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Bose-Mesner Algebras Related to Type II Matrices and Spin Models

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Abstract. A *type II matrix* is a square matrix W with non-zero complex entries such that the entrywise quotient of any two distinct rows of W sums to zero. Hadamard matrices and character tables of abelian groups are easy examples, and other examples called *spin models* and satisfying an additional condition can be used as basic data to construct invariants of links in 3-space. Our main result is the construction, for every type II matrix W, of a *Bose-Mesner algebra* $N(W)$, which is a commutative algebra of matrices containing the identity I, the all-one matrix J, closed under transposition and under Hadamard (i.e., entrywise) product. Moreover, if W is a spin model, it belongs to $N(W)$. The transposition of matrices W corresponds to a classical notion of *duality* for the corresponding Bose-Mesner algebras $N(W)$. Every Bose-Mesner algebra encodes a highly regular combinatorial structure called an *association scheme,* and we give an explicit construction of this structure. This allows us to compute $N(W)$ for a number of examples.

Keywords: spin model, link invariant, association scheme, Bose-Mesner algebra

1. Introduction

The main motivation for the present work comes from the study of *spin models for link invariants*. Such a spin model can be viewed as a square matrix with complex entries which satisfies two equations, the *type II* and *type III* equations. Any solution to these equations yields an invariant of knots and links in 3-space via a construction due to V. Jones [21] for symmetric matrices and generalized in [26] to arbitrary matrices. The problem of the classification of solutions to the type II and type III equations seems to be very difficult, and it was soon realized (see [17]) that it is deeply related with a classical topic in algebraic combinatorics, namely the study of *association schemes*.

Roughly speaking, an association scheme on a (finite) set X is a partition of $X \times X$ into relations which exhibits nice regularity properties. A standard example is given by *distance-regular graphs* (see [7]): X is the vertex-set and the relations correspond to the possible values of the distance function. Another standard example is given by transitive actions of finite groups on X (see [5]): the relations are the orbits of the corresponding action on $X \times X$. The regularity properties of the relations are conveniently described in terms of their adjacency matrices: they span an algebra of matrices with unit I closed under transposition. This algebra is called the *Bose-Mesner algebra* of the association scheme. It is also an algebra with unit J (the all-one matrix) for the Hadamard (i.e., entrywise) product of matrices. We always assume here that the ordinary matrix product is commutative. In this case, Bose-Mesner algebras and association schemes are equivalent concepts.

In October 1994 the first author proved, using the topological interpretation of the type II and type III equations, that every symmetric solution W to these equations belongs to some Bose-Mesner algebra [20]. Moreover, this algebra has a *duality*, i.e., a linear automorphism which, up to multiplication by a scalar, exchanges the ordinary and Hadamard product, and whose square is a scalar multiple of the transposition map. This duality has a nice expression in terms of W, or equivalently W is given by a solution to the so-called *modular invariance equation* corresponding to this duality (see [4]).

Immediately afterwards, the third author gave a purely algebraic proof that any symmetric matrix W satisfying the type II equation defines some Bose-Mesner algebra $N(W)$, which will contain W if the type III equation is also satisfied [31]. This approach is obviously more general than that of [20], and this is particularly significant since solutions to the type II equation (which we call *type II matrices*) are of great interest in the theory of Von Neumann algebras and of some natural generalizations (see [1, 23]). Another advantage of this approach is that it can be generalized to non-symmetric matrices, and such a generalization is the main content of the present paper.

To sum up, we shall generalize both the results of [20, 31] by associating with every type II matrix W a Bose-Mesner algebra $N(W)$, which contains W when this matrix also satisfies the type III equation. The Bose-Mesner algebra $N({}^tW)$, where ${}^tW$ denotes the transpose of W, is *dual to* $N(W)$, which means that there is a linear isomorphism from $N(W)$ to $N({}^tW)$ which converts the ordinary (respectively: Hadamard) product in $N(W)$ into the Hadamard (respectively: ordinary) product of $N({}^tW)$. In particular, if W can be transformed into a symmetric matrix by multiplying row and columns by non-zero scalars (this occurs for instance if the type III equation holds), $N(W) = N({}^tW)$ has a duality.

We give also a direct and explicit construction of the association scheme corresponding to $N(W)$, and of its symmetrization, in terms of certain graphs associated with W. This allows us to study effectively the tensor product construction for type II matrices, and a number of examples: character tables of abelian groups, Hadamard matrices of size 16, type II matrices of size 4, spin models for the Kauffman polynomial of links (see [17]), and the spin models defined on Hadamard graphs by the third author (see [29]).

2. Preliminaries

2.1. Association schemes and Bose-Mesner algebras

For more details concerning this section, the reader is referred to [5, 6, 8–10, 32].

Let X be a non-empty finite set. A *d-class association scheme* on X is a partition of $X \times X$ into $d+1$ non-empty relations R_i, $i = 0, \ldots, d$, such that the following properties hold.

(1) $R_0 = \{(x, x) \mid x \in X\}$.
(2) For every $i \in \{0, \ldots, d\}$ there exists $i' \in \{0, \ldots, d\}$ such that $\{(y, x) \mid (x, y) \in R_i\} = R_{i'}$.
(3) There exist *intersection numbers* p_{ij}^k $(i, j, k \in \{0, \ldots, d\})$ such that, for every $(x, y) \in R_k$, $|\{z \in X \mid (x, z) \in R_i,\ (z, y) \in R_j\}| = p_{ij}^k$.
(4) $p_{ij}^k = p_{ji}^k$ for every $i, j, k \in \{0, \ldots, d\}$.

Remark Since all association schemes which appear in the present work satisfy property (4), we have incorporated this property in our definition. Thus, we agree with the terminology of [10], and our association schemes are commutative in the terminology of [5].

We denote by M_X the set of square matrices with rows and columns indexed by X and entries in **C**. The (x, y)-entry of $A \in M_X$ is denoted by $A(x, y)$. We denote by I the identity matrix, by J the matrix with all entries equal to 1, by ${}^tA$ the transpose of A, and by AB the (ordinary) matrix product of A and B. The Hadamard product of A and B is denoted by $A \circ B$ and defined by $(A \circ B)(x, y) = A(x, y)B(x, y)$.

For every i in $\{0, \ldots, d\}$, let $A_i \in M_X$ be defined by: $A_i(x, y) = 1$ if $(x, y) \in R_i$, $A_i(x, y) = 0$ otherwise.

The facts that the A_i have entries 0 or 1 and that the R_i form a partition of $X \times X$ into non-empty relations are translated as follows.

(5) $A_i \neq 0$, $A_i \circ A_j = \delta(i, j) A_i$ (where δ is the Kronecker symbol).
(6) $\sum_{i=0}^d A_i = J$.

Properties (1), (2), (3) and (4) now become

(7) $A_0 = I$.
(8) ${}^tA_i = A_{i'}$.
(9) $A_i A_j = A_j A_i = \sum_{k=0}^d p_{ij}^k A_k$.

Let $\mathcal{A}$ be the **C**-linear span of $\{A_i \mid i = 0, \ldots, d\}$. By (5) and (6), $\mathcal{A}$ is a (commutative) algebra under Hadamard product, with unity element J, and $\{A_i \mid i = 0, \ldots, d\}$ is a basis of orthogonal idempotents for this algebra. By (7) and (9), $\mathcal{A}$ is also a commutative algebra under matrix product, with unity element I. Finally, (8) means that $\mathcal{A}$ is closed under transposition. The vector subspace $\mathcal{A}$ of M_X is called the *Bose-Mesner algebra* of the association scheme $\{R_i \mid i = 0, \ldots, d\}$ on X.

Conversely, it is not difficult to show that every vector subspace of M_X containing I and J, closed under transposition, which is a commutative algebra under Hadamard product and also under matrix product, is the Bose-Mesner algebra of some association scheme on X. The only non-trivial step is the existence of a basis of orthogonal idempotents for the Hadamard product, for which the reader is referred to [7], Theorem 2.6.1 (i) (the proof given there obviously works for non-symmetric matrices as well).

A Bose-Mesner algebera $\mathcal{A}$ (or the corresponding association scheme) is *symmetric* if every matrix in $\mathcal{A}$ is symmetric. For every Bose-Mesner algebra $\mathcal{A}$, its *symmetrization* $\tilde{\mathcal{A}}$ is the set of symmetric matrices in $\mathcal{A}$. Clearly, $\tilde{\mathcal{A}}$ is a symmetric Bose-Mesner algebra. Its Hadamard idempotents are the A_i for all i with $i = i'$ and the $A_i + A_{i'}$ for all i with $i \neq i'$.

Let $\mathcal{A}$ be a Bose-Mesner algebra on X with basis of Hadamard idempotents $\{A_i \mid i = 0, \ldots, d\}$.

Since the A_i are real matrices, $\mathcal{A}$ is closed under complex conjugation, and hence also under conjugate transposition. Then the commutativity of the matrix product on $\mathcal{A}$ implies the existence of a unitary matrix U such that $U^{-1}\mathcal{A}U$ consists of diagonal matrices. Note that on this algebra the ordinary matrix product and the Hadamard product coincide. It follows that $\mathcal{A}$ has a basis $\{E_i \mid i = 0, \ldots, d\}$ such that the diagonal matrices $\Delta_i = U^{-1}E_iU$ satisfy the identity $\Delta_i \circ \Delta_j = \delta(i, j)\Delta_i$ (see again [7] Theorem 2.6.1 (i)), or equivalently $\Delta_i\Delta_j = \delta(i, j)\Delta_i$. Thus $\{E_i \mid i = 0, \ldots, d\}$ is a basis of orthogonal idempotents for the matrix product:

(10) $E_iE_j = \delta(i, j)E_i$.

Since I belongs to $\mathcal{A}$,

(11) $\sum_{i=0}^{d} E_i = I$.

Since $JA = AJ$ is a scalar multiple of J for all A in $\mathcal{A}$, one easily shows that $(1/|X|)J \in \{E_i \mid i = 0, \ldots, d\}$, and the notation is chosen so that

(12) $E_0 = \frac{1}{|X|}J$.

Since the Δ_i have entries 0 or 1, the E_i are Hermitian matrices, and the uniqueness of the basis of orthogonal idempotents shows that

(13) For every i in $\{0, \ldots, d\}$ there exists $\hat{i}$ in $\{0, \ldots, d\}$ such that ${}^tE_i = \bar{E}_i = E_{\hat{i}}$.

Finally the fact that $\mathcal{A}$ is closed under Hadamard product implies the existence of numbers q_{ij}^k ($i, j, k \in \{0, \ldots, d\}$) called the *Krein parameters* such that

(14) $E_i \circ E_j = \frac{1}{|X|}\sum_{k=0}^{d} q_{ij}^k E_k$.

We now introduce several notions of isomorphism for association schemes and Bose-Mesner algebras.

Two association schemes $\{R_i \mid i = 0, \ldots, d\}$ and $\{S_i \mid i = 0, \ldots, d\}$ on X are *isomorphic* if there exist permutations $\rho : X \to X$ and $\sigma : \{0, \ldots, d\} \to \{0, \ldots, d\}$ such that $(\rho(x), \rho(y))$ belongs to $S_{\sigma(i)}$ if and only if (x, y) belongs to R_i.

Two Bose-Mesner algebras $\mathcal{A}$ and $\mathcal{B}$ in M_X are *combinatorially isomorphic* if there exists a permutation matrix P in M_X such that $\mathcal{B} = P^{-1}\mathcal{A}P$. It is easy to see that $\mathcal{A}$ and $\mathcal{B}$ are combinatorially isomorphic if and only if the corresponding association schemes are isomorphic.

For $\mathcal{A}$ and $\mathcal{B}$ as above, a *Bose-Mesner isomorphism from $\mathcal{A}$ to $\mathcal{B}$* is a linear isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\varphi(AA') = \varphi(A)\varphi(A')$ and $\varphi(A \circ A') = \varphi(A) \circ \varphi(A')$ for every A, A' in $\mathcal{A}$. In particular, for every permutation matrix P, setting $\varphi(A) = P^{-1}AP$ defines a *combinatorial Bose-Mesner isomorphism* from $\mathcal{A}$ to $P^{-1}\mathcal{A}P$. However, a Bose-Mesner isomorphism need not be combinatorial. For instance, a 2-class symmetric association scheme (R_0, R_1, R_2) is determined by the strongly regular graph corresponding to the relation R_1, and the Bose-Mesner algebras of two non-isomorphic strongly regular graphs with the same parameters (i.e., intersection numbers) such as the Shrikhande graph and the lattice graph $L_2(4)$ (see [7]) will be related by a non-combinatorial Bose-Mesner isomorphism.

