kappa/2-1}\langle \xi, \mathcal{V}_N\xi\rangle + CN^{3\beta/2+\kappa/2}.
\end{aligned}$$

We proceed similarly as above for the terms ζ_4 and ζ_5 which yields

$$\begin{aligned}
&|\langle \xi, \zeta_4\xi\rangle| + |\langle \xi, \zeta_5\xi\rangle| \\
&\leq \frac{CN^\kappa}{N^2} \sum^{*}_{\substack{u\in\Lambda^*, r\in P_H,\\ v,w,q\in P_L: v-u\in P_L}} |q|^{-1}|q-u| \|a_{v+r}(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)^{-1/2}a_{w-r}a_{q-u}\xi\| \\
&\quad \times |\eta_r||q||q-u|^{-1}\|a_w(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)^{1/2}a_{v-u}a_q\xi\| \\
&\quad + \frac{CN^\kappa}{N^2} \sum^{*}_{\substack{u\in\Lambda^*, r\in P_H,\\ v,w,q\in P_L:\\ v+r+u\in P_L}} \Big(|q||v|^{-1}\|a_{v+r+u}a_q a_{w-r}\xi\|\Big)\Big(|\eta_r||q|^{-1}|v|\|a_{q+u}a_v a_w\xi\|\Big) \\
&\leq CN^{5\beta/2+2\kappa-\alpha/2-1}\langle \xi, \mathcal{K}_{\leq 3N^\beta}(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)\xi\rangle \\
&\leq CN^{\beta+\kappa-1}\langle \xi, \mathcal{K}_{\leq 3N^\beta}(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)\xi\rangle,
\end{aligned} \tag{7.61}$$

where, for ζ_5, we used that $v+r+u \in P_L$ implies that $|u| \geq \frac{3}{4}N^\alpha$, and thus $|q+u| \geq \frac{1}{2}N^\alpha$, whenever $v,q \in P_L$, $r \in P_H$ and $N \in \mathbb{N}$ sufficiently large (otherwise $|v+r+u| \geq \frac{1}{4}N^\alpha - N^\beta > N^\beta$ for large enough $N \in \mathbb{N}$). Finally, ζ_6 can be controlled by

$$\begin{aligned}
&|\langle \xi, \zeta_6 \xi\rangle| \\
&\quad = \Big| \frac{1}{N} \sideset{}{^*}\sum_{\substack{r\in P_H,\\ v,w,q\in P_L}} \int_{\Lambda^2} N^{2-2\kappa} V(N^{1-\kappa}(x-y)) e^{-ivx-iqy} \eta_r \langle \xi, a^*_{v+r} a^*_{w-r} a^*_q a_w \check{a}_x \check{a}_y \xi\rangle \Big| \\
&\quad \leq C N^{\beta/2+\kappa-\alpha/2-1/2} \| \mathcal{V}_N^{1/2} \xi\| \Big(N^{\kappa-1} \int_\Lambda dx \sideset{}{^*}\sum_{\substack{r\in P_H,\\ w,q\in P_L}} |q| \Big\| \sum_{v\in P_L} e^{-ivx} a_{v+r} a_{w-r} a_q \xi \Big\|^2 \Big)^{1/2} \\
&\quad \leq C N^{\beta/2+\kappa/2-1/2} \| \mathcal{V}_N^{1/2}\xi\| \| \mathcal{K}_L^{1/2} (\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)^{1/2} \xi\|
\end{aligned}$$

In summary, the previous estimates show that

$$\begin{aligned}
\pm \sum_{j=1}^{6} \big(\zeta_j + \text{h.c.}\big) &\leq \delta \mathcal{V}_N + C N^{3\beta/2+\kappa/2-1} \mathcal{V}_N + C N^{\alpha+\beta+2\kappa-1} \mathcal{K}_{\leq 2N^\beta} + C N^{2\beta+\kappa} \\
&\quad + C(1+\delta^{-1}) N^{\beta+\kappa-1} \mathcal{K}_{\leq 3N^\beta} (\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)
\end{aligned} \tag{7.62}$$

for all $\delta > 0$. On the other hand, by Lemma 7.3, we also know that

$$\begin{aligned}
&\pm \Bigg[\frac{1}{N} \sideset{}{^*}\sum_{\substack{u\in\Lambda^*, v,w\in P_L:\\ v+u, w-u\in P_L}} N^\kappa (\widehat{V}(./N^{1-\kappa}) * \eta/N)(u) \int_0^s dt\, e^{-tD} a^*_{v+u} a^*_{w-u} a_v a_w e^{tD} \\
&\qquad - \frac{s}{N} \sideset{}{^*}\sum_{\substack{u\in\Lambda^*, v,w\in P_L:\\ v+u, w-u\in P_L}} N^\kappa (\widehat{V}(./N^{1-\kappa}) * \eta/N)(u) a^*_{v+u} a^*_{w-u} a_v a_w \Bigg] \\
&\quad \leq C N^{-3\beta-3\kappa} \mathcal{K} + C N^{3\beta+\kappa-2} + C N^{\beta+\kappa-1} \mathcal{K}_L (\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)\,.
\end{aligned} \tag{7.63}$$

Now, going back to (7.55), the bounds (7.62) and (7.63) imply that

$$\begin{aligned}
e^{-sD} \mathrm{W} e^{sD} &= \mathrm{W} + \frac{s}{N} \sideset{}{^*}\sum_{\substack{u\in\Lambda^*, v,w\in P_L:\\ v+u, w-u\in P_L}} N^\kappa (\widehat{V}(./N^{1-\kappa}) * \eta/N)(u) a^*_{v+u} a^*_{w-u} a_v a_w \\
&\quad + \mathcal{E}_1(s) + \mathcal{E}_2(s,\delta),
\end{aligned} \tag{7.64}$$

where the self-adjoint operators $\mathcal{E}_1(s)$ and $\mathcal{E}_2(s)$ are bounded by

$$\pm \mathcal{E}_1(s) \leq C(N^{\alpha+\beta+2\kappa-1} + N^{-3\beta-3\kappa}) \mathcal{K} + C N^{2\beta+\kappa},$$

as well as

$$\begin{aligned}
\pm \mathcal{E}_2(s,\delta) &\leq C N^{\beta+\kappa-1} \mathcal{K}_L (\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1) + C(\delta + C N^{3\beta/2+\kappa/2-1}) \int_0^s dt\, e^{-tD} \mathcal{V}_N e^{tD} \\
&\quad + C(1+\delta^{-1}) N^{\beta+\kappa-1} \int_0^s dt\, e^{-tD} \mathcal{K}_{\leq 2N^\beta} (\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1) e^{tD},
\end{aligned}$$

for all $\delta > 0$ and uniformly in $s \in [-1;1]$. Defining $\mathcal{E}_2(s) = \mathcal{E}_2(s, N^{-\beta-\kappa})$, this concludes the proof. $\square$

Equipped with Proposition 7.11, we go back to (7.50) and conclude that

$$\begin{aligned} \mathcal{M}_N^{(4)} \geq \mathcal{H}_N &- \frac{1}{2N} \sum_{\substack{r\in\Lambda^*, v,w\in P_L:\\ v+r, w-r\neq 0}} \widehat{V}(r/N^{1-\kappa})\big(a^*_{v+r}a^*_{w-r}a_v a_w + \text{h.c.}\big) \\ &- \frac{1}{2N} \sum^*_{\substack{u\in\Lambda^*, v,w\in P_L:\\ v+u, w-u\in P_L}} N^\kappa(\widehat{V}(./N^{1-\kappa}) * \eta/N)(u) a^*_{v+u}a^*_{w-u}a_v a_w \\ &- \frac{1}{8}\mathcal{K} - CN^{2\beta+\kappa} + \int_0^1 ds\, \mathcal{E}_2(s) + \int_0^1 ds\, e^{-sD}\Big(\mathcal{E}_{[\mathcal{V}_N,D]} + \mathcal{E}_{[\mathcal{K},D]}\Big)e^{sD}, \end{aligned} \tag{7.65}$$

for all $\alpha \geq 3\beta + 2\kappa \geq 0$ with $\alpha + \beta + 2\kappa - 1 < 0$, $0 \leq \kappa < \beta$ and $N \in \mathbb{N}$ large enough.

