Romanian Reports in Physics, Vol. 67, No. 1, P. 5–50, 2015

BOSE-EINSTEIN CONDENSATION: TWENTY YEARS AFTER

V. S. BAGNATO[1], D. J. FRANTZESKAKIS[2], P. G. KEVREKIDIS[3,4],
B. A. MALOMED[5,*], D. MIHALACHE[6]

[1]Instituto de Física de Sao Carlos, Universidade de Sao Paulo, Caixa Postal 369,
13560-970 Sao Carlos, Sao Paulo, Brazil
[2]Department of Physics, University of Athens, Panepistimiopolis, Zografos, Athens 157 84, Greece
[3]Department of Mathematics and Statistics, University of Massachusetts,
Amherst, MA 01003-9305, USA
[4]Center for Nonlinear Studies and Theoretical Division, Los Alamos National Laboratory,
Los Alamos, NM 87544, USA
[5]Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering,
Tel Aviv University, Tel Aviv 69978, Israel
[6]Horia Hulubei National Institute for Physics and Nuclear Engineering, P. O. Box MG-6,
RO-077125 Magurele, Romania
[*]Corresponding author *E-mail*: malomed@post.tau.ac.il

Abstract. The aim of this introductory article is two-fold. First, we aim to offer a general introduction to the theme of Bose-Einstein condensates, and briefly discuss the evolution of a number of relevant research directions during the last two decades. Second, we introduce and present the articles that appear in this Special Volume of *Romanian Reports in Physics* celebrating the conclusion of the second decade since the experimental creation of Bose-Einstein condensation in ultracold gases of alkali-metal atoms.

1. INTRODUCTION

1.1. ATOMIC BOSE-EINSTEIN CONDENSATES

The Bose-Einstein condensate (BEC) is a macroscopic quantum state of matter, which was predicted theoretically by Bose and Einstein 90 years ago [1]. Atomic BECs were created experimentally in ultracold vapors of ^{87}Rb [2], ^{23}Na [3] and ^{7}Li [4] 70 years later. The aim of this Special Issue is to celebrate the twentieth anniversary of this remarkable achievement, which was also marked by the Nobel Prize in Physics for 2001, awarded to E. A. Cornell, W. Ketterle, and C. E. Weiman [5].

In the BEC state, all atoms in the bosonic gas fall ("condense") into a single quantum-mechanical ground state. The transition to the BEC occurs if the atomic density, n, and the de Broglie wavelength, λ, corresponding to the characteristic velocity of the thermal motion of the atoms, satisfy the following condition [6, 7]:

$$n\lambda^3 > 2.612, \tag{1}$$

which implies that λ is comparable to or larger than the mean distance between atoms, thus making the gas a macroscopic (degenerate) quantum state. In the above-mentioned atomic gases, this theoretical condition is met at temperatures T which are a small fraction of milli-Kelvin (mK), hence the atomic BECs created in the laboratories are, as a matter of fact, the coldest objects existing in the universe (the early BEC-experiments achieving the condensed state around 100 nK). Their creation became possible after the development of appropriate experimental techniques needed to reach the necessary ultra-low temperatures (see, e.g., Ref. [8]). The required extreme cooling is achieved in two stages. First, the method of *laser cooling* (which was also rewarded with the Nobel Prize in Physics for 1997 [9]) is applied to the gas loaded into a magneto-optical trap, which makes it possible to create a moderately cool state, at temperature $\sim 100\ \mu$K. Next, this state undergoes forced *evaporative cooling*, losing $\sim 90\%$ of atoms, and the remaining atomic cloud spontaneously forms the BEC. In the experiments, the number of atoms in the BEC typically ranges between $1,500$ and $1,000,000$ (although both smaller and larger numbers are, in principle, possible), and the size of the domain in which the gas is trapped is $\sim 100\ \mu$m. A characteristic time scale relevant to the experiments is measured in milliseconds, while the lifetime of the condensate can be easily raised to several seconds.

The applicability of the laser-cooling method to particular atomic species depends on the peculiarities of their electron configuration. As a result, this technique has made it possible to achieve the Bose-Einstein condensation in vapors of alkaline, alkaline-earth, and lanthanoid metals: ^{7}Li, ^{23}Na, ^{39}K, ^{41}K, ^{85}Rb, ^{87}Rb, ^{133}Cs, ^{52}Cr, ^{40}Ca, ^{84}Sr, ^{86}Sr, ^{88}Sr, ^{174}Yb, ^{164}Dy, and ^{168}Er. Perhaps especially interesting, among the more recent developments, is the creation of BEC in the gases of chromium [10] and dysprosium [11], where atoms carry large magnetic moments, which makes it possible to predict and observe many effects produced by long-range dipole-dipole interactions [12]. The challenging aim of creating BEC in the gas of spin-polarized hydrogen atoms has been finally achieved too, with a specially devised technique which made it possible to cool the gas to $50\ \mu$K [13].

The BECs created in the laboratory constitute a prototypical manifestly quantum-macroscopic state of matter available in the experiments. In other settings, where low temperatures are crucially important too, macroscopic quantum effects, such as superconductivity in metals and superfluidity in liquid helium, are well known too, but they correspond to "implicit" quantum states. For instance, a superconducting metallic sample as a whole is not a macroscopic quantum object. The same pertains to the recently created out-of-equilibrium BEC of quasi-particles in condensed matter, namely, exciton-polaritons [14] and magnons [15], which have drawn a great deal of attention in the past decade (see Refs. [16–18]). Also, a considerable attention was drawn to the topic of localization of exciton-polaritons in semiconductor microcavities [19–22]; for an excellent recent review focused on several physical phenomena exhibited by exciton-polariton condensates see Ref. [23]. The condensation of effectively massive photons trapped in a microcavity was reported too [24], the peculiarity of these settings being the nonconservation of the total number of the quasi-particles or photons.

Surveys of the broad subject of Bose-Einstein condensation and numerous related areas are provided by many review articles and books [6, 7], [12], [25]-[41]. It is important to mention that this list of surveys on the topic of BEC is, of course, far from being exhaustive. This is a clear indication of the impact of this research theme to almost all branches of contemporary physics.

The goal of the present article is to offer a broad picture of some of the past and currently active research areas in the realm of BEC (admittedly biased towards the particular research interests of the authors) and to overview the scientific literature, akin to a Resource Letter of *American Journal of Physics*.

1.2. MEAN-FIELD DESCRIPTION AND NONLINEAR DYNAMICS OF BEC

From a theoretical standpoint, and for many experimentally relevant conditions, the static and dynamical properties of a BEC can be described by means of an effective mean-field equation known as the Gross-Pitaevskii (GP) equation [6, 7]. This is a variant of the famous nonlinear Schrödinger equation (NLSE), incorporating an external potential used to confine the condensate; NLSE is known to be a universal model describing the evolution of complex field envelopes in nonlinear dispersive media [42–44]. In the case of BECs, the nonlinearity in the GP (NLSE) model is introduced by the interatomic interactions, accounted for through an effective mean field. Thus, an inherent feature of the BEC dynamics is its *nonlinearity*, which is induced by collisions between atoms, in spite of the fact that the density of the quantum bosonic gases is very low.

The studies of the matter waves in the presence of the nonlinearity drive a vast research area known as "nonlinear atom optics" (see, e.g., Refs. [45]). Importantly, many collective excitations, including self-trapped localized states supported by the condensate's intrinsic nonlinearity (e.g., solitons), are less straightforward to create (and difficult to describe by adequate models) in dense media featuring macroscopic quantum phenomena, such as liquid helium. An exception are phase solitons (fluxons), i.e., quanta of magnetic flux trapped in long Josephson junctions formed by superconductors [46], whose experimental and theoretical studies are relatively straightforward and have been developed in detail [47]. Nevertheless, atomic BECs constitute an ideal setting for studies of such macroscopic nonlinear excitations, as is explained in more detail below.

In the atomic BEC with intrinsic self-attraction (e.g., ^{7}Li or ^{85}Rb BECs), the creation of effectively one-dimensional (1D) matter-wave *bright solitons* (both isolated ones and multi-soliton sets) in *cigar-shaped* configurations, which are tightly confined by external potentials in the transverse plane, was successfully reported in condensates of ^{7}Li [48–50] and ^{85}Rb [51, 52] (see also the reviews in Ref. [53]). Collisions between moving quasi-1D solitons also admit accurate experimental implementation [54] and theoretical analysis [54, 55].

More typical is the repulsive sign of the inter-atomic interactions (as in the cases of ^{87}Rb or ^{23}Na BECs), which lends the BEC the effective self-repulsive nonlinearity. This kind of the intrinsic interaction readily creates *dark solitons*, which were predicted theoretically [56] and created experimentally [57, 58] in BECs loaded into a cigar-shaped trap. In fact, dark solitons were first created [57] prior to the realization of the above-mentioned bright solitons in the self-attractive condensates, placed into the same type of the trapping potential (a review of the topic of dark solitons in BEC was given in article [36]). Similar to the case of bright solitons, not only single-soliton [57] states, but also multiple dark solitons were created [58], while their interactions and collisions were also studied both in theory [58, 59] and experiments [58, 60].

Later, stable *dark-bright soliton* complexes in binary BEC were predicted in theory [61] and observed in experiments [62] as well. In more recent works, multiple dark-bright solitons [63], as well as *dark-dark solitons* [64] were also experimentally created. In addition, in the same setting of the self-repulsive nonlinearity, not only dark solitons, but also bright solitons are possible too: in particular, if – instead of the usual parabolic trap – a periodic (optical lattice) potential [31] is used to confine the condensate, then *gap solitons* can be formed. Experimentally, matter-wave gap solitons built of ~ 250 atoms in a ^{87}Rb condensate, were reported in Ref. [65].

On the other hand, in the case of multidimensional BEC geometry, there has been an intense theoretical and experimental activity on vortices and vortex structures in BECs with the self-repulsive intrinsic nonlinearity (see Refs. [66] for reviews on this topic). This is due to the fact that vortices are intimately related to the superfluid properties of BECs, and play an important role in transport, dissipative dynamics and quantum turbulence (see, e.g., Refs. [67]). Historically, the first observation of vortices in BECs was achieved by *phase imprinting* [68] (a technique that was also used to create dark solitons [57]). Nevertheless, there exist other techniques that have been used in experiments to nucleate vortices in BECs: these include *stirring* the condensate above a certain critical angular speed [69] (this method was used to create *vortex lattices* [70]), *nonlinear interference* between different condensate fragments [71] (this technique was also employed for the creation of dark solitons [58]), by forcing superfluid flow around a repulsive Gaussian obstacle within the BEC [72], and through the *Kibble-Zurek mechanism* [73] (the latter was originally proposed for the formation of large-scale structures in the universe by means of a quench through a phase transition [74]).

It is also important to mention that there still exist other notable results concerning nonlinear phenomena in BECs. These include the realization of few-vortex clusters and complex structures such as vortex dipoles, vortex tripoles, parallel vortex rings, etc., and the study of their dynamics [69, 72, 75–88], the observation of quantum shock waves [89], the realization and study of "hybrid" soliton-vortex structures [90], the observation of Josephson oscillations [91] and spontaneous symmetry breaking transitions [92] in BECs loaded into a symmetric double-well trapping potential, and so on.

Many of the nonlinear phenomena mentioned above can be successfully studied in the framework of mean-field theory. Nevertheless, there exist other phenomena (where the underlying intrinsic BEC nonlinearity is also important) that can not be described by means of the mean-field theory. Examples of such situations, along with cases where quantum and/or thermal fluctuations come into play, will be discussed below.

1.3. WHY ARE BOSE-EINSTEIN CONDENSATES ALWAYS "IN VOGUE"?

A great advantage offered by the BEC in low-density atomic gases is that these media can be easily and very efficiently controlled by means of external magnetic and optical fields. This circumstance enables various experiments, and provides a framework for very accurate theoretical models. As a result, the ultracold gases can be used for *emulation* of many phenomena which originate, e.g., in condensed matter physics, but take a very complex form in the original settings, due to the strong interactions in them, while in atomic BECs similar effects can be simulated in a much simpler ("clean") form.

An example which has recently drawn a great deal of interest is the emulation of the *spin-orbit coupling* (SOC), i.e., the interaction between the motion of an electron or hole and their spin, or, in other words, the linear mixing between two components of the electron's or hole's spinor wave function. It is a fundamentally important phenomenon in semiconductors, known in two distinct forms, as the *Dresselhaus* and *Rashba* SOC [93]. In binary atomic BECs, created as a mixture of two different hyperfine states of atoms of ^{87}Rb, the SOC has been experimentally implemented as a similar linear coupling between the two atomic components, with the help of appropriate laser beams illuminating the condensate, and a dc magnetic field applied to it [94].

There are many other prominent examples of such an emulation provided by the atomic-gas BEC. One such example concerns the direct observation of the transition between the superfluid and Mott-insulator states formed in the BEC loaded into an "egg-carton" periodic potential for individual atoms, which is, in turn, imposed by an optical-lattice structure [95]. Other examples include resonantly-enhanced tunneling in periodic potentials [96], artificial gauge fields [41, 97], topological insulators [41], the fractional quantum Hall effect [97], and so on. Recently, the general topic of using the BEC in atomic gases as the "quantum simulator", and the related theme of using ultra-cold fermionic gases for the same general purpose, have been reviewed in a series of articles [98] and in a recent book [39].

As concerns applications of BEC, the most straightforward one is, arguably, the use of matter waves for interferometry. With the help of the intrinsic nonlinearity, they may feature superb accuracy, in comparison with traditional optical interferometric schemes [99]. Especially promising are interferometric schemes of the Mach-Zehnder type, based on splitting and subsequent recombination of bright matter-wave solitons [53, 100, 101]. More generally, nonlinear waves that arise in BECs play a principal role in different applications, in addition to their intrinsic interest. For instance, bright solitons have been argued to provide the potential for 100-fold improved sensitivity for interferometers and their lifetime of a few seconds enables precise force sensing applications [102] (see also Ref. [103]). For repulsively interacting BECs too, the inter-atomic interactions have also been suggested to increase the sensitivity to phase shifts, precisely due to the emergence of dark solitons which enable (e.g. through their oscillation in the confining trap) better detection schemes [104]. Furthermore, nonlinear atom interferometers can overcome the limitations of the current state-of-the-art standard based on the so-called Ramsey spectroscopy [105, 106], due to their ability to surpass the classical precision limit. Finally, vortices present their own potential for applications. An example is the so-called "analogue gravity" [107], whereby they may play a role similar to spinning black holes. This allows to observe, in experimentally controllable environments, associated phenomena such as the celebrated Hawking radiation or even simpler ones such as the super-radiant amplification of sonic waves scattered from black holes [108].

BECs have also been experimentally demonstrated to be usable for performing remarkable tasks such as the implementation of the Gaussian sum algorithm for factoring numbers [109], by exploiting higher order quantum momentum states, improving in this way the algorithm's accuracy, once again, beyond its classical implementation. This is in line with the development of the Shor algorithm as an efficient quantum mechanical way to factorize large numbers, a task thought to be classically intractable [110, 111].

Another potentially promising application is the use of BEC as a resource for the implementation of quantum computing. In this context, one possibility is to use trapped droplets of the condensate as qubits [112, 113]. In optical lattices also, atomic analogs of semiconductor electronic circuits (the so-called "atomtronics") have been proposed, in order to realize quantum devices such as diodes and transistors [114]. On the other hand, collision between quantum matter-wave solitons can be used to induce entanglement between them [115], which is a prerequisite necessary for the design of soliton-based quantum-information-processing schemes.

Also promising is the development of atom-wave lasers, which should be able to emit high-intensity coherent matter-wave beams, in continuous-wave (CW) or pulsed (soliton-like) regimes. Such beams may be very useful, in particular, for precision measurements. The first experimental realization of a CW matter-wave laser was reported in Ref. [116], which was followed by the development of a design with a separated BEC reservoir and the beam-emitting cavity [117]. These experimental works used condensates consisting of ^{87}Rb atoms. A review of experimental results on the topic of matter-wave lasers was recently published in Ref. [118]. Models of matter-wave lasers operating in a pulsed regime were developed theoretically [119].

2. MODELS, SETTINGS AND BASIC RESULTS

2.1. THE GROSS-PITAEVSKII EQUATION

As mentioned above, the fundamental model which provides for an accurate description of the BEC in dilute degenerate gases of bosonic atoms is based on lowest-order mean-field theory. According to this approach, the gas is described by means of the Gross-Pitaevskii equation (GPE) for the single-particle wave function, $\Psi(x,y,z,t)$, where (x,y,z) and t are the coordinates and time [as mentioned above, "degenerate" means that the de Broglie wavelength of atoms moving with the thermal velocity in the dilute gas is large enough in comparison with the mean inter-atomic distance – see Eq. (1)]. The three-dimensional (3D) form of the GPE is written, in physical units, as:

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x^2}\right)\Psi+U(x,y,z;t)\Psi+\frac{4\pi\hbar^2}{m}a_s(x,y,z;t)|\Psi|^2\Psi, \tag{2}$$

where m is the atomic mass, $U(x,y,z;t)$ is the external potential acting on individual atoms and thus confining the condensate as a whole (U may depend on time too, which is often called *management* of the potential [120]), and $a_s(x,y,z;t)$ is the scattering length which determines collisions between the atoms: $a_s > 0$ and $a_s < 0$ correspond to repulsive and attractive interactions, respectively. The spatial and temporal dependence of a_s, which is important for many predictions of nontrivial dynamical states in the BEC (see below), may be induced by means of the *Feshbach-resonance* (FR) *management* technique [120]. FR implies the formation of a quasi-bound state of two atoms in the course of the collision between them in the presence of an external magnetic field [121], or under appropriate laser illumination [122]. The FR can be induced too by combined magneto-optical settings [123]. Making use of the very accurate tunability of the FR [124], spatially non-uniform and/or temporally variable external fields, controlling the FR, can be employed to induce spatially and/or time-dependent nonlinearity coefficients, accounted for by $a_s(x,y,z;t)$. Further, the wave function is subject to the normalization condition, which is determined by the total number, N, of atoms in the condensate:

$$\int\int\int |\Psi(x,y,z;t)|^2 dxdydz = N. \tag{3}$$

Note that, alternatively, the wave function may be defined with the unitary norm, $\int\int\int |\Psi(x,y,z;t)|^2 dxdydz = 1$, replacing a_s in Eq. (2) by Na_s.