A *duality* from $\mathcal{A}$ to $\mathcal{B}$ is a linear isomorphism $\Psi : \mathcal{A} \to \mathcal{B}$ such that $\Psi(AA') = \Psi(A) \circ \Psi(A')$ and $\Psi(A \circ A') = (1/|X|)\Psi(A)\Psi(A')$ for every A, A' in $\mathcal{A}$. If there exists a duality Ψ from $\mathcal{A}$ to $\mathcal{B}$, $\mathcal{B}$ is called a *dual to* $\mathcal{A}$. Then one easily shows that $|X|\Psi^{-1}$ is a duality from $\mathcal{B}$ to $\mathcal{A}$, and consequently $\mathcal{A}$ is a dual to $\mathcal{B}$. We shall say that $(\mathcal{A}, \mathcal{B})$ is a *dual pair*.

Remarks

(i) For the sake of simplicity, we have adopted a non-standard terminology: if there exists a duality Ψ from $\mathcal{A}$ to $\mathcal{B}$, one usually says that $\mathcal{B}$ is "formally dual" to $\mathcal{A}$. This is to distinguish the following situation corresponding to an "actual duality": X is an abelian group, and $\mathcal{A}$ is contained in the Bose-Mesner algebra $\mathcal{A}_X$ of X (the set of matrices A in M_X such that, writing X additively, $A(i+x, j+x) = A(i, j)$ for all i, j, x in X). It is well known (see Section 5.1) that there exist dualities $\Psi : \mathcal{A}_X \to \mathcal{A}_X$, and, for any such duality Ψ, $(\mathcal{A}, \Psi(\mathcal{A}))$ is a dual pair.

(ii) Not every Bose-Mesner algebra $\mathcal{A}$ admits a dual Bose-Mesner algebra $\mathcal{B}$. Indeed a duality from $\mathcal{A}$ to $\mathcal{B}$ sends the basis of ordinary idempotents of $\mathcal{A}$ to the basis of Hadamard idempotents of $\mathcal{B}$. By comparing (9) and (14) one sees that the Krein parameters of $\mathcal{A}$ must correspond to the intersection numbers of $\mathcal{B}$, and hence be integers, which is not true in general.

(iii) It is clear from (5), (7), (8) and (9) that $p_{ij}^0 \neq 0$ iff $j = i'$. It easily follows that every Bose-Mesner isomorphism commutes with the transposition map. Similarly, since the trace of $E_i E_j$ equals the sum of entries of $E_i \circ {}^tE_j$, and the sum of entries of E_i is $\delta(i, 0)|X|$ by (12), we obtain from (13) and (14) that $q_{ij}^0 \neq 0$ iff $j = \hat{i}$. This implies that every duality commutes with the transposition map.

Finally, let $\Psi : \mathcal{A} \to \mathcal{A}$ be a duality. To conform to the standard terminology, we shall call it a *duality of* $\mathcal{A}$ only if $\Psi^2 = |X|\tau_{\mathcal{A}}$, where $\tau_{\mathcal{A}}$ is the transposition map on $\mathcal{A}$. We shall say that $\mathcal{A}$ is *self-dual* if it has a duality in this sense.

Remark We do not know any example of a Bose-Mesner algebra $\mathcal{A}$ which is not self-dual but has a duality $\Psi : \mathcal{A} \to \mathcal{A}$. The question of the existence of such examples is raised by A. Munemasa in [16].

2.2. *Spin models for link invariants*

In [21], Vaughan Jones has introduced a construction of invariants of links in 3-space based on the statistical mechanical concept of *spin model*. The main idea is to represent every link by a connected plane diagram whose regions are colored black or white in such a way that adjacent regions receive different colors. Then one defines states of the diagram as mappings from the set of black regions to some fixed finite set X of spins. There are two types of crossings, positive and negative, and each one is assigned a weight matrix W^+ or W^- in M_X. For every state σ and crossing v incident with the black regions r, r', let $\langle\sigma \mid v\rangle = W^{\pm}(\sigma(r), \sigma(r'))$ be the local weight of σ at v. Let $\langle\sigma\rangle = \prod_v \langle\sigma \mid v\rangle$ be the weight of σ, where the product is over all crossings. Then the partition function Z is $\sum_\sigma \langle\sigma\rangle$, where the sum is over all states.

It is shown in [21] that if the matrices W^+, W^- satisfy certain *invariance equations*, then, after suitable normalization, Z defines a link invariant. The construction in [21] assumes that W^+, W^- are symmetric. This restriction was removed in [26], which gives the following invariance equations:

(15) (i) $W^+ \circ I = aI,\ JW^+ = W^+J = Da^{-1}J.$

(ii) $W^- \circ I = a^{-1}I,\ JW^- = W^-J = DaJ.$

(16) $W^+W^- = |X|I,\ W^+ \circ {}^tW^- = J.$

(17) For every α, β, γ in X,

$$\sum_{x\in X} W^+(x,\alpha)W^+(x,\beta)W^-(\gamma,x) = DW^+(\alpha,\beta)W^-(\beta,\gamma)W^-(\gamma,\alpha),$$

where $a \in \mathbf{C}^*$ and $D^2 = |X|$.

Remarks

(i) When W^+, W^- are symmetric, this reduces to the invariance equations in [21].
(ii) Assuming (16), Eq. (17) is equivalent to other similar equations usually chosen to define spin models (see [26], Proposition 2.1).
(iii) (16) and (17) imply that (15) holds for some $a \in \mathbf{C}^*$.

We shall also refer to (15), (16) and (17) as the type I, type II and type III conditions respectively, since they correspond to Reidemeister moves of these types (see [21, 26]). The types also correspond to the degrees of the equations in terms of the entries of W^+, W^-. It will be convenient to reformulate the above equations in terms of $W = W^+$ alone. In particular, we shall say that a matrix W in M_X with non-zero entries is a *type II matrix* if it satisfies one of the following conditions, each of which is equivalent with (16) for $W^+ = W$:

(18) (i) $\sum_{x\in X} \frac{W(b,x)}{W(c,x)} = |X|\delta(b,c)$ for all (b,c) in X,

(ii) $\sum_{x\in X} \frac{W(x,b)}{W(x,c)} = |X|\delta(b,c)$ for all (b,c) in X.

2.3. Spin models and commuting squares

Let D_X consist of all diagonal matrices in M_X and let W be an invertible matrix in M_X. We consider the following *square*

$$\begin{array}{ccc} D_X & \subset & M_X \\ \cup & & \cup \\ \mathbf{C} & \subset & W^{-1}D_X W \end{array}$$

which is a diagram of inclusions of algebras under the matrix product (here $\mathbf{C}$ is identified with the algebra of scalar matrices). M_X is endowed with the normalized trace $\mathrm{tr} : M_X \to \mathbf{C}$, with $\mathrm{tr}(A) = (1/|X|)\mathrm{Trace}(A)$. Then the above square is *commuting* (we describe here a particular case of a very general object—see [12, 1]) if $\mathrm{tr}(AB) = \mathrm{tr}(A)\mathrm{tr}(B)$ for every $A \in D_X$, $B \in W^{-1}D_X W$. This situation can also be described in terms of orthogonal pairs of algebras (see [14], Section 1.5 of [27, 28]). An easy computation shows that the square is commuting if and only if $W^{-1}(i, j)W(j, i) = (1/|X|)$ for all i, j in X, that is if and only if W is a type II matrix.

The case when W is unitary is of special importance for the study of subfactors in the theory of Von Neumann algebras (see [1, 12, 23]). This occurs exactly when all entries of W have absolute value $1/\sqrt{n}$, where $n = |X|$.

3. A dual pair of Bose-Mesner algebras

3.1. Construction of the dual pair

Let W be a type II matrix in M_X. We introduce for each pair $(b, c) \in X \times X$ two column vectors Y_{bc} and Y'_{bc} indexed by X and defined as follows:

(19) $Y_{bc}(x) = \frac{W(x,b)}{W(x,c)}$,

(20) $Y'_{bc}(x) = \frac{W(b,x)}{W(c,x)}$.

Let

$$N(W) = \{A \in M_X \mid Y_{bc} \text{ is an eigenvector of } A \text{ for all } b, c \in X\},$$

and

$$N'(W) = \{A \in M_X \mid Y'_{bc} \text{ is an eigenvector of } A \text{ for all } b, c \in X\}.$$

Note that $N'(W) = N({}^tW)$. In the sequel we write N for $N(W)$, N' for $N'(W)$.

For every A in N, let $\Psi(A) \in M_X$ be defined by

(21) $AY_{bc} = (\Psi(A))(b, c)Y_{bc}$ for all $(b, c) \in X \times X$.

Similarly, for every A' in N', we define $\Psi'(A') \in M_X$ by

(22) $A'Y'_{bc} = (\Psi'(A'))(b,c)Y'_{bc}$ for all $(b,c) \in X \times X$.

We denote by τ_N (respectively, $\tau_{N'}$) the restriction of the transposition map to N (respectively, N').

Theorem 1 *N and N' are Bose-Mesner algebras. Moreover, $\Psi(N) = N'$, $\Psi'(N') = N$, and $\Psi : N \to N'$, $\Psi' : N' \to N$ are dualities such that $\Psi'\Psi = |X|\tau_N$, $\Psi\Psi' = |X|\tau_{N'}$. Hence (N, N') is a dual pair of Bose-Mesner algebras. Moreover when $N = N'$ and $\Psi = \Psi'$, N is a self-dual Bose-Mesner algebra.*

Proof: It is clear from (21) that N is a vector subspace of M_X and that $\Psi : N \to M_X$ is a linear map. We observe that

(23) For every c in X, $\{Y_{bc} \mid b \in X\}$ is a basis of column vectors.

Indeed, the matrix with (x,b)-entry equal to $Y_{bc}(x)$ is ΔW, where $\Delta = \mathrm{Diag}[1/W(x,c)]_{x\in X}$, and both Δ and W are invertible.

It follows that Ψ is injective.

Note that I belongs to N, with $\Psi(I) = J$.

Moreover, the type II property (18(ii)) can be written $JY_{bc} = |X|\delta(b,c)\mathbf{1}$, where $\mathbf{1}$ is the all-one column vector. Since $Y_{bb} = \mathbf{1}$ for every b in X, this means that J belongs to N, with $\Psi(J) = |X|I$.

Let now A, B be two matrices in N. It is immediate from (21) that, for every (b,c) in $X \times X$,

$$ABY_{bc} = BAY_{bc} = (\Psi(A))(b,c)(\Psi(B))(b,c)Y_{bc}.$$

This, together with (23), shows that N is a commutative algebra under matrix product, and that $\Psi(AB) = \Psi(A) \circ \Psi(B)$.

We now show that $\Psi(N) \subseteq N'$.

For A in N and a, b, c in X the a-entry of $\Psi(A)Y'_{bc}$ is

$$(\Psi(A)Y'_{bc})(a) = \sum_{x\in X}(\Psi(A))(a,x)\frac{W(b,x)}{W(c,x)}.$$

By (21), $(\Psi(A))(a,x)Y_{ax} = AY_{ax}$, and by considering c-entries we obtain

$$(\Psi(A))(a,x)\frac{W(c,a)}{W(c,x)} = \sum_{y\in X} A(c,y)\frac{W(y,a)}{W(y,x)}$$

and hence

$$(\Psi(A))(a,x)\frac{W(b,x)}{W(c,x)} = \sum_{y\in X} A(c,y)\frac{W(y,a)W(b,x)}{W(c,a)W(y,x)}.$$

It follows that

$$\begin{aligned}(\Psi(A)Y'_{bc})(a) &= \sum_{y\in X} A(c,y)\frac{W(y,a)}{W(c,a)}\left(\sum_{x\in X}\frac{W(b,x)}{W(y,x)}\right)\\ &= \sum_{y\in X} A(c,y)\frac{W(y,a)}{W(c,a)}|X|\delta(b,y) \quad \text{(by (18)(i))}\\ &= |X|A(c,b)\frac{W(b,a)}{W(c,a)} = |X|A(c,b)Y'_{bc}(a).\end{aligned}$$

Thus $\Psi(A)Y'_{bc} = |X|A(c,b)Y'_{bc}$.