Next, let us analyse the error terms related to $\mathcal{E}_2(s)$ and $\mathcal{E}_{[\mathcal{V}_N,D]}$ further. The bounds (7.53) and (7.21) (with $\delta = cN^{-\beta-\kappa}$ for a sufficiently small $c > 0$; this choice guarantees that we can extract the term $\mathcal{V}_{N,L}$ in (7.66), with an error that can be absorbed in $\mathcal{K}$) imply, together with Lemmas 7.1, 7.2, Corollaries 7.4 and 7.6 and with the assumption (5.6) on the exponents α, β, that

$$\begin{aligned} &\int_0^1 ds\,\Big(e^D\mathcal{E}_2(s)e^{-D} + e^{(1-s)D}\mathcal{E}_{[\mathcal{V}_N,D]}e^{-(1-s)D}\Big) \\ &\quad \geq -CN^{2\beta+2\kappa-1}\mathcal{K}_L(\mathcal{N}_{\geq\frac{1}{2}N^\alpha}+1) - \widetilde{C}N^{-\beta-\kappa}(\mathcal{V}_N + \mathcal{V}_{N,L}) - CN^{2\beta}(\mathcal{N}_{\geq\frac{1}{2}N^\alpha}+1) \\ &\qquad - CN^{4\beta+2\kappa-1}(\mathcal{N}_{\geq\frac{1}{2}N^\alpha}+1)^2 \end{aligned}$$

for all $N \in \mathbb{N}$ large enough and for an arbitrarily small constant $\widetilde{C} > 0$. With Corollary 7.4 and (7.65), we conclude that

$$\begin{aligned} \mathcal{M}_N^{(4)} \geq \mathcal{H}_N &- \frac{1}{2N} \sum_{\substack{r\in\Lambda^*, v,w\in P_L:\\ v+r, w-r\neq 0}} \widehat{V}(r/N^{1-\kappa})\big(a^*_{v+r}a^*_{w-r}a_v a_w + \text{h.c.}\big) \\ &- \frac{1}{2N} \sum^*_{\substack{u\in\Lambda^*, v,w\in P_L:\\ v+u, w-u\in P_L}} N^\kappa(\widehat{V}(./N^{1-\kappa}) * \eta/N)(u) a^*_{v+u}a^*_{w-u}a_v a_w \\ &- \frac{1}{4}\mathcal{K} - CN^{2\beta+\kappa} - \widetilde{C}N^{-\beta-\kappa}\mathcal{V}_{N,L} + \int_0^1 ds\, e^{-sD}\mathcal{E}_{[\mathcal{K},D]}e^{sD} + \mathcal{E}^{(41)}_{\mathcal{M}_N}, \end{aligned} \tag{7.66}$$

where the error $\mathcal{E}^{(41)}_{\mathcal{M}_N}$ is such that

$$\begin{aligned} e^D\mathcal{E}^{(41)}_{\mathcal{M}_N}e^{-D} \geq &-CN^{2\beta+2\kappa-1}\mathcal{K}_L(\mathcal{N}_{\geq\frac{1}{2}N^\alpha}+1) - CN^{-\beta-\kappa}\mathcal{V}_N \\ &- CN^{2\beta}\mathcal{N}_{\geq\frac{1}{2}N^\alpha} - CN^{4\beta+2\kappa-1}\mathcal{N}^2_{\geq\frac{1}{2}N^\alpha} \end{aligned}$$

Applying Lemmas 4.2, 4.3 and Corollary 4.5, we deduce with the operator inequality $\mathcal{N}_{\geq \frac{1}{2}N^\alpha} \leq 4N^{-2\alpha}\mathcal{K}$ that

$$e^A e^D \mathcal{E}^{(41)}_{\mathcal{M}_N} e^{-D} e^{-A} \geq -CN^{-\beta}\mathcal{K} - CN^{-\beta-\kappa}\mathcal{V}_N - CN^{2\beta+2\kappa-1} - CN^{2\beta+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq \frac{1}{2}N^\alpha} \tag{7.67}$$

for all $N \in \mathbb{N}$ large enough.

Now, we switch to the contribution containing the operator $\mathcal{E}_{[\mathcal{K},D]}$ on the r.h.s. of the lower bound (7.66). We recall once again that

$$\int_0^1 ds\; e^{-sD}\mathcal{E}_{[\mathcal{K},D]}e^{sD} = \int_0^1 ds\; \sum_{j=1}^3 e^{-sD}\big(\Sigma_j + \text{h.c.}\big)e^{sD},$$

where the operators Σ_1, Σ_2 and Σ_3 were defined in (7.26). It turns out that Σ_2 and Σ_3 are negligible errors while Σ_1 still contains an important contribution of leading order. We start with the analysis of the contribution related to Σ_1.

Proposition 7.12. *Assume the exponents α, β satisfy (5.6). Then, we have that*

$$\begin{aligned}&\frac{1}{2N}\sum_{\substack{u\in P_H^c\cup\{0\},p,q\in P_L:\\ p+u,q-u\neq 0}} N^\kappa\big(\widehat{V}(/N^{1-\kappa}) * \widehat{f}_N\big)(u)e^{-sD}\big(a^*_{p+u}a^*_{q-u}a_pa_q + \text{h.c.}\big)e^{sD}\\ &\quad= \frac{1}{2N}\sum_{\substack{u\in P_H^c\cup\{0\},p,q\in P_L:\\ p+u,q-u\neq 0}} N^\kappa\big(\widehat{V}(/N^{1-\kappa}) * \widehat{f}_N\big)(u)\big(a^*_{p+u}a^*_{q-u}a_pa_q + \text{h.c.}\big) + \mathcal{E}_3(s)\end{aligned} \tag{7.68}$$

and there exists a constant $C > 0$ such that

$$\begin{aligned}&\pm\, e^A e^D \mathcal{E}_3(s) e^{-D} e^{-A}\\ &\quad\leq CN^{\alpha+\beta+2\kappa-1}\mathcal{K} + CN^{\alpha+\beta+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq\frac{1}{2}N^\alpha} + CN^{4\beta+2\kappa} + CN^{\alpha+3\beta+2\kappa-1}\end{aligned} \tag{7.69}$$

for all $s \in [-1; 1]$ and for all $N \in \mathbb{N}$ sufficiently large.

Proof. We proceed as in Proposition 7.11 and recall $\Sigma_1 : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$ to be

$$\Sigma_1 = \frac{1}{2N}\sum_{\substack{u\in P_H^c\cup\{0\},p,q\in P_L:\\ p+u,q-u\neq 0}} N^\kappa\big(\widehat{V}(/N^{1-\kappa}) * \widehat{f}_N\big)(u)\big(a^*_{p+u}a^*_{q-u}a_pa_q + \text{h.c.}\big).$$

We then have

$$e^{-sD}\Sigma_1 e^{sD} - \Sigma_1 = \int_0^s dt\; e^{-tD}[\Sigma_1, D_1]e^{tD} + \text{h.c.} \tag{7.70}$$

Similarly as in (7.58) and (7.2.3), we find that

$$[\Sigma_1, D_1] + \text{h.c.} = \sum_{i=1}^8 (\Gamma_i + \text{h.c.}), \tag{7.71}$$

where

$$\Gamma_1 = \frac{1}{N^2} \sum^{*}_{\substack{u\in P_H^c\cup\{0\},r\in P_H,v,w\in P_L:\\ v+u+r,w-u-r\in P_L}} N^\kappa\big(\widehat{V}(./N^{1-\kappa}) * \widehat{f}_N\big)(u)\eta_r a^*_{v+u+r}a^*_{w-u-r}a_v a_w,$$

$$\Gamma_2 = -\frac{1}{2N^2} \sum^{*}_{\substack{u\in P_H^c\cup\{0\},r\in P_H,\\ v,w\in P_L:\\ w-u,v+u\in P_L}} N^\kappa\big(\widehat{V}(./N^{1-\kappa}) * \widehat{f}_N\big)(u)\eta_r a^*_{v+r}a^*_{w-r}a_{w-u} a_{v+u},$$

$$\Gamma_3 = -\frac{1}{2N^2} \sum^{*}_{\substack{u\in P_H^c\cup\{0\},r\in P_H,\\ v,w\in P_L}} N^\kappa\big(\widehat{V}(./N^{1-\kappa}) * \widehat{f}_N\big)(u)\eta_r a^*_{v+r}a^*_{w-r}a_{w-u} a_{v+u},$$