The GPE and its variants constitute a relatively simple mathematical framework, which admits precise simulations and the use of effective analytical approaches. The latter include the Thomas-Fermi approximation (TFA), which neglects the terms of the second derivative (the kinetic energy of the quantum particles) in Eq. (2) [7], and a more accurate and versatile variational approximation, which has found a great number of applications to BECs [125, 126]. The TFA is relevant for $a_s > 0$ (the self-repulsive nonlinearity), for solutions which, in the simplest case, do not include a nontrivial phase structure; in such a case, the solution with chemical potential $\mu > 0$ is found by using the ansatz $\Psi = e^{-i\mu t}\Phi_{\mathrm{TFA}}(x,y,z)$, where the density $|\Phi_{\mathrm{TFA}}(x,y,z)|^2$ is approximated as:

$$|\Phi_{\mathrm{TFA}}(x,y,z)|^2 = \frac{m}{4\pi\hbar^2 a_s(x,y,z)} \begin{cases} \mu - U(x,y,z), & \text{for } U(x,y,z) < \mu; \\ 0, & \text{for } U(x,y,z) > \mu. \end{cases} \tag{4}$$

The TFA may be easily generalized for vortex states, which are sought for as solutions to Eq. (2) in the cylindrical coordinates ($\rho \equiv \sqrt{x^2+y^2},\ \theta,\ z$), assuming that $U = U(\rho,z)$ and $a_s = a_s(\rho,z)$ are subject to the cylindrical symmetry:

$$\Psi = e^{-i\mu t + iS\theta}\Phi(\rho,z), \tag{5}$$

with real *integer* vorticity S, and function Φ satisfying the respective stationary equation:

$$\mu\Phi = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{S^2}{\rho^2} + \frac{\partial^2}{\partial z^2}\right)\Phi + U(\rho,z)\Phi + \frac{4\pi\hbar^2}{m}a_s(\rho,z)|\Phi|^2\Phi. \tag{6}$$

The TFA neglects all the terms with ρ- and z-derivatives in Eq. (6), which yields the following generalization of solution (4) [127, 128]:

$$|\Phi_{\mathrm{TFA}}|^2 = \frac{m}{4\pi\hbar^2 a_s(\rho,z)} \begin{cases} \mu - \left[U(\rho,z) + \frac{\hbar^2 S^2}{2m}\rho^{-2}\right], & \text{for } U(\rho,z) + \frac{\hbar^2 S^2}{2m}\rho^{-2} < \mu; \\ 0, & \text{for } U(\rho,z) + \frac{\hbar^2 S^2}{2m}\rho^{-2} > \mu. \end{cases} \tag{7}$$

Techniques for numerical treatment of GPEs have been developed in great detail too. They include methods for the solution of boundary-value problems, aimed at finding stationary states trapped in external potentials, or self-trapped due to the nonlinearity (solitons), as well as direct simulations of the GPE in real or imaginary time (the latter approach helps to generate stationary solutions for ground states [129], due to the relaxational character of the dynamics). In particular, the semi-implicit split-step Crank-Nicolson algorithm [130] has become a method of choice for solving GPEs in many settings. The unconditional stability of this algorithm makes it especially useful in studies of real-time dynamics of BECs, although it can be equally well used to produce ground states of relevant experimental setups, including fast-rotating BECs with many vortices. The readily available Fortran and C codes [131], which implement the Crank-Nicolson approach in 1D, 2D, and 3D geometries of BECs with different symmetries, are well tested and highly optimized. Furthermore, the C codes are parallelized using the OpenMP approach, which speeds up execution of numerical simulations of BECs significantly, up to one order of magnitude on modern computers with multi-core CPUs. Apart from the imaginary-time propagation implemented in the framework of GPEs, ground states of a BEC (and of other quantum systems described by linear and nonlinear Schrödinger equations) can also be calculated numerically using the higher-order effective-action approach [132], as demonstrated in Refs. [133]. Other popular methods used to solve the GPE by means of pseudospectral and finite-difference methods are detailed in Refs. [134–137]. A survey of analytical and numerical methods used in the studies of BEC models was given in review article [32].

For the BEC composed of chromium [10] or dysprosium [11] atoms, with magnetic moments, μ, polarized in certain direction by an external uniform dc magnetic field, the GPE includes a nonlinear term accounting for the long-range dipole-dipole interactions:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + U(\mathbf{r};t)\Psi + \frac{\mu_0\mu^2}{4\pi}\Psi(\mathbf{r})\int\left|\Psi(\mathbf{r}')\right|^2\frac{1-3\cos^2\theta}{|\mathbf{r}-\mathbf{r}'|^3}d\mathbf{r}'. \tag{8}$$

In Eq. (8), ∇^2 stands for the 3D Laplacian, the term $\sim a_s$, which represents the usual contact interaction (see Eq. (2)), is dropped on grounds that it is small in comparison to the dipole-dipole interaction, μ_0 is the magnetic permeability of vacuum, and θ is the angle between vector $\mathbf{r}-\mathbf{r}'$ and the polarization direction of the atomic magnetic moments.

Beyond the mean-field approximation, quantum fluctuations and interaction of the condensate with the thermal component of the gas are described within the framework of the Hartree-Fock-Bogoliubov equations, which are essentially more cumbersome than the relatively simple GPE [138]. However, it is only under somewhat special conditions (i.e., small atom numbers below $N \approx 1000$ or large enough temperatures, of the order of many tens or hundreds nK) that fluctuations play a crucial role for coherent matter-wave patterns, including solitons in the case of experiments. Nevertheless, such settings are becoming of increasing interest in both theoretical and experimental studies [37, 40].

2.2. TWO-COMPONENT SYSTEMS

Shortly after the experimental realization of the single-component BEC, advances in trapping techniques opened the possibility to simultaneously confine atomic clouds in different hyperfine spin states. The first such experiment, the so-called *pseudospinor* condensate, was achieved in mixtures of two magnetically trapped hyperfine states of ^{87}Rb [139]. Subsequently, experiments in optically trapped ^{23}Na [140] were able to produce multicomponent condensates for different Zeeman sublevels of the same hyperfine level, the so-called *spinor condensates*. In addition to these two classes of experiments, mixtures of two different species of condensates have also been created by sympathetic cooling (i.e. condensing one species and allowing the other one to condense by taking advantage of the coupling with the first species) – see examples of such BEC mixtures below.

Regarding the modelling of multi-component BECs, it is natural to proceed from the single-component GPE to the corresponding system of coupled GPEs. In particular, for the simplest case of two-component mixtures, Eq. (2) is replaced by the following system of equations for mean-field wave functions, Ψ_1 and Ψ_2, of the two components:

$$i\hbar\frac{\partial\Psi_1}{\partial t} = -\frac{\hbar^2}{2m_1}\nabla^2\Psi_1 + U_1(\mathbf{r};t)\,\Psi_1 + \frac{4\pi\hbar^2}{m_1}\left(a_s^{(1)}\left|\Psi_1\right|^2 + a_s^{(12)}\left|\Psi_2\right|^2\right)\Psi_1, \quad (9)$$

$$i\hbar\frac{\partial\Psi_2}{\partial t} = -\frac{\hbar^2}{2m_2}\nabla^2\Psi_2 + U_2(\mathbf{r};t)\,\Psi_2 + \frac{4\pi\hbar^2}{m_2}\left(a_s^{(2)}\left|\Psi_2\right|^2 + a_s^{(12)}\left|\Psi_1\right|^2\right)\Psi_2, \quad (10)$$

where $a_s^{(12)}$ is the scattering length for collisions between atoms belonging to the two different species, for which the trapping potentials, U_1 and U_2, induced by the same external field, may be, generally speaking, different. Masses m_1 and m_2 are different for BEC mixtures composed by different atom species – also-called *heteronuclear* systems – such as ^{85}Rb$-^{87}$Rb [141] and ^{87}Rb$-^{133}$Cs [142] binary BEC mixtures. On the other hand, $m_1 = m_2$ for mixtures of different hyperfine states of the same atomic species; such mixtures were experimentally realized for the first time with ^{87}Rb atoms [143]. In the case of repulsive intra- and inter-species interactions, when all the scattering lengths in Eqs. (9) and (10) are positive, that is, $a_s^{(1)} > 0$, $a_s^{(2)} > 0$, and $a_s^{(12)} > 0$, the condition for the *immiscibility* of the two components – and the onset of the separation between them – is given by [144]:

$$a_s^{(1)}a_s^{(2)} < \left(a_s^{(12)}\right)^2, \quad (11)$$

which implies that the repulsion between atoms belonging to the different components is stronger than the repulsion between atoms in each component. Condition (11) pertains to the free infinite space, while the pressure of the trapping potential makes the binary BEC more miscible, shifting the critical point, $a_s^{(1)}a_s^{(2)} = \left(a_s^{(12)}\right)^2$, to larger values of positive $a_s^{(12)}$ [145]. Immiscible two-component condensates, loaded into a trap, form domain walls separating the two components [146]. Such domain walls were observed in experiments [147].

Numerical analyses of realistic trapped states of binary BECs have revealed, in the case of cigar-shaped geometries, two distinct immiscible stationary configurations: a segregated one, in which the two components face one another, being separated by a domain wall, and a symbiotic state, in which one component effectively traps the other [148]. Both configurations do not directly obey the aforementioned simple miscibility criteria, although they can be transformed into a miscible configuration when the condensate is subject to a resonant drive [148]. Finally, we mention that, while the symbiotic state is not a soliton, numerous types of solitons are known to exist in spinor condensates (see, e.g., [149, 150]).

In the case when the two wave-function components represent different hyperfine states of the same atomic species, an external resonant radiofrequency field (with frequencies in the GHz range) may add linear mixing, with strength κ (alias *Rabi coupling*), to the system, which is accounted for by extra terms $\kappa\Psi_2$ and $\kappa\Psi_1$, added to equations (9) and (10), respectively [151]. The interplay of the Rabi coupling with the repulsive interactions causes a shift of the miscibility-immiscibility transition (11) [152] (see relevant experimental results in Ref. [153]).

2.3. TRAPPING POTENTIALS

Coming back to the single GPE (2), it is relevant to stress that two most common types of the confining potential are the harmonic-oscillator (HO)

$$U_{\mathrm{HO}} = \frac{m}{2}\left(\Omega_x^2 x^2 + \Omega_y^2 y^2 + \Omega_z^2 z^2\right) \tag{12}$$

and optical-lattice (OL)

$$U_{\mathrm{OL}} = U_x \sin^2(k_x x) + U_y \sin^2(k_y y) + U_z \sin^2(k_z z) \tag{13}$$

ones, where $\Omega_{x,y,z}^2$ are *trapping frequencies* of the (generally, anisotropic) HO potential, and $U_{x,y,z}$ represent the depths of the periodic OL potential. The OL is built as the classical interference pattern by pairs of counterpropagating mutually coherent laser beams illuminating the condensate, with respective wavelengths $\lambda_{x,y,z} = 2\pi/k_{x,y,z}$.

The OL is made attractive or repelling for individual atoms by red- or blue-detuning of the illuminating light with respect to the frequency of the dipole transition in the atoms. Typically, the OL wavelength $\lambda \sim 1\ \mu$m is used in experiments. Usually, the depth U of the OL potential is measured in natural units of the recoil energy,

$E_R = (\hbar k)^2 / (2m)$. The use of the OL potentials for the creation of matter-wave patterns in BEC was proposed in [154] and relevant applications, such as macroscopic quantum interference, immediately ensued [155] – see also reviews [156] and [157]. A well-known example of the effect induced by the OL is the transition from the bosonic superfluid to the Mott insulator [95].

Generally, the technique based on the use of OLs is similar to that which was proposed [158] and implemented in the form of photonic lattices in photorefractive media, producing a number of spectacular results, including 1D and 2D optical solitons and vortices in 2D [159] – see reviews [160].

2.4. THE DISCRETE SYSTEM

In both the BEC and photonic settings, a very deep (compared to the chemical potential) OL potential effectively splits the mean-field matter-wave function, or the optical electromagnetic field, into a set of nodes (each one representing one well) weakly interacting between them via tunneling coupling. In this case, using the expansion of the continuum field over a set of Wannier modes localized around local wells, GPE (2) can be reduced to a discrete nonlinear Schrödinger equation (NLSE) [161, 162]. In a properly scaled form, its 3D version is

$$\begin{aligned} i\frac{d\Psi_{j,k,l}}{dt} &= -\frac{1}{2}\left(\Psi_{j+1,k,l} + \Psi_{j-1,k,l} + \Psi_{j,k+1,l} + \Psi_{k,k-1,l}\right. \\ &\quad \left. + \Psi_{j,k,l+1} + \Psi_{j,k,l-1} - 6\Psi_{j,k,l}\right) - \left|\Psi_{j,k,l}\right|^2 \Psi_{j,k,l}, \end{aligned} \tag{14}$$

where j,k,l are discrete coordinates on the lattice, and $\Psi_{j,k,l}$ are amplitudes of trapped matter-wave fragments, the 2D and 1D versions being produced by obvious reductions of Eq. (14). The present form of the discrete NLSE implies that the on-site nonlinearity is self-attractive. However, unlike the continuous model, in the discrete one the self-repulsive nonlinearity may be transformed into its self-attractive counterpart by means of the well-known *staggering transformation* [162], $\Psi_{j,k,l}(t) \equiv (-1)^{j+k+l} e^{-12it} \tilde{\Psi}^*_{j,k,l}(t)$, where the asterisk stands for the complex conjugate.

The discrete NLSE gives rise to many species of solitons [162], especially interesting ones being *discrete localized vortices*, which were predicted theoretically [163] and created experimentally (as nearly discrete objects) in photonics using waveguide arrays built in a photorefractive material [159]. Thus, while 1D and 2D versions of Eq. (14) apply to the photonic settings [160], the 3D discrete system is meaningful solely in the BEC context. In the latter case, it generates complex stable localized modes, such as, e.g., *discrete Skyrmions* [164], diamonds, octupoles, oblique vortices, and vortex cubes [165], among many others.

The discrete NLSE suggests a direct transition to the fully quantum system in the form of the Bose-Hubbard (BH) model, replacing the mean-field (classical) lattice wave functions in Eq. (14) by quantum operators, b_j (for simplicity, discussed here in the 1D setting). The corresponding Hamiltonian is

$$H = \sum_j \left[-J b_j^\dagger (b_{j+1} + b_{j-1}) + \frac{1}{2} U n_j (n_j - 1) \right] , \tag{15}$$

where $n_j = b_j^\dagger b_j$ is the operator of the on-site number of particles, J is the inter-site-hopping constant, and U is the constant of the on-site interaction ($U > 0$ and $U < 0$ correspond, as before, to the self-attraction and self-repulsion, respectively). A famous result produced by Hamiltonian (15) is the phase diagram separating the quantum superfluid and Mott insulator (see Ref. [166]). In terms of applications, the BH model is a natural tool for the theoretical analysis of operations of BEC-based qubits. Reviews of the topic of BH in connection to its realization in BEC can be found in articles [34] and [113]. The well-elaborated numerical technique for the analysis of the BH model and its modifications is based on the density-matrix-renormalization-group method [167].

2.5. REDUCTION TO LOWER-DIMENSIONAL SETTINGS

The above-mentioned nearly-1D cigar-shaped traps are represented by the potentials which include tight confinement in the transverse plane, i.e., large $\Omega_y^2 = \Omega_z^2 \equiv \Omega_\perp^2$ in Eq. (12), and an arbitrary weak potential, $U(x,t)$, acting in the axial direction, x. In this case, the 3D wave function can be approximated by the factorized *Ansatz* (see, e.g., Ref. [168]),

$$\Psi(x,y,z,t) = \frac{1}{\sqrt{\pi} a_\perp} \exp\left(-i\Omega_\perp t - \frac{y^2+z^2}{2a_\perp^2} \right) \psi(x,t), \tag{16}$$

where the transverse part represents the ground state of the two-dimensional HO in the transverse plane, with the respective oscillator length

$$a_\perp = \sqrt{\hbar/(m\Omega_\perp)} \tag{17}$$

(typical values relevant to the experiments are $a_\perp \simeq 3$ μm), while the axial wave function, $\psi(x,t)$, subject to the normalization condition $\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = N$, see Eq. (3), satisfies the 1D equation obtained by averaging in the transverse plane:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + U(x;t)\psi + \frac{4\hbar^2}{m a_\perp^2} a_s(x;t) |\psi|^2 \psi, \tag{18}$$

which has the form of the one-dimensional NLSE, with an external potential, U. Essentially the same equation occurs in many other physical settings, such as the light propagation in planar waveguides, in which case t is actually the propagation distance, while $-U(x)$ represents the confining profile of the local refractive index

[43, 44, 169]. Accordingly, Eq. (18) with $a_s < 0$ and $a_s > 0$ gives rise, respectively, to the commonly known bright- and dark-soliton solutions. Interestingly, the NLSE in 1D is integrable in the case of $U = 0$ and $a_s = \mathrm{const}$ [170].

An interesting ramification of this setting is the toroidal quasi-1D trap, which is described by Eq. (18) with periodic boundary conditions in x. Such toroidal traps, realized by means of several different techniques, are available in the experiment [171].

A quasi-2D pancake-shaped (oblate) configuration, with strong confinement acting in the transverse 1D direction, corresponds to large $\Omega_z^2 \equiv \Omega_\perp^2$ in Eq. (12), combined with a general relatively weak potential, $U(x,y)$, acting in the pancake's plane. This configuration is approximated by the respective factorized ansatz,

$$\Psi(x,y,z,t) = \frac{1}{\pi^{1/4}\sqrt{a_\perp}} \exp\left(-\frac{i}{2}\Omega_\perp t - \frac{z^2}{2a_\perp^2}\right)\psi(x,y,t), \tag{19}$$

which leads to the effective 2D equation,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi + U(x,y;t)\psi + \frac{2\sqrt{2\pi}\hbar^2}{ma_\perp}a_s(x;t)|\psi|^2\psi. \tag{20}$$

The reduction of the 3D GPE (2) to its 1D version (18) on the basis of factorized *Ansatz* (16) with the fixed transverse localization radius, $a_\perp$, is relevant in the limit of low density. For higher density, the reduction is also based on ansatz (18), in which $a_\perp$ is allowed to be a variable parameter. Then, the reduction to 1D is performed by means of the variational approximation, which leads to a system of 1D equations for $\psi(x,t)$ and $a_\perp(x)$ [172], that can be reduced to a single effective equation for the 1D wave function with a nonpolynomial nonlinearity. The resulting "nonpolynomial NLSE" (abbreviated as NPSE [172]), and the respective local expression for $a_\perp$, are given (in a scaled form) by

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + U(x,t)\psi + \frac{1+(3/2)g|\psi|^2}{\sqrt{1+g|\psi|^2}}\psi, \tag{21}$$

and

$$a_\perp^4 = 1 + g|\psi|^4,\; g \equiv 2a_s/a_\perp < 0. \tag{22}$$

However, the relevant reduction from 3D to 1D may be done in multiple ways (e.g., working at the level of underlying Lagrangian/Hamiltonian structure and of the corresponding action or at that of the equations of motion). On a related direction, using the standard adiabatic approximation and accurate results for the local chemical potential, one obtains an alternative equation with a nonpolynomial nonlinearity [173]:

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + U(x,t)\psi + \sqrt{1+2g|\psi|^2}\psi, \tag{23}$$

where g is the same as in Eq. (22).

2.6. COLLAPSE OF ATTRACTIVE CONDENSATES

In the free space ($U = 0$), with a constant negative scattering length, corresponding to the self-attractive nonlinearity ($a_s < 0$), a rescaled form of Eq. (20) amounts to the 2D version of the NLSE:

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi - |\psi|^2\psi. \tag{24}$$

A well-known fact is that Eq. (24) gives rise to a family of isotropic, so-called *Townes' solitons* [174],

$$\psi = \exp(-i\mu t)\,\phi(r), \quad r \equiv \sqrt{x^2+y^2}, \tag{25}$$

with arbitrary chemical potential $\mu < 0$, and real function ϕ obeying the equation:

$$\mu\phi + (1/2)\left(\phi'' + r^{-1}\phi'\right) + \phi^3 = 0. \tag{26}$$

The family of the Townes' solitons is degenerate, in the sense that their norm takes a single value which does not depend on μ:

$$N_{\mathrm{T}} = 2\pi\int_0^\infty \phi^2(r)rdr \approx 5.85. \tag{27}$$

Note that an analytical variational approximation for this norm predicts $N_{\mathrm{T}} = 2\pi$, with relative error $\approx 7\%$ [175].