This shows that $\Psi(A) \in N'$ with $\Psi'(\Psi(A)) = |X|\,{}^tA$.

Replacing W by ${}^tW$, we see that all results obtained so far are also valid if we interchange N and N'. Let us sum up these results.

(i) N, N' are vector subspaces of M_X containing I, J.
(ii) $\Psi : N \to N'$ and $\Psi' : N' \to N$ are linear injective maps.
(iii) N, N' are commutative algebras under matrix product, $\Psi(AB) = \Psi(A) \circ \Psi(B)$ (A, B in N), and $\Psi'(A'B') = \Psi'(A') \circ \Psi'(B')$ (A', B' in N').
(iv) $\Psi'(\Psi(A)) = |X|\,{}^tA$ for A in N, $\Psi(\Psi'(A')) = |X|\,{}^tA'$ for A' in N'.

By (ii), both Ψ and Ψ' are bijective.

Let A, B belong to N. There exists A', B' in N' such that $A = \Psi'(A')$, $B = \Psi'(B')$. By (iv) we have $\Psi(A) = |X|\,{}^tA'$, $\Psi(B) = |X|\,{}^tB'$. Now, by (iii), $A \circ B = \Psi'(A') \circ \Psi'(B') = \Psi'(A'B')$ belongs to N. Thus N (and similarly N') is closed under Hadamard product.

Also note that by (iv) ${}^tA = |X|^{-1}\Psi'(\Psi(A))$ belongs to N. Thus N (and similarly N') is closed under transposition.

It follows that N and N' are Bose-Mesner algebras. By (iv), $\Psi'\Psi = |X|\tau_N$, $\Psi\Psi' = |X|\tau_{N'}$.

Finally,

$$\begin{aligned}\Psi(A \circ B) &= \Psi(\Psi'(A'B')) = |X|^t(A'B') = |X|^t(B'A')\\ &= \frac{1}{|X|}(|X|\,{}^tA')(|X|\,{}^tB') = \frac{1}{|X|}\Psi(A)\Psi(B)\end{aligned}$$

shows, together with (iii), that Ψ (and similarly Ψ') is a duality. Thus (N, N') is a dual pair. If $N = N'$ and $\Psi = \Psi'$, $\Psi^2 = |X|\tau_N$ and N is self-dual. □

Remarks

(i) We do not know if $N = N'$ is a sufficient condition for the self-duality of N (see the remark at the end of Section 2.1).
(ii) The idea of the construction of Theorem 1 comes from [31]. Actually, Theorem 1 generalizes the main result of [31] which states that (for symmetric W) the set of symmetric matrices in $N(W)$ is a symmetric Bose-Mesner algebra.

(iii) The algebra N' has an interesting interpretation in the context of commuting squares. A commuting square associated with a type II matrix has a certain *Markovian property* which allows the application of the *basic construction* to this square. This construction produces an infinite grid of commuting squares, with the initial square situated in the left and lowest corner. Our algebra N' can be described in terms of this initial square and the adjacent one on its right:

$$\begin{array}{ccccc} D_X = A_{1,0} & \subset & M_X = A_{1,1} & \subset & A_{1,2} \\ \cup & & \cup & & \cup \\ \mathbf{C} = A_{0,0} & \subset & W^{-1} D_X W = A_{0,1} & \subset & A_{0,2} \end{array}$$

The *second relative commutant* associated with the initial square is the algebra

$$A'_{1,0} \cap A_{0,2} = \{a \in A_{0,2} \mid ab = ba \text{ for all } b \in A_{1,0}\}$$

(note that ab and ba are well defined elements of $A_{1,2}$). It is shown in [1] that this second relative commutant can be identified with the subalgebra $N' = N'(W)$ of M_X via some appropriate isomorphism. As a consequence, some of the results and examples to follow may have some interest in the study of towers of algebras and subfactors (see [23]).

3.2. Equivalences and the symmetric case

If W is a type II matrix and Δ, Δ' are invertible diagonal matrices in M_X, clearly $\Delta W \Delta'$ is also a type II matrix, and we shall say that this matrix is obtained from W by *scaling*.

In the sequel, W_1 and W_2 are type II matrices.

Proposition 2 *If W_2 is obtained from W_1 by scaling, $N(W_2) = N(W_1)$ and $N'(W_2) = N'(W_1)$.*

Proof: The effect of scaling on each vector Y_{bc} or Y'_{bc} defined by (19) and (20) is a multiplication by a non-zero scalar. □

Now if P, P' are permutation matrices in M_X, PWP' is also a type II matrix, and we shall say that it is obtained from W by *permutation*.

Proposition 3 *If W_2 is obtained from W_1 by permutation, $N(W_2)$ is combinatorially isomorphic to $N(W_1)$ and $N'(W_2)$ is combinatorially isomorphic to $N'(W_1)$.*

Proof: There exist permutations α, β of X such that $W_2(x, y) = W_1(\alpha(x), \beta(y))$. For all x, b, c in X, let

$$Y^i_{bc}(x) = \frac{W_i(x, b)}{W_i(x, c)}, \quad (i = 1, 2).$$

Thus $Y^2_{bc}(x) = Y^1_{\beta(b)\beta(c)}(\alpha(x))$. Hence there exists a permutation matrix P such that

$$\left\{Y^2_{bc} \,\middle|\, (b, c) \in X \times X\right\} = \left\{PY^1_{bc} \,\middle|\, (b, c) \in X \times X\right\}.$$

It follows that $N(W_2) = PN(W_1)P^{-1}$, so that $N(W_2)$ is combinatorially isomorphic to $N(W_1)$. Working with ${}^tW_1$ and ${}^tW_2$ we also obtain that $N'(W_2)$ is combinatorially isomorphic to $N'(W_1)$. □

Remarks

(i) The equivalence of type II matrices generated by scaling and transposition corresponds to a natural notion of isomorphism of commuting squares [12]. Propositions 2 and 3 give only a special case of the result that higher relative commutants are invariants of commuting squares [12].
(ii) If W has entries ± 1, it is a Hadamard matrix. It is easy to see that in this case the equivalence generated by scaling and permutation corresponds to the usual notion of Hadamard equivalence. Hence the combinatorial types of $N(W)$ and $N'(W)$ are invariants of Hadamard matrices W under Hadamard equivalence.

Assume now that W is a symmetric type II matrix. Thus $Y_{bc} = Y'_{bc}$ for every b, c in X. Hence $N = N'$ and $\Psi = \Psi'$. As a consequence of Theorem 1 and of the proof of Proposition 2, we obtain:

Proposition 4 *If a type II matrix W is obtained from some symmetric matrix by scaling, $N(W) = N'(W)$ is a self-dual Bose-Mesner algebra.*

A matrix W is *symmetrizable* if it can be obtained from some symmetric matrix by scaling (this definition is clearly equivalent to the one given in Section 2.1 of [24]). It is easy to see that a matrix is symmetrizable iff it can be obtained from its transpose by some scaling (which must be conjugation by a diagonal matrix).

3.3. *Graph descriptions of the dual pair*

Let W be a type II matrix in M_X. The following idea was first introduced by Vaughan Jones [22]. We shall associate with W two undirected graphs G and H on the vertex set $X \times X$ and use them to describe the dual pair (N, N'). These graphs will have no multiple edges (but possibly loops) and two vertices (possibly equal) will be said to be adjacent if they are joined by an edge.

Given two column vectors T, T' indexed by X, we write $\langle T, T'\rangle$ for their usual scalar product $\sum_{x\in X} T(x)T'(x)$, and $\langle\!\langle T, T'\rangle\!\rangle$ for their Hermitian product $\sum_{x\in X} T(x)\overline{T'(x)} = \langle T, \overline{T'}\rangle$.

Two vertices (b, c), (d, e) will be adjacent in G (respectively, H) iff $\langle Y_{bc}, Y_{de}\rangle \neq 0$ (respectively, $\langle\!\langle Y_{bc}, Y_{de}\rangle\!\rangle \neq 0$). Thus H has a loop incident with every vertex, and G may have loops incident with some vertices. We denote by G^2 the (proper) squared graph of G:

two vertices are adjacent in G^2 if there is a vertex to which they are both adjacent in G (this implies the existence in G^2 of a loop incident with every non-isolated vertex of G).

Let $C_1, \ldots, C_p$ (respectively, $K_1, \ldots, K_q$; $L_1, \ldots, L_r$) be the connected components of G (respectively, G^2; H).

Let $A(C_i)$ (respectively, $A(K_i)$; $A(L_i)$) be the matrix in M_X with (b, c)-entry equal to 1 if $(b, c) \in C_i$ (respectively, K_i; L_i) and to 0 otherwise.

Let $V(C_i)$ (respectively, $V(K_i)$; $V(L_i)$) be the $\mathbf{C}$-linear span of the set of vectors Y_{bc} such that (b, c) belongs to C_i (respectively, K_i; L_i). We denote by V the space of column vectors indexed by X.

Theorem 5

(i) *$V = \oplus_{i=1}^{p} V(C_i)$. Let $E(C_i)$ $(i = 1, \ldots, p)$ be the matrices in M_X, which represent the projections $V \to V(C_i)$ in the canonical basis of V. Then $\{E(C_i) \mid i = 1, \ldots, p\}$ is the basis of ordinary idempotents of $\tilde{N}$, and $\{A(C_i) \mid i = 1, \ldots, p\}$ is the basis of Hadamard idempotents of $\tilde{N}'$.*

(ii) *$V = \oplus_{i=1}^{q} V(K_i)$. Let $E(K_i)$ $(i = 1, \ldots, q)$ be the matrices in M_X which represent the projections $V \to V(K_i)$ in the canonical basis of V. Then $\{E(K_i) \mid i = 1, \ldots, q\}$ is the basis of ordinary idempotents of N, and $\{A(K_i) \mid i = 1, \ldots, q\}$ is the basis of Hadamard idempotents of N'.*

(iii) *$V = \oplus_{i=1}^{r} V(L_i)$. Let $E(L_i)$ $(i = 1, \ldots, r)$ be the matrices in M_X which represent the projections $V \to V(L_i)$ in the canonical basis of V. Then $\{E(L_i) \mid i = 1, \ldots, r\}$ is the basis of ordinary idempotents of N, and $\{A(L_i) \mid i = 1, \ldots, r\}$ is the basis of Hadamard idempotents of N'.*

As a consequence, $q = r \geq p$ with full equality if and only if N is symmetric.

Proof: By (23), $\{Y_{bc} \mid (b, c) \in X \times X\}$ spans V, and hence

$$V = \sum_{i=1}^{p} V(C_i) = \sum_{i=1}^{q} V(K_i) = \sum_{i=1}^{r} V(L_i).$$

Let us show that these sums are direct. Note that by definition the $V(C_i)$ are mutually orthogonal with respect to the usual scalar product, and the $V(L_i)$ are mutually orthogonal with respect to the Hermitian product. Since these products are non-degenerate,

$$V(C_i) \cap \sum_{j \neq i} V(C_j) = \{0\} \quad \text{and} \quad V(L_i) \cap \sum_{j \neq i} V(L_j) = \{0\}$$

for all i, and hence $V = \oplus_{i=1}^{p} V(C_i) = \oplus_{i=1}^{r} V(L_i)$.

Let us now relate the connected components of G^2 with those of G.

If a connected component of G is non-bipartite (this occurs for instance if some vertex is incident with a loop), any two of its vertices can be joined in G by a path of even length (possibly with repeated vertices and edges) and hence it also defines a connected component of G^2.

On the other hand, a bipartite connected component of G splits into two connected components of G^2, each one corresponding to an independent set in G. Let K_i, K_j be two connected components of G^2 corresponding in this way to a bipartition of a connected component C_k of G. Since K_i is independent in G, $V(K_i)$ is orthogonal to itself with respect to the usual scalar product, and similarly for $V(K_j)$. Hence, $V(K_i) \cap V(K_j)$ is orthogonal to $V(C_k)$, and also to $\oplus_{\ell \neq k} V(C_\ell)$. It follows that $V(K_i) \cap V(K_j) = \{0\}$ and $V(C_k) = V(K_i) \oplus V(K_j)$. We conclude that $V = \oplus_{i=1}^{q} V(K_i)$.