$$\Gamma_4 = -\frac{1}{N^2} \sum^{*}_{\substack{u\in P_H^c\cup\{0\},r\in P_H,\\ v,w,q\in P_L:\\ v-u\in P_L}} N^\kappa\big(\widehat{V}(./N^{1-\kappa}) * \widehat{f}_N\big)(u)\eta_r a^*_{v+r}a^*_{w-r}a^*_{q-u}a_w a_{v-u}a_q,$$

$$\Gamma_5 = \frac{1}{N^2} \sum^{*}_{\substack{u\in P_H^c\cup\{0\},r\in P_H,\\ v,w,q\in P_L:\\ v+r+u\in P_L}} N^\kappa\big(\widehat{V}(./N^{1-\kappa}) * \widehat{f}_N\big)(u)\eta_r a^*_{v+r+u}a^*_q a^*_{w-r}a_{q+u}a_v a_w,$$

$$\Gamma_6 = -\frac{1}{N^2} \sum^{*}_{\substack{u\in P_H^c\cup\{0\},r\in P_H,\\ v,w,q\in P_L}} N^\kappa\big(\widehat{V}(./N^{1-\kappa}) * \widehat{f}_N\big)(u)\eta_r a^*_{v+r}a^*_{w-r}a^*_q a_w a_{v-u}a_{q+u}.$$

The operators Γ_1 to Γ_6 can be bounded similarly as in the proof of Proposition 7.11. Let us start with Γ_1. Applying as usual Cauchy-Schwarz implies that

$$\begin{aligned}
&|\langle\xi,\Gamma_1\xi\rangle|\\
&\quad\le \frac{CN^\kappa}{N^2}\sum^{*}_{\substack{u\in P_H^c\cup\{0\},r\in P_H,v,w\in P_L:\\ v+u+r,w-u-r\in P_L}}\Big(|v|^{-1}\|a_{v+u+r}a_{w-u-r}\xi\|\Big)\Big(|\eta_r||v|\|a_v a_w\xi\|\Big)\\
&\quad\le CN^{\alpha/2+5\beta/2+2\kappa-1/2}\|\xi\|\|\mathcal{K}_L^{1/2}\xi\|\le CN^{\alpha+\beta+2\kappa-1}\langle\xi,\mathcal{K}_L\xi\rangle+CN^{4\beta+2\kappa}\|\xi\|^2
\end{aligned}$$

where we used that $v+u+r\in P_L$ implies $|u|\ge N^\alpha-3N^\beta$ and $|r|\le N^\alpha+3N^\beta$ whenever $u\in P_H^c, r\in P_H$ and $v\in P_L$ (otherwise $|u+r+v|\ge|r|-|u|-N^\beta\ge 2N^\beta>N^\beta$ if either $|u|\le N^\alpha-3N^\beta$ or $|r|\ge N^\alpha+3N^\beta$, in contradiction to $u+r+v\in P_L$) for $N\in\mathbb{N}$ sufficiently large. Notice in addition that $\sum_{N^\alpha-3N^\beta\le|u|\le N^\alpha}\le CN^{2\alpha+\beta}$.

The term Γ_2 can be estimated exactly as the term ζ_2 in (7.60), that is

$$|\langle\xi,\Gamma_2\xi\rangle|\le CN^{\alpha+\beta+2\kappa-1}\langle\xi,\mathcal{K}_{\le 2N^\beta}\xi\rangle+CN^{2\beta+\kappa-1}\langle\xi,(\mathcal{N}_{\ge\frac12 N^\alpha}+1)\xi\rangle.$$

The contribution Γ_3 can be controlled by

$$|\langle\xi,\Gamma_3\xi\rangle| \le \frac{CN^\kappa}{2N^2} \sum^{*}_{\substack{u\in P_H^c\cup\{0\},r\in P_H,\\ v,w\in P_L}} |\eta_r| \|a_{v+r}(\mathcal{N}_{\ge\frac12 N^\alpha}+1)^{-1/2}a_{w-r}\xi\| \\ \|a_{w-u}(\mathcal{N}_{\ge\frac12 N^\alpha}+1)^{1/2}a_{v+u}\xi\| \\ \le CN^{\alpha+3\beta+2\kappa-1}\langle\xi,(\mathcal{N}_{\ge\frac12 N^\alpha}+1)\xi\rangle.$$

The terms Γ_4 and Γ_5 can be bounded exactly as in (7.61). We find

$$|\langle\xi,\Gamma_4\xi\rangle| + |\langle\xi,\Gamma_5\xi\rangle| \le CN^{\beta+\kappa-1}\langle\xi,\mathcal{K}_{\le 2N^\beta}(\mathcal{N}_{\ge\frac12 N^\alpha}+1)\xi\rangle,$$

Finally, the last contribution Γ_6 is bounded by

$$|\langle\xi,\Gamma_6\xi\rangle| \le \frac{CN^\kappa}{N^2} \sum^{*}_{\substack{u\in P_H^c\cup\{0\},r\in P_H,\\ v,w,q\in P_L}} \Big(|q||w|^{-1}\|a_{v+r}(\mathcal{N}_{\ge\frac12 N^\alpha}+1)^{-1/2}a_{w-r}a_q\xi\|\Big) \\ \times\Big(|\eta_r||w||q|^{-1}\|a_{v-u}(\mathcal{N}_{\ge\frac12 N^\alpha}+1)^{1/2}a_w a_{q+u}\xi\|\Big) \\ \le CN^{\alpha+\beta+2\kappa-1}\langle\xi,\mathcal{K}_L(\mathcal{N}_{\ge\frac12 N^\alpha}+1)\xi\rangle.$$

In conclusion, the above estimates imply that

$$\pm\sum_{i=1}^{6}\big(\Gamma_i+\text{h.c.}\big) \le CN^{\alpha+\beta+2\kappa-1}\mathcal{K}_{\le 2N^\beta} + CN^{\alpha+\beta+2\kappa-1}\mathcal{K}_{\le 2N^\beta}(\mathcal{N}_{\ge\frac12 N^\alpha}+1) \\ + CN^{\alpha+3\beta+2\kappa-1}(\mathcal{N}_{\ge\frac12 N^\alpha}+1) + CN^{4\beta+2\kappa}$$

for all $\alpha > 3\beta+2\kappa \ge 0$ and for all $N\in\mathbb{N}$ sufficiently large. Combining this estimate with the identites (7.70) and (7.71), and applying Lemmas 4.2, 4.3, 7.1 as well as Lemma 7.2 together with the operator inequality $\mathcal{N}_{\ge\frac12 N^\alpha} \le 4N^{-2\alpha}\mathcal{K}$ proves the proposition. □

Applying Proposition 7.12 to the lower bound (7.66) and defining $\mathcal{E}^{(42)}_{\mathcal{M}_N} = \int_0^1 ds\, \mathcal{E}_3(s)$ with $\mathcal{E}(s)$ from Proposition 7.12, we conclude that

$$\mathcal{M}_N^{(4)} \ge \mathcal{H}_N - \frac{1}{2N}\sum_{\substack{r\in\Lambda^*,v,w\in P_L:\\ v+r,w-r\neq 0}} \widehat{V}(r/N^{1-\kappa})\big(a^*_{v+r}a^*_{w-r}a_v a_w+\text{h.c.}\big) \\ - \frac{1}{2N}\sum^{*}_{\substack{u\in\Lambda^*,v,w\in P_L:\\ v+u,w-u\in P_L}} N^\kappa(\widehat{V}(./N^{1-\kappa})*\eta/N)(u)a^*_{v+u}a^*_{w-u}a_v a_w \\ + \frac{1}{2N}\sum_{\substack{u\in P_H^c\cup\{0\},p,q\in P_L:\\ p+u,q-u\neq 0}} N^\kappa\big(\widehat{V}(/N^{1-\kappa})*\widehat{f}_N\big)(u)\big(a^*_{p+u}a^*_{q-u}a_p a_q+h.c.\big) \\ - \frac14\mathcal{K} - CN^{-\beta-\kappa}\mathcal{V}_{N,L} + \mathcal{E}^{(41)}_{\mathcal{M}_N} + \mathcal{E}^{(42)}_{\mathcal{M}_N} + \int_0^1 ds\, e^{-sD}\big(\Sigma_2+\Sigma_3+\text{h.c.}\big)e^{sD}, \tag{7.72}$$

where $\mathcal{E}^{(41)}_{\mathcal{M}_N}$ satisfies the lower bound (7.67), $\mathcal{E}^{(42)}_{\mathcal{M}_N}$ satisfies the bound (7.69) and where the operators Σ_2 and Σ_3 were defined in (7.26).