On the other hand, the three-dimensional GPE (2) in the free space, with the uniform self-attractive nonlinearity, $U = 0$ and $a_s < 0$, gives rise to a family of isotropic soliton solutions in the form given by Eq. (25). Unlike their 2D counterparts in the form of the above-mentioned Townes' solitons, the norm of the 3D solitons depends on μ, $N = \mathrm{const}\cdot(-\mu)^{-1/2}$, cf. Eq. (27). The celebrated Vakhitov-Kolokolov (VK) necessary stability criterion [176, 177], $dN/d\mu < 0$, does not hold for this $N(\mu)$ dependence, hence the entire family of the 3D free-space solitons is *unstable*, which is completely corroborated by the full analysis of the stability [177]. For the 2D Townes solitons, Eq. (27) formally predicts neutral VK stability, $dN/d\mu = 0$, but in reality the Townes solitons are unstable too. However, their instability is nonlinear, i.e., it is not accounted form by any unstable eigenvalue in the spectrum of eigenmodes computed around the stationary soliton, using the respective Bogoliubov-de Gennes equations (BdGEs). The eigenvalue associated with the instability in this special case is at 0, corresponding to a special invariance arising in this critical case, namely the so-called conformal invariance [43], which allows a rescaling of the solitary wave. In fact, the instability of the multidimensional solitons is explained by the fact that the NLSE with the self-attractive cubic nonlinearity gives rise to the dynamical *collapse*, i.e., the self-similar formation of a true singularity after a finite evolution time. In the 2D space, the collapse is *critical*, which implies, *inter alia*, that the norm of collapsing solutions must exceed a threshold (minimum) value, which is precisely the Townes-soliton norm (27), while the 3D collapse is supercritical, as its threshold norm is zero [177]. In the experiments with the self-attractive BEC, the onset of the collapse was readily observed (the first time in ^{85}Rb [178]), as spontaneous explosion of the condensate (the so-called "Bose nova"). It is interesting to mention that the small part of the condensate surviving the explosion, can form a stable soliton in ^{85}Rb [51].

2.7. MODELS FOR NON-BEC ULTRACOLD GASES

The above discussion pertains to quantum bosonic gases. Ultracold fermionic gases have also been created in the experiment [179], which was followed by the observation of their condensation into the bosonic gas of Bardeen-Cooper-Schrieffer (BCS) pairs [180]. The theoretical description of the Fermi gases is more complex, because, in the general case, the Pauli principle prevents the application of the mean-field approximation to fermions, making it necessary to treat such gases directly as systems with many degrees of freedom of individual particles (see, e.g., the works [181] and the review [182]).

An approach to sufficiently dense Fermi gases is possible in terms of a hydrodynamic description, which, in a sense, is a variety of the mean-field theory. This approach starts from the famous works by Yang and Lee who derived the energy density for a weakly coupled BCS superfluid [183]. In the spirit of the hydrodynamic approach, an effective equation for an order parameter of the Fermi gas, $\Psi\left(x,y,x,t\right)$ was derived, which seems as the NLSE with the self-repulsive term of power $7/3$ [184]:

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m_{\mathrm{eff}}}\nabla^2\Psi+U\left(x,y,z\right)\Psi+\frac{\hbar^2}{2m}\left|\Psi\right|^{4/3}\Psi, \tag{28}$$

where m_{eff} is the effective mass, which may be different from the particle's mass. This equation is valid for a slowly varying order-parameter field, under the condition that the local Fermi energy is much larger than all other local energy scales, such as potential $U\left(x,y,z\right)$. Equation (28) and its 1D and 2D reductions can be used to predict various stationary and quasi-stationary density patterns in the Fermi gas [184]. Related equations for the dynamics of Fermi gases are discussed in Ref. [185].

Similarly to the case of bosonic gases, nonlinear excitations of Fermi gases have attracted attention. In particular, dark solitons in a Fermi gas were predicted near the BEC-BCS transition, using the description in terms of the BdGEs [186], while their dynamical properties were studied in several works [187]. It is worth noting that dark solitons [188] and hybrid soliton-vortex structures [189] (the so-called solitonic vortices) were recently observed in experiments with Fermi gases in the BEC-BCS crossover.

Another well-known example of a dilute quantum medium different from BEC is the 1D Tonks-Girardeau (TG) gas, formed by hard-core bosons, a solution for which may be mapped into that for a gas of non-interacting fermions [190]. In particular, this mapping (also known as "Bose-Fermi mapping") makes it possible to produce a solution for a dark soliton in the TG gas [191]. Importantly, the TG gas was realized experimentally using ultracold ^{87}Rb atoms loaded into a tightly confined quasi-1D trap [192].

An equation for the order parameter of the TG gas was derived in work [193], in the form of the 1D NLSE with the self-repulsive quintic term:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + U\left(x\right)\Psi + \frac{\pi\hbar^2}{2m}\left|\Psi\right|^4\Psi. \tag{29}$$

Similar to Eq. (28), Eq. (29) is valid only as a quasi-stationary one and it does not provide correct description of dynamical effects in the TG gas [194]. There exist examples of a relevant use of Eq. (29), including the prediction of solitons supported by the long-range dipole-dipole attraction between atoms forming the gas [195], and the study of dark soliton oscillations [196]. Interestingly, the result for the soliton oscillation frequency, which was analytically found in Ref. [196] to be equal to the axial trap frequency, was in agreement with numerical predictions obtained in Ref. [197] via the Bose-Fermi mapping.

3. SOME SPECIAL TOPICS THAT HAVE ATTRACTED INTEREST

3.1. TEMPORAL AND SPATIAL MANAGEMENT

The *management* concept can be applied for the trapping potentials by making the HO or OL strengths time-dependent. A typical example of results produced by the time-periodic management of the HO potential (12), with $\Omega^2 = \Omega_0^2 + \Omega_1^2 \sin\left(\omega t\right)$ (in the simplest 1D setting), is the prediction of parametric resonances of self-trapped matter-wave packets (solitons) in the latter setting [198]. On the other hand, for the OL the management was experimentally realized [199] in the form of a "rocking" OL, by introducing a small wavelength mismatch between the two laser beams building the OL, which is made a periodic function of time: $\Delta\lambda = \Delta\lambda_0 \sin\left(\omega t\right)$, i.e., the lattice as a whole performs periodic oscillatory motion. In particular, the rocking OL potential may effectively suppress the matter-wave tunneling across the lattice [199].

One particular application of the periodic modulation of the strength of the HO potential deals with the appearance of regular patterns in the density profile of the condensate through a modulational instability. A prototypical example is the emergence of Faraday patterns in cigar-shaped BECs [200] through the periodic modulation of the strength of the radial component of the magnetic trap, similar experimental results being known for ^{4}He [201]. As another aspect of theoretical results, we mention numerous studies of Faraday waves in models of the condensates with short-range interactions [202–204], dipolar condensates [205], binary condensates with short-range interactions [206], Fermi-Bose mixtures [207], and superfluid Fermi gases [208]. As a related topic, let us mention that it has been shown theoretically that Faraday waves can be suppressed in condensates subject either to resonant parametric modulations [209] or space- and time-modulated potentials [210, 211], and that pattern-forming modulational instabilities lead to chaotic density profiles [202] akin to those of turbulent BECs [67, 212]. Apart from the use of parametric excitations, the formation of density waves has been studied in expanding ultra-cold Bose gases (either fully [213] or only partly condensed [214, 215]), and the spontaneous formation of density waves has been reported for antiferromagnetic BECs [216].

As predicted theoretically (see Ref. [217] for a review), many possibilities for the creation of matter-wave patterns in BEC are offered by various patterns of spatial [218] and temporal [219] modulation of the local scattering length, a_s, as implied by the general form of GPE (2). Experimentally, spatial "landscapes" of the scattering length can be induced, via the FR mechanism, by the corresponding spatially periodic distributions of the magnetic field (*magnetic lattices*), created with the help of periodic structures built of ferromagnetic materials [220]. Another possibility is the local modulation of the scattering length, which may be imposed, via the optical FR, by time-average patterns "painted" by rapidly moving laser beams [221]. Spatial modulation of interatomic interactions has also been demonstrated at the submicron level via pulsed optical standing waves in an ytterbium BEC [222]. Once again in this context, it is relevant to point out that some of these possibilities, such as the temporal modulation of the nonlinearity have been also realized in parallel, in other contexts such as, e.g., nonlinear optics [223].

Employing such magnetically or optically induced Feshbach resonances, via the above-mentioned FR-management technique, indeed constitutes one of the most promising methods for manipulating BECs. Such a possibility to control the effective nonlinearity of the condensate, has given rise to many theoretical and experimental studies. Probably the most well-known example between these, is the formation of bright matter-wave solitons and soliton trains in attractive condensates [48]-[52], by switching the interatomic interactions from repulsive to attractive.

This inspired many theoretical works studying the BEC dynamics under temporal and/or spatial modulation of the nonlinearity. In particular, the application of FR-management technique, with the low-frequency modulation of the strength of the magnetic field causing the nonlinearity to periodically switch between attraction and repulsion, can be used to stabilize 2D solitons in the free space [219] (in reality, a weak two-dimensional HO trapping potential is necessary in the experiment). However, this method does not work in 3D, nor for 2D vortex solitons. On the other hand, the same technique can be used for the generation of robust matter-wave breathers [224].

On the other hand, the so-called "collisionally inhomogeneous condensates" (a term coined in Refs. [218]) controlled by the *spatially modulated nonlinearity*, have been predicted to support a variety of new phenomena. This new regime can be achieved by means of magnetically or optically controlled Feshbach resonances. The magnetic Feshbach resonances is a well-established experimental method, which has been used to study the formation of ultracold molecules [225], the BEC-BCS crossover [226], and the production of Efimov trimer states [227], but the inhomogeneity length scale of the necessary magnetic field is usually larger than the size of the atomic sample, therefore this method is not very efficient in reaching the collisionally inhomogeneous regime. However, the optical Feshbach resonance has been shown to allow fine spatial control of the scattering length, and recent experimental results demonstrate controllable modulations of the s-wave scattering length on the scale of hundreds of nanometers [222].

The range of new nonlinear phenomena specific to the inhomogeneous-nonlinearity regime includes adiabatic compression of matter waves [218, 228], Bloch oscillations of matter-wave solitons [218], emission of the solitons and design of atom-beam lasers [229], dynamical trapping of matter-wave solitons [230, 231] enhancement of transmissivity of matter waves through barriers, [231, 232], creation of stable condensates exhibiting both attractive and repulsive interatomic interactions [230], competition between incommensurable linear and nonlinear lattices [233], the generation of dark solitons and vortex rings [234], control of Faraday waves [235], and many others. Importantly, the Feshbach resonance was used in recent experiments to induce real spatial inhomogeneities of the scattering length [221, 222], which paves the way for implementation of the above-mentioned phenomena in the experiment.

3.2. MULTIDIMENSIONAL LOCALIZED STRUCTURES

3.2.1. Attractive BEC

The stabilization of multidimensional solitons is a problem of great interest not only to BEC, but also to nonlinear optics and related research areas [236–240]. It was predicted theoretically, but not as yet demonstrated experimentally, that the use of OL potentials is a universal means for the stabilization of such solitons [241, 242] (a similar stabilization mechanism was predicted for 2D optical solitons supported by the Kerr self-focusing nonlinearity in photonic-crystal fibers [243]). In particular, the stabilization of 2D solitons is explained by the fact that the norm of such solitons, trapped in the OL potential, takes values *below* the threshold value (27) corresponding to the Townes' soliton, hence the solitons have no chance to start collapsing. The same property, $N < N_{\mathrm{T}}$, explains the stabilization of solitons trapped in the HO potential (12) [244, 245]. For a comprehensive study of stability of two-dimensional elliptic vortices in self-attractive Bose-Einstein condensates, trapped by an anisotropic harmonic trapping potential, see Ref. [246].

On a different but related direction let us mention that OLs of the Bessel type can support soliton rotation, see Refs. [247–250] for the theoretical works and Ref. [251] for the experimental verification, a nonlinear phenomenon which has been investigated in both BEC and nonlinear optical settings.

Still more challenging objects are multidimensional vortex solitons (i.e., self-trapped modes with embedded vorticity). In addition to the possibility of the collapse, they are still more unstable to fragmentation by azimuthal perturbations [217, 236]. An accurate analysis of the stability of three-dimensional solitons with vorticity $S = 1$ in self-attractive Bose-Einstein condensates, trapped in an anisotropic three-dimensional HO potential was reported in Ref. [252]. The analysis predicts that vortex solitons can also be stabilized by OL potentials [241]. Of course, the lattice breaks the axial symmetry, but, nevertheless, the vorticity embedded into a localized state may be defined in this case too [241, 253]. In the simplest form, stable vortex solitons with topological charge (integer vorticity) S can be constructed as N-peaked ring-shaped patterns with the vorticity represented by the phase circulation along the ring, with phase shift $\Delta\phi = 2\pi/N$ between adjacent peaks (see the straightforward definition (5) of the vorticity for isotropic settings). OLs may stabilize solitons even if the lattice's dimension is smaller by one than the dimension of the physical space, including 1D [254] and 2D [254, 255] OLs in the 2D and 3D settings, respectively.

A new approach to the creation of stable 2D solitons supported by the cubic self-attraction, which was considered impossible until recently, was put forward in theoretical work [256]. It is based on the system of two GPEs, linearly coupled by first-derivative terms representing the above-mentioned SOC of the Rashba type, and by nonlinear terms accounting for collisions between atoms belonging to the two different atomic states, which underlie the coupled system. In the scaled form, the system is given by

$$i\frac{\partial\psi_1}{\partial t} = -\frac{1}{2}\nabla^2\psi_1 - (|\psi_1|^2 + \gamma|\psi_2|^2)\psi_1 + \lambda\left(\frac{\partial\psi_2}{\partial x} - i\frac{\partial\psi_2}{\partial y}\right) \tag{30}$$

and

$$i\frac{\partial\psi_2}{\partial t} = -\frac{1}{2}\nabla^2\psi_2 - (|\psi_2|^2 + \gamma|\psi_1|^2)\psi_2 - \lambda\left(\frac{\partial\psi_1}{\partial x} + i\frac{\partial\psi_1}{\partial y}\right), \tag{31}$$

where ∇^2 is the Laplacian acting on coordinates (x, y), the real-valued λ is the strength of the SOC, and γ is the relative strength of inter-component nonlinearity in comparison with the intra-component self-attraction. Due to the specific form of the

SOC terms, composite solitons are generated by the system (30)-(31) as bound states of a fundamental localized state in one mode and a vortex in the other mode (*semi-vortices*, also referred to as filled vortices [257] or vortex-bright solitary waves [258]), or mixtures of fundamental and vortical components in both modes. As mentioned above, solitons trapped in the OL potential are stabilized by it because their norm drops below the threshold necessary for the onset of the collapse, while it was believed that this is impossible in the 2D free space. The new feature of the composite solitons produced by the system (30)-(31) is that their norm also takes values *below the threshold*, without the help of any trapping potential. The full stability is provided by the comparison of values of the Hamiltonian of Eqs. (30)-(31),

$$\begin{aligned} H &= \int\int \left\{ \frac{1}{2}\left(|\nabla\psi_1|^2 + |\nabla\psi_2|^2\right) - \frac{1}{2}\left(|\psi_1|^4 + |\psi_2|^4\right) - \gamma|\psi_1|^2|\psi_2|^2 \right. \\ &\left. + \frac{\lambda}{2}\left[\psi_1^*\left(\frac{\partial\psi_2}{\partial x} - i\frac{\partial\psi_2}{\partial y}\right) + \psi_2^*\left(-\frac{\partial\psi_1}{\partial x} - i\frac{\partial\psi_1}{\partial y}\right)\right] + \text{c.c.} \right\} dxdy, \quad (32) \end{aligned}$$

where c.c., as well as the asterisk, stands for the complex conjugate, for the composite semi-vortex and mixed-mode solitons. The self-trapped modes of the former and latter types are stable and realize the system's ground state (i.e., they minimize energy (32)) at $\gamma < 1$ and $\gamma > 1$, respectively.

3.2.2. Repulsive BEC

For the repulsive interaction between atoms, it has been predicted that stable matter-wave vortices are supported by condensates loaded into OLs [259] and that gap solitons and gap-soliton vortices also exist if the condensate is loaded into the OL of the same dimension [260]. On the other hand, the application of an OL, or of a magnetic lattice [220], to impose spatial modulation of the scattering length, $a_s(x, y, z)$ in Eq. (2), induces an effective nonlinear lattice, which can readily support 1D solitons, while the stabilization of their 2D counterparts by nonlinear periodic potentials is a difficult problem [217].

New perspectives for the creation of stable complex 3D localized modes, such as vortex rings, vortex-antivortex hybrids, and *Hopfions* (twisted rings, featuring two independent topological numbers), which were unavailable in other physical media, were recently predicted by a model with the local strength of the self-repulsive cubic nonlinearity growing from the center to periphery at any rate faster than r^3 [128, 261, 262]. Realization of these settings in BEC is a challenge to the experiment. Below we will focus on some of the more standard settings involving vortices and related structures in higher-dimensional BECs without spatial or temporal modulation of the scattering length.

3.3. VORTICES AND VORTEX CLUSTERS

Arguably, one of the most striking features of BECs is the possibility of supporting vortices, which have been observed in many experiments by means of a variety of methods. Vortices are characterized by their non-zero topological charge S, whereby the phase of the wavefunction has a phase jump of $2\pi S$ along a closed contour surrounding the core of the vortex. The width of single-charge vortices in BECs is of the order $\mathcal{O}(\xi)$ – where ξ is the healing length of the condensate (see, e.g., [7]) – while higher-charge vortices, with $|S| > 1$, have cores wider than the healing length. Such higher-charge vortices are generally unstable in the homogeneous background case; nevertheless, they may be stabilized by employing external impurities [263] (the latter can be used to confine the so-called persistent current; see, e.g., the recent discussion of relevant experiments in [264] and references therein), or by using external potentials [265]. Notice that, when unstable, higher-charge vortices typically split in multiple single-charge vortices, since the system has no other way to dispose of the topological charge [266].

The fact that single-charge vortices carry topological charge renders them extremely robust objects: indeed, continuous deformation of the vortex profile cannot eliminate the 2π phase jump. An exception is a case where the background condensate density is close to zero, and that is why, in BEC stirring experiments, vortices are nucleated at the periphery of the harmonically trapped condensate [267].

Vortices are prone to motion caused by gradients in the density (and phase) of the background, induced by an external potential (as, e.g., in the case of a trapped BEC) or by the presence of other vortices. The motion of the vortex in such cases can be studied by means of the matched asymptotic expansion method [268]. The same method can also be used to study the effect of vortex precession induced by the external trap (see, e.g., the review [66]), also in the presence of collisional inhomogeneities and dissipative perturbations [269]. Note that in the simplest case of a single vortex in a 2D BEC confined in a harmonic trap, a Bogoliubov-de Gennes analysis reveals the connection of the vortex precession frequency with characteristic eigenfrequencies of the spectrum and – in particular – with the *anomalous mode* [270] (for the latter, the integral of the norm $\times$ energy product is negative [7]). Indeed, the negative-energy mode bifurcates in the linear limit from the dipole mode (which has a constant magnitude equal to the trap frequency); then, as the chemical potential increases, the anomalous mode eigenfrequency decreases, and becomes equal to the precession frequency of the vortex in the Thomas-Fermi limit (see, e.g., Refs. [269, 271]).