It is clear from their definition that the $E(C_i)$ (respectively, $E(K_i)$; $E(L_i)$) are orthogonal idempotents in M_X. These idempotents belong to N, since $E(C_i)Y_{bc} = Y_{bc}$ if $(b, c) \in C_i$, $E(C_i)Y_{bc} = 0$ otherwise, and similarly for $E(K_i)$ and $E(L_i)$. This also shows that $\Psi(E(C_i)) = A(C_i)$, $\Psi(E(K_i)) = A(K_i)$, $\Psi(E(L_i)) = A(L_i)$. Then, in view of Theorem 1, the proof of Theorem 5 will be completed if we show that each of $\{E(K_i) \mid i = 1, \ldots, q\}$ and $\{E(L_i) \mid i = 1, \ldots, r\}$ spans N, and that $\{E(C_i) \mid i = 1, \ldots, p\}$ spans $\tilde{N}$ (the equality $\Psi(\tilde{N}) = \tilde{N}'$ comes from the equality $\Psi\tau_N = \tau_{N'}\Psi$ which follows immediately from Theorem 1—see also Remark (iii) in Section 2.1).

Note that

$$\langle Y_{bc},\ Y_{cb}\rangle = \sum_{x \in X} \frac{W(x, b)}{W(x, c)} \cdot \frac{W(x, c)}{W(x, b)} = |X| \neq 0$$

for every (b, c) in $X \times X$ and hence each $A(C_i)$ is symmetric. It follows that $E(C_i) \in \tilde{N}$ for $i = 1, \ldots, p$ (using again the equality $\Psi\tau_N = \tau_{N'}\Psi$).

If (b, c) and (d, e) are adjacent vertices of G, for every matrix A in N,

$$\begin{aligned}\Psi(A)(b, c)\langle Y_{bc},\ Y_{de}\rangle &= \langle AY_{bc},\ Y_{de}\rangle = \langle Y_{bc},\ {}^tAY_{de}\rangle \\ &= (\Psi({}^tA))(d, e)\langle Y_{bc},\ Y_{de}\rangle\end{aligned}$$

implies that $\Psi(A)(b, c) = \Psi({}^tA)(d, e)$.

In particular, if $A \in \tilde{N}$, $\Psi(A)(b, c) = \Psi(A)(d, e)$ whenever (b, c) and (d, e) belong to the same connected component of G. It follows that $\Psi(A)$ belongs to the linear span of the $A(C_i)$, $i = 1, \ldots, p$, and hence A belongs to the linear span of the $E(C_i)$, $i = 1, \ldots, p$.

In general, $\Psi(A)(b, c) = \Psi(A)(d, e)$ whenever (b, c) and (d, e) belong to the same connected component of G^2. Then the same argument shows that every matrix A in N belongs to the linear span of the $E(K_i)$, $i = 1, \ldots, q$.

Finally, if (b, c) and (d, e) are adjacent vertices in H, for every matrix A in N,

$$\begin{aligned}\Psi(A)(b, c)\langle\langle Y_{bc},\ Y_{de}\rangle\rangle &= \langle\langle AY_{bc},\ Y_{de}\rangle\rangle = \langle\langle Y_{bc},\ {}^tAY_{de}\rangle\rangle \\ &= \overline{(\Psi({}^t\overline{A}))(d, e)}\langle\langle Y_{bc},\ Y_{de}\rangle\rangle\end{aligned}$$

implies that $\Psi(A)(b, c) = \overline{\Psi({}^t\overline{A})}(d, e)$. Now $\Psi(\overline{A}) = \overline{\Psi(A)}$ for every A in N, as can be easily checked by expressing A in the basis of ordinary idempotents and using (13) and the fact that Hadamard idempotents are real.

It follows that $\Psi(A)(b, c) = \Psi(A)(d, e)$ whenever (b, c) and (d, e) belong to the same connected component of H. Hence A belongs to the linear span of the $E(L_i)$, $i = 1, \ldots, r$.
□

It is clear from the above proof that N is non-symmetric if and only if some component of G splits into two components of G^2, or equivalently into two components of H. Moreover in such a splitting there exist b, c in X such that (b, c) is in one of the new components and (c, b) is in the other. Hence we obtain the following result.

Proposition 6 *N is non-symmetric if and only if G has a bipartite connected component. Moreover, if N is non-symmetric, there exist b, c in X such that the following properties hold:*

(i) $\sum_{x\in X} \frac{W(x,b)^2}{W(x,c)^2} = 0, \quad \sum_{x\in X} \frac{W(x,c)^2}{W(x,b)^2} = 0$

(ii) $\sum_{x\in X} \frac{W(x,b)\overline{W(x,c)}}{W(x,c)\overline{W(x,b)}} = 0$

Consequently, if W is a real matrix, N is symmetric.

Proof:

(i) is equivalent to $\langle Y_{bc}, Y_{bc}\rangle = \langle Y_{cb}, Y_{cb}\rangle = 0$, i.e., to the fact that there is no loop in G incident to (b, c) or (c, b). This must hold if (b, c) and (c, b) are not adjacent in G^2, since they are adjacent in G.

(ii) is equivalent to $\langle\langle Y_{bc}, Y_{cb}\rangle\rangle = 0$, i.e., to the fact that (b, c) and (c, b) are not adjacent in H. □

Remark When all entries of W have the same absolute values (i.e., W is unitary up to a factor) (i) and (ii) are equivalent.

Remark As suggested by one of the referees, we might have defined type II matrices with different sets of rows and columns. However, the type III condition can be defined only if we identify these sets. For the sake of simplicity, we choose the same set X for both rows and columns. Here we state briefly what occurs if we distinguish the set X of rows and the set X' of columns with the same cardinality. We regard W as a linear map from $\mathbf{C}[X]$ to $\mathbf{C}[X']$, where $\mathbf{C}[X]$ denotes the vector space with basis X. Then, the type II condition becomes the existence of $W^- \in \mathrm{Hom}(\mathbf{C}[X'], \mathbf{C}[X])$ such that

$$W \circ {}^tW^- = J \in \mathrm{Hom}(\mathbf{C}[X], \mathbf{C}[X'])$$
$$WW^- = n \cdot I \in \mathrm{Hom}(\mathbf{C}[X'], \mathbf{C}[X']).$$

Now $N(W)$ is a subalgebra of Hom $(\mathbf{C}[X'], \mathbf{C}[X'])$ and $N'(W)$ is a subalgebra of $\mathrm{Hom}(\mathbf{C}[X], \mathbf{C}[X])$, which are dual to each other. Every result in Sections 3 and 4 that uses only the type II condition for W is still valid. On the contrary, any statement concerning the type I, III conditions or symmetricity of W, like Propositions 4, 8–10, 12, and Theorem 11, requires an identification of X and X'.

4. Some applications

4.1. Tensor products

Given two finite sets X_1, X_2, we consider the bilinear map from $M_{X_1} \times M_{X_2}$ to $M_{X_1\times X_2}$ which associates with $A_1 \in M_{X_1}$, $A_2 \in M_{X_2}$ their Kronecker product $A_1 \otimes A_2$ defined by:

$$A_1 \otimes A_2((x_1, x_2), (y_1, y_2)) = A_1(x_1, y_1)A_2(x_2, y_2) \quad \text{for all } x_1, y_1 \in X_1, x_2, y_2 \in X_2.$$

This establishes an isomorphism of vector spaces between $M_{X_1\times X_2}$ and the tensor product $M_{X_1} \otimes M_{X_2}$ and legitimates the use in what follows of the symbol $\otimes$ for the Kronecker product of matrices as well as for the tensor product of vector spaces of matrices.

We note that if $\mathcal{A}_1, \mathcal{A}_2$ are Bose-Mesner algebras in M_{X_1}, M_{X_2} respectively, then $\mathcal{A}_1 \otimes \mathcal{A}_2$ (which is the linear span of the matrices $A_1 \otimes A_2$, $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$) is a Bose-Mesner algebra in $M_{X_1\times X_2}$ since $(A_1 \otimes A_2)(B_1 \otimes B_2) = (A_1B_1) \otimes (A_2B_2)$, $(A_1 \otimes A_2) \circ (B_1 \otimes B_2) = (A_1 \circ B_1) \otimes (A_2 \circ B_2)$ and ${}^t(A_1 \otimes A_2) = {}^tA_1 \otimes {}^tA_2$ for every A_1, B_1 in $\mathcal{A}_1$ and A_2, B_2 in $\mathcal{A}_2$.

Let now $W_1 \in M_{X_1}$, $W_2 \in M_{X_2}$ be two type II matrices. It is easy to check that $W = W_1 \otimes W_2$ is also a type II matrix.

Using formula (19), we associate with W, W_1, W_2 the vectors $Y_{(b_1,b_2)(c_1,c_2)}$, $Y^1_{b_1c_1}$, $Y^2_{b_2c_2}$ respectively, for all b_1, c_1 in X_1 and b_2, c_2 in X_2.

Proposition 7
(i) $N(W) = N(W_1) \otimes N(W_2)$.
(ii) $\widetilde{N(W)} \neq \widetilde{N(W_1)} \otimes \widetilde{N(W_2)}$ *if and only if both* $N(W_1)$ *and* $N(W_2)$ *are non-symmetric.*

Proof: It follows immediately from (19) that

$$Y_{(b_1,b_2)(c_1,c_2)}((x_1, x_2)) = Y^1_{b_1c_1}(x_1)Y^2_{b_2c_2}(x_2).$$

Hence, for every $A_1 \in N(W_1)$, $A_2 \in N(W_2)$, $Y_{(b_1,b_2)(c_1,c_2)}$ is an eigenvector of $A_1 \otimes A_2$.

This implies that $N(W_1) \otimes N(W_2) \subseteq N(W)$ and $\widetilde{N(W_1)} \otimes \widetilde{N(W_2)} \subseteq \widetilde{N(W)}$. Let G, H be the graphs associated with W as in Section 3.3 and let G_i, H_i $(i = 1, 2)$ be the corresponding graphs for W_1, W_2. Clearly

$$\left\langle Y_{(b_1,b_2)(c_1,c_2)},\ Y_{(d_1,d_2)(e_1,e_2)}\right\rangle = \left\langle Y^1_{b_1c_1},\ Y^1_{d_1e_1}\right\rangle\left\langle Y^2_{b_2c_2},\ Y^2_{d_2e_2}\right\rangle$$

and similarly for the Hermitian product $\langle\!\langle\ ,\ \rangle\!\rangle$.

Hence $((b_1, b_2), (c_1, c_2))$, $((d_1, d_2), (e_1, e_2))$ are adjacent in G (respectively, H) if and only if (b_1, c_1), (d_1, e_1) are adjacent in G_1 (respectively, H_1) and (b_2, c_2), (d_2, e_2) are adjacent in G_2 (respectively, H_2).

If we identify each vertex $((b_1, b_2), (c_1, c_2))$ of G with the pair $((b_1, c_1), (b_2, c_2))$ formed with one vertex of G_1 and one vertex of G_2 we see that G is the *categorical product* $G_1 \cdot G_2$ of G_1 and G_2 (see [34]). Similarly $H = H_1 \cdot H_2$.

We shall need the following graph-theoretical result (see [34]): the categorical product of two connected graphs is disconnected if and only if each of these graphs is bipartite. From this it easily follows that the number of connected components of the categorical product of two graphs is different from the product of the number of connected components of these graphs if and only if each of these graphs has a bipartite component.

Let r, r_1, r_2 be the numbers of connected components of H, H_1, H_2. Each vertex in H_1 or H_2 is incident with a loop, hence H_1 and H_2 have no bipartite components. It follows that $r = r_1 r_2$. By Theorem 5, this means that $\dim N(W) = (\dim N(W_1))(\dim N(W_2)) = \dim N(W_1) \otimes N(W_2)$ and hence $N(W) = N(W_1) \otimes N(W_2)$.

Now let p, p_1, p_2 be the numbers of connected components of G, G_1, G_2. The same argument shows that $\widetilde{N(W)} \neq \widetilde{N(W_1)} \otimes \widetilde{N(W_2)}$ iff $p \neq p_1 p_2$, that is iff both G_1, G_2 have some bipartite component. By Proposition 6, this occurs if and only if both $N(W_1)$ and $N(W_2)$ are non-symmetric. □

4.2. *The type I property and expressions for duality*

Let W^+ be a type II matrix in M_X. We define W^- so that the equations

(16) $W^+ W^- = |X| I, \quad W^+ \circ {}^t W^- = J$

hold.