Let us finally estimate the size of the error in the last line of (7.72), involving the two operators Σ_2 and Σ_3. Using the estimate (7.31) together with Lemmas 4.2, 4.3, 7.1 and 7.2, we find for $\mathcal{E}^{(43)}_{\mathcal{M}_N} = \int_0^1 ds\, e^{-sD}\big(\Sigma_2 + \text{h.c.}\big)e^{sD}$

$$e^A e^D \mathcal{E}^{(43)}_{\mathcal{M}_N} e^{-D} e^{-A} \geq -CN^{-\beta-1}\mathcal{K}\mathcal{N}_{\geq \frac{1}{2}N^\alpha} - CN^{-5\beta-4\kappa}\mathcal{K} - CN^\beta. \quad (7.73)$$

Finally, consider the operator $\mathcal{E}^{(44)}_{\mathcal{M}_N} = \int_0^1 ds\ e^{-sD}\big(\Sigma_3 + \text{h.c.}\big)e^{sD}$, with Σ_3 defined in (7.26). Let $m_0 \in \mathbb{R}$ be such that $m_0\beta = \alpha$ (in particular, $\lfloor m_0 \rfloor \geq 3$). Here, we use the bound (7.32) to find first of all that

$$\begin{aligned}\mathcal{E}^{(44)}_{\mathcal{M}_N} \geq - \int_0^1 ds\ \|\mathcal{K}^{1/2}e^{sD}\xi\| \Big(& N^{-1/2}\big\|\mathcal{K}_L^{1/2}(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)^{1/2}\xi\big\| \\ & + N^{\beta-1}\big\|(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)^{3/2}\xi\big\|\Big)\end{aligned}$$

for any $\xi \in \mathcal{F}_+^{\leq N}$ with $\|\xi\| = 1$. Notice that we applied once again Lemmas 7.1 and 7.2 in the second factor. With Corollary 7.8, the first factor is bounded by

$$\begin{aligned}&\mathcal{E}^{(44)}_{\mathcal{M}_N} \\ &\quad \geq -C\bigg(\|\mathcal{K}^{1/2}\xi\| + \|\mathcal{V}_N^{1/2}\xi\| + \|\mathcal{V}_{N,L}^{1/2}\xi\| + N^{5\beta/8+\kappa/2}\|\mathcal{K}^{1/2}_{\leq N^{3\beta/2}}\xi\| \\ &\qquad + N^{-1/2}\big\|\mathcal{K}_L^{1/2}(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)^{1/2}\xi\big\| + N^{3\beta/2+\kappa/2} \\ &\qquad + \sum_{j=3}^{2\lfloor m_0 \rfloor -1} N^{j\beta/4+3\beta/4+\kappa-1/2}\Big[\big\|\mathcal{K}_L^{1/2}(\mathcal{N}_{\geq \frac{1}{2}N^{j\beta/2}}+1)^{1/2}\xi\big\| \\ &\qquad + N^\beta\big\|(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)^{1/2}(\mathcal{N}_{\geq \frac{1}{2}N^{j\beta/2}}+1)^{1/2}\xi\big\|\Big] \\ &\qquad + N^{\alpha/2+\beta/2+\kappa-1/2}\Big[\big\|\mathcal{K}_L^{1/2}(\mathcal{N}_{\geq \frac{1}{2}N^{\lfloor m_0 \rfloor\beta}}+1)^{1/2}\xi\big\| \\ &\qquad + N^\beta\big\|(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)^{1/2}(\mathcal{N}_{\geq \frac{1}{2}N^{\lfloor m_0 \rfloor\beta}}+1)^{1/2}\xi\big\|\Big]\bigg) \\ &\qquad \times \Big(N^{-1/2}\big\|\mathcal{K}_L^{1/2}(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)^{1/2}\xi\big\| + N^{\beta-1}\big\|(\mathcal{N}_{\geq \frac{1}{2}N^\alpha}+1)^{3/2}\xi\big\|\Big)\end{aligned}$$

for all exponents α, β satisfying (5.6) and $N \in \mathbb{N}$ sufficiently large. It follows that

$$\mathcal{E}^{(44)}_{\mathcal{M}_N} \geq \mathcal{E}^{(441)}_{\mathcal{M}_N} + \mathcal{E}^{(442)}_{\mathcal{M}_N} + \mathcal{E}^{(443)}_{\mathcal{M}_N}, \quad (7.74)$$

where

$$\mathcal{E}^{(441)}_{\mathcal{M}_N} = -\frac{1}{8}\mathcal{K} - \widetilde{C}N^{-\alpha}\mathcal{V}_{N,L} - CN^{3\beta+\kappa}, \qquad \mathcal{E}^{(442)}_{\mathcal{M}_N} = N^{-\alpha}\mathcal{V}_N \quad (7.75)$$

with an arbitrarily small constant $\widetilde{C} > 0$ and where after an additional application of Lemmas 4.2, 4.3, 7.1 and 7.2 together with the operator bound $\mathcal{N}_{\geq\Theta} \leq \Theta^{-2}\mathcal{K}$, the error $\mathcal{E}^{(443)}_{\mathcal{M}_N}$ is such that

$$\begin{aligned} &e^A e^D \mathcal{E}^{(443)}_{\mathcal{M}_N} e^{-D} e^{-A} \\ &\quad \geq -CN^{\alpha+\beta+2\kappa-1}\mathcal{K} - CN^{\alpha-1}\mathcal{K}\mathcal{N}_{\geq \frac{1}{2}N^\alpha} - CN^{\alpha+3\beta+2\kappa-1} \\ &\qquad - C\sum_{j=3}^{2\lfloor m_0\rfloor -1} N^{j\beta/2+\beta/2+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq \frac{1}{2}N^{j\beta/2}} - CN^{\alpha+\beta+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq \frac{1}{2}N^{\lfloor m_0\rfloor\beta}} \end{aligned} \tag{7.76}$$

for all exponents α, β satisfying (5.6) and $N \in \mathbb{N}$ sufficiently large.

Choosing $\widetilde{C} > 0$ sufficiently large (but independently of $N \in \mathbb{N}$) and arguing as right before (7.66), we deduce that

$$\begin{aligned} &e^A\Big(\widetilde{C}N^{-\alpha}e^D\mathcal{V}_{N,L}e^{-D} + e^D\mathcal{E}^{(442)}_{\mathcal{M}_N}e^{-D}\Big)e^{-A} \\ &\quad \geq -CN^{-\alpha}\mathcal{V}_N - CN^{-3\beta-\kappa}\mathcal{N}_+ - CN^{-2\beta-\kappa-1}\mathcal{K}\mathcal{N}_{\geq\frac{1}{2}N^\alpha} \end{aligned} \tag{7.77}$$

for all α, β satisfying (5.6) and $N \in \mathbb{N}$ sufficiently large. This follows through another application of Corollaries 4.5, 7.4 and 7.6, together with Lemmas 4.2, 4.3, 7.1 and 7.2. We summarize these bounds in the following corollary.