On the other hand, the motion induced on a vortex by another vortex is tantamount to the one observed in fluid vortices (see, e.g. Ref. [272]): this way, vortices with same charge travel parallel to each other at constant speed, while vortices of opposite charges rotate about each other at constant angular speed. Following Helmholtz' and Kirchhoff's considerations, one may treat vorticity as a sum of point vortices and determine the velocity field created by the vortices (this velocity field induces the vortex motion) by means of the Biot-Savart law. This way, one may find a set of ordinary differential equations (ODEs) for the location of the vortices, that describe vortex-vortex interactions [273]. A more subtle consideration in this context is the "screening" effect of the vortex-vortex interaction by the inhomogeneities in the density (due to the presence of the external potential). The latter effect has been incorporated in some of the above works by an effective renormalization of the interaction prefactor, but a more systematic study of this effect is still lacking despite the fact that the relevant more complicated dynamical equations can still be systematically derived [274].

In a 2D (disk-shaped) BEC confined in a parabolic trap, it is then possible to employ variational arguments [275] and combine both effects: vortex precession (induced by the trap) and vortex-vortex interactions. This way, the effective dynamics of a small cluster of interacting vortices (of potentially same or different charges) is described by a system of ODEs for the centers of the vortices. The relevant dynamical system possesses two integrals of motion (Hamiltonian and angular momentum) and, thus, it is completely integrable for vortex dipoles composed by two counter-rotating or co-rotating vortices. Importantly, a theoretical description of vortex trajectories was found to be in excellent agreement with pertinent experimental findings [77]. Notice that apart from vortex dipoles, the same methodology has been used in cases of vortex clusters composed by more than two vortices. Clusters involving e.g. 4, 6, 8 vortices of alternating charges in polygonal form [271] or 4, 5, 6 vortices in a linear configuration are currently challenging to produce experimentally (and are found to be dynamically unstable when possessing more than 4 vortices in a polygonal shape, or 3 or more vortices in a linear configuration [276]). On the other hand, producing such clusters with a controllable number of vortices of the same charge is straightforward (see the second item in Ref. [77]). Please note that the system possesses a number of intriguing symmetry-breaking bifurcations [277], which can be explained even in analytical form through a linear stability analysis [278].

Apart from single vortices and small vortex clusters, there has been much interest in *vortex lattices* in rapidly rotating condensates. Such configurations consist of a large number of ordered lattices of vortices, arranged in triangular configurations, the so-called Abrikosov lattices [279]. The first BEC experiments reported observation of vortex lattices consisted of just a few (< 15) vortices [280], but later on it was possible to nucleate experimentally and maintain vortex lattices with over 100 vortices [281, 282]. Subsequent efforts enabled the observation of intriguing phenomena associated with these vortex lattices, including their collective (so-called Tkachenko) oscillations, as well as their structural phase transitions either under multi-component interaction (a transition from hexagonal to square lattice was observed experimentally in [283]) or in the presence of external potentials (a similar transition was theoretically reported in the presence of a square optical lattice in [284]). Lastly, it is relevant to mention here that such multi-vortex configurations are at the heart of ongoing studies of phenomena including vortex turbulence and more generally non-equilibrium dynamics of atomic condensates [67].

Admittedly, the above discussion contains a rather partial perspective of a field that has truly boomed in a remarkable number of research threads and has done so via a rather unique cross-pollination with other fields of physics that is simply impossible to capture within the confines of the present chapter. Nevertheless, we hope to have conveyed some of the main areas of the pertinent studies and the ever-expanding (in terms of research groups and themes of study) enthusiasm surrounding this area. We now turn our attention to the specific contributions associated with this special volume, which touch upon many of the above-mentioned topics.

4. A SYNOPSIS OF THE ARTICLES INCLUDED IN THE PRESENT SPECIAL ISSUE

As mentioned above, the current theoretical and experimental work on BEC and related topics covers a vast research area. Of course, the papers selected for this Special Issue cannot survey all aspects of this work. Most of the papers present theoretical results, in compliance with the obvious trend that many more original theoretical papers on BEC, than experimental ones, appear in the scientific literature. Nevertheless, some articles from the Special Issue present experimental results too, as briefly recapitulated below. Some papers deal directly with basic aspects of the studies of BEC, while others address different but related topics, such as fermion quantum gases, few-boson models, etc. The different settings and problems addressed in the articles may be categorized as more physical or more mathematical ones.

(1.) M. A. Caracanhas, E. A. L. Henn, and V. S. Bagnato, *Quantum turbulence in trapped BEC: New perspectives for a long lasting problem*

BEC in atomic gases provides the most natural testbed for exploring turbulent dynamics of superfluids. This article [285] offers a review of recent experimental and theoretical results on this topic, and a discussion on the directions for the further development of studies dedicated to quantum-liquid turbulence.

(2.) A. Vardi, *Chaos, ergodization, and thermalization with few-mode Bose-Einstein condensates*

This article [286] considers a system with few degrees of freedom, which represents “small” Bose-Hubbard (BH) models, namely, BH dimers and trimers, the former one being reduced to a classical kicked top. In the framework of these systems, the analysis is focused on aspects of classical dynamical chaos in them, including the problems of the onset of ergodicity and thermalization. The energy diffusion in the systems' phase space is explored by means of the Fokker-Planck equation.

(3.) R. Radha and P. S. Vinayagam, *An analytical window into the world of ultracold atoms*

The paper [287] addresses BEC models based on GPEs, in terms of the possible integrability of these models. The approach develops the known method of transforming the standard integrable form of the NLSE into seemingly complex, but still integrable ones, by means of explicit transformations of the wave functions and variables (x, t). In particular, considered are models which may be integrable, while they include complex ingredients, such as the time dependence (management) of the scattering length, and a parabolic potential (expulsive, i.e., of the anti-harmonic-oscillator type, rather than the trapping one), with a constant or time-dependent strength. In addition to the single-component models, two-component systems are considered too, with both nonlinear and linear couplings between the components. A number of exact solutions are found in such models, including bright and dark solitons.

(4.) A. I. Nicolin, M. C. Raportaru, and A. Balaž, *Effective low-dimensional polynomial equations for Bose-Einstein condensates*

The article [288] addresses the derivation and analysis of effective equations with reduced (1D and 2D) dimensions for prolate and oblate (cigar-shaped and pancake-shaped) condensates, respectively. The equations with the reduced dimensionality are derived from the full 3D GPE, under the condition of strong confinement in the transverse direction(s). The effective equations with polynomial nonlinearities are derived in this context.

(5.) V. I. Yukalov and E. P. Yukalova, *Statistical models of nonequilibrium Bose gases*

The analysis in this article [289] addresses strongly perturbed BEC, i.e., it goes far beyond the limits of the near-equilibrium mean-field theory. In particular, this paper makes a contact with the considerations presented in the article by M. A. Caracanhas, E. A. L. Henn, and V. S. Bagnato in the same Special Issue [285], as the analysis develops a description of strongly excited BEC in terms of a statistical model of grain turbulence.

(6.) H.-S. Tao, W. Wu, Y.-H. Chen, and W.-M. Liu, *Quantum phase transitions of cold atoms in honeycomb optical lattices*

Optical lattices with different geometries (in particular, 2D honeycomb lattices considered in this article) help to support quantum gases (both bosonic and fermionic) in highly-correlated states. The article [290] aims to review recent results for quantum phase transitions of cold fermionic atoms in these lattices. In that sense, it is an essential addition to the collection of topical chapters on the theme of BEC, as results are presented for quantum Fermi gases, rather than for bosons. The analysis combines mean-field considerations with quantum Monte-Carlo computations, with the aim to calculate various properties of the systems under consideration, such as the density of states, the Fermi surface, etc. Also considered in this article are bilayer lattices, in addition to the monolayer ones, and effects of the spin-orbit interaction between the fermion components.

(7.) T. He, W. Li, L. Li, J. Liu, and Q. Niu, *Stationary solutions for nonlinear Schrödinger equation with ring trap and their evolution under the periodic kick force*

This work [291] addresses solutions of the nonlinear Schrödinger equation for a periodically kicked quantum rotator. The model may also be relevant for a BEC trapped in a toroidal quasi-1D trap, with a periodically applied potential profile. The analysis is focused on the especially interesting cases of quantum anti-resonance and quantum resonance, using analytical stationary solutions of the nonlinear Schrödinger equation with periodic boundary conditions.

(8.) D. A. Zezyulin and V. V. Konotop, *Stationary vortex flows and macroscopic Zeno effect in Bose-Einstein condensates with localized dissipation*

This article [292] addresses an interesting topic of nonlinear BEC in dissipative media. A specific setting is considered with flow of the superfluid towards the central part of the 2D system, where the loss is concentrated. The model does not include any explicit gain, but, nevertheless, it gives rise to stationary global patterns, including those with embedded vorticity, due to the balance between the influx from the reservoir at infinity and the effectively localized dissipation. The solution is interpreted in terms of the Zeno effect in the dissipative BEC.

(9.) V. Achilleos, D. J. Frantzeskakis, P. G. Kevrekidis, P. Schmelcher, and J. Stockhofe, *Positive and negative mass solitons in spin-orbit coupled Bose-Einstein condensates*

The article [293] addresses the currently hot topic of solitons in the two-component BEC realizing the spin-orbit-coupling effect. The analysis is developed for the 1D geometry, and relies on the reduction of the underlying two-component GPE system to a single NLSE, by means of the multiscale-expansion method. In this way, apart from the usual positive-mass bright and dark solitons, negative mass structures, namely bright (dark) solitons for repulsive (attractive) interactions are predicted as well. The analytical predictions are confirmed by numerical simulations.

(10.) A. I. Yakimenko, S. I. Vilchinskii, Y. M. Bidasyuk, Y. I. Kuriatnikov, K. O. Isaieva, and M. Weyrauch, *Generation and decay of persistent currents in a toroidal Bose-Einstein condensate*

The topic of persistent superfluid flows in toroidal traps is theoretically addressed in the article, being motivated by recent experimental observations of this effect. This article [294] offers a review of theoretical results on this topic, recently produced by the present authors. In particular, special attention is paid to the phenomenon of hysteresis in this setting. The analysis is performed by means of numerical solutions of the 3D GPE. Some related 2D settings are considered too.

(11.) M. Galante, G. Mazzarella, and L. Salasnich, *Analytical results on quantum correlations of few bosons in a double-well trap*

This work [295] deals not with BEC proper, but rather with sets of few bosons, the number of which is $N = 2$, 3, or 4. The bosons are trapped in a double-well potential, which is also used in many experimental and theoretical studies of BEC, such as those dealing with bosonic Josephson oscillations. Eventually, the system is reduced to the simplest two-site truncation of the Bose-Hubbard model, which has something in common with the setting considered in another article included into this Special Issue, the one by Vardi [286]. The analysis aims to find exact ground states for these few-boson sets, and study variation of their characteristics, such as the energy and entanglement entropy, as functions of system's parameters.

(12.) V. Bolpasi and W. von Klitzing, *Adiabatic potentials and atom lasers*

The article [296] addresses the topic of the design of atom-beam lasers. A detailed analytic model of the trap is presented and the flux of the atom laser is determined. The analytical results are found to be in good agreement with recent experimental data. The analysis is focused on the harmonic-oscillator trapping potential for the BEC, from which the laser beams are emitted. Gravity is taken into regard too.

REFERENCES

1. S. N. Bose, "Plancks Gesetz und Lichtquantenhypothese", Z. Physik **26**, 178 (1924);
A. Einstein, "Quantentheorie des einatomigen idealen Gases", Sitzber. Kgl. Preuss. Akad. Wiss. **1**, 3 (1925).
2. M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, "Observation of Bose-Einstein condensation in a dilute atomic vapor", Science **269**, 198 (1995).
3. K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, "Bose-Einstein condensation in a gas of sodium atoms", Phys. Rev. Lett. **75**, 3969 (1995).
4. C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, "Evidence of Bose-Einstein condensation in an atomic gas with attractive interactions", Phys. Rev. Lett. **75**, 1687 (1995);
C. C. Bradley, C. A. Sackett, and R. G. Hulet, "Bose-Einstein condensation of lithium: Observation of limited condensate number", Phys. Rev. Lett. **78**, 985 (1997).
5. E. A. Cornell and C. E. Wieman, "Nobel Lecture: Bose-Einstein condensation in a dilute gas, the first 70 years and some recent experiments", Rev. Mod. Phys. **74**, 875 (2002);
W. Ketterle, "Nobel Lecture: When atoms behave as waves: Bose-Einstein condensation and the atom laser", Rev. Mod. Phys. **74**, 1131 (2002).
6. C. J. Pethick and H. Smith, *Bose-Einstein Condensation in Dilute Gases* (Cambridge University Press, Cambridge, 2002).
7. L. P. Pitaevskii and A. Stringari, *Bose-Einstein Condensation* (Clarendon Press, Oxford, 2003).
8. C. J. Foot, *Atomic Physics* (Oxford University Press, Oxford, 2005).
9. S. Chu, "Nobel Lecture: The manipulation of neutral particles", Rev. Mod. Phys. **70**, 685 (1998);

C. N. Cohen-Tannoudji, “Nobel Lecture: Manipulating atoms with photons”, Rev. Mod. Phys. **70**, 707 (1998);
W. D. Phillips, “Nobel Lecture: Laser cooling and trapping of neutral atoms”, Rev. Mod. Phys. **70**, 721 (1998).

10. T. Lahaye, T. Koch, B. Fröhlich, M. Fattori, J. Metz, A. Griesmaier, S. Giovanazzi, and T. Pfau, “Strong dipolar effects in a quantum ferrofluid”, Nature **448**, 672 (2007);
Q. Beaufils, R. Chicireanu, T. Zanon, B. Laburthe-Tolra, E. Maréchal, L. Vernac, J.-C. Keller, and O. Gorceix, “All-optical production of chromium Bose-Einstein condensates”, Phys. Rev. A **77**, 061601(R) (2008).
11. M. Lu, N. Q. Burdick, S. H. Youn, and B. L. Lev, “Strongly dipolar Bose-Einstein condensate of dysprosium”, Phys. Rev. Lett. **107**, 190401 (2011).
12. T. Lahaye, C. Menotti, L. Santos, M. Lewenstein, and T. Pfau, “The physics of dipolar bosonic quantum gases”, Rep. Prog. Phys. **72**, 126401 (2009).
13. D. G. Fried, T. C. Killian, L. Willmann, D. Landhuis, S. C. Moss, D. Kleppner, and T. J. Greytak, “Bose-Einstein condensation of atomic hydrogen”, Phys. Rev. Lett. **81**, 3811 (1998).
14. J. Kasprzak, M. Richard, S. Kundermann, A. Baas, P. Jeambrun, J. M. J. Keeling, F. M. Marchetti, M. H. Szymańska, R. André, J. L. Staehli, V. Savona, P. B. Littlewood, B. Deveaud, and L. S. Dang, “Bose-Einstein condensation of exciton polaritons”, Nature **443**, 409 (2006);
K. G. Lagoudakis, B. Pietka, M. Wouters, R. André, and B. Deveaud-Plédran, “Coherent oscillations in an exciton-polariton Josephson Junction”, Phys. Rev. Lett. **105**, 120403 (2010).
15. S. O. Demokritov, V. E. Demidov, O. Dzyapko, G. A. Melkov, A. A. Serga, B. Hillebrands, and A. N. Slavin, “Bose-Einstein condensation of quasi-equilibrium magnons at room temperature under pumping”, Nature **443**, 430 (2006).
16. A. Imamoglu, R. J. Ram, S. Pau, and Y. Yamamoto, “Nonequilibrium condensates and lasers without inversion: Exciton-polariton lasers”, Phys. Rev. A **53**, 4250 (1996).
17. J. D. Plumhof, T. Stöferle, L. Mai, U. Scherf, and R. F. Mahrt, “Room-temperature Bose-Einstein condensation of cavity exciton-polaritons in a polymer”, Nature Materials **13**, 247 (2014).
18. J.-H. Jiang and S. John, “Photonic crystal architecture for room-temperature equilibrium Bose-Einstein condensation of exciton polaritons”, Phys. Rev. X **4**, 031025 (2014).
19. O. A. Egorov, A. V. Gorbach, F. Lederer, and D. V. Skryabin, “Two-dimensional localization of exciton polaritons in microcavities”, Phys. Rev. Lett. **105**, 073903 (2010).
20. E. A. Ostrovskaya, J. Abdullaev, A. S. Desyatnikov, M. D. Fraser, and Y. S. Kivshar, “Dissipative solitons and vortices in polariton Bose-Einstein condensates”, Phys. Rev. A **86**, 013636 (2012).
21. M. Sich, D. N. Krizhanovskii, M. S. Skolnick, A. V. Gorbach, R. Hartley, D. V. Skryabin, E. A. Cerda-Méndez, K. Biermann, R. Hey, and P. V. Santos, “Observation of bright polariton solitons in a semiconductor microcavity”, Nature Photonics **6**, 50 (2012).
22. S. Barland, M. Giudici, G. Tissoni, J. R. Tredicce, M. Brambilla, L. Lugiato, F. Prati, S. Barbay, R. Kuszelewicz, T. Ackemann, W. J. Firth, and G.-L. Oppo, “Solitons in semiconductor microcavities”, Nature Photonics **6**, 204 (2012).
23. T. Byrnes, N. Y. Kim, and Y. Yamamoto, “Exciton-polariton condensates”, Nature Physics **10**, 803 (2014).
24. J. Klaers, J. Schmitt, F. Vewinger, and M. Weitz, “Bose-Einstein condensation of photons in an optical microcavity”, Nature **468**, 545 (2010).
25. J. Weiner, V. S. Bagnato, S. Zilio, and P. S. Julienne, “Experiments and theory in cold and ultracold collisions”, Rev. Mod. Phys. **71**, 1 (1999).
26. F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, “Theory of Bose-Einstein condensation in trapped gases”, Rev. Mod. Phys. **71**, 453 (1999).