Proposition 8

(i) *If $I \circ W^+ = aI$ for some $a \in \mathbf{C}^*$, $\Psi(A) = a^{-1} W^+ \circ (({}^t W^+ \circ A)\, {}^t W^-)$ for every A in $N(W^+)$.*

(ii) *If $W^- J = DaJ$ for some a and D in $\mathbf{C}^*$, $\Psi(A) = D^{-1} a^{-1} ({}^t W^+ \circ (W^- A))\, {}^t W^-$ for every A in $N(W^+)$.*

Proof:

(i) Considering b-entries in the equation

(21) $AY_{bc} = (\Psi(A))(b, c) Y_{bc},$

we obtain

$$\sum_{x \in X} A(b, x) W^+(x, b) W^-(c, x) = (\Psi(A))(b, c) W^+(b, b) W^-(c, b)$$

and hence

$$(\Psi(A))(b, c) = a^{-1} W^+(b, c) \sum_{x \in X} A(b, x) W^+(x, b) W^-(c, x).$$

It is easy to see that the right-hand side is the (b, c)-entry of $a^{-1}\, W^+ \circ (({}^t W^+ \circ A)\, {}^t W^-)$.

(ii) Considering b-entries in the equation

$$W^- AY_{bc} = (\Psi(A))(b,c)\, W^-\, Y_{bc},$$

we obtain

$$\begin{aligned}&\sum_{y\in X} W^-(b,y)\sum_{x\in X} A(y,x)W^+(x,b)W^-(c,x)\\ &\quad= (\Psi(A))(b,c)\sum_{y\in X} W^-(b,y)W^+(y,b)W^-(c,y)\\ &\quad= (\Psi(A))(b,c)\sum_{y\in X} W^-(c,y) = Da(\Psi(A))(b,c).\end{aligned}$$

It is easy to see that the left-hand side is the (b,c)-entry of $({}^tW^+ \circ (W^-A))\, {}^tW^-$. □

Remarks

(i) Many other similar expressions for Ψ or Ψ' can be obtained under each of the hypotheses of Proposition 8.
(ii) Each of these hypotheses can be realized by using scaling.

4.3. Spin models for link invariants

We keep the notations of the preceding section.

Proposition 9 *The following properties are equivalent:*
(i) $W^+ \in N(W^+)$
(ii) W^+ *satisfies the type III condition* (17) *for some* $D \in \mathbf{C}^*$.

Proof: (17) can be written

$${}^tW^+Y_{\beta\gamma} = DW^-(\beta,\gamma)Y_{\beta\gamma} \quad \text{(for all } \beta, \gamma \text{ in } X).$$

Thus (17) is equivalent to the property that ${}^tW^+ \in N(W^+)$ with $\Psi({}^tW^+) = DW^-$, or, by Theorem 1, to the property that $W^+ \in N(W^+)$ with $\Psi(W^+) = D\,{}^tW^-$.

Conversely, if $W^+ \in N(W^+)$, or equivalently ${}^tW^+ \in N(W^+)$, let $F = \Psi({}^tW^+)$. Then the equality ${}^tW^+Y_{\beta\gamma} = F(\beta,\gamma)Y_{\beta\gamma}$ can be written

$$\sum_{x\in X} W^+(x,\alpha)W^+(x,\beta)W^-(\gamma,x) = F(\beta,\gamma)W^+(\alpha,\beta)W^-(\gamma,\alpha) \quad \text{(for all } \alpha \in X).$$

When $\alpha = \gamma$ this becomes

$$\sum_{x\in X} W^+(x,\beta) = F(\beta,\alpha)W^+(\alpha,\beta)W^-(\alpha,\alpha).$$

Since $N(W^+)$ is a Bose-Mesner algebra, there exist constants a, D in $\mathbf{C}^*$ such that $JW^+ = Da^{-1}J$, $I \circ W^- = a^{-1}I$ (note that W^- has non-zero entries, so that a diagonal element is of the form a^{-1} for some $a \in \mathbf{C}^*$, and W^+ is invertible, so that JW^+ is non-zero).

Thus we obtain $Da^{-1} = F(\beta,\alpha)W^+(\alpha,\beta)a^{-1}$, i.e., $F(\beta,\alpha) = DW^-(\beta,\alpha)$. Hence $\Psi({}^tW^+) = F = DW^-$ as required. □

Remark If (17) holds for some arbitrary D in $\mathbf{C}^*$, we may multiply W^+ by a suitable constant to obtain the same property with $D^2 = |X|$. This normalization is needed to realize the topological invariance of the partition function.

Proposition 10 *If the type III condition* (17) *holds then* W^+ *is symmetrizable.*

Proof: The exchange of α, β in (17) leaves the left-hand side invariant. Consideration of the right-hand side leads to the identity

$$W^+(\alpha,\beta)W^-(\beta,\gamma)W^-(\gamma,\alpha) = W^+(\beta,\alpha)W^-(\alpha,\gamma)W^-(\gamma,\beta),$$

which for any fixed γ in X is equivalent to the equation

$$\Delta W^+\Delta' = \Delta'\,{}^tW^+\Delta$$

where $\Delta = \mathrm{Diag}[W^-(\gamma,x)]_{x\in X}$, $\Delta' = \mathrm{Diag}[W^-(x,\gamma)]_{x\in X}$. Since Δ and Δ' are invertible and $\Delta W^+\Delta'$ is symmetric, W^+ is symmetrizable. □

We may now state the following result.

Theorem 11 *Let W be a type II matrix. Then $W \in N(W)$ if and only if some scalar multiple W^+ of W gives a solution to the invariance equations* (15), (16), (17) *and hence defines a link invariant. In this case*, $N(W) = N(W^+)$ *is a self-dual Bose-Mesner algebra*, *with duality Ψ given by*

$$\begin{aligned}\Psi(A) &= a^{-1}W^+ \circ ({}^tW^-({}^tW^+ \circ A))\\ &= D^{-1}a^{-1}\,{}^tW^-({}^tW^+ \circ (W^-A)) \quad (\textit{for all } A \in N(W^+)).\end{aligned}$$

Proof: This is an immediate consequence of Propositions 9, 10, 4 and 8 using the commutativity of the matrix product of $N(W^+)$, and the remark that W^- and ${}^tW^-$ belong to $N(W^+)$ since ${}^tW^-$ is the inverse of W^+ under Hadamard product. □

Remark Theorem 11 generalizes the main result of [20], which states that symmetric matrices W^+, W^- satisfying (15), (16) and (17) belong to some symmetric self-dual Bose-Mesner algebra with duality expressed in a similar way as in Theorem 11. The Bose-Mesner algebra in [20] is the image of a certain algebra of tangles under a matrix-valued partition function map. It can be shown that this algebra is contained in $N(W^+)$ (see [20], Proposition 6). The proof relies on a diagrammatic description of $N(W^+)$ given in [1] for the second relative commutant of a commuting square associated with a spin model. The two algebras need not be equal, even if they are both symmetric, as shown by the following example. If a symmetric Hadamard matrix W^+ satisfies the type III property (17), one can show that the algebra from [20] has dimension 3 (when $|X| > 4$). This is because $W^+ = W^-$ and hence the partition function map "forgets" the spatial structure of tangles. On the other hand, we shall give in Section 5.2 examples of Hadamard matrices W in M_X, where $|X|$ is any even power of 2, which yield an algebra $N(W)$ of dimension $|X|$ containing W.

We now point out that the expression of the duality Ψ given in Theorem 11 means that W^+ is given by a solution of the *modular invariance equation* for the Bose-Mesner algebra $N(W^+)$ (see [4] where slightly different notations are used).

Let P be the matrix of Ψ in the basis of ordinary idempotents $\{E_i \mid i = 0, \dots, d\}$ of $N(W^+)$. Thus the indices of the Hadamard idempotents A_j, $j = 0, \dots, d$ can be chosen so that $A_j = \sum_{i=0}^d P(i, j)E_i = \Psi(E_j)$ for $i = 0, \dots, d$ (and P is a *first eigenmatrix* of $N(W^+)$). We write ${}^tW^-$ in the form $D\sum_{i=0}^d t_i E_i$ and reformulate the identity

$$\Psi(A) = a^{-1}W^+ \circ ({}^tW^-({}^tW^+ \circ A)) \quad (A \in N(W^+))$$

in terms of the t_i. Let $T = \mathrm{Diag}[t_i]_{i\in\{0,\dots,d\}}$.

Proposition 12 *The identity*

$$\Psi(A) = a^{-1}W^+ \circ ({}^tW^-({}^tW^+ \circ A)) \quad (A \in N(W^+))$$

is equivalent to the equation $(PT)^3 = aD^3 I$.

Proof: Recall from the proof of Proposition 9 that $\Psi({}^tW^+) = DW^-$ and hence $\Psi(W^-) = DW^+$ (this also follows easily from the above identity applied to $A = W^-$). Hence ${}^tW^+ = \sum_{i=0}^d t_i A_i$. Clearly T is the matrix of the map $A \to {}^tW^+ \circ A$ in the basis of Hadamard idempotents. Similarly, the matrix of the map $A \to {}^tW^- A$ in the basis of ordinary idempotents is DT. Hence, the matrix of this map in the basis of Hadamard idempotents is $DP^{-1}TP$.

Since ${}^t({}^tW^+ \circ {}^tA) = W^+ \circ A$, the matrix of the map $A \to W^+ \circ A$ in the basis of Hadamard idempotents is RTR where R is the matrix of the transposition map in this basis.

Finally, the matrix of Ψ in the basis of Hadamard idempotents is $P^{-1}PP = P$.

Thus the identity of Theorem 11 translates into

$$P = a^{-1}(RTR)(DP^{-1}TP)T.$$

Since $P^2 = D^2 R$, this becomes $P = a^{-1}D^{-1}RTPTPT$ or $(PT)^3 = aD^3 I$. □

This is the modular invariance equation considered in [4], whose origin is to be found in [2].

We conclude that spin models for link invariants in the sense of [26] can be classified in terms of solutions of the modular invariance equations for self-dual Bose-Mesner algebras.

Note, however, that there exist solutions of the modular invariance equations which do not satisfy the type III condition (see [4]).

5. Examples

5.1. Abelian group schemes

Let X be a finite abelian group written additively.

For all i in X we define A_i in M_X by the identity $A_i(x, y) = \delta(y-x, i)$. Then properties (5)–(7) trivially hold, and properties (8), (9) also hold with $i' = -i$, $p_{ij}^k = \delta(k, i+j)$. Thus the A_i are the Hadamard idempotents of a Bose-Mesner algebra $\mathcal{A}_X$ (the corresponding association scheme is the *group scheme* of X) of dimension $|X|$. For convenience we replace the index set $\{0, \ldots, d\}$ (where $d = |X| - 1$) by X.

Let $\{E_i \mid i \in X\}$ be the basis of ordinary idempotents of $\mathcal{A}_X$, and write $A_j = \sum_{i\in X} P(i, j)E_i$ for all j in X: this defines the *first eigenmatrix* P of $\mathcal{A}_X$ (for some choice of indexes of the idempotents).

The equality $A_j A_k = A_{j+k}$ translates into the identity $P(i, j)P(i, k) = P(i, j+k)$. Thus each row of P represents a character of X. It is well known that there exists exactly $|X|$ such characters, and since P is invertible each one appears as a row of P. In the sequel we write $P(i, j) = \chi_i(j)$ $(i, j \in X)$, and $\{\chi_i \mid i \in X\}$ is the set of characters of X. Property (18(ii)) for $W = P$ reduces to the classical identity $\sum_{i\in X} \chi_i(b-c) = |X|\delta(b, c)$, or equivalently property (18(i)) reduces to the orthogonality relations of characters. Thus P is a type II matrix.

By (19), $Y_{b,c}(i) = \chi_i(b-c)$ for all i, b, c in X. Hence the set of vectors $Y_{b,c}$ (b, c in X) is identical to the set of columns of P. It follows that $N(P)$ is the set of matrices A in M_X such that $AP = P\Delta$ for some diagonal matrix Δ and hence has dimension $|X|$. Moreover, for A in $N(P)$, the Eq. (21) $AY_{bc} = (\Psi(A))(b, c)Y_{bc}$ shows that $\Psi(A) \in \mathcal{A}_X$ since Y_{bc} only depends on $b - c$. It then follows from Theorem 1 that $N'(P) = \mathcal{A}_X$.