Corollary 7.13. *Let $m_0 \in \mathbb{R}$ be such that $m_0\beta = \alpha$ and let $\mathcal{M}_N^{(4)}$ be defined as in (7.35). For every $\widetilde{C} > 0$, there exists a constant $C > 0$ such that*

$$\begin{aligned} \mathcal{M}_N^{(4)} &\geq \frac{1}{2}\mathcal{K} + \mathcal{V}_N - \frac{1}{2N}\sum_{\substack{r\in\Lambda^*, v,w\in P_L:\\ v+r, w-r\neq 0}} \widehat{V}(r/N^{1-\kappa})\big(a^*_{v+r}a^*_{w-r}a_va_w + h.c.\big) \\ &\quad - \frac{1}{2N}\sum^{*}_{\substack{u\in\Lambda^*, v,w\in P_L:\\ v+u, w-u\in P_L}} N^\kappa(\widehat{V}(./N^{1-\kappa}) * \eta/N)(u)a^*_{v+u}a^*_{w-u}a_va_w \\ &\quad + \frac{1}{2N}\sum_{\substack{u\in P_H^c\cup\{0\}, p,q\in P_L:\\ p+u, q-u\neq 0}} N^\kappa\big(\widehat{V}(/N^{1-\kappa}) * \widehat{f}_N\big)(u)\big(a^*_{p+u}a^*_{q-u}a_pa_q + \text{h.c.}\big) \\ &\quad - \widetilde{C}N^{-\beta-\kappa}\mathcal{V}_{N,L} + \mathcal{E}^{(4)}_{\mathcal{M}_N} \end{aligned} \tag{7.78}$$

where

$$\begin{aligned} &e^Ae^D\mathcal{E}^{(4)}_{\mathcal{M}_N}e^{-D}e^{-A} \\ &\quad \geq -CN^{-\beta}\mathcal{K} - CN^{-\beta-\kappa}\mathcal{V}_N - CN^{\alpha+\beta+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq\frac{1}{2}N^{\lfloor m_0\rfloor\beta}} \\ &\qquad - C\sum_{j=3}^{2\lfloor m_0\rfloor-1} N^{j\beta/2+\beta/2+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq\frac{1}{2}N^{j\beta/2}} - CN^{2\beta+\kappa} \end{aligned} \tag{7.79}$$

for all exponents α, β satisfying (5.6) and for all $N \in \mathbb{N}$ sufficiently large.

Proof. The proof follows from defining $\mathcal{E}^{(4)}_{\mathcal{M}_N} = \sum_{j=1}^3 \mathcal{E}^{(4j)}_{\mathcal{M}_N} + \sum_{j=1}^3 \mathcal{E}^{(44j)}_{\mathcal{M}_N}$ and combining (7.67), (7.72), (7.69), (7.73), (7.74), (7.75), (7.77), (7.76) and the operator bound $\mathcal{N}_+ \leq (2\pi)^{-2}\mathcal{K}$ in $\mathcal{F}_+^{\leq N}$. □

7.3. Proof of Proposition 5.1

Recall from (7.34) the decomposition

$$\mathcal{M}_N = 4\pi\mathfrak{a}_0 N^{1+\kappa} - 4\pi\mathfrak{a}_0 N^{\kappa-1}\mathcal{N}_+^2/N + \mathcal{M}_N^{(2)} + \mathcal{M}_N^{(3)} + \mathcal{M}_N^{(4)}$$

Collecting the results of Propositions 7.9, 7.10 and Corollary 7.13, we deduce that

$$\begin{aligned}
\mathcal{M}_N \geq\ & 4\pi\mathfrak{a}_0 N^{1+\kappa} - 4\pi\mathfrak{a}_0 N^{\kappa-1}\mathcal{N}_+^2 + 8\pi\mathfrak{a}_0 N^{\kappa} \sum_{p\in P_H^c} \Big[b_p^* b_p + \frac{1}{2} b_p^* b_{-p}^* + \frac{1}{2} b_p b_{-p}\Big] \\
& + \frac{8\pi\mathfrak{a}_0 N^{\kappa}}{\sqrt{N}} \sum_{\substack{p\in P_H^c, q\in P_L:\\ p+q\neq 0}} \big[b_{-p}^* a_{p+q}^* a_q + \text{h.c.}\big] + \frac{1}{2}\mathcal{K} \\
& + \mathcal{V}_N - \frac{1}{2N} \sum_{\substack{r\in\Lambda^*, v,w\in P_L:\\ v+r, w-r\neq 0}} \widehat{V}(r/N^{1-\kappa})\big(a_{v+r}^* a_{w-r}^* a_v a_w + \text{h.c.}\big) \\
& - \frac{1}{2N} \sum^{*}_{\substack{r\in\Lambda^*, v,w\in P_L:\\ v+r, w-r\in P_L}} N^{\kappa}(\widehat{V}(./N^{1-\kappa}) * \eta/N)(r) a_{v+r}^* a_{w-r}^* a_v a_w \\
& + \frac{1}{2N} \sum_{\substack{r\in P_H^c\cup\{0\}, v,w\in P_L:\\ v+r, w-r\neq 0}} N^{\kappa}\big(\widehat{V}(./N^{1-\kappa}) * \widehat{f}_N\big)(r)\big(a_{v+r}^* a_{w-r}^* a_v a_w + \text{h.c.}\big) \\
& - \widetilde{C} N^{-\beta-\kappa}\mathcal{V}_{N,L} + \mathcal{E}'_{\mathcal{M}_N},
\end{aligned} \tag{7.80}$$

where $\mathcal{E}'_{\mathcal{M}_N}$ satisfies the lower bound

$$\begin{aligned}
e^A e^D \mathcal{E}'_{\mathcal{M}_N} e^{-D} e^{-A} \geq\ & -CN^{-\beta}\mathcal{K} - CN^{-\beta-\kappa}\mathcal{V}_N - CN^{\alpha+\beta+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq\frac{1}{2}N^{\lfloor m_0\rfloor\beta}} \\
& - C\sum_{j=3}^{2\lfloor m_0\rfloor-1} N^{j\beta/2+\beta/2+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq\frac{1}{2}N^{j\beta/2}} - CN^{\alpha+\beta/2+2\kappa}
\end{aligned} \tag{7.81}$$

for all $N\in\mathbb{N}$ sufficiently large.

We combine next the terms on the third, fourth and fifth lines in (7.3). We first notice that

$$\begin{aligned}
& \frac{1}{2N} \sum_{\substack{r\in\Lambda^*, v,w\in P_L:\\ v+r, w-r\neq 0}} \widehat{V}(r/N^{1-\kappa})\big(a_{v+r}^* a_{w-r}^* a_v a_w + a_v^* a_w^* a_{w-r} a_{v+r}\big) \\
&\quad = \frac{1}{2N} \sum_{\substack{r\in\Lambda^*, v,w\in\Lambda_+^*:\\ v,w\in P_L,\\ v+r, w-r\neq 0}} \widehat{V}(r/N^{1-\kappa}) a_{v+r}^* a_{w-r}^* a_v a_w \\
&\qquad + \frac{1}{2N} \sum_{\substack{r\in\Lambda^*, v,w\in\Lambda_+^*:\\ v+r, w-r\in P_L}} \widehat{V}(r/N^{1-\kappa}) a_{v+r}^* a_{w-r}^* a_v a_w
\end{aligned}$$

$$= \frac{1}{2N} \sum^{*}_{\substack{r\in\Lambda^*, v,w\in\Lambda^*_+:\\ (v,w)\in P_L^2 \text{ or } (v+r,w-r)\in P_L^2}} \widehat{V}(r/N^{1-\kappa}) a^*_{v+r} a^*_{w-r} a_v a_w$$
$$+ \frac{1}{2N} \sum^{*}_{\substack{r\in\Lambda^*, v,w\in\Lambda^*_+:\\ (v,w,v+r,w-r)\in P_L^4}} \widehat{V}(r/N^{1-\kappa}) a^*_{v+r} a^*_{w-r} a_v a_w \tag{7.82}$$