27. A. J. Leggett, “Bose-Einstein condensation in the alkali gases: Some fundamental concepts”, Rev. Mod. Phys. **73**, 307 (2001).
28. J. O. Andersen, “Theory of the weakly interacting Bose gas”, Rev. Mod. Phys. **76**, 599 (2004).
29. V. A. Brazhnyi and V. V. Konotop, “Theory of nonlinear matter waves in optical lattices”, Mod. Phys. Lett. B **18**, 627 (2004).
30. K. Bongs and K. Sengstock, “Physics with coherent quantum gases”, Rep. Prog. Phys. **67**, 907 (2004).
31. O. Morsch and M. Oberthaler, “Dynamics of Bose-Einstein condensates in optical lattices”, Rev. Mod. Phys. **78**, 179 (2006).
32. R. Carretero-González, D. J. Frantzeskakis, and P. G. Kevrekidis, “Nonlinear waves in Bose-Einstein waves condensates: physical relevance and mathematical techniques”, Nonlinearity **21**, R139 (2008).
33. P. G. Kevrekidis, D. J. Frantzeskakis, and R. Carretero-González (Eds.), *Emergent Nonlinear Phenomena in Bose-Einstein Condensates* (Springer, Berlin, 2008).
34. I. Bloch, J. Dalibard, and W. Zwerger, “Many-body physics with ultracold gases”, Rev. Mod. Phys. **80**, 885 (2008).
35. H. K. Stoof, K. B. Gubbels, and D. B. M. Dickerscheid, *Ultracold Quantum Fields* (Springer, Dordrecht, 2009).
36. D. J. Frantzeskakis, “Dark solitons in atomic Bose-Einstein condensates: from theory to experiments”, J. Phys. A: Math. Theor. **43**, 213001 (2010).
37. L. D. Carr (Ed.), *Understanding Quantum Phase Transitions* (Taylor and Francis, Boca Raton, 2010).
38. M. A. Cazalilla, R. Citro, T. Giamarchi, E. Orignac, and M. Rigol, “One dimensional bosons: From condensed matter systems to ultracold gases”, Rev. Mod. Phys. **83**, 1405 (2011).
39. M. Lewenstein, A. Sanpera, and V. Ahufinger, *Ultracold Atoms in Optical Lattices: Simulating Quantum Many-Body Systems* (Oxford University Press, Oxford, 2012).
40. N. P. Proukakis, S. A. Gardiner, M. Davis and M. Szymańska (Eds.), *Quantum Gases: Finite Temperature and Non-Equilibrium Dynamics* (Imperial College Press, London, 2013).
41. J. Dalibard, F. Gerbier, G. Juzeliūnas, and P. Öhberg, “Colloquium: Artificial gauge potentials for neutral atoms”, Rev. Mod. Phys. **83**, 1523 (2011);
N. Goldman, G. Juzeliūnas, P. Öhberg, and I. B. Spielman, “Light-induced gauge fields for ultracold atoms”, Rep. Prog. Phys. **77**, 126401 (2014).
42. R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, *Solitons and Nonlinear Wave Equations* (Academic, New York, 1983).
43. C. Sulem and P. L. Sulem, *The Nonlinear Schrödinger Equation* (Springer, New York, 1999).
44. M. J. Ablowitz, B. Prinari, and A. D. Trubatch, *Discrete and Continuous Nonlinear Schrödinger Systems* (Cambridge University Press, Cambridge, 2004).
45. S. L. Rolston and W. D. Phillips, “Nonlinear and quantum atom optics”, Nature **416**, 219 (2002);
B. P. Anderson and P. Meystre, “Nonlinear atom optics”, Contemporary Physics **44**, 473 (2003).
46. A. Wallraff, A. Lukashenko, J. Lisenfeld, A. Kemp, M. V. Fistul, Y. Koval, and A. V. Ustinov, “Quantum dynamics of a single vortex”, Nature **425**, 155 (2003).
47. Y. S. Kivshar and B. A. Malomed, “Dynamics of solitons in nearly integrable systems”, Rev. Mod. Phys. **61**, 763 (1989).
48. K. E. Strecker, G. B. Partridge, A. G. Truscott, and R. G. Hulet, “Formation and propagation of matter-wave soliton trains”, Nature **417**, 150 (2002).
49. L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L. D. Carr, Y. Castin, and C. Salomon, “Formation of a matter-wave soliton”, Science **296**, 1290 (2002).

50. K. E. Strecker, G. B. Partridge, A. G. Truscott, and R. G. Hulet, “Bright matter-wave solitons in Bose-Einstein condensates”, New J. Phys. **5**, 73 (2003).
51. S. L. Cornish, S. T. Thompson, and C. E. Wieman, “Formation of bright matter-wave solitons during the collapse of attractive Bose-Einstein condensates”, Phys. Rev. Lett. **96**, 170401 (2006).
52. A. L. Marchant, T. P. Billam, T. P. Wiles, M. M. H. Yu, S. A. Gardiner, and L. Cornish, “Controlled formation and reflection of a bright solitary matter-wave”, Nature Communications **4**, 1865 (2013).
53. F. Kh. Abdullaev, A. Gammal, A. M. Kamchatnov, and L. Tomio, “Dynamics of bright matter wave solitons in a Bose-Einstein condensate”, Int. J. Mod. Phys. B **19**, 3415 (2005);
T. P. Billam, A. L. Marchant, S. L. Cornish, S. A. Gardiner, and N. G. Parker, “Bright solitary matter waves: Formation, stability and interactions”, in: *Spontaneous Symmetry Breaking, Self-Trapping, and Josephson Oscillations*, ed. by B. A. Malomed (Springer, Berlin and Heidelberg, 2013), pp. 403-456.
54. J. H. V. Nguyen, P. Dyke, D. Luo, B. A. Malomed, and R. G. Hulet, “Collisions of matter-wave solitons”, Nature Physics **10**, 918 (2014).
55. N. G. Parker, A. M. Martin, S. L. Cornish, and C. S. Adams, “Collisions of bright solitary matter waves”, J. Phys. B **41**, 045303 (2008).
56. R. Dum, J. I. Cirac, M. Lewenstein, and P. Zoller, ”Creation of dark solitons and vortices in Bose-Einstein condensates”, Phys. Rev. Lett. **80**, 2972 (1998).
57. S. Burger, K. Bongs, S. Dettmer, W. Ertmer, K. Sengstock, A. Sanpera, G. V. Shlyapnikov, and M. Lewenstein, “Dark solitons in Bose-Einstein condensates”, Phys. Rev. Lett. **83**, 5198 (1999);
J. Denschlag, J. E. Simsarian, D. L. Feder, C. W. Clark, L. A. Collins, J. Cubizolles, L. Deng, E. W. Hagley, K. Helmerson, W. P. Reinhardt, S. L. Rolston, B. I. Schneider, and W. D. Phillips, “Generating solitons by phase engineering of a Bose-Einstein condensate”, Science **287**, 97 (2000).
58. A. Weller, J. P. Ronzheimer, C. Gross, J. Esteve, M. K. Oberthaler, D. J. Frantzeskakis, G. Theocharis, and P. G. Kevrekidis, “Experimental observation of oscillating and interacting matter wave dark solitons”, Phys. Rev. Lett. **101**, 130401 (2008);
G. Theocharis, A. Weller, J. P. Ronzheimer, C. Gross, M. K. Oberthaler, P. G. Kevrekidis, and D. J. Frantzeskakis, “Multiple atomic dark solitons in cigar-shaped Bose-Einstein condensates”, Phys. Rev. A **81**, 063604 (2010).
59. G. Huang, M. G. Velarde, and V. A. Makarov, “Dark solitons and their head-on collisions in Bose-Einstein condensates”, Phys. Rev. A **64**, 013617 (2001).
60. S. Stellmer, C. Becker, P. Soltan-Panahi, E.-M. Richter, S. Dörscher, M. Baumert, J. Kronjäger, K. Bongs, and K. Sengstock, “Collisions of dark solitons in elongated Bose-Einstein condensates”, Phys. Rev. Lett. **101**, 120406 (2008).
61. Th. Busch and J. R. Anglin, “Dark-bright solitons in inhomogeneous Bose-Einstein condensates”, Phys. Rev. Lett. **87**, 010401 (2001).
62. C. Becker, S. Stellmer, P. Soltan-Panahi, S. Dörscher, M. Baumert, E.-M. Richter, J. Krönjager, K. Bongs, and K. Senhstock, “Oscillations and interactions of dark and dark-bright solitons in Bose-Einstein condensates”, Nature Physics **4**, 496 (2008);
S. Middelkamp, J. J. Chang, C. Hamner, R. Carretero-González, P. G. Kevrekidis, V. Achilleos, D. J. Frantzeskakis, P. Schmelcher, and P. Engels, “Dynamics of dark-bright solitons in cigar-shaped Bose-Einstein condensates”, Phys. Lett. A **375**, 642 (2011).
63. C. Hamner, J. J. Chang, P. Engels, and M. A. Hoefer, “Generation of dark-dright soliton trains in superfluid-superfluid counterflow”, Phys. Rev. Lett. **106**, 065302 (2011);
D. Yan, J. J. Chang, C. Hamner, P. G. Kevrekidis, P. Engels, V. Achilleos, D. J. Frantzeskakis, R. Carretero-González, and P. Schmelcher, “Multiple dark-bright solitons in atomic Bose-Einstein

condensates", Phys. Rev. A **84**, 053630 (2011).

64. M. A. Hoefer, J. J. Chang, C. Hamner, and P. Engels, "Dark-dark solitons and modulational instability in miscible two-component Bose-Einstein condensates", Phys. Rev. A **84**, 041605(R) (2011);
D. Yan, J. J. Chang, C. Hamner, M. Hoefer, P. G. Kevrekidis, P. Engels, V. Achilleos, D. J. Frantzeskakis, and J. Cuevas, "Beating dark-dark solitons in Bose-Einstein condensates", J. Phys. B: At. Mol. Opt. Phys. **45**, 115301 (2012).
65. B. Eiermann, Th. Anker, M. Albiez, M. Taglieber, P. Treutlein, K.-P. Marzlin, and M. K. Oberthaler, "Bright Bose-Einstein gap solitons of atoms with repulsive interaction", Phys. Rev. Lett. **92**, 230401 (2004).
66. A. L. Fetter and A. A. Svidzinsky, "Vortices in a trapped dilute Bose-Einstein condensate", J. Phys.-Cond. Matt. **13**, R135 (2001);
P. Engels, I. Coddington, V. Schweikhard, and E. A. Cornell, "Vortex lattice dynamics in a dilute-gas BEC", J. Low Temp. Phys. **134**, 683 (2004);
P. G. Kevrekidis, R. Carretero-González, D. J. Frantzeskakis, and I. G. Kevrekidis, "Vortices in Bose-Einstein condensates: Some recent developments", Mod. Phys. Lett. B **18**, 1481 (2004);
A. L. Fetter, "Rotating trapped Bose-Einstein condensates", Rev. Mod. Phys. **81**, 647 (2009).
67. E. A. L. Henn, J. A. Seman, G. Roati, K. M. F. Magalhaes, and V. S. Bagnato, "Emergence of turbulence in an oscillating Bose-Einstein condensate", Phys. Rev. Lett. **103**, 045301 (2009);
T. W. Neely, A. S. Bradley, E. C. Samson, S. J. Rooney, E. M. Wright, K. J. H. Law, R. Carretero-González, P. G. Kevrekidis, M. J. Davis, and B. P. Anderson, "Characteristics of two-dimensional quantum turbulence in a compressible superfluid", Phys. Rev. Lett. **111**, 235301 (2013).
68. M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wieman, and E. A. Cornell, "Vortices in Bose-Einstein condensate", Phys. Rev. Lett. **83**, 2498 (1999).
69. K. W. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, "Vortex formation in a stirred Bose-Einstein condensate", Phys. Rev. Lett. **84**, 806 (2000).
70. J. R. Abo-Shaeer, C. Raman, J. M. Vogels, and W. Ketterle, "Observation of vortex lattices in Bose-Einstein condensates", Science **292**, 476 (2001).
71. D. R. Scherer, C. N. Weiler, T. W. Neely, and B. P. Anderson, "Vortex formation by merging of multiple trapped Bose-Einstein condensates", Phys. Rev. Lett. **98**, 110402 (2007);
R. Carretero-González, B. P. Anderson, P. G. Kevrekidis, D. J. Frantzeskakis, and C. N. Weiler, "Dynamics of vortex formation in merging Bose-Einstein condensate fragments", Phys. Rev. A **77**, 033625 (2008).
72. T. W. Neely, E. C. Samson, A. S. Bradley, M. J. Davis, and B. P. Anderson, "Observation of vortex dipoles in an oblate Bose-Einstein condensate", Phys. Rev. Lett. **104**, 160401 (2010).
73. C. N. Weiler, T. W. Neely, D. R. Scherer, A. S. Bradley, M. J. Davis, and B. P. Anderson, "Spontaneous vortices in the formation of Bose-Einstein condensates", Nature **455**, 948 (2008).
74. T. W. B. Kibble, "Topology of cosmic domains and strings", J. Phys. A: Math. Gen. **9**, 1387 (1976);
W. H. Zurek, "Cosmological experiments in superfluid helium?", Nature **317**, 505 (1985).
75. D. V. Freilich, D. M. Bianchi, A. M. Kaufman, T. K. Langin, and D. S. Hall, "Real-time dynamics of single vortex lines and vortex dipoles in a Bose-Einstein condensate", Science **320**, 1182 (2010);
76. J. A. Seman, E. A. L. Henn, M. Haque, R. F. Shiozaki, E. R. F. Ramos, M. Caracanhas, P. Castilho, C. Castelo Branco, P. E. S. Tavares, F. J. Poveda-Cuevas, G. Roati, K. M. F. Magalhaes, and V. S. Bagnato, "Three-vortex configurations in trapped Bose-Einstein condensates" Phys. Rev. A **82**, 033616 (2010).

77. S. Middelkamp, P. J. Torres, P. G. Kevrekidis, D. J. Frantzeskakis, R. Carretero-González, P. Schmelcher, D. V. Freilich, and D. S. Hall, “Guiding-center dynamics of vortex dipoles in Bose-Einstein condensates”, Phys. Rev. A **84**, 011605(R) (2011);
R. Navarro, R. Carretero-González, P. J. Torres, P. G. Kevrekidis, D. J. Frantzeskakis, M. W. Ray, E. Altuntas, and D. S. Hall, “Dynamics of a few corotating vortices in Bose-Einstein condensates”, Phys. Rev. Lett. **110**, 225301 (2013).
78. L. C. Crasovan, G. Molina-Terriza, J. P. Torres, L. Torner, V. M. Pérez-García, and D. Mihalache, “Globally linked vortex clusters in trapped wave fields”, Phys. Rev. E **66**, 036612 (2002).
79. L. C. Crasovan, V. Vekslerchik, V. M. Pérez-García, J. P. Torres, D. Mihalache, and L. Torner, “Stable vortex dipoles in nonrotating Bose-Einstein condensates”, Phys. Rev. A **68**, 063609 (2003).
80. M. Möttönen, S. M. M. Virtanen, T. Isoshima, and M. M. Salomaa, “Stationary vortex clusters in nonrotating Bose-Einstein condensates”, Phys. Rev. A **71**, 033626 (2005).
81. V. Pietilä, M. Möttönen, T. Isoshima, J. A. M. Huhtamäki, and S. M. M. Virtanen, “Stability and dynamics of vortex clusters in nonrotated Bose-Einstein condensates”, Phys. Rev. A **74**, 023603 (2006).
82. W. Li, M. Haque, and S. Komineas, “Vortex dipole in a trapped two-dimensional Bose-Einstein condensate”, Phys. Rev. A **77**, 053610 (2008).
83. P. J. Torres, P. G. Kevrekidis, D. J. Frantzeskakis, R. Carretero-González, P. Schmelcher, and D. S. Hall, “Dynamics of vortex dipoles in confined Bose-Einstein condensates”, Phys. Lett. A **375**, 3044 (2011).
84. P. Kuopanportti, J. A. M. Huhtamäki, and M. Möttönen, “Size and dynamics of vortex dipoles in dilute Bose-Einstein condensates”, Phys. Rev. A **83**, 011603 (2011).
85. T. Aioi, T. Kadokura, and H. Saito, “Penetration of a vortex dipole across an interface of Bose-Einstein condensates”, Phys. Rev. A **85**, 023618 (2012).
86. V. M. Lashkin, A. S. Desyatnikov, E. A. Ostrovskaya, and Y. S. Kivshar, “Azimuthal vortex clusters in Bose-Einstein condensates”, Phys. Rev. A **85**, 013620 (2012).
87. L. C. Crasovan, V. M. Pérez-García, I. Danaila, D. Mihalache, and L. Torner, “Three-dimensional parallel vortex rings in Bose-Einstein condensates”, Phys. Rev. A **70**, 033605 (2004).
88. L. Wu, L. Li, J. F. Zhang, D. Mihalache, B. A. Malomed, and W. M. Liu, “Exact solutions of the Gross-Pitaesvkii equation for stable vortex modes in two-dimensional Bose-Einstein condensates”, Phys. Rev. A **81**, 061805 (2010).
89. Z. Dutton, M. Budde, C. Slowe, and L. V. Hau, “Observation of quantum shock waves created with ultra-compressed slow light pulses in a Bose-Einstein condensate”, Science **293**, 663 (2001).
90. N. S. Ginsberg, J. Brand, and L. V. Hau, “Observation of hybrid soliton vortex-ring structures in Bose-Einstein condensates”, Phys. Rev. Lett. **94**, 040403 (2005);
I. Shomroni, E. Lahoud, S. Levy, and J. Steinhauer, “Evidence for an oscillating soliton/vortex ring by density engineering of a Bose-Einstein condensate”, Nature Physics **5**, 193 (2009);
C. Becker, K. Sengstock, P. Schmelcher, P. G. Kevrekidis, and R. Carretero-González, “Inelastic collisions of solitary waves in anisotropic Bose-Einstein condensates: sling-shot events and expanding collision bubbles”, New J. Phys. **15**, 113028 (2013);
S. Donadello, S. Serafini, M. Tylutki, L. P. Pitaevskii, F. Dalfovo, G. Lamporesi, and G. Ferrari, “Observation of solitonic vortices in Bose-Einstein condensates”, Phys. Rev. Lett. **113**, 065302 (2014);
M. J.-H. Ku, W. Ji, B. Mukherjee, E. Guardado-Sanchez, L. W. Cheuk, T. Yefsah, and M. W. Zwierlein, “Motion of a solitonic vortex in the BEC-BCS crossover”, Phys. Rev. Lett. **113**, 065301 (2014).

91. F. S. Cataliotti, S. Burger, C. Fort, P. Maddaloni, F. Minardi, A. Trobettoni, A. Smerzi, and M. Inguscio, “Josephson junction arrays with Bose-Einstein condensates”, Science **293**, 843 (2001);
M. Albiez, R. Gati, J. Folling, S. Hunsmann, M. Critiani, and M. K. Oberthaler, “Direct observation of tunneling and nonlinear self-trapping in a single bosonic Josephson junction”, Phys. Rev. Lett. **95**, 010402 (2005);
S. Levy, E. Lahoud, I. Shomroni, and J. Steinhauer, “The a.c. and d.c. Josephson effects in a Bose-Einstein condensate”, Nature **449**, 579 (2007).
92. T. Zibold. E. Nicklas, C. Gross, M. K. Oberthaler, “Classical bifurcation at the transition from Rabi to Josephson dynamics”, Phys. Rev. Lett. **105**, 204101 (2010).
93. G. Dresselhaus, “Spin-orbit coupling effects in Zinc Blende structures”, Phys. Rev. **100**, 580 (1955);
Y. A. Bychkov and E. I. Rashba, “Oscillatory effects and the magnetic susceptibility of carriers in inversion layers”, J. Phys. C **17**, 6039 (1984).
94. Y. J. Lin, K. Jimenez-Garcia, and I. B. Spielman, “Spin-orbit-coupled Bose-Einstein condensates”, Nature **471**, 83 (2011);
J.-Y. Zhang, S.-C. Ji, Z. Chen, L. Zhang, Z.-D. Du, B. Yan, G.-S. Pan, B. Zhao, Y. Deng, H. Zhai, S. Chen, and J.-W. Pan, “Collective dipole oscillations of a spin-orbit coupled Bose-Einstein condensate”, Phys. Rev. Lett. **109**, 115301 (2012);
C. Hamner, C. Qu, Y. Zhang, J. Chang, M. Gong, C. Zhang, and P. Engels, “Dicke-type phase transition in a spin-orbit coupled Bose-Einstein condensate”, Nature Communications **5**, 4023 (2014);
A. J. Olson, S.-J. Wang, R. J. Niffenegger, C.-H. Li, C. H. Greene, and Y. P. Chen, “Tunable Landau-Zener transitions in a spin-orbit-coupled Bose-Einstein condensate”, Phys. Rev. A **90**, 013616 (2014).
95. M. Greiner, O. Mandel, T. Esslinger, T. W. Hansch, and I. Bloch,“Quantum phase transition from a superfluid to a Mott insulator in a gas of ultracold atoms”, Nature **415**, 39 (2002).
96. A. Zenesini, C. Sias, H. Lignier, Y. Singh, D. Ciampini, O. Morsch, R. Mannella, E. Arimondo, A. Tomadin, and S. Wimberger, “Resonant tunneling of Bose–Einstein condensates in optical lattices”, New J. Phys. **10**, 053038 (2008).
97. Y. J. Lin, R. L. Compton, K. Jimenez-Garcia, J. V. Porto, and I. B. Spielman, “Synthetic magnetic fields for ultracold neutral atoms”, Nature **462**, 628 (2009).
98. P. Hauke, F. M. Cucchietti, L. Tagliacozzo, I. Deutsch, and M. Lewenstein, “Can one trust quantum simulators?”, Rep. Prog. Phys. **75**, 082401 (2012);
I. Bloch, J. Dalibard, and S. Nascimbène, “Quantum simulations with ultracold quantum gases”, Nature Physics **8**, 267 (2012);
T. H. Johnson, S. R. Clark, and D. Jaksch, “What is a quantum simulator?”, EPJ Quantum Technology **1**, 10 (2014);
I. M. Georgescu, S. Ashhab, and F. Nori, “Quantum simulation”, Rev. Mod. Phys. **86**, 153 (2014).
99. N. Veretenov, Yu. Rozhdestvenskya, N. Rosanov, V. Smirnov, and S. Fedorov, “Interferometric precision measurements with Bose-Einstein condensate solitons formed by an optical lattice”, Eur. Phys. J. D **42**, 455 (2007);
J. Grond, U. Hohenester, I. Mazets, and J. Schmiedmayer, “Atom interferometry with trapped Bose-Einstein condensates: impact of atom-atom interactions”, New J. Phys. **12**, 065036 (2010);
J. Grond, U. Hohenester, J. Schmiedmayer, and A. Smerzi, “Mach-Zehnder interferometry with interacting trapped Bose-Einstein condensates”, Phys. Rev. A **84**, 023619 (2011).
100. J. Cuevas, P. G. Kevrekidis, B. A. Malomed, P. Dyke, and R. G. Hulet, “Interactions of solitons with a Gaussian barrier: Splitting and recombination in quasi-one-dimensional and three-dimensional settings”, New J. Phys. **15**, 063006 (2013).