$N(P)$ itself is not in general equal to $\mathcal{A}_X$. However it is possible to choose the indexing of the characters of X so that $\chi_i(j) = \chi_j(i)$ for all i, j in X (see for instance [7] Proposition 2.10.7). Then P is symmetric and $N(P) = N'(P)$, $\Psi = \Psi'$.

Remark One can show that dualities of $\mathcal{A}_X$ correspond exactly to the indexings of the idempotents for which P is symmetric. These dualities are classified in [6].

Assume now that P is symmetric.

In [4] it is shown that the modular invariance equation $(PT)^3 = \lambda I$, where $\lambda \neq 0$ and $T = \mathrm{Diag}[t_i]_{i\in X}$, is equivalent to the identity $\chi_i(j)t_i t_j = t_{i+j}$ (for some appropriate normalization). Let then $W = \sum_{i\in X} t_i A_i \in \mathcal{A}_X$.

One easily checks that

$$\frac{W(x, b)}{W(x, c)} = \frac{t_b}{t_c}\chi_x(c-b)$$

and it follows that W is a type II matrix. Moreover, one shows as above that $N(W) = \mathcal{A}_X$.

Now it follows from Theorem 11 that (after suitable normalization) W defines a link invariant (see also [4, 18] for other proofs of this fact).

Remarks

(i) If for instance X is a cyclic group of odd order n, ω is a primitive nth root of unity and $\chi_i(j) = \omega^{2ij}$, setting $t_i = \omega^{i^2}$ we obtain a symmetric matrix $W = \sum_{i\in X} t_i A_i$ such that $N(W)$ is the non-symmetric Bose-Mesner algebra $\mathcal{A}_X$.

(ii) We leave it to the reader to reprove the above results using the graph descriptions of Theorem 5.

Conversely, suppose now that $N(W)$ has dimension $|X| = n$ for a type II matrix W in M_X. We shall show that W is obtained by scaling from a character table for some additive group structure on X.

By Theorem 1, $N'(W)$ has also dimension n.

Let $A'_0, \ldots, A'_{n-1}$ be the Hadamard idempotents of $N'(W)$. These matrices commute with J (by (6) and (9)) and hence have constant (non-zero) row sum. Since $\sum_{i=0}^{n-1} A'_i = J$ by (6), each A'_i is a permutation matrix. It then easily follows from (9), (7) and (8) that these n permutation matrices form an abelian group with identity element A'_0. From now on we identify $\{0, \ldots, n-1\}$ with X, so that the basis of Hadamard idempotents of $N'(W)$ is $\{A'_x \mid x \in X\}$. We equip X with the additive group structure such that $A'_x(y, z) = \delta(z-y, x)$ for every x, y, z in X, and we denote by $e \in X$ the zero element of this group.

Using appropriate scaling we may assume that $W(x, e) = W(e, x) = 1$ for every $x \in X$.

For any two elements y, z of X, consider the vertices $(y, y+z)$ and (e, z) of the graph H defined in Section 3.3. Clearly these vertices belong to the connected component of H corresponding to the Hadamard idempotent A'_z as stated in Theorem 5 (iii). Hence each of $Y_{y,y+z}$ and $Y_{e,z}$ is orthogonal (with respect to the Hermitian product) to all vectors of the form $Y_{z',e}$, where $z' \neq -z$, and, by (23), $Y_{y,y+z}$ and $Y_{e,z}$ are collinear. Comparing e-entries we obtain that $Y_{y,y+z} = Y_{e,z}$. By (19) this gives $W(x, y)W(x, y+z)^{-1} = W(x, z)^{-1}$ for every $x \in X$. Thus, for every x, y, z in X, $W(x, y+z) = W(x, y)W(x, z)$. This means that for every x in X, the mapping $y \to W(x, y)$ from X to $\mathbf{C}^*$ is a character of the additive group X. There are n such characters and n rows of W. Since W is invertible, each character of X appears exactly once as a row of W, and W is a character table for the additive group X.

5.2. *Hadamard matrices*

Keeping the notations of the preceding section, if X is an elementary abelian 2-group its characters take their values in $\{+1, -1\}$ and hence P is a Hadamard matrix (called a Sylvester matrix). Thus there exists Hadamard matrices W in M_X such that $\dim N(W) = |X|$ whenever $|X|$ is a power of 2.

Moreover, if $|X|$ is an even power of 2, let us identify X with $GF(2)^{2m}$ for some $m \geq 1$, and let Q be a quadratic form on X associated with some symplectic form B. That is, B is a non-degenerate bilinear form on X with $B(x, x) = 0$ for all $x \in X$, and $Q : X \to GF(2)$

satisfies the identity

$$Q(x) + Q(y) + Q(x + y) = B(x, y).$$

The characters of X can be labeled such that $\chi_i(j) = (-1)^{B(i,j)}$ for all i, j in X, and then $\chi_i(j) = \chi_j(i)$. Moreover, if we set $t_i = (-1)^{Q(i)}$ for every i in X, the modular invariance equation $\chi_i(j)t_i t_j = t_{i+j}$ is satisfied. Hence $W = \sum_{i \in X} t_i A_i$ is a Hadamard matrix with $W \in N(W) = \mathcal{A}_X$.

When $|X| = 16$ and X is identified with $GF(2)^4$, our computations show that for every Hadamard matrix W in M_X, $N(W)$ is contained in $\mathcal{A}_X$ (up to combinatorial isomorphism) and thus we have dual pairs of Bose-Mesner algebras coming from an "actual duality" as described in Remark (i) at the end of Section 2.1. More precisely, it is well known that there exist exactly 5 Hadamard equivalence classes, called *Hall's Classes I–V* (see [33], p. 420), of Hadamard matrices of order 16. Using Theorem 5 we have computed $N'(W)$ for one representative matrix W in each class. The resulting Bose-Mesner algebras can be described as follows (up to combinatorial isomorphism).

(i) As shown above, there is a symmetric Hadamard matrix W with $W \in N(W) = N'(W) = \mathcal{A}_X$. This matrix W belongs to Hall's Class I.

(ii) Write $X = X_1 \times X_2$, where $X_1 \simeq X_2 \simeq GF(2)^2$, and identify $\mathcal{A}_X$ with $\mathcal{A}_{X_1} \otimes \mathcal{A}_{X_2}$ (see Section 4.1). A second symmetric Hadamard matrix W (which belongs to Hall's Class II) yields the Bose-Mesner algebra $N'(W) = \mathcal{A}_{X_1} \otimes I + J \otimes \mathcal{A}_{X_2}$. If Ψ_1 and Ψ_2 are dualities of $\mathcal{A}_{X_1}$ and $\mathcal{A}_{X_2}$ respectively, $(\Psi_1 \otimes \Psi_2)(N'(W)) = \mathcal{A}_{X_1} \otimes J + I \otimes \mathcal{A}_{X_2}$. It is now easy to show, using appropriate combinatorial isomorphisms from $\mathcal{A}_{X_1} \otimes \mathcal{A}_{X_2}$ to $\mathcal{A}_{X_2} \otimes \mathcal{A}_{X_1}$, and between $\mathcal{A}_{X_1}$ and $\mathcal{A}_{X_2}$, that $N'(W)$ is self-dual.

(iii) For a third symmetric Hadamard matrix W (which belongs to Hall's Class III), $N'(W)$ is the linear span of the matrices I, J, A_i, E_j in $\mathcal{A}_X$, where A_i is a Hadamard idempotent distinct from I, E_j is an ordinary idempotent distinct from J, and $A_i E_j = E_j$. It is easy to show that this Bose-Mesner algebra is self-dual.

(iv) Finally, we have a Hadamard matrix W (which belongs to Hall's Class IV) which is not equivalent to its transpose (which belongs to Hall's Class V). The Bose-Mesner algebra $N'(W)$ is the linear span of I, J, A_i, where A_i is a Hadamard idempotent of $\mathcal{A}_X$ distinct from I (or equivalently, the Bose-Mesner algebra of the graph formed by 8 disjoint edges), and $N'({}^tW)$ is the linear span of $I, J, \Psi(A_i)$ for some duality Ψ of $\mathcal{A}_X$ (or equivalently, the Bose-Mesner algebra of the complete bipartite graph $K_{8,8}$). This example is used in [23] to construct a subfactor which is not self-dual.

In contrast with the above case $|X| = 16$, one can show that when $n \equiv 4 \pmod 8$, $n \geq 12$, and W is a Hadamard matrix of order n, $N(W) = N'(W)$ is always the linear span of I and J (see [1]). Indeed this follows easily from Theorem 5 and from the fact that

$$\langle Y_{bc}, Y_{de} \rangle = \sum_{x \in X} \frac{W(x, b)}{W(x, c)} \frac{W(x, d)}{W(x, e)}$$

is non-zero whenever b, c, d, e are all distinct. To check this last statement, we may assume without loss of generality that $W(x, b) = 1$ for all x. Let A, B, C, D be the sets of row indices x such that $(W(x, c),\ W(x, d))$ equals $(1, 1), (1, -1), (-1, 1), (-1, -1)$ respectively. It is well known and easy to prove that $|A| = |B| = |C| = |D| = n/4$. Now let A^+, B^+, C^+, D^+ be the sets of row indices x in A, B, C, D respectively such that $W(x, e) = +1$. The orthogonality of the column e with the columns b, c, d gives $|A^+| + |B^+| + |C^+| + |D^+| = n/2$, $|A^+| + |B^+| + |C - C^+| + |D - D^+| = n/2$, $|A^+| + |C^+| + |B - B^+| + |D - D^+| = n/2$, and hence $|A^+| + |B^+| = |C^+| + |D^+| = |A^+| + |C^+| = |B^+| + |D^+| = n/4$. On the other hand, $\langle Y_{bc}, Y_{de}\rangle = 2|A^+| - |A| + 2|D^+| - |D| - (2|B^+| - |B|) - (2|C^+| - |C|) = 2(|A^+| + |D^+| - |B^+| - |C^+|)$. This is non-zero since otherwise $|A^+| + |D^+| = |B^+| + |C^+| = n/4$ and the three numbers $|A^+| + |B^+|$, $|B^+| + |C^+|$, $|C^+| + |A^+|$ are odd, a contradiction.

5.3. *Type II matrices of size four*

Let $X = \{1, 2, 3, 4\}$. Consider the following symmetric matrix $U(\lambda)$ in M_X for each complex number $\lambda \neq 0$:

$$U(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \lambda & -\lambda \\ 1 & -1 & -\lambda & \lambda \end{pmatrix},$$

As easily shown, $U(\lambda)$ satisfies the type II condition (18).

Proposition 13

(i) *Every type II matrix $W \in M_X$ is obtained by scaling and permutation from $U(\lambda)$ for some λ.*

(ii) *$U(\lambda)$ is obtained from $U(\mu)$ by scaling and permutation if and only if $\mu = \pm\lambda^{\pm 1}$.*

(iii) *Set $N = N(U(\lambda))$. We have* $\dim N = 4$ *if and only if* $\lambda^4 = 1$ (*otherwise* $\dim N = 3$), *and we have* $\dim \tilde{N} = 4$ *if and only if* $\lambda^2 = 1$ (*otherwise* $\dim \tilde{N} = 3$).

Proof:

(i) We may assume $W(1, x) = W(x, 1) = 1$ for all $x \in X$ by scaling. Then the type II condition implies

(24) $$\textstyle\sum_{x\in X} W(b, x) = \sum_{x\in X} \frac{1}{W(b,x)} = \sum_{x\in X} W(x, b) = \sum_{x\in X} \frac{1}{W(x,b)} = 0$$

for all $b \in \{2, 3, 4\}$.

Claim For $b \in \{2, 3, 4\}$, $W(b, d) = W(e, b) = -1$ holds for some $d, e \in X$.