Arguing in the same way for the contribution on the fifth line in (7.3), using that $(\widehat{f}_N - \eta/N)(p) = \delta_{p,0}$ for all $p \in \Lambda^*_+$, and using that $v \in P_L$ and $v+r \in P_L$ implies in particular that $r \in P_H^c$, we therefore obtain that

$$\begin{aligned}
&\mathcal{V}_N - \frac{1}{2N} \sum^{*}_{\substack{r\in\Lambda^*, v,w\in\Lambda^*_+:\\ (v,w)\in P_L^2 \text{ or } (v+r,w-r)\in P_L^2}} \widehat{V}(r/N^{1-\kappa}) a^*_{v+r} a^*_{w-r} a_v a_w \\
&\quad - \frac{1}{2N} \sum_{\substack{r\in\Lambda^*, v,w\in\Lambda^*_+:\\ (v,w,v+r,w-r)\in P_L^4}} \widehat{V}(r/N^{1-\kappa}) a^*_{v+r} a^*_{w-r} a_v a_w \\
&\quad - \frac{1}{2N} \sum^{*}_{\substack{r\in\Lambda^*, v,w\in P_L:\\ v+r,w-r\in P_L}} N^\kappa (\widehat{V}(./N^{1-\kappa}) * \eta/N)(r) a^*_{v+r} a^*_{w-r} a_v a_w \\
&\quad + \frac{1}{2N} \sum_{\substack{r\in P_H^c\cup\{0\}, v,w\in P_L:\\ v+r,w-r\neq 0}} N^\kappa \big(\widehat{V}(/N^{1-\kappa}) * \widehat{f}_N\big)(r) \big(a^*_{v+r} a^*_{w-r} a_v a_w + \text{h.c.}\big) \\
&= \mathcal{V}_N - \frac{1}{2N} \sum^{*}_{\substack{r\in\Lambda^*, v,w\in\Lambda^*_+:\\ (v,w)\in P_L^2 \text{ or } (v+r,w-r)\in P_L^2}} \widehat{V}(r/N^{1-\kappa}) a^*_{v+r} a^*_{w-r} a_v a_w \\
&\quad + \frac{1}{2N} \sum^{*}_{\substack{r\in P_H^c\cup\{0\}, v,w\in P_L:\\ (v,w)\in P_L^2 \text{ or } (v+r,w-r)\in P_L^2}} N^\kappa \big(\widehat{V}(/N^{1-\kappa}) * \widehat{f}_N\big)(r) a^*_{v+r} a^*_{w-r} a_v a_w.
\end{aligned} \tag{7.83}$$

Now, notice furthermore that

$$\begin{aligned}
&\mathcal{V}_N - \frac{1}{2N} \sum^{*}_{\substack{r\in\Lambda^*, v,w\in\Lambda^*_+:\\ (v,w)\in P_L^2 \text{ or } (v+r,w-r)\in P_L^2}} \widehat{V}(r/N^{1-\kappa}) a^*_{v+r} a^*_{w-r} a_v a_w \\
&\quad = \frac{1}{2N} \sum^{*}_{\substack{r\in\Lambda^*, v,w\in\Lambda^*_+:\\ (v,w)\in (P_L^2)^c \text{ and}\\ (v+r,w-r)\in (P_L^2)^c}} \widehat{V}(r/N^{1-\kappa}) a^*_{v+r} a^*_{w-r} a_v a_w,
\end{aligned}$$

such that, after switching to position space, the pointwise positivity $V \geq 0$ implies

$$\begin{aligned}
&\mathcal{V}_N - \frac{1}{2N} \sum^{*}_{\substack{r\in\Lambda^*, v,w\in\Lambda^*_+:\\ (v,w)\in P_L^2 \text{ or } (v+r,w-r)\in P_L^2}} \widehat{V}(r/N^{1-\kappa}) a^*_{v+r} a^*_{w-r} a_v a_w \\
&\quad = \int_{\Lambda^2} dxdy\, N^{2-2\kappa} V(N^{1-\kappa}(x-y)) \\
&\qquad \times \Big[a^*\big((\check{\chi}_{P_L^c})_x\big) a^*\big((\check{\chi}_{P_L^c})_y\big) + a^*\big((\check{\chi}_{P_L})_x\big) a^*\big((\check{\chi}_{P_L^c})_y\big) + a^*\big((\check{\chi}_{P_L^c})_x\big) a^*\big((\check{\chi}_{P_L})_y\big) \Big] \\
&\qquad \times \Big[a\big((\check{\chi}_{P_L^c})_x\big) a\big((\check{\chi}_{P_L^c})_y\big) + a\big((\check{\chi}_{P_L})_x\big) a\big((\check{\chi}_{P_L^c})_y\big) + a\big((\check{\chi}_{P_L^c})_x\big) a\big((\check{\chi}_{P_L})_y\big) \Big] \\
&\quad \geq 0.
\end{aligned} \tag{7.84}$$

Here, we used that $\Lambda^*_+ = P_L \cup P_L^c$ and we denote by $\check{\chi}_S$ the distribution which has Fourier transform χ_S, the characteristic function of the set $S \subset \Lambda^*_+$.

Combining (7.3), (7.3), (7.3) and (7.84), it follows that

$$\begin{aligned}
\mathcal{M}_N \geq\; & 4\pi\mathfrak{a}_0 N^{1+\kappa} - 4\pi\mathfrak{a}_0 N^{\kappa-1}\mathcal{N}_+^2 + 8\pi\mathfrak{a}_0 N^{\kappa} \sum_{p\in P_H^c} \Big[b^*_p b_p + \frac{1}{2} b^*_p b^*_{-p} + \frac{1}{2} b_p b_{-p} \Big] \\
& + \frac{8\pi\mathfrak{a}_0 N^{\kappa}}{\sqrt{N}} \sum_{\substack{p\in P_H^c, q\in P_L:\\ p+q\neq 0}} \big[b^*_{-p} a^*_{p+q} a_q + \text{h.c.} \big] + \frac{1}{2}\mathcal{K} \\
& + \frac{1}{2N} \sum^{*}_{\substack{r\in P_H^c\cup\{0\}, v,w\in P_L:\\ (v,w)\in P_L^2 \text{ or } (v+r,w-r)\in P_L^2}} N^{\kappa} \big(\widehat{V}(./N^{1-\kappa}) * \widehat{f}_N \big)(r) a^*_{v+r} a^*_{w-r} a_v a_w \\
& - \widetilde{C} N^{-\beta-\kappa} \mathcal{V}_{N,L} + \mathcal{E}'_{\mathcal{M}_N}
\end{aligned} \tag{7.85}$$

Using Lemma 3.1, part ii), we have $\big(\widehat{V}(./N^{1-\kappa}) * \widehat{f}_N\big)(0) = 8\pi\mathfrak{a}_0 + \mathcal{O}(N^{\kappa-1})$. This implies

$$\begin{aligned}
\mathcal{M}_N \geq\; & 4\pi\mathfrak{a}_0 N^{1+\kappa} + 8\pi\mathfrak{a}_0 N^{\kappa} \sum_{p\in P_H^c} \Big[b^*_p b_p + \frac{1}{2} b^*_p b^*_{-p} + \frac{1}{2} b_p b_{-p} \Big] \\
& + \frac{8\pi\mathfrak{a}_0 N^{\kappa}}{\sqrt{N}} \sum_{\substack{p\in P_H^c, q\in P_L:\\ p+q\neq 0}} \big[b^*_{-p} a^*_{p+q} a_q + \text{h.c.} \big] + \frac{1}{2}\mathcal{K} \\
& + \frac{1}{2N} \sum^{*}_{\substack{r\in P_H^c, v,w\in P_L:\\ (v,w)\in P_L^2 \text{ or } (v+r,w-r)\in P_L^2}} N^{\kappa} \big(\widehat{V}(./N^{1-\kappa}) * \widehat{f}_N \big)(r) a^*_{v+r} a^*_{w-r} a_v a_w \\
& - \widetilde{C} N^{-\beta-\kappa} \mathcal{V}_{N,L} + \mathcal{E}''_{\mathcal{M}_N},
\end{aligned} \tag{7.86}$$

where, by (7.81) and Lemmas 4.2 and 7.1,

$$e^A e^D \mathcal{E}''_{\mathcal{M}_N} e^{-D} e^{-A} \geq -CN^{-\beta}\mathcal{K} - CN^{-\beta-\kappa}\mathcal{V}_N - CN^{\alpha+\beta+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq \frac{1}{2}N^{\lfloor m_0 \rfloor \beta}} - C\sum_{j=3}^{2\lfloor m_0 \rfloor -1} N^{j\beta/2+\beta/2+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq \frac{1}{2}N^{j\beta/2}} - CN^{\alpha+\beta/2+2\kappa} \tag{7.87}$$

Similarly, for $r \in P_H^c$, we know that

$$\left|\left(\widehat{V}(./N^{1-\kappa}) * \widehat{f}_N\right)(r) - 8\pi\mathfrak{a}_0\right| \leq CN^{\alpha+\kappa-1}.$$