101. G. D. McDonald, C. C. N. Kuhn, K. S. Hardman, S. Bennetts, P. J. Everitt, P. A. Altin, J. E. Debs, J. D. Close, and N. P. Robin, "Bright solitonic matter-wave interferometer", Phys. Rev. Lett. **113**, 013002 (2014).
102. See the presentation of Mark Kasevich at the NASA Quantum Future Technologies Conference, Jan. 17-21 2012, available at: *http://quantum.nasa.gov/agenda.html*.
103. A. D. Martin and J. Ruostekoski, "Quantum dynamics of atomic bright solitons under splitting and recollision, and implications for interferometry", New J. Phys. **14**, 043040 (2012).
104. A. Negretti and C. Henkel, "Enhanced phase sensitivity and soliton formation in an integrated BEC interferometer", J. Phys. B: At. Mol. Opt. Phys. **37**, L385 (2004).
105. C. Gross, T. Zibold, E. Nicklas, J. Esteve, and M. K. Oberthaler, "Nonlinear atom interferometer surpasses classical precision limit", Nature **464**, 1165 (2010).
106. C. Gross, H. Strobel, E. Nicklas, T. Zibold, N. Bar-Gill, G. Kurizki, and M. K. Oberthaler, "Atomic homodyne detection of continuous variable entangled twin-atom states", Nature **480**, 219 (2011).
107. C. Barceló, S. Liberati, and M. Visser, "Analogue gravity", arXiv:gr-qc/0505065v3.
108. T. R. Slatyer and C. M. Savage, "Superradiant scattering from a hydrodynamic vortex", Class. Quantum Grav. **22**, 3833 (2005).
109. M. Sadgrove, S. Kumar, and K. Nakagawa, "Enhanced factoring with a Bose-Einstein condensate", Phys. Rev. Lett. **101**, 180502 (2008).
110. P. W. Shor in *Proceedings of the 35th Annual Symposium on Foundations of Computer Science*, Santa Fe, New Mexico, 1994, edited by Shafi Goldwasser (IEEE Computer Society Press, Los Alamitos, 1994), p. 124.
111. M. A. Nielsen and I. L. Chuang, *Quantum Computing and Quantum Information* (Cambridge University Press, Cambridge, 2000).
112. J. V. Porto, S. Rolston, B. L. Tolra, C. J. Williams, and W. D. Phillips, "Quantum information with neutral atoms as qubits", Phil. Trans. Royal Soc. L. A: Math. Phys. Eng. Sci. **361**, 1417 (2003);
J. K. Pachos and P. L. Knight, "Quantum computation with a one-dimensional optical lattice", Phys. Rev. Lett. **91**, 107902 (2003);
T. Calarco, U. Dorner, P. S. Julienne, C. J. Williams, and P. Zoller, "Quantum computations with atoms in optical lattices: Marker qubits and molecular interactions", Phys. Rev. A **70**, 012306 (2004);
T. Byrnes, K. Wen, and Y. Yamamoto, "Macroscopic quantum computation using Bose-Einstein condensates", Phys. Rev. **85**, 040306 (2012).
113. D. Jaksch and P. Zoller, "The cold atom Hubbard toolbox", Annals of Physics **315**, 52 (2005).
114. R. A. Pepino, J. Cooper, D. Z. Anderson, and M. J. Holland, "Atomtronic circuits of diodes and transistors", Phys. Rev. Lett. **103**, 140405 (2009).
115. M. Lewenstein and B. A. Malomed, "Entanglement generation by collisions of quantum solitons in the Born's approximation", New J. Phys. **11**, 113014 (2009).
116. W. Guerin, J.-F. Riou, J. P. Gaebler, V. Josse, P. Bouyer, and A. Aspect, "Guided quasicontinuous atom laser", Phys. Rev. Lett. **97**, 200402 (2006).
117. N. P. Robins, C. Figl, M. Jeppesen, G. R. Dennis, and J. D. Close, "A pumped atom laser", Nature Physics **4**, 731 (2008).
118. N. P. Robins, P. A. Altin, J. E. Debs, and J. D. Close, "Atom lasers: Production, properties and prospects for precision inertial measurement", Phys. Rep. **529**, 265 (2013).
119. L. D. Carr and J. Brand, "Pulsed atomic soliton laser", Phys. Rev. A 70, 033607 (2004);
P. Y. P. Chen and B. A. Malomed, "Stable circulation modes in a dual-core matter-wave laser", J.

Phys. B: At. Mol. Opt. Phys. **39**, 2803 (2006);
A. V. Carpentier, H. Michinel, M. I. Rodas-Verde, and V. M. Pérez-García, "Analysis of an atom laser based on the spatial control of the scattering length", Phys. Rev. A **74**, 013619 (2006);
T. Wasak, V. V. Konotop, and M. Trippenbach, "Atom laser based on four-wave mixing with Bose-Einstein condensates in nonlinear lattices", Phys. Rev. A **74**, 013619 (2013).

120. B. A. Malomed, *Soliton Management in Periodic Systems* (Springer, New York, 2006).
121. S. Inouye, M. R. Andrews, J. Stenger, H. J. Miesner, D. M. Stamper-Kurn, and W. Ketterle, "Observation of Feshbach resonances in a Bose-Einstein condensate", Nature **392**, 151 (1998).
122. P. O. Fedichev, Yu. Kagan, G. V. Shlyapnikov, and J. T. M. Walraven, "Influence of nearly resonant light on the scattering length in low-temperature atomic gases", Phys. Rev. Lett. **77**, 2913 (1996);
M. Yan, B. J. DeSalvo, B. Ramachandhran, H. Pu, and T. C. Killian, "Controlling condensate collapse and expansion with an optical Feshbach resonance", Phys. Rev. Lett. **110**, 123201 (2013).
123. D. M. Bauer, M. Lettner, C. Vo, G. Rempe, and S. Dürr, "Control of a magnetic Feshbach resonance with laser light", Nature Physics **5**, 339 (2009).
124. G. Roati, M. Zaccanti, C. D'Errico, J. Catani, M. Modugno, A. Simoni, M. Inguscio, and G. Modugno, "^{39}K Bose-Einstein condensate with tunable interactions", Phys. Rev. Lett. **99**, 010403 (2007);
S. E. Pollack, D. Dries, M. Junker, Y. P. Chen, T. A. Corcovilos, and R. G. Hulet, "Extreme tunability of interactions in a ^{7}Li Bose-Einstein condensate", Phys. Rev. Lett. **102**, 090402 (2009).
125. V. M. Pérez-García, H. Michinel, J. I. Cirac, M. Lewenstein, and P. Zoller, "Low energy excitations of a Bose-Einstein condensate: A time-dependent variational analysis", Phys. Rev. Lett. **77**, 5320 (1996);
V. M. Pérez-García, H. Michinel, J. I. Cirac, M. Lewenstein, and P. Zoller, "Dynamics of Bose-Einstein condensates: Variational solutions of the Gross-Pitaevskii equations", Phys. Rev. A **56**, 1424 (1997);
A. Trombettoni and A. Smerzi, "Variational dynamics of Bose-Einstein condensates in deep optical lattices", J. Phys. B: At. Mol. Opt. Phys. **34**, 4711 (2001);
F. K. Abdullaev, A. Gammal, and L. Tomio, "Dynamics of bright matter-wave solitons in Bose-Einstein condensates inhomogeneous scattering length", J. Phys. B: At. Mol. Opt. Phys. **37**, 635 (2004);
V. M. Pérez-García, "Self-similar solutions and collective-coordinate method for nonlinear Schrödinger equations", Physica D **191**, 211 (2004).
126. B. A. Malomed, "Variational methods in nonlinear fiber optics and related field", Progr. Optics **43**, 71 (2002).
127. D. L. Feder, C. W. Clark, and B. I. Schneider, "Vortex stability of interacting Bose-Einstein condensates confined in anisotropic harmonic traps", Phys. Rev. Lett. **82**, 4956 (1999).
128. R. Driben, Y. V. Kartashov, B. A. Malomed, T. Meier, and L. Torner, "Soliton gyroscopes in media with spatially growing repulsive nonlinearity", Phys. Rev. Lett. **112**, 020404 (2014).
129. M. L. Chiofalo, S. Succi, and M. P. Tosi, "Ground state of trapped interacting Bose-Einstein condensates by an explicit imaginary-time algorithm", Phys. Rev. E **62**, 7438 (2000);
W. Bao and Q. Du, "Computing the ground state solution of Bose-Einstein condensates by a normalized gradient flow", SIAM J. Sci. Comput. **25**, 1674 (2004).
130. J. Crank and P. Nicolson, "A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type", Proc. Camb. Phil. Soc. **43**, 50 (1947).
131. P. Muruganandam and S. K. Adhikari, "Fortran programs for the time-dependent Gross-Pitaevskii equation in a fully anisotropic trap", Comput. Phys. Commun. **180**, 1888 (2009);
D. Vudragović, I. Vidanović, A. Balaž, P. Muruganandam, and S. K Adhikari, "C programs for

solving the time-dependent Gross-Pitaevskii equation in a fully anisotropic trap", Comput. Phys. Commun. **183**, 2021 (2012).

132. A. Balaž, A. Bogojević, I. Vidanović, and A. Pelster, "Recursive Schrödinger equation approach to faster converging path integrals", Phys. Rev. E **79**, 036701 (2009);
A. Bogojević, I. Vidanović, A. Balaž, and A. Belić, "Fast convergence of path integrals for many-body systems", Phys. Lett. A **372**, 3341 (2008);
D. Stojiljković, A. Bogojević, and A. Balaž, "Efficient calculation of energy spectra using path integrals", Phys. Lett. A **360**, 205 (2006);
J. Grujić, A. Bogojević, and A. Balaž, "Energy estimators and calculation of energy expectation values in the path integral formalism", Phys. Lett. A **360**, 217 (2006);
A. Bogojević, A. Balaž, and A. Belić, "Jaggedness of path integral trajectories", Phys. Lett. A **345**, 258 (2005);
A. Bogojević, A. Balaž, and A. Belić, "Generalization of Euler's summation formula to path integrals", Phys. Lett. A **344**, 84 (2005);
A. Bogojević, A. Balaž, and A. Belić, "Asymptotic properties of path integral ideals", Phys. Rev. E **72**, 36128 (2005);
A. Bogojević, A. Balaž, and A. Belić, "Systematic speedup of path integrals of a generic N-fold discretized theory", Phys. Rev. B **72**, 064302 (2005);
A. Bogojević, A. Balaž, and A. Belić, "Systematically accelerated convergence of path integrals", Phys. Rev. Lett. **94**, 180403 (2005).

133. A. Balaž, I. Vidanović, D. Stojiljković, D. Vudragović, A. Belić, and A. Bogojević, "SPEEDUP code for calculation of transition amplitudes via the effective action approach", Commun. Comput. Phys. **11** 739 (2012);
A. Balaž, I. Vidanović, A. Bogojević, A. Belić, and A. Pelster, "Fast converging path integrals for time-dependent potentials: II. Generalization to many-body systems and real-time formalism", J. Stat. Mech. P03005 (2011);
A. Balaž, I. Vidanović, A. Bogojević, A. Belić, and A. Pelster, "Fast converging path integrals for time-dependent potentials: I. Recursive calculation of short-time expansion of the propagator", J. Stat. Mech. P03004 (2011);
A. Balaž, I. Vidanović, A. Bogojević, and A. Pelster, "Ultra-fast converging path-integral approach for rotating ideal Bose-Einstein condensates", Phys. Lett. A **374**, 1539 (2010);
I. Vidanović, A. Bogojević, A. Balaž, and A. Belić, "Properties of quantum systems via diagonalization of transition amplitudes. II. Systematic improvements of short-time propagation", Phys. Rev. E **80**, 066706 (2009);
I. Vidanović, A. Bogojević, and A. Belić, "Properties of quantum systems via diagonalization of transition amplitudes. I. Discretization Effects", Phys. Rev. E **80**, 066705 (2009).

134. S. K. Adhikari, "Numerical solution of the two-dimensional Gross-Pitaevskii equation for trapped interacting atoms", Phys. Lett. A **265**, 91 (2000).

135. S. K. Adhikari, "Numerical study of the spherically symmetric Gross-Pitaevskii equation in two space dimensions", Phys. Rev. E **62**, 2937 (2000).

136. S. K. Adhikari and P. Muruganandam, "Bose-Einstein condensation dynamics from the numerical solution of the Gross-Pitaevskii equation", J. Phys. B: At. Mol. Opt. Phys. **35**, 2831 (2002).

137. P. Muruganandam and S. K. Adhikari, "Bose-Einstein condensation dynamics in three dimensions by the pseudospectral and finite-difference methods", J. Phys. B: At. Mol. Opt. Phys. **36**, 2501 (2003).

138. R. Ozeri, N. Katz, J. Steinhauer, and N. Davidson, "Bulk Bogoliubov excitations in a Bose-Einstein condensate", Rev. Mod. Phys. **77**, 187 (2005);
A. M. Rey, B. L. Hu, E. Calzetta, A. Roura, and C. W. Clark, "Nonequilibrium dynamics of op-

tical lattice-loaded Bose-Einstein-condensate atoms: Beyond the Hartree-Fock-Bogoliubov approximation", Phys. Rev. A **69**, 033610 (2004);
L. Salasnich, "Beyond mean-field theory for attractive bosons under transverse harmonic confinement", J. Phys. B: At. Mol. Opt. Phys. **39**, 1743 (2006).

139. C. J. Myatt, E. A. Burt, R. W. Ghrist, E. A. Cornell, and C. E. Wieman, "Production of two overlapping Bose-Einstein condensates by sympathetic cooling", Phys. Rev. Lett. **78**, 586 (1997).
140. J. Stenger, S. Inouye, D. M. Stamper-Kurn, H.-J. Miesner, A. P. Chikkatur, and W. Ketterle, "Spin domains in ground-state Bose-Einstein condensates", Nature **396**, 345 (1998).
141. S. B. Papp, J. M. Pino, and C. E. Wieman, "Tunable miscibility in a dual-species Bose-Einstein condensate", Phys. Rev. Lett. **101**, 040402 (2008).
142. D. J. McCarron, H. W. Cho, D. L. Jenkin, M. P. Köppinger, and S. L. Cornish, "Dual-species Bose-Einstein condensate of ^{87}Rb and ^{133}Cs", Phys. Rev. A **84**, 011603(R) (2011).
143. D. S. Hall, M. R. Matthews, J. R. Ensher, C. E. Wieman, and E. A. Cornell, "Dynamics of component separation in a binary mixture of Bose-Einstein condensates", Phys. Rev. Lett. **81**, 1539 (1998).
144. V. P. Mineev, "The theory of the solution of two near-ideal Bose gases", Zh. Eksp. Teor. Fiz. **67** 263 (1974) [Sov. Phys. JETP **40** 132 (1974)].
145. L. Wen, W. M. Liu, Y. Cai, J. M. Zhang, and J. Hu, "Controlling phase separation of a two-component Bose-Einstein condensate by confinement", Phys. Rev. A **85**, 043602 (2012).
146. M. Trippenbach, K. Góral, K. Rzażewski, B. Malomed, and Y. B. Band, "Structure of binary Bose-Einstein condensates", J. Phys. B: At. Mol. Opt. **33**, 4017 (2000);
R. W. Pattinson, T. P. Billam, S. A. Gardiner, D. J. McCarron, H. W. Cho, S. L. Cornish, N. G. Parker, and N. P. Proukakis, "Equilibrium solutions for immiscible two-species Bose-Einstein condensates in perturbed harmonic traps", Phys. Rev. A **87**, 013625 (2013).
147. N. Gemelke, X. Zhang, C.-L. Hung, and C. Chin, "In situ observation of incompressible Mott-insulating domains in ultracold atomic gases", Nature **460**, 995 (2009);
S. Tojo, Y. Taguchi, Y. Masuyama, T. Hayashi, H. Saito, and T. Hirano, "Controlling phase separation of binary Bose-Einstein condensates via mixed-spin-channel Feshbach resonance", Phys. Rev. A **82**, 033609 (2010).
148. A. Balaz and A. I. Nicolin, "Faraday waves in binary nonmiscible Bose-Einstein condensates", Phys. Rev. A **85**, 023613 (2012).
149. L. Li, Z. Li, B. A. Malomed, D. Mihalache, and W. M. Liu, "Exact soliton solutions and nonlinear modulational instability in spinor Bose-Einstein condensates", Phys. Rev. A **72**, 033611 (2005).
150. L. Li, B. A. Malomed, D. Mihalache, and W. M. Liu, "Exact soliton-on-plane-wave solutions for two-component Bose-Einstein condensates", Phys. Rev. E **73**, 066610 (2006).
151. R. J. Ballagh, K. Burnett, and T. F. Scott, "Theory of an output coupler for Bose-Einstein condensed atoms", Phys. Rev. Lett. **78**, 1607 (1997).
152. M. I. Merhasin, B. A. Malomed, and R. Driben, "Transition to miscibility in a binary Bose-Einstein condensate induced by linear coupling", J. Phys. B: At. Mol. Opt. Phys. **38**, 877 (2005).
153. E. Nicklas, H. Strobel, T. Zibold, C. Gross, B. A. Malomed, P. G. Kevrekidis, and M. K. Oberthaler, "Rabi flopping induces spatial demixing dynamics", Phys. Rev. Lett. **107**, 193001 (2011).
154. D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, "Cold bosonic atoms in optical lattices", Phys. Rev. Lett. **81**, 3108 (1998).
155. B. P. Anderson and M. A. Kasevich, "Macroscopic quantum interference from atomic tunnel arrays", Science **282**, 1686 (1998).
156. I. Bloch, "Ultracold quantum gases in optical lattices", Nature Physics **11**, 23 (2005).