This can be shown as follows. By (24) we have

$$1 + W(b, 2) + W(b, 3) = -W(b, 4) \quad \text{and} \quad 1 + \frac{1}{W(b, 2)} + \frac{1}{W(b, 3)} = -\frac{1}{W(b, 4)}.$$

Multiplying these two equations, we obtain

$$(1 + W(b,2) + W(b,3))\left(1 + \frac{1}{W(b,2)} + \frac{1}{W(b,3)}\right) = 1,$$

and this is equivalent to

$$(W(b,2) + W(b,3))\left(1 + \frac{1}{W(b,2)}\right)\left(1 + \frac{1}{W(b,3)}\right) = 0.$$

So if $W(b,2) \neq -1$ and $W(b,3) \neq -1$, then we must have $W(b,2) + W(b,3) = 0$, and hence $W(b,4) = -1$. The proof that $W(e,b) = -1$ for some $e \in \{2,3,4\}$ is similar.

By the above claim, we may assume $W(2,2) = -1$. We distinguish the following three cases for $W(2,3)$.

First let us consider the case $W(2,3) = 1$. In this case we have $W(2,4) = -1$ by (24) and also $W(3,3) = W(4,3) = -1$ by the above claim and (24). Let us set $W(3,2) = \lambda$. Then we have $W(4,2) = W(3,4) = -\lambda$ and $W(4,4) = \lambda$ by (24). Then W is obtained from $U(\lambda)$ by exchanging the second column and the third column.

The case $W(2,3) = -1$ (in this case we have $W(2,4) = 1$ by (24)) can be reduced to the above case by exchanging the third column and the fourth column.

Next let us consider the case $W(2,3) \neq \pm 1$. Let us set $W(2,3) = \lambda$. Then $W(2,4) = -\lambda$ by (24). By the above claim, one of $W(3,3)$ and $W(4,3)$ must be equal to -1. We may assume $W(3,3) = -1$ (exchange the third and the fourth row if necessary). Then $W(4,3) = -\lambda$ by (24). Let us consider the fourth column. Since $W(2,4) \neq -1$, one of $W(3,4)$ and $W(4,4)$ must be -1 by the claim. If $W(4,4) = -1$, then $W(3,4) = \lambda$ by (24), and the type II condition applied to the third column and the fourth column implies $\lambda^2 = 1$, contradicting $W(2,3) \neq \pm 1$. Therefore we must have $W(3,4) = -1$ and $W(4,4) = \lambda$. Then $W(3,2) = 1$ and $W(4,2) = -1$ by (24). Now W is obtained from $U(\lambda)$ by exchanging the second row and the third row. This completes the proof of (i).

(ii) Assume $\mu = \pm\lambda^{\pm 1}$. When $\mu = -\lambda$, $U(\lambda)$ can be obtained from $U(\mu)$ by exchanging the third row and the fourth row. When $\mu = \lambda^{-1}$, $U(\lambda)$ can be obtained from $U(\mu)$ by the following steps: multiply the third row by λ and the fourth row by $-\lambda$, exchange the first column with the third column, exchange the second column with the fourth column, and then multiply the second row by -1. The case $\mu = -\lambda^{-1}$ is reduced to the above two cases.

To show the converse, we introduce the following set $\Lambda(W)$ for each type II matrix W:

$$\Lambda(W) = \left\{ \frac{W(b,d)W(c,e)}{W(b,e)W(c,d)} \,\middle|\, b,\ c,\ d,\ e \in X \right\}.$$

Clearly $\Lambda(W_1) = \Lambda(W_2)$ holds if W_1 is obtained from W_2 by permutation and scaling.

As easily shown, we have $\Lambda(U(\lambda)) = \{\pm 1, \pm\lambda^{\pm 1}\}$. Hence, if $U(\lambda)$ is obtained from $U(\mu)$ by permutation and scaling, we have $\{\pm 1, \pm\lambda^{\pm 1}\} = \{\pm 1, \pm\mu^{\pm 1}\}$. This implies $\mu = \pm\lambda^{\pm 1}$.

Table 1.

(b,c)				Entries of Y_{bc}			
(1,1)	(2,2)	(3,3)	(4,4)	1	1	1	1
(1,2)	(2,1)	(3,4)	(4,3)	1	1	-1	-1
(1,3)	(2,4)			1	-1	λ^{-1}	$-\lambda^{-1}$
(3,1)	(4,2)			1	-1	λ	$-\lambda$
(1,4)	(2,3)			1	-1	$-\lambda^{-1}$	λ^{-1}
(4,1)	(3,2)			1	-1	$-\lambda$	λ

(iii) By Theorem 5, the dimension of $\tilde{N}$ (respectively, N) is given by the number of connected components of the graph G (respectively, H) on the vertex set $X \times X$. The vectors Y_{bc} (see (19)) are given in Table 1.

Let us determine the connected components of the graph G. Clearly

$$C_1 = \{(1,1),\ (2,2),\ (3,3),\ (4,4)\} \quad \text{and} \quad C_2 = \{(1,2),\ (2,1),\ (3,4),\ (4,3)\}$$

are components of G. Since $\langle Y_{13},\ Y_{31}\rangle = \langle Y_{14},\ Y_{41}\rangle = 4 \neq 0$,

$$C_3 = \{(1,3),\ (3,1),\ (2,4),\ (4,2)\} \quad \text{and} \quad C_4 = \{(1,4),\ (4,1),\ (2,3),\ (3,2)\}$$

induce connected subgraphs in G. Moreover, we have $\langle Y_{13},\ Y_{14}\rangle = 2(1-\lambda^{-2})$. So when $\lambda \neq \pm 1$, $(1,3)$ is adjacent to $(1,4)$, and hence $C_3 \cup C_4$ forms a connected component of G. When $\lambda = \pm 1$, it is easy to show that there is no edge between C_3 and C_4, and so C_3 and C_4 are connected components of G. Therefore we have $\dim \tilde{N} = 4$ when $\lambda = \pm 1$, and $\dim \tilde{N} = 3$ otherwise.

Next let us determine the connected components of the graph H. Clearly, C_1 and C_2 are connected components of H. The values of some Hermitian products $\langle\!\langle Y_{bc},\ Y_{de}\rangle\!\rangle$ are given in Table 2.

Therefore, we have $\langle\!\langle Y_{13},\ Y_{31}\rangle\!\rangle = \langle\!\langle Y_{14},\ Y_{41}\rangle\!\rangle = 0$ if and only if $\lambda\sqrt{-1}$ is real, we have $\langle\!\langle Y_{13},\ Y_{41}\rangle\!\rangle = \langle\!\langle Y_{31},\ Y_{14}\rangle\!\rangle = 0$ if and only if λ is real, and we have $\langle\!\langle Y_{13},\ Y_{14}\rangle\!\rangle = \langle\!\langle Y_{31},\ Y_{41}\rangle\!\rangle = 0$ if and only if $|\lambda| = 1$. Then it can be easily shown that the set $X \times$

Table 2.

	Y_{31}	Y_{14}	Y_{41}
Y_{13}	$2(1+\lambda^{-1}\overline{\lambda})$	$2(1-\lambda^{-1}\overline{\lambda}^{-1})$	$2(1-\lambda^{-1}\overline{\lambda})$
Y_{31}		$2(1-\lambda\overline{\lambda}^{-1})$	$2(1-\lambda\overline{\lambda})$
Y_{14}			$2(1+\lambda^{-1}\overline{\lambda})$

$X - (C_1 \cup C_2)$ is not connected (and splits into two connected components) if and only if $\lambda^4 = 1$. Thus we have dim $N = 4$ if $\lambda^4 = 1$, and dim $N = 3$ otherwise. □

Remarks

(i) It is not difficult to see that there is a unique type II matrix of size $n = 2$ or 3 up to permutation and scaling.

(ii) When $\lambda = \pm 1$, N is the Bose-Mesner algebra of the group $(\mathbf{Z}/2\mathbf{Z})^2$, and when $\lambda = \pm\sqrt{-1}$, N is the Bose-Mesner algebra of the group $\mathbf{Z}/4\mathbf{Z}$. Otherwise N is the Bose-Mesner algebra of the square (i.e., the cycle on 4 vertices, which is a strongly regular graph).

5.4. Spin models for the Kauffman polynomial

The Kauffman polynomial is an invariant of links which can be characterized by a certain *exchange equation* (see [25]). It is shown in [13, 17] that one can obtain symmetric spin models whose associated link invariant is an evaluation of the Kauffman polynomial by adding to the invariance equations (15)–(17) the following matrix version of the exchange equation:

(25) $W^+ - W^- = z(DI - J)$.

Here z is some parameter which is related to the parameters a, D appearing in (15)–(17) by the equation

(26) $a - a^{-1} = z(D - 1)$

((26) is obtained by considering the diagonal entries in (25)).

Remark There is another version of (25) where the minus signs are replaced by plus signs, but it is essentially equivalent to the previous one. To simplify the exposition we shall consider only (25).

We now study $N(W^+)$ when W^+, W^- are symmetric and satisfy (15)–(17) and (25). Let $\mathcal{A}$ be the linear span of $\{I, J, W^+\}$. It is easy to show (see [17]) that $\mathcal{A}$ is a (symmetric) self-dual Bose-Mesner algebra.

If $\mathcal{A}$ has dimension 2, W^+ is a linear combination of I and J: we have a Potts model, and the associated link invariant is the Jones polynomial. The algebra $N(W^+)$ for this case is studied in [1].

From now on we assume that $\mathcal{A}$ has dimension 3 and hence $|X| \geq 4$.

If $z = 0$, $W^+ = W^-$ is a Hadamard matrix (by (25) and (16)). Some examples of such matrices satisfying (15)–(17) with $a = 1$ and $D = \pm 2^m$ for some integer m have already been described in Section 5.2 in terms of a quadratic form on $GF(2)^{2m}$ (the proof that one can take $D = \pm 2^m$ in (15) and (17) is easy and left to the reader). If W^+ is such a matrix, $N(W^+)$ is the Bose-Mesner algebra of (the group scheme of) $GF(2)^{2m}$ and hence has dimension 2^{2m}. We do not know if other types of symmetric Hadamard matrices $W^+ = W^-$ satisfying (15)–(17) can exist.

From now on we assume that $z \neq 0$.

Let us write $W^+ = aI + t_1A_1 + t_2A_2$, where $A_0 = I$, A_1, A_2 are the Hadamard idempotents of $\mathcal{A}$. Then, by (16), $W^- = a^{-1}I + t_1^{-1}A_1 + t_2^{-1}A_2$. Considering the non-diagonal entries in (25) yields $z = t_1^{-1} - t_1 = t_2^{-1} - t_2$. Since $\mathcal{A}$ is three-dimensional, $t_1 \neq t_2$ and hence $t_2 = -t_1^{-1}$. If $|X| = 4$, let us assume without loss of generality that A_1 is the adjacency matrix of the square. Then it is shown in [17] that we have a spin model for the Kauffman polynomial as soon as $a = t_1^{-1}$. In this case one easily shows that W^+ can be obtained from $U(t_1^{-4})$ (see Section 5.3) by scaling and permutation. By Proposition 13 (iii), $\dim N(W^+) = 4$ if and only if $t_1^{16} = 1$ (otherwise $\dim N(W^+) = 3$), and $\dim(N(\widetilde{W^+})) = 4$ if and only if $t_1^8 = 1$ (otherwise $\dim N(\widetilde{W^+}) = 3$).

From now on we assume $|X| > 4$.

Let us compute

$$\langle Y_{bc}, Y_{bd} \rangle = \sum_{x \in X} \frac{W^+(x,b)W^+(x,b)}{W^+(x,c)W^+(x,d)}$$

for every $b, c, d \in X$.