Therefore, proceeding exactly as between (7.27) and (7.30), with $\left(\widehat{V}(./N^{1-\kappa}) * \widehat{f}_N\right)(r)$ replaced by $\left|\left(\widehat{V}(/N^{1-\kappa}) * \widehat{f}_N\right)(r) - 8\pi\mathfrak{a}_0\right|$, we deduce that

$$\begin{aligned}\mathcal{M}_N \geq\ & 4\pi\mathfrak{a}_0 N^{1+\kappa} + \frac{1}{2}\mathcal{K} + 8\pi\mathfrak{a}_0 N^\kappa \sum_{p\in P_H^c}\left[b_p^* b_p + \frac{1}{2}b_p^* b_{-p}^* + \frac{1}{2}b_p b_{-p}\right] \\ & + \frac{8\pi\mathfrak{a}_0 N^\kappa}{\sqrt{N}} \sum_{\substack{p\in P_H^c, q\in P_L:\\ p+q\neq 0}} \left[b_{-p}^* a_{p+q}^* a_q + \text{h.c.}\right] \\ & + \frac{4\pi\mathfrak{a}_0 N^\kappa}{N} \sum^*_{\substack{r\in P_H^c, v,w\in P_L:\\ (v,w)\in P_L^2\\ \text{or } (v+r,w-r)\in P_L^2}} a_{v+r}^* a_{w-r}^* a_v a_w - \widetilde{C}N^{-\beta-\kappa}\mathcal{V}_{N,L} + \mathcal{E}'''_{\mathcal{M}_N},\end{aligned} \tag{7.88}$$

with $\mathcal{E}'''_{\mathcal{M}_N}$ satisfying the same bound (7.87) as $\mathcal{E}''_{\mathcal{M}_N}$. Here we used Lemmas 4.2, 4.3, 7.1 and 7.2, as well as the assumption (5.6).

Finally, recalling the definition (5.1) and the identity (5.2), we find

$$\begin{aligned}\mathcal{M}_N \geq\ & 4\pi\mathfrak{a}_0 N^{1+\kappa} + \frac{1}{2}\mathcal{K} + 8\pi\mathfrak{a}_0 N^\kappa \sum_{p\in P_H^c}\left[b_p^* b_p + \frac{1}{2}b_p^* b_{-p}^* + \frac{1}{2}b_p b_{-p}\right] \\ & + 8\pi\mathfrak{a}_0 N^\kappa \sum_{p\in P_H^c}\left[b_{-p}^* e_{-p} + e_{-p}^* b_{-p} + b_{-p}^* e_p^* + e_p b_{-p} + b_{-p}^* c_p^* + c_p b_{-p}\right] \\ & + \frac{4\pi\mathfrak{a}_0 N^\kappa}{N} \sum^*_{\substack{r\in P_H^c, v,w\in P_L:\\ (v,w)\in P_L^2\\ \text{or } (v+r,w-r)\in P_L^2}} a_{v+r}^* a_{w-r}^* a_v a_w - \widetilde{C}N^{-\beta-\kappa}\mathcal{V}_{N,L} + \mathcal{E}'''_{\mathcal{M}_N}.\end{aligned} \tag{7.89}$$

To express also the first term in the third line of (7.89) in terms of the modified creation and annihilation fields defined in (5.1), we first observe that

$$\begin{aligned}
&\frac{4\pi\mathfrak{a}_0 N^\kappa}{N} \sideset{}{^*}\sum_{\substack{r\in P_H^c, v,w\in P_L:\\ (v,w)\in P_L^2 \\ \text{or } (v+r,w-r)\in P_L^2}} a^*_{v+r}a^*_{w-r}a_v a_w \\
&\quad= \frac{4\pi\mathfrak{a}_0 N^\kappa}{N}\sum_{r\in P_H^c} \sideset{}{^*}\sum_{\substack{v,w\in P_L:\\ (v,w)\in P_L^2 \\ \text{or } (v+r,w-r)\in P_L^2}} a^*_{v+r}a_v a^*_{w-r}a_w - \frac{4\pi\mathfrak{a}_0 N^\kappa}{N} \sideset{}{^*}\sum_{\substack{r\in P_H^c, v\in P_L:\\ (v,v+r)\in P_L^2}} a^*_{v+r}a_{v+r} \\
&\quad\geq \frac{4\pi\mathfrak{a}_0 N^\kappa}{N}\sum_{r\in P_H^c} \sideset{}{^*}\sum_{\substack{v,w\in P_L:\\ (v,w)\in P_L^2 \\ \text{or } (v+r,w-r)\in P_L^2}} a^*_{v+r}a_v a^*_{w-r}a_w - CN^{3\beta+\kappa-1}\mathcal{N}_+ - C.
\end{aligned}$$

Then, for a fixed $r \in P_H^c$, we have that

$$\big\{(v,w)\in\Lambda_+^*\times\Lambda_+^* : (v,w)\in P_L^2 \text{ or } (v+r,w-r)\in P_L^2\big\} = \bigcup_{j=1}^{7} S_j,$$

where

$$\begin{aligned}
S_1 &= \big\{(v,w)\in\Lambda_+^*\times\Lambda_+^* : v\in P_L, w\in P_L, v+r\in P_L, w-r\in P_L\big\},\\
S_2 &= \big\{(v,w)\in\Lambda_+^*\times\Lambda_+^* : v\in P_L, w\in P_L, v+r\in P_L, w-r\in P_L^c\big\},\\
S_3 &= \big\{(v,w)\in\Lambda_+^*\times\Lambda_+^* : v\in P_L, w\in P_L, v+r\in P_L^c, w-r\in P_L\big\},\\
S_4 &= \big\{(v,w)\in\Lambda_+^*\times\Lambda_+^* : v\in P_L, w\in P_L, v+r\in P_L^c, w-r\in P_L^c\big\},\\
S_5 &= \big\{(v,w)\in\Lambda_+^*\times\Lambda_+^* : v\in P_L^c, w\in P_L, v+r\in P_L, w-r\in P_L\big\},\\
S_6 &= \big\{(v,w)\in\Lambda_+^*\times\Lambda_+^* : v\in P_L, w\in P_L^c, v+r\in P_L, w-r\in P_L\big\},\\
S_7 &= \big\{(v,w)\in\Lambda_+^*\times\Lambda_+^* : v\in P_L^c, w\in P_L^c, v+r\in P_L, w-r\in P_L\big\}.
\end{aligned}$$

In particular, the union $\bigcup_{j=1}^7 S_j$ is a disjoint union. As a consequence, we find that

$$\begin{aligned}
&\frac{4\pi\mathfrak{a}_0 N^\kappa}{N}\sum_{r\in P_H^c} \sideset{}{^*}\sum_{\substack{v,w\in P_L:\\ (v,w)\in P_L^2 \\ \text{or } (v+r,w-r)\in P_L^2}} a^*_{v+r}a_v a^*_{w-r}a_w \\
&\quad= 8\pi\mathfrak{a}_0 N^\kappa \sum_{r\in P_H^c}\Big[e_r^*c_{-r}^* + c_{-r}e_r + \frac{1}{2}d_r^*e_{-r}^* + \frac{1}{2}e_{-r}e_r + \frac{1}{2}c_r^*c_{-r}^* + \frac{1}{2}c_{-r}c_r\Big]\\
&\qquad + 8\pi\mathfrak{a}_0 N^\kappa \sum_{r\in P_H^c}\Big[e_r^*e_r + c_r^*e_r + e_r^*c_r\Big].
\end{aligned}$$

Inserting in (7.88), we obtain

$$\begin{aligned} \mathcal{M}_N \geq 4\pi\mathfrak{a}_0 N^{1+\kappa} + \frac{1}{2}\mathcal{K} + 8\pi\mathfrak{a}_0 N^\kappa \sum_{r\in P_H^c} \big(b_r^* + c_r^* + e_r^*\big)\big(b_r + c_r + e_r\big) \\ + 4\pi\mathfrak{a}_0 N^\kappa \sum_{r\in P_H^c} \Big[\big(b_r^* + c_r^* + e_r^*\big)\big(b_{-r}^* + c_{-r}^* + e_{-r}^*\big) + \text{h.c.}\Big] \\ - 8\pi\mathfrak{a}_0 N^\kappa \sum_{r\in P_H^c} [c_r^* c_r + b_r^* c_r + c_r^* b_r] - \widetilde{C} N^{-\beta-\kappa} \mathcal{V}_{N,L} + \mathcal{E}_{\mathcal{M}_N}''' \end{aligned} \tag{7.90}$$

with

$$\begin{aligned} e^A e^D \mathcal{E}_{\mathcal{M}_N}''' e^{-D} e^{-A} \geq -CN^{-\beta}\mathcal{K} - CN^{-\beta-\kappa}\mathcal{V}_N - CN^{\alpha+\beta+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq \frac{1}{2}N^{\lfloor m_0\rfloor\beta}} \\ - C\sum_{j=3}^{2\lfloor m_0\rfloor -1} N^{j\beta/2+\beta/2+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq\frac{1}{2}N^{j\beta/2}} - CN^{\alpha+\beta/2+2\kappa} \end{aligned}$$