157. P. Windpassinger and K. Sengstock, "Engineering novel optical lattices", Rep. Prog. Phys. **76**, 086401 (2013).
158. N. K. Efremidis, S. Sears, D. N. Christodoulides, J. W. Fleischer, and M. Segev, "Discrete solitons in photorefractive optically induced photonic lattices", Phys. Rev. E **66**, 046602 (2002);
N. K. Efremidis, J. Hudock, D. N. Christodoulides, J. W. Fleischer, O. Cohen, and M. Segev, "Two-Dimensional optical lattice solitons", Phys. Rev. Lett. **91**, 213906 (2003).
159. D. N. Neshev, T. J. Alexander, E. A. Ostrovskaya, Y. S. Kivshar, H. Martin, I. Makasyuk, and Z. Chen, "Observation of discrete vortex solitons in optically induced photonic lattices", Phys. Rev. Lett. **92**, 123903 (2004);
J. W. Fleischer, G. Bartal, O. Cohen, O. Manela, M. Segev, J. Hudock, and D. N. Christodoulides, "Observation of vortex-ring *discrete* solitons in 2D photonic lattices", Phys. Rev. Lett. **92**, 123904 (2004).
160. J. W. Fleischer, G. Bartal, O. Cohen, T. Schwartz, O. Manela, B. Freedman, M. Segev, H. Buljan, and N. K. Efremidis, "Spatial photonics in nonlinear waveguide arrays", Opt. Exp. **13**, 1780 (2005);
F. Lederer, G. I. Stegeman, D. N. Christodoulides, G. Assanto, M. Segev, and Y. Silberberg, "Discrete solitons in optics", Phys. Rep. **463**, 1 (2008);
Y. V. Kartashov, V. A. Vysloukh, and L. Torner, "Soliton shape and mobility control in optical lattices", Progr. Opt. **52**, 63 (2009).
161. A. Trombettoni and A. Smerzi, "Discrete solitons and breathers with dilute Bose-Einstein condensates", Phys. Rev. Lett. **86**, 2353 (2001);
F. Kh. Abdullaev, B. B. Baizakov, S. A. Darmanyan, V. V. Konotop, and M. Salerno, "Nonlinear excitations in arrays of Bose-Einstein condensates", Phys. Rev. A **64**, 043606 (2001);
N. K. Efremidis and D. N. Christodoulides, "Lattice solitons in Bose-Einstein condensates", Phys. Rev. A **67**, 063608 (2003).
162. P. G. Kevrekidis, *The Discrete Nonlinear Schrödinger Equation: Mathematical Analysis, Numerical Computations, and Physical Perspectives* (Springer, Berlin and Heidelberg, 2009).
163. M. Johansson, S. Aubry, Yu. B. Gaididei, P. L. Christiansen, and K. Ø. Rasmussen, "Dynamics of breathers in discrete nonlinear Schrödinger models", Physica D **119**, 115 (1998);
B. A. Malomed and P. G. Kevrekidis, "Discrete vortex solitons", Phys. Rev. E **64**, 026601 (2001).
164. P. G. Kevrekidis, R. Carretero-González, D. J. Frantzeskakis, B. A. Malomed, and F. K. Diakonos, "Skyrmion-like states in two- and three-dimensional dynamical lattices", Phys. Rev. E **75**, 026603 (2007).
165. R. Carretero-González, P. G. Kevrekidis, B. A. Malomed, and D. J. Frantzeskakis, "Three-dimensional nonlinear lattices: From oblique vortices and octupoles to discrete diamonds and vortex cubes", Phys. Rev. Lett. **94**, 203901 (2005);
M. Lukas, D. E. Pelinovsky, and P. G. Kevrekidis, "Lyapunov-Schmidt reduction algorithm for three-dimensional discrete vortices", Physica D **237**, 339 (2008).
166. M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, "Boson localization and the superfluid-insulator transition", Phys. Rev. B **40**, 546 (1989).
167. S. R. White, "Density matrix formulation for quantum renormalization groups", Phys. Rev. Lett. **69**, 2863 (1992).
168. V. M. Pérez-García, H. Michinel, and H. Herrero, "Bose-Einstein solitons in highly asymmetric traps", Phys. Rev. A **57**, 3837 (1998).
169. Y. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals* (Academic Press, San Diego, 2003).
170. V. E. Zakharov, S. V. Manakov, S. P. Novikov, and L. P. Pitaevskii, *Theory of Solitons: Methods of the Inverse Scattering Transform* (Moscow, Nauka Publishers, 1980; English translation:

Consultants Bureau, New York, 1984).

171. S. Gupta, K. W. Murch, K. L. Moore, T. P. Purdy, and D. M. Stamper-Kurn, "Bose-Einstein condensation in a circular Waveguide", Phys. Rev. Lett. **95**, 143201 (2005);
A. S. Arnold, C. S. Garvie, and E. Riis, "Large magnetic storage ring for Bose-Einstein condensates", Phys. Rev. A **73**, 041606(R) (2006);
I. Lesanovsky and W. von Klitzing, "Time-averaged adiabatic potentials: Versatile matter-wave guides and atom traps", Phys. Rev. Lett. **99**, 083001 (2007);
C. Ryu, M. F. Andersen, P. Cladé, V. Natarajan, K. Helmerson, and W. D. Phillips, "Observation of persistent flow of a Bose-Einstein condensate in a toroidal trap", Phys. Rev. Lett. **99**, 260401 (2007).
172. L. Salasnich, A. Parola, and L. Reatto, "Effective wave equations for the dynamics of cigar-shaped and disk-shaped Bose condensates", Phys. Rev. A **65**, 043614 (2002).
173. A. Muñoz Mateo and V. Delgado, "Effective mean-field equations for cigar-shaped and disk-shaped Bose-Einstein condensates", Phys. Rev. A **77**, 013617 (2008).
174. R. Y. Chiao, E. Garmire, and C. H. Townes, "Self-trapping of optical beams", Phys. Rev. Lett. **13**, 479 (1964).
175. M. Desaix, D. Anderson, and M. Lisak, "Variational approach to collapse of optical pulses", J. Opt. Soc. Am. B **8**, 2082 (1991).
176. N. G. Vakhitov and A. A. Kolokolov, "Stationary solutions of the wave equation in a medium with nonlinearity saturation", Izv. Vuz. Radiofiz. **16**, 1020 (1973) [Soviet Radiophys. and Quantum Electronics **16**, 783 (1975)].
177. L. Bergé, "Wave collapse in physics: principles and applications to light and plasma waves", Phys. Rep. **303**, 260 (1998).
178. E. A. Donley, N. R. Claussen, S. L. Cornish, J. L. Roberts, E. A. Cornell, and C. E. Wieman, "Dynamics of collapsing and exploding Bose-Einstein condensates", Nature **412**, 295 (2001).
179. B. DeMarco and D. S. Jin, "Onset of Fermi degeneracy in a trapped atomic gas", Science **285**, 1703 (1999).
180. M. Greiner, C. A. Regal, and D. S. Jin, "Emergence of a molecular Bose-Einstein condensate from a Fermi gas", Nature **426**, 537 (2003).
181. D. A. Butts and D. S. Rokhsar, "Trapped Fermi gases", Phys. Rev. A **55**, 4346 (1997);
T. Karpiuk, M. Brewczyk, and K. Rzazewski, "Ground state of two-component degenerate fermionic gases", Phys. Rev. A **69**, 043603 (2004);
M. Salerno, "Matter-wave quantum dots and antidots in ultracold atomic Bose-Fermi mixtures", Phys. Rev. A **72**, 063602 (2005).
182. S. Giorgini, L. P. Pitaevskii, and S. Stringari, "Theory of ultracold atomic Fermi gases", Rev. Mod. Phys. **80**, 1215 (2008).
183. K. Huang and C. N. Yang, "Quantum-mechanical many-body problem with hard-sphere interaction", Phys. Rev. **105**, 767 (1957);
T. D. Lee and C. N. Yang, "Many-body problem in quantum mechanics and quantum statistical mechanics", Phys. Rev. **105**, 1119 (1957).
184. S. K. Adhikari, "Superfluid Fermi-Fermi mixture: Phase diagram, stability, and soliton formation", Phys. Rev. A **76**, 053609 (2007);
S. Adhikari and B. A. Malomed, "Tightly bound gap solitons in a Fermi gas", Europhys. Lett. **79**, 50003 (2007);
S. K. Adhikari, "Nonlinear Schrödinger equation for a superfluid Fermi gas in the BCS-BEC crossover", Phys. Rev. A **77**, 045602 (2008).
185. L. Salasnich and F. Toigo, "Extended Thomas-Fermi density functional for the unitary Fermi

gas", Phys. Rev. A **78**, 053626 (2008);
L. Salasnich, "Hydrodynamics of Bose and Fermi superfluids at zero temperature: the superfluid nonlinear Schrödinger equation", Laser Phys. **19**, 642 (2009);
S. K. Adhikari and L. Salasnich, "Effective nonlinear Schrödinger equations for cigar-shaped and disk-shaped Fermi superfluids at unitarity", New J. Phys. **11**, 023011 (2009).

186. M. Antezza, F. Dalfovo, L. P. Pitaevskii, and S. Stringari, "Dark solitons in a superfluid Fermi gas", Phys. Rev. A **76**, 043610 (2007);
R. G. Scott, F. Dalfovo, L. P. Pitaevskii, and S. Stringari, "Dynamics of dark solitons in a trapped superfluid Fermi gas", Phys. Rev. Lett. **106**, 185301 (2011).
187. A. Spuntarelli, L. D. Carr, P. Pieri, and G. C. Strinati, "Gray solitons in a strongly interacting superfluid Fermi gas", New J. Phys. **13**, 035010 (2011);
D. Yan, P. G. Kevrekidis, and D. J. Frantzeskakis, "Dark Solitons in a Gross-Pitaevskii equation with a power-law nonlinearity: Application to ultracold Fermi gases near the Bose-Einstein condensation regime", J. Phys. A: Math. Theor. **44**, 415202 (2011);
R. G. Scott, F. Dalfovo, L. P. Pitaevskii, S. Stringari, O. Fialko, R. Liao, and J. Brand, "The decay and collisions of dark solitons in superfluid Fermi gases", New J. Phys. **14**, 023044 (2012).
188. T. Yefsah, A. T. Sommer, M. J. H. Ku, L. W. Cheuk, W. Ji, W. S. Bakr, and M. W. Zwierlein, "Heavy solitons in a Fermionic superfluid", Nature **499**, 426 (2013).
189. M. J. H. Ku, W. Ji, B. Mukherjee, E. Guardado-Sanchez, L. W. Cheuk, T. Yefsah, and M. W. Zwierlein, " Motion of a solitonic vortex in the BEC-BCS crossover", Phys. Rev. Lett. **113**, 065301 (2014).
190. L. Tonks, "The complete equation of state of one, two and three-dimensional gases of hard elastic spheres", Phys. Rev. **50**, 955 (1936);
M. Girardeau, "Relationship between systems of impenetrable Bosons and Fermions in one dimension", J. Math. Phys. **1**, 516 (1960);
E. H. Lieb and W. Liniger, "Exact analysis of an interacting Bose gas. I. The general solution and the ground state", Phys. Rev. **130**, 1605 (1963).
191. M. D. Girardeau and E. M. Wright, "Dark solitons in a one-dimensional condensate of hard core bosons", Phys. Rev. Lett. **84**, 5691 (2000).
192. B. Paredes, A. Widera, V. Murg, O. Mandel, S. Fölling, I. Cirac, G. V. Shlyapnikov, T. W. Hänsch, and I. Bloch, "Tonks-Girardeau gas of ultracold atoms in an optical lattice", Nature **429**, 277 (2004);
T. Kinoshita, T. Wenger, and D. S. Weiss, "Observation of a one-dimensional Tonks-Girardeau gas", Science **305**, 1125 (2004).
193. E. B. Kolomeisky, T. J. Newman, J. P. Straley, and X. Qi, "Low-dimensional Bose liquids: beyond the Gross-Pitaevskii approximation", Phys. Rev. Lett. **85**, 1146 (2000).
194. M. D. Girardeau and E. M. Wright, "Breakdown of time-dependent mean-field theory for a one-dimensional condensate of impenetrable bosons", Phys. Rev. Lett. **84**, 5239 (2000).
195. B. B. Baizakov, F. Kh. Abdullaev, B. A. Malomed, and M. Salerno, "Solitons in Tonks-Girardeau gas with dipolar interactions", J. Phys. B: At. Mol. Opt. Phys. **42**, 175302 (2009).
196. D. J. Frantzeskakis, N. P. Proukakis, and P. G. Kevrekidis, "Dynamics of shallow dark solitons in a trapped gas of impenetrable bosons", Phys. Rev. A **70**, 015601 (2004).
197. Th. Busch and G. Huyet, "Low-density, one-dimensional quantum gases in a split trap", J. Phys. B: At. Mol. Opt. Phys. **36**, 2553 (2003).
198. B. B. Baizakov, G. Filatrella, B. A. Malomed, and M. Salerno, "Double parametric resonance for matter-wave solitons in a time-modulated trap", Phys. Rev. E **71**, 036619 (2005).
199. H. Lignier, C. Sias, D. Ciampini, Y. Singh, A. Zenesini, O. Morsch, and E. Arimondo, "Dynami-

cal control of matter-wave tunneling in periodic potentials", Phys. Rev. Lett. **99**, 220403 (2007).

200. P. Engels, C. Atherton, and M. A. Hoefer, "Observation of Faraday waves in a Bose-Einstein condensate", Phys. Rev. Lett. **98**, 095301 (2007).
201. H. Abe, T. Ueda, M. Morikawa, Y. Saitoh, R. Nomura, and Y. Okuda, "Faraday instability of superfluid surface", Phys. Rev. E **76**, 046305 (2007);
T. Ueda, H. Abe, Y. Saitoh, R. Nomura, and Y. Okuda, "Faraday instability on a free surface of superfluid ^{4}He", J. Low Temp. Phys. **148**, 553 (2007).
202. K. Staliunas, S. Longhi, and G. J. de Valcárcel, "Faraday patterns in Bose-Einstein condensates", Phys. Rev. Lett. **89**, 210406 (2002).
203. K. Staliunas, S. Longhi, and G. J. de Valcárcel, "Faraday patterns in low-dimensional Bose-Einstein condensates", Phys. Rev. A **70**, 011601(R) (2004);
M. Kramer, C. Tozzo, and F. Dalfovo, "Parametric excitation of a Bose-Einstein condensate in a one-dimensional optical lattice", Phys. Rev. A **71**, 061602(R) (2005);
M. Modugno, C. Tozzo, and F. Dalfovo, "Detecting phonons and persistent currents in toroidal Bose-Einstein condensates by means of pattern formation", Phys. Rev. A **74**, 061601(R) (2006).
204. A. I. Nicolin, R. Carretero-González, and P. G. Kevrekidis, "Faraday waves in Bose-Einstein condensates", Phys. Rev. A **76**, 063609 (2007);
A. I. Nicolin and M. C. Raportaru, "Faraday waves in high-density cigar-shaped Bose-Einstein condensates", Physica A **389**, 4663 (2010);
A. I. Nicolin and M. C. Raportaru, "Faraday waves in one-dimensional Bose-Einstein condensates", Proc. Romanian Acad. A **12**, 209 (2011);
A. I. Nicolin, "Faraday waves in Bose-Einstein condensates subject to anisotropic transverse confinement", Rom. Rep. Phys. **63**, 1329 (2011);
A. I. Nicolin, "Variational treatment of Faraday waves in inhomogeneous Bose-Einstein condensates", Physica A **391**, 1062 (2012).
205. R. Nath and L. Santos, "Faraday patterns in two-dimensional dipolar Bose-Einstein condensates", Phys. Rev. A **81**, 033626 (2010);
L. Kazimierz, N. Rejish, and L. Santos, "Faraday patterns in coupled one-dimensional dipolar condensates", Phys. Rev. A **86**, 023620 (2012);
A. I. Nicolin, "Density waves in dipolar Bose-Einstein condensates", Proc. Romanian Acad. A **14**, 35 (2013).
206. M. C. Raportaru, "Formation of Faraday waves in driven Bose-Einstein condensates", Rom. Rep. Phys. **64**, 105 (2012).
207. F. Kh. Abdullaev, M. Ogren, and M. P. Sorensen, "Faraday waves in quasi-one-dimensional superfluid Fermi-Bose mixtures", Phys. Rev. A **87**, 023616 (2013).
208. P. Capuzzi and P. Vignolo, "Faraday waves in elongated superfluid fermionic clouds", Phys. Rev. A **78**, 043613 (2008);
R. A. Tang, H. C. Li, and J. K., Xue, "Faraday instability and Faraday patterns in a superfluid Fermi gas", J. Phys. B **44**, 115303 (2011).
209. A. I. Nicolin, "Resonant wave formation in Bose-Einstein condensates", Phys. Rev. E **84**, 056202 (2011).
210. P. Capuzzi, M. Gattobigio, and P. Vignolo, "Suppression of Faraday waves in a Bose-Einstein condensate in the presence of an optical lattice", Phys. Rev. A **83**, 013603 (2011).
211. K. Staliunas, "Removal of excitations of Bose-Einstein condensates by space- and time-modulated potentials", Phys. Rev. A **84**, 013626 (2011).
212. V. S. Bagnato *et al.*, *Characteristics and Perspectives of Quantum Turbulence in Atomic Bose-Einstein Condensates*, in *Physics of Quantum Fluids. New Trends and Hot Topics in Atomic and Polariton Condensates*, pp. 301-314, edited by A. Bramati and M. Modugno (Springer, New

York, 2013).

213. L. Salasnich, N. Manini, F. Bonelli, M. Korbman, and A. Parola, "Self-induced density modulations in the free expansion of Bose-Einstein condensates", Phys. Rev A **75**, 043616 (2007).
214. A. Imambekov, I. E. Mazets, D. S. Petrov, V. Gritsev, S. Manz, S. Hofferberth, T. Schumm, E. Demler, and J. Schmiedmayer, "Density ripples in expanding low-dimensional gases as a probe of correlations", Phys. Rev. A **80**, 033604 (2009).
215. S. Manz, R. Bücker, T. Betz, Ch. Koller, S. Hofferberth, I. E. Mazets, A. Imambekov, E. Demler, A. Perrin, J. Schmiedmayer, and T. Schumm, "Two-point density correlations of quasicondensates in free expansion", Phys. Rev. A **81**, 031610(R) (2010).
216. J. Kronjäger, C. Becker, P. Soltan-Panahi, K. Bongs, and K. Sengstock, "Spontaneous pattern formation in an antiferromagnetic quantum gas", Phys. Rev. Lett. **105**, 090402 (2010).
217. Y. V. Kartashov, B. A. Malomed, and L. Torner, "Solitons in nonlinear lattices", Rev. Mod. Phys. **83**, 247 (2011).
218. G. Theocharis, P. Schmelcher, P. G. Kevrekidis, and D. J. Frantzeskakis, "Matter-wave solitons of collisionally inhomogeneous condensates", Phys. Rev. A **72**, 033614 (2005);
G. Theocharis, P. Schmelcher, P. G. Kevrekidis, and D. J. Frantzeskakis, "Dynamical trapping and transmission of matter-wave solitons in a collisionally inhomogeneous environment", Phys. Rev. A **74**, 053614 (2006).
219. F. Kh. Abdullaev, J. G. Caputo, R. A. Kraenkel, and B. A. Malomed, "Controlling collapse in Bose-Einstein condensation by temporal modulation of the scattering length", Phys. Rev. A **67**, 013605 (2003).
H. Saito and M. Ueda, "Dynamically stabilized bright solitons in a two-dimensional Bose-Einstein condensate", Phys. Rev. Lett. **90**, 040403 (2003).
220. S. Ghanbari, T. D. Kieu, A. Sidorov, and P. Hannaford, "Permanent magnetic lattices for ultracold atoms and quantum degenerate gases," J. Phys. B **39**, 847 (2006);
O. Romero-Isart, C. Navau, A. Sanchez, P. Zoller, and J. I. Cirac, "Superconducting vortex lattices for ultracold atoms", Phys. Rev. Lett. **111**, 145304 (2013);
S. Jose, P. Surendran, Y. Wang, I. Herrera, L. Krzemien, S. Whitlock, R. McLean, A. Sidorov, and P. Hannaford, "Periodic array of Bose-Einstein condensates in a magnetic lattice", Phys. Rev. A **89**, 051602 (2014).
221. K. Henderson, C. Ryu, C. MacCormick, and M. G. Boshier, "Experimental demonstration of painting arbitrary and dynamic potentials for Bose-Einstein condensates", New J. Phys. **11**, 043030 (2009);
N. R. Cooper, "Optical flux lattices for ultracold atomic gases", Phys. Rev. Lett. **106**, 175301 (2011).
222. R. Yamazaki, S. Taie, S. Sugawa, and Y. Takahashi, "Submicron spatial modulation of an interatomic interaction in a Bose-Einstein condensate", Phys. Rev. Lett. **105**, 050405 (2010).
223. M. Centurion, M. A. Porter, P. G. Kevrekidis, and D. Psaltis, "Nonlinearity management in optics: Experiment, theory, and simulation", Phys. Rev. Lett. **97**, 033903 (2006);
M. Centurion, M. A. Porter, Y. Pu, P. G. Kevrekidis, D. J. Frantzeskakis, and D. Psaltis, "Modulational instability in a layered Kerr medium: Theory and experiment", Phys. Rev. Lett. **97**, 234101 (2006).
224. P. G. Kevrekidis, G. Theocharis, D. J. Frantzeskakis, and B. A. Malomed, "Feshbach resonance management for Bose-Einstein condensates", Phys. Rev. Lett. **90**, 230401 (2003);
D. E. Pelinovsky, P. G. Kevrekidis, and D. J. Frantzeskakis, "Averaging for solitons with nonlinearity management", Phys. Rev. Lett. **91**, 240201 (2003);
D. E. Pelinovsky, P. G. Kevrekidis, D. J. Frantzeskakis, and V. Zharnitsky, "Hamiltonian averaging for solitons with nonlinearity management", Phys. Rev. E **70**, 047604 (2004);

Z. X. Liang, Z. D. Zhang, and W. M. Liu, "Dynamics of a bright soliton in Bose-Einstein condensates with time-dependent atomic scattering length in an expulsive parabolic potential", Phys. Rev. Lett. **94**, 050402 (2005);
M. Matuszewski, E. Infeld, B. A. Malomed, and M. Trippenbach, "Fully three dimensional breather solitons can be created using Feshbach resonances", Phys. Rev. Lett. **95**, 050403 (2005).

225. T. Kohler, K. Goral, and P. S. Julienne, "Production of cold molecules via magnetically tunable Feshbach resonances", Rev. Mod. Phys. **78**, 1311 (2006).
226. M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, C. Chin, J. Hecker Denschlag, and R. Grimm, "Collective excitations of a degenerate gas at the BEC-BCS crossover", Phys. Rev. Lett. **92**, 203201 (2004).
227. T. Kraemer, M. Mark, P. Waldburger, J. G. Danzl, C. Chin, B. Engeser, A. D. Lange, K. Pilch, A. Jaakkola, H. C. Nagerl, and R. Grimm, "Evidence for Efimov quantum states in an ultracold gas of caesium atoms", Nature **440**, 315 (2006).
228. F. Kh. Abdullaev and M. Salerno, "Adiabatic compression of soliton matter waves", J. Phys. B **36**, 2851 (2003).
229. M. I. Rodas-Verde, H. Michinel, and V. M. Pérez-García, "Controllable soliton emission from a Bose-Einstein condensate", Phys. Rev. Lett. **95**, 153903 (2005).
230. H. Sakaguchi and B. A. Malomed, "Matter-wave solitons in nonlinear optical lattices", Phys. Rev. E **72**, 046610 (2005).
231. P. Niarchou, G. Theocharis, P. G. Kevrekidis, P. Schmelcher, and D. J. Frantzeskakis, "Soliton oscillations in collisionally inhomogeneous attractive Bose-Einstein condensates", Phys. Rev. A **76**, 023615 (2007).
232. J. Garnier and F. Kh. Abdullaev, "Transmission of matter-wave solitons through nonlinear traps and barriers", Phys. Rev. A **74**, 013604 (2006).
233. H. Sakaguchi and B. A. Malomed, "Solitons in combined linear and nonlinear lattice potentials", Phys. Rev. A **81**, 013624 (2010).
234. C. Wang, P. G. Kevrekidis, T. P. Horikis, and D. J. Frantzeskakis, "Collisional-inhomogeneity-induced generation of matter-wave dark solitons", Phys. Lett. A **374**, 3863 (2010);
F. Pinsker, N. G. Berloff, and V. M. Pérez-García, "Nonlinear quantum piston for the controlled generation of vortex rings and soliton trains", Phys. Rev. A **87**, 053624 (2013).
235. A. Balaz, R. Paun, A. I. Nicolin, S. Balasubramanian, and R. Ramaswamy, "Faraday waves in collisionally inhomogeneous Bose-Einstein condensates", Phys. Rev. A **89**, 023609 (2014).
236. B. A. Malomed, D. Mihalache, F. Wise, and L. Torner, "Spatiotemporal optical solitons", J. Opt. B: Quantum Semiclass. Opt. **7**, R53 (2005).
237. Z. Chen, M. Segev, and D. N. Christodoulides, "Optical spatial solitons: historical overview and recent advances", Rep. Prog. Phys. **75**, 086401 (2012).
238. D. Mihalache, "Linear and nonlinear light bullets: Recent theoretical and experimental studies", Rom. J. Phys. **57**, 352 (2012).
239. H. Leblond and D. Mihalache, "Models of few optical cycle solitons beyond the slowly varying envelope approximation", Phys. Rep. **523**, 61 (2013).
240. D. Mihalache, "Multidimensional localized structures in optics and Bose-Einstein condensates: A selection of recent studies", Rom. J. Phys. **59**, 295 (2014).
241. B. B. Baizakov, B. A. Malomed, and M. Salerno, "Multidimensional solitons in periodic potentials", Europhys. Lett. **63**, 642 (2003).
242. D. Mihalache, D. Mazilu, F. Lederer, B. A. Malomed, L. C. Crasovan, Y. V. Kartashov, and L. Torner, "Stable three-dimensional solitons in attractive Bose-Einstein condensates loaded in an optical lattice", Phys. Rev. A **72**, 021601 (2005).

243. A. Ferrando, M. Zacarès, P. Fernandez de Cordoba, D. Binosi, and J. Mosoriu, “Spatial soliton formation in photonic crystal fibers”, Opt. Exp. **11**, 452 (2003).
244. R. J. Dodd, “Approximate solutions of the nonlinear Schrödinger equation for ground and excited states of Bose-Einstein condensates”, J. Res. Natl. Inst. Stand. Technol. **101**, 545 (1996);
T. J. Alexander and L. Bergé, “Ground states and vortices of matter-wave condensates and optical guided waves”, Phys. Rev. E **65**, 026611 (2002).
245. D. Mihalache, D. Mazilu, B. A. Malomed, and F. Lederer, “Vortex stability in nearly-two-dimensional Bose-Einstein condensates with attraction”, Phys. Rev. A **73**, 043615 (2006).
246. F. W. Ye, L. W. Dong, B. A. Malomed, D. Mihalache, and B. B. Hu, “Elliptic vortices in optical waveguides and self-attractive Bose-Einstein condensates”, J. Opt. Soc. Am. B **27**, 757 (2010).
247. Y. V. Kartashov, V. A. Vysloukh, and L. Torner, “Rotary solitons in Bessel optical lattices”, Phys. Rev. Lett. **93**, 093904 (2004).
248. Y. V. Kartashov, R. Carretero-González, B. A. Malomed, V. A. Vysloukh, and L. Torner, “Multipole-mode solitons in Bessel optical lattices”, Opt. Exp. **13**, 10703 (2005).
249. Y. J. He, B. A. Malomed, and H. Z. Wang, “Steering the motion of rotary solitons in radial lattices”, Phys. Rev. A **76**, 053601 (2007).
250. Y. J. He, B. A. Malomed, D. Mihalache, and H. Z. Wang, “Tunable rotary orbits of matter-wave nonlinear modes in attractive Bose-Einstein condensates”, J. Phys. B: At. Mol. Opt. Phys. **41**, 055301 (2008).
251. X. Wang, Z. Chen, and P. G. Kevrekidis, “Observation of discrete solitons and soliton rotation in optically induced periodic ring lattices”, Phys. Rev. Lett. **96**, 083904 (2006).
252. B. A. Malomed, F. Lederer, D. Mazilu, and D. Mihalache, “On stability of vortices in three-dimensional self-attractive Bose-Einstein condensates”, Phys. Lett. A **361**, 336 (2007).
253. J. Yang and Z. H. Musslimani, ”Fundamental and vortex solitons in a two-dimensional photonic lattice”, Opt. Lett. **28**, 2094 (2003).
254. B. B. Baizakov, B. A. Malomed and M. Salerno, “Multidimensional solitons in a low-dimensional periodic potential”, Phys. Rev. A **70**, 053613 (2004).
255. D. Mihalache, D. Mazilu, F. Lederer, Y. V. Kartashov, L.-C. Crasovan, and L. Torner, “Stable three-dimensional solitons in a two-dimensional photonic lattices”, Phys. Rev. E **70**, 055603 (2004).
256. H. Sakaguchi, B. Li, and B. A. Malomed, “Creation of two-dimensional composite solitons in spin-orbit-coupled self-attractive Bose-Einstein condensates in free space”, Phys. Rev. E **89**, 032920 (2014).
257. B. P. Anderson, P. C. Haljan, C. E. Wieman, and E. A. Cornell, “Vortex precession in Bose-Einstein condensates: Observations with filled and empty cores”, Phys. Rev. Lett. **85**, 2857 (2000).
258. K. J. H. Law, P. G. Kevrekidis, and L.S. Tuckerman, “Stable vortex-bright-soliton structures in two-component Bose-Einstein condensates”, Phys. Rev. Lett. **105**, 160405 (2010).
259. T. J. Alexander, E. A. Ostrovskaya, A. A. Sukhorukov, and Y. S. Kivshar, “Three-dimensional matter-wave vortices in optical lattices”, Phys. Rev. A **72**, 043603 (2005).
260. G. L. Alfimov, V. V. Konotop, and M. Salerno, “Matter solitons in Bose-Einstein condensates with optical lattices”, Europhys. Lett. **58**, **7** (2002);
E. A. Ostrovskaya and Y. S. Kivshar, “Matter-wave solitons in atomic band-gap structures”, Phys. Rev. Lett. **90**, 160407 (2003);
H. Sakaguchi and B. A. Malomed, “Two-dimensional loosely and tightly bound solitons in optical lattices and inverted traps”, J. Phys. B **37**, 2225 (2004).
261. O. V. Borovkova, Y. V. Kartashov, L. Torner, and B. A. Malomed, “Bright solitons from defocus-

ing nonlinearities", Phys. Rev. E **84**, 035602 (R) (2011).

262. R. Driben, Y. V. Kartashov, B. A. Malomed, T. Meier, and L. Torner, "Three-dimensional hybrid vortex solitons", New J. Phys. **16**, 063035 (2014);
Y. V. Kartashov, B. A. Malomed, Y. Shnir, and L. Torner, "Twisted toroidal vortex-solitons in inhomogeneous media with repulsive nonlinearity", Phys. Rev. Lett. **113**, 264101 (2014).
263. T. P. Simula, S. M. M. Virtanen, and M. M. Salomaa, "Stability of multiquantum vortices in dilute Bose-Einstein condensates", Phys. Rev. A **65**, 033614 (2002).
264. K. J. H. Law, T. W. Neely, P. G. Kevrekidis, B. P. Anderson, A. S. Bradley, and R. Carretero-González, "Dynamic and energetic stabilization of persistent currents in Bose-Einstein condensates", Phys. Rev. A **89**, 053606 (2014).
265. H. Pu, C. K. Law, J. H. Eberly, and N. P. Bigelow, "Coherent disintegration and stability of vortices in trapped Bose condensates", Phys. Rev. A **59**, 1533 (1999).
266. Y. Shin, M. Saba, M. Vengalattore, T. A. Pasquini, C. Sanner, A. E. Leanhardt, M. Prentiss, D. E. Pritchard, and W. Ketterle, "Dynamical instability of a doubly quantized vortex in a Bose-Einstein condensate", Phys. Rev. Lett. **93**, 160406 (2004).
267. A. A. Penckwitt, R. J. Ballagh, and C. W. Gardiner, "Nucleation, growth, and stabilization of Bose-Einstein condensate vortex lattices", Phys. Rev. Lett. **89**, 260402 (2002);
K. Kasamatsu, M. Tsubota, and M. Ueda, "Nonlinear dynamics of vortex lattice formation in a rotating Bose-Einstein condensate", Phys. Rev. A **67**, 033610 (2003);
C. Lobo, A. Sinatra, and Y. Castin, "Vortex lattice formation in Bose-Einstein condensates", Phys. Rev. Lett. **92**, 020403 (2004).
268. B. Y. Rubinstein and L. M. Pismen, "Vortex motion in the spatially inhomogenous conservative Ginzburg-Landau model", Physica D **78**, 1 (1994).
269. S. Middelkamp, P. G. Kevrekidis, D. J. Frantzeskakis, R. Carretero-González and P. Schmelcher, "Stability and dynamics of matter-wave vortices in the presence of collisional inhomogeneities and dissipative perturbations", J. Phys. B **43**, 155303 (2010).
270. D. L. Feder, A. A. Svidzinsky, A. L. Fetter, and C. W. Clark, "Anomalous modes drive vortex dynamics in confined Bose-Einstein condensates", Phys. Rev. Lett. **86**, 564 (2001).
271. S. Middelkamp, P. G. Kevrekidis, D. J. Frantzeskakis, R. Carretero-González, and P. Schmelcher, "Bifurcations, stability, and dynamics of multiple matter-wave vortex states", Phys. Rev. A **82**, 013646 (2010).
272. L. M. Pismen, *Vortices in Nonlinear Fields* (Oxford Science, Oxford, 1999).
273. P. K. Newton and G. Chamoun, "Vortex lattice theory: A particle interaction perspective", SIAM Rev. **51**, 501 (2009).
274. S. McEndoo and Th. Busch, "Small numbers of vortices in anisotropic traps", Phys. Rev. A **79**, 053616 (2009).
275. D. E. Pelinovsky and P. G. Kevrekidis, "Variational approximations of trapped vortices in the large-density limit", Nonlinearity **24**, 1271 (2011).
276. A. M. Barry and P. G. Kevrekidis, "Few-particle vortex cluster equilibria in Bose-Einstein condensates: existence and stability", J. Phys. A **46**, 445001 (2013).
277. A. V. Zampetaki, R. Carretero-González, P. G. Kevrekidis, F. K. Diakonos, and D. J. Frantzeskakis, "Exploring rigidly rotating vortex configurations and their bifurcations in atomic Bose-Einstein condensates", Phys. Rev. E **88**, 042914 (2013).
278. T. Kolokolnikov, P. G. Kevrekidis, and R. Carretero-González, "A tale of two distributions: from few to many vortices in quasi-two-dimensional Bose-Einstein condensates", Proc. Roy. Soc. A **470**, 20140048 (2014).
279. A. A. Abrikosov, "On the magnetic properties of superconductors of the second group", Sov.

Phys. JETP **5**, 1174 (1957).
280. C. Raman, J. R. Abo-Shaeer, J. M. Vogels, K. Xu, and W. Ketterle, "Vortex nucleation in a stirred Bose-Einstein condensate", Phys. Rev. Lett. **87**, 210402 (2001).
281. P. Engels, I. Coddington, P. C. Haljan, V. Schweikhard, and E. A. Cornell, "Observation of long-lived vortex aggregates in rapidly rotating Bose-Einstein condensates", Phys. Rev. Lett. **90**, 170405 (2003).
282. I. Coddington, P. Engels, V. Schweikhard, and E. A. Cornell, "Observation of Tkachenko oscillations in rapidly rotating Bose-Einstein condensates", Phys. Rev. Lett. **91**, 100402 (2003).
283. V. Schweikhard, I. Coddington, P. Engels, S. Tung, and E. A. Cornell, "Vortex-lattice dynamics in rotating spinor Bose-Einstein condensates", Phys. Rev. Lett. **93**, 210403 (2004).
284. H. Pu, L. O. Baksmaty, S. Yi, and N. P. Bigelow, "Structural phase transitions of vortex matter in an optical lattice", Phys. Rev. Lett. **94**, 190401 (2005).
285. M. A. Caracanhas, E. A. L. Henn, and V. S. Bagnato, "Quantum turbulence in trapped BEC: New perspectives for a long lasting problem", Rom. Rep. Phys. **67**, 51 (2015).
286. A. Vardi, "Chaos, ergodization, and thermalization with few-mode Bose-Einstein condensates", Rom. Rep. Phys. **67**, 67 (2015).
287. R. Radha and P. S. Vinayagam, "An analytical window into the world of ultracold atoms", Rom. Rep. Phys. **67**, 89 (2015).
288. A. I. Nicolin, M. C. Raportaru, and A. Balaž, "Effective low-dimensional polynomial equations for Bose-Einstein condensates", Rom. Rep. Phys. **67**, 143 (2015).
289. V. I. Yukalov and E. P. Yukalova, "Statistical models of nonequilibrium Bose gases", Rom. Rep. Phys. **67**, 159 (2015).
290. H.-S. Tao, W. Wu, Y.-H. Chen, and W.-M. Liu, "Quantum phase transitions of cold atoms in honeycomb optical lattices", Rom. Rep. Phys. **67**, 187 (2015).
291. T. He, W. Li, L. Li, J. Liu, and Q. Niu, "Stationary solutions for nonlinear Schrödinger equation with ring trap and their evolution under the periodic kick force", Rom. Rep. Phys. **67**, 207 (2015).
292. D. A. Zezyulin and V. V. Konotop, "Stationary vortex flows and macroscopic Zeno effect in Bose-Einstein condensates with localized dissipation", Rom. Rep. Phys. **67**, 223 (2015).
293. V. Achilleos, D. J. Frantzeskakis, P. G. Kevrekidis, P. Schmelcher, and J. Stockhofe, "Positive and negative mass solitons in spin-orbit coupled Bose-Einstein condensates", Rom. Rep. Phys. **67**, 235 (2015).
294. A. I. Yakimenko, S. I. Vilchinskii, Y. M. Bidasyuk, Y. I. Kuriatnikov, K. O. Isaieva, and M. Weyrauch, "Generation and decay of persistent currents in a toroidal Bose-Einstein condensate", Rom. Rep. Phys. **67**, 249 (2015).
295. M. Galante, G. Mazzarella, and L. Salasnich, "Analytical results on quantum correlations of few bosons in a double-well trap", Rom. Rep. Phys. **67**, 273 (2015).
296. V. Bolpasi and W. von Klitzing, "Adiabatic potentials and atom lasers", Rom. Rep. Phys. **67**, 295 (2015).