Let us now estimate the remaining terms on the last line of (7.90). For $\xi \in \mathcal{F}_+^{\leq N}$, we have

$$\begin{aligned} &\Big| 8\pi\mathfrak{a}_0 N^\kappa \sum_{r\in P_H^c} \langle \xi, c_r^* c_r \xi\rangle \Big| \\ &\quad\leq \frac{CN^\kappa}{N} \sum^*_{\substack{r\in P_H^c, v,w\in P_L: \\ v\in P_L, r+v\in P_L^c, \\ w\in P_L, w+r\in P_L^c}} \Big(|w||v|^{-1}\|a_{r+v}a_w\xi\|\Big)\Big(|v||w|^{-1}\|a_v a_{w+r}\xi\|\Big) \\ &\quad\leq CN^{\beta+\kappa-1}\langle \xi, \mathcal{K}_L(\mathcal{N}_{\geq N^\beta}+1)\xi\rangle, \end{aligned} \tag{7.91}$$

and

$$\begin{aligned} \Big| 8\pi\mathfrak{a}_0 N^\kappa \sum_{r\in P_H^c} \langle\xi, (b_r^* c_r + c_r^* b_r)\xi\rangle\Big| &\leq \frac{1}{4}\sum_{r\in P_H^c}\langle\xi, b_r^* b_r\xi\rangle + CN^{2\kappa}\sum_{r\in P_H^c}\langle\xi, c_r^* c_r\xi\rangle \\ &\leq \frac{1}{4}\mathcal{K} + CN^{\beta+2\kappa-1}\langle\xi, \mathcal{K}_L(\mathcal{N}_{\geq N^\beta}+1)\xi\rangle, \end{aligned} \tag{7.92}$$

Similarly, we can bound

$$\begin{aligned} N^{-\beta-\kappa}\langle\xi, \mathcal{V}_{N,L}\xi\rangle &\leq CN^{-\beta-1}\sum_{\substack{u\in\Lambda^*, p,q\in\Lambda_+^*: \\ p+u, q+u, p,q\in P_L}} \|a_{p+u}a_q\xi\|\|a_p a_{q+u}\xi\| \\ &\leq CN^{-\beta-1}\sum_{\substack{u\in\Lambda^*, p,q\in\Lambda_+^*: \\ p+u, q+u, p,q\in P_L}} \frac{|q|^2}{|p|^2}\|a_{p+u}a_q\xi\|^2 \\ &\leq CN^{-1}\|\mathcal{K}^{1/2}\mathcal{N}_+^{1/2}\xi\|^2 \leq C\|\mathcal{K}^{1/2}\xi\|^2 \end{aligned}$$

Thus, choosing the constant $\widetilde{C} > 0$ small enough and applying Lemmas 7.2, 4.3 and 4.2 to the r.h.s. of (7.91) and to the second term on the r.h.s. of (7.92), we conclude that

$$\mathcal{M}_N \geq 4\pi\mathfrak{a}_0 N^{1+\kappa} + \frac{1}{4}\mathcal{K} + 8\pi\mathfrak{a}_0 N^\kappa \sum_{r\in P_H^c} \big(b_r^* + c_r^* + e_r^*\big)\big(b_r + c_r + e_r\big) + 4\pi\mathfrak{a}_0 N^\kappa \sum_{r\in P_H^c} \Big[\big(b_r^* + c_r^* + e_r^*\big)\big(b_{-r}^* + c_{-r}^* + e_{-r}^*\big) + \text{h.c.}\Big] + \mathcal{E}''''_{\mathcal{M}_N} \tag{7.93}$$

where $\mathcal{E}''''_{\mathcal{M}_N}$ is such that

$$\begin{aligned} e^A e^D \mathcal{E}''''_{\mathcal{M}_N} e^{-A} e^{-D} \geq & - CN^{-\beta}\mathcal{K} - CN^{-\beta-\kappa}\mathcal{V}_N \\ & - CN^{\beta+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq N^\beta} - CN^{\alpha+\beta+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq \frac{1}{2}N^{\lfloor m_0\rfloor\beta}} \\ & - C\sum_{j=3}^{2\lfloor m_0\rfloor -1} N^{j\beta/2+\beta/2+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq \frac{1}{2}N^{j\beta/2}} - CN^{\alpha+\beta/2+2\kappa} \end{aligned} \tag{7.94}$$

We introduce the operators

$$g_r^* = b_r^* + c_r^* + e_r^*, \qquad g_r = b_r + c_r + e_r.$$

With the algebraic identity

$$\sum_{r\in P_H^c} \Big[g_r^* g_r + \frac{1}{2} g_r^* g_{-r}^* + \frac{1}{2} g_{-r} g_r\Big] = \frac{1}{2}\sum_{r\in P_H^c} \big(g_r^* + g_{-r}\big)\big(g_r + g_{-r}^*\big) - \frac{1}{2}\sum_{r\in P_H^c} [g_r, g_r^*],$$

we conclude that

$$\mathcal{M}_N \geq 4\pi\mathfrak{a}_0 N^{1+\kappa} + \frac{1}{4}\mathcal{K} - 4\pi\mathfrak{a}_0 N^\kappa \sum_{r\in P_H^c} [g_r, g_r^*] + \mathcal{E}''''_{\mathcal{M}_N}$$

Since

$$[b_r, c_r^*] = [b_r, e_r^*] = [c_r, b_r^*] = [e_r, b_r^*] = [c_r, e_r^*] = [e_r, c_r^*] = 0,$$

we obtain that

$$\begin{aligned} [g_r, g_r^*] = & \frac{N - \mathcal{N}_+}{N} - \frac{1}{N} a_r^* a_r + \frac{1}{N} \sum_{\substack{v\in\Lambda_+^* : v\in P_L, \\ v+r\in P_L^c}} a_v^* a_v - \frac{1}{N} \sum_{\substack{v\in\Lambda_+^* : v\in P_L, \\ v+r\in P_L^c}} a_{v+r}^* a_{v+r} \\ & + \frac{1}{4N} \sum_{\substack{v\in\Lambda_+^* : v\in P_L, \\ v+r\in P_L}} a_v^* a_v - \frac{1}{4N} \sum_{\substack{v\in\Lambda_+^* : v\in P_L, \\ v+r\in P_L}} a_{v+r}^* a_{v+r}. \end{aligned}$$

A straightforward computation then shows that

$$-4\pi\mathfrak{a}_0 N^\kappa \sum_{p\in P_H^c} [g_r, g_r^*] \geq -CN^{3\alpha+\kappa}(1 - \mathcal{N}_+/N) - CN^{3\alpha+\kappa}\mathcal{N}_+/N \geq -CN^{3\alpha+\kappa}.$$

Thus

$$\mathcal{M}_N \geq 4\pi\mathfrak{a}_0 N^{1+\kappa} + \frac{1}{4}\mathcal{K} + \mathcal{E}_{\mathcal{M}_N}$$

where $\mathcal{E}_{\mathcal{M}_N}$ satisfies

$$\begin{aligned} e^A e^D \mathcal{E}_{\mathcal{M}_N} e^{-A} e^{-D} \geq & -CN^{-\beta}\mathcal{K} - CN^{-\beta-\kappa}\mathcal{V}_N \\ & - CN^{\beta+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq N^\beta} - CN^{\alpha+\beta+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq \frac{1}{2}N^{\lfloor m_0 \rfloor \beta}} \\ & - C\sum_{j=3}^{2\lfloor m_0 \rfloor -1} N^{j\beta/2+\beta/2+2\kappa-1}\mathcal{K}\mathcal{N}_{\geq \frac{1}{2}N^{j\beta/2}} - CN^{3\alpha+\kappa} \end{aligned}$$

This concludes the proof of Proposition 5.1. □

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