By (25), (15) and (16) we have

$$W^+ \circ W^+ = W^+ \circ (W^- + zDI - zJ) = J + zDaI - zW^+$$

and hence

$$W^+(x,b)^2 = 1 + zDa\delta(x,b) - zW^+(x,b).$$

This gives

$$\langle Y_{bc}, Y_{bd} \rangle = \sum_{x \in X} \frac{1}{W^+(x,c)W^+(x,d)} + \frac{zDa}{W^+(b,c)W^+(b,d)} - z\sum_{x \in X} \frac{W^+(x,b)}{W^+(x,c)W^+(x,d)}.$$

Using (16) this becomes

$$(27)\quad \langle Y_{bc}, Y_{bd} \rangle = \sum\nolimits_{x \in X} W^-(x,c)W^-(x,d) + zDaW^-(b,c)W^-(b,d) - z\sum\nolimits_{x \in X} W^+(x,b)W^-(x,c)W^-(x,d).$$

The first term of (27) is the (c,d)-entry of $(W^-)^2$. By (25), (15) and (16),

$$(W^-)^2 = W^-(W^+ - zDI + zJ) = |X|I - zDW^- + zDaJ,$$

and hence the first term of (27) is $|X|\delta(c,d) - zDW^-(c,d) + zDa$. It is easy to show that if (16) and (17) hold for the matrices W^+, W^-, they also hold if we exchange W^+ and W^-. Hence the third term of (27) is equal to $-zDW^-(c,d)W^+(b,c)W^+(b,d)$. Thus finally

$$\begin{aligned}\langle Y_{bc}, Y_{bd}\rangle &= |X|\delta(c,d) - zDW^-(c,d) + zDa + zDaW^-(b,c)W^-(b,d)\\ &\quad - zDW^-(c,d)W^+(b,c)W^+(b,d).\end{aligned}$$

Assume now that $A_i(b,c) = A_i(b,d) = A_j(c,d) = 1$ for some $i, j \in \{1,2\}$. Then

$$\begin{aligned}\langle Y_{bc}, Y_{bd}\rangle &= zD\left(-t_j^{-1} + a + at_i^{-2} - t_i^2 t_j^{-1}\right)\\ &= zD\left(1 + t_i^2\right)\left(at_i^{-2} - t_j^{-1}\right).\end{aligned}$$

We may now distinguish the following cases.

(i) $\left(1 + t_i^2\right)\left(at_i^{-2} - t_j^{-1}\right) \neq 0$ for all $i, j \in \{1,2\}$.

Let G be the graph defined on the vertex set $X \times X$ as in Section 3.3: (b,c) and (d,e) are adjacent iff $\langle Y_{bc}, Y_{de}\rangle \neq 0$. Let G_i $(i = 0,1,2)$ be the subgraph of G induced by $\{(b,c) \in X \times X \mid A_i(b,c) = 1\}$. Clearly G_0 is connected (actually G_0 is a clique and forms a connected component of G).

Since $zD \neq 0$, our hypothesis implies that if (b,c), (b,d) are distinct vertices of G_i $(i = 1,2)$ they are adjacent. Since (b,c) and (c,b) are also adjacent, it easily follows that G_1 and G_2 are connected.

Hence by Theorem 5 (i), $N(\widetilde{W^+})$ has dimension at most 3. By Proposition 9, and since W^+ is symmetric, $W^+ \in N(\widetilde{W^+})$ and hence $\mathcal{A} \subseteq N(\widetilde{W^+})$. It follows that $N(\widetilde{W^+}) = \mathcal{A}$.

Note that if both A_1 and A_2 have row sum at least 3, G_1 and G_2 have triangles induced by sets of vertices of the form $\{(b,c), (b,d), (b,e)\}$. Since $|X| > 4$, G_0, G_1, G_2 are non bipartite and, by Proposition 6, $N(W^+) = N(\widetilde{W^+}) = \mathcal{A}$.

Otherwise, since it is shown in [17] that A_1, A_2 are adjacency matrices of complementary strongly regular graphs which are connected for $|X| \geq 5$, we must have $|X| = 5$ (then A_1, A_2 correspond to complementary pentagons). This case is settled using Section 3.6.3 of [17] and Remark (i) of Section 5.1: $N(W^+)$ is the Bose-Mesner algebra of the group scheme of $\mathbf{Z}/5\mathbf{Z}$.

(ii) $t_1^2 = -1$ or $t_2^2 = -1$.

Since $t_2 = -t_1^{-1}$, we have $t_1 = t_2$, and this is ruled out since $\mathcal{A}$ has dimension 3.

(iii) $at_1^{-2} = t_1^{-1}$ or $at_2^{-2} = t_2^{-1}$.

In this case, $a = t_1$ or $a = t_2$, and $z = a^{-1} - a$. By (26) we get $D = 0$, which is excluded.

(iv) $at_1^{-2} = t_2^{-1}$ or $at_2^{-2} = t_1^{-1}$.

Up to the exchange of A_1 and A_2 we may assume that $at_1^{-2} = t_2^{-1}$. Then $a = -t_1^3$, $z = t_1^{-1} - t_1$. We are in the situation described in Section 3.6.4 of [17] (where t_1 is denoted by $-t$): A_1 is the adjacency matrix of a lattice graph with $(t_1^2 + t_1^{-2} + 2)^2$ vertices. Then one can show that W^+ is a tensor product of a Potts model with itself. Hence $N(W^+)$ can be determined from Proposition 7 (i) and the results in [1].

5.5. *Spin models on Hadamard graphs*

Let H be a Hadamard matrix of size $4m$. Then one can construct from H a distance-regular graph Γ with $16m$ vertices, called a *Hadamard graph*, with intersection array (see [7] Theorem 1.8.1)

$$\{4m,\ 4m-1,\ 2m,\ 1;\ 1,\ 2m,\ 4m-1,\ 4m\}.$$

Thus Γ is a (simple and undirected) connected graph of diameter 4, and the relations R_i $(i = 0, \ldots, 4)$ on the vertex set X of Γ, defined by $R_i = \{(x, y) \mid \partial(x, y) = i\}$ (where $\partial(x, y)$ denotes the usual distance of x and y in the graph Γ), form a symmetric association scheme whose non-zero intersection numbers of the form p_{i1}^k are

$$p_{11}^0 = 4m,\quad p_{21}^1 = 4m-1,\quad p_{31}^2 = 2m,\ \ p_{41}^3 = 1,$$
$$p_{01}^1 = 1,\quad p_{11}^2 = 2m,\quad p_{21}^3 = 4m-1,\quad p_{31}^4 = 4m.$$

Let $A_i\,(i = 0, \ldots, 4)$ be the Hadamard idempotents of the Bose-Mesner algebra $\mathcal{A}$ of this association scheme.

For $q, \omega \in \mathbf{C}^*$ such that

$$q^4 + q^{-4} + 2 = 4m,\ \ \omega^4 = 1,$$

let us define a matrix W by $W = \sum_{i=0}^{4} t_i A_i$, where

$$t_0 = -q^{-3},\ t_1 = \omega,\ t_2 = q,\ t_3 = -t_1,\ t_4 = t_0.$$

It is shown in [29] (see also [30] for an alternative proof) that $W^+ = W$ satisfies (with appropriate W^-, a and D) the invariance Eqs. (15)–(17), and the corresponding invariant of links is determined in [18, 19].

The purpose of this section is to show that $N(W) = \widetilde{N(W)} = \mathcal{A}$.

Remark that ω^2 and q^2 are real. Remark also that the graph Γ is bipartite, so that we have a bipartition $X = X_1 \cup X_2$, $|X_1| = |X_2| = 8m$, with $R_0 \cup R_2 \cup R_4 = (X_1 \times X_1) \cup (X_2 \times X_2)$ and $R_1 \cup R_3 = (X_1 \times X_2) \cup (X_2 \cup X_1)$.

Lemma 14 *W is obtained by scaling from a real matrix, and hence $N(W) = \widetilde{N(W)}$. Moreover, when $m = 1$, W is obtained by scaling from a Hadamard matrix.*

Proof: For $(x, y) \in (X_1 \times X_1) \cup (X_2 \times X_2)$ we have $W(x, y) \in \{t_0, t_2, t_4\} = \{-q^{-3}, q\}$, and for $(x, y) \in (X_1 \times X_2) \cup (X_2 \times X_1)$ we have $W(x, y) \in \{t_1, t_3\} = \{\omega, -\omega\}$. Multiply the x th row of W by ωq for all $x \in X_1$, multiply the y th column by ω^{-1} for all $y \in X_1$, and multiply the y th column by q for all $y \in X_2$. Then the resulting matrix W' has its entries in $\{-q^{-2}, q^2, \pm 1, \pm\omega^2 q^2\}$, so that W' is a real matrix. Thus $N(W) = N(W')$ is symmetric by Propositions 2 and 6, and hence $\widetilde{N(W)} = N(W)$. When $m = 1$, we have $q^4 = 1$, so that W' has entries ± 1, and hence is a Hadamard matrix. □

When $m = 1$, W is obtained by scaling from a Hadamard matrix of size 16 by Lemma 14, and so $N(W)$ can be determined using Section 5.2.

From now on we assume $m > 1$. Remark that we have $q^8 \neq 1$ in this case.

Lemma 15 $\mathcal{A} \subset N(W)$.

Proof: We have $W \in N(W)$ by Proposition 9 since W satisfies the type III condition (17). Consider the ℓth power of W with respect to the Hadamard product: $W_\ell = \sum_{i=0}^{4} t_i^\ell A_i$ $(\ell = 1, \ldots, 4)$. Since $t_1, \ldots, t_4$ are easily seen to be distinct, $I, W_1, \ldots, W_4$ are linearly independent (this is shown by a non-zero Vandermonde determinant). Hence $\mathcal{A}$ is generated by I and W with respect to Hadamard product. Clearly, this implies $\mathcal{A} \subset N(W)$. □

Let G be the graph on $X \times X$ defined in Section 3.3 with respect to the usual scalar product. In the following we shall show that R_i is a connected subgraph of G $(i = 0, \ldots, 4)$. This will imply $\dim \widetilde{N(W)} \leq 5$ by Theorem 5, and hence $\mathcal{A} = \widetilde{N(W)} = N(W)$ by Lemmas 14 and 15.

Clearly R_0 is connected, so we start with R_1. In the sequel we use implicitly the fact that $\langle Y_{bc}, Y_{cb}\rangle \neq 0$ for all b, c in X.

Lemma 16 *For (b, c) and (b, d) in R_1 with $c \neq d$, $\langle Y_{bc}, Y_{bd}\rangle \neq 0$.*

Proof: We have

$$
\begin{aligned}
\langle Y_{bc}, Y_{bd}\rangle &= \sum_{x \in X} \frac{W(x,b)W(x,b)}{W(x,c)W(x,d)} \\
&= \sum_{i,j,k \in \{0,\ldots,4\}} \sum_{x \in \Gamma_i(b)\cap\Gamma_j(c)\cap\Gamma_k(d)} \frac{W(x,b)W(x,b)}{W(x,c)W(x,d)} \\
&= \sum_{i,j,k \in \{0,\ldots,4\}} P_{ijk} \frac{t_i t_i}{t_j t_k},
\end{aligned}
$$

where as usual $\Gamma_\ell(y)$ denotes the set of vertices at distance ℓ from y and P_{ijk} denotes the size of $\Gamma_i(b) \cap \Gamma_j(c) \cap \Gamma_k(d)$. The non-zero values of P_{ijk} are the following constants (see [29]):

$$
\begin{aligned}
&P_{011} = P_{102} = P_{120} = P_{324} = P_{342} = P_{433} = 1, \\
&P_{122} = P_{322} = 4m - 2, \quad P_{211} = P_{233} = 2m - 1, \quad P_{213} = P_{231} = 2m.
\end{aligned}
$$

From these values and the relations $t_3 = -t_1$, $t_4 = t_0$, we obtain

$$\langle Y_{bc},\ Y_{bd}\rangle = 2\left(\frac{t_0t_0}{t_1t_1} + \frac{t_1t_1}{t_0t_2} + \frac{t_1t_1}{t_2t_0}\right) + 2(4m-2)\frac{t_1t_1}{t_2t_2} - 2\frac{t_2t_2}{t_1t_1}.$$

Since $t_0 = -q^{-3}$, $t_1 = \omega$, $t_2 = q$, $4m - 2 = q^4 + q^{-4}$ and $\omega^2 = \omega^{-2}$, we obtain

$$\langle Y_{bc},\ Y_{bd}\rangle = \pm 4q^{-6}(q^8 - 1) \neq 0. \qquad \square$$

From Lemma 16, any two directed edges (b, c), (b, d) in Γ, which have a common initial vertex b, are adjacent in G. This implies that R_1 is connected in G since Γ is connected.

Next let us consider R_2.

Lemma 17 *For (b, c) and (b, d) in R_2 with $\partial(c, d) = 2$, $\langle Y_{bc},\ Y_{bd}\rangle \neq 0$.*

Proof: The non-zero values of $P_{ijk} = |\Gamma_i(b) \cap \Gamma_j(c) \cap \Gamma_k(d)|$ are

$$\begin{aligned}
&P_{022} = P_{202} = P_{220} = P_{224} = P_{242} = P_{422} = 1,\\
&P_{111} = P_{113} = P_{131} = P_{133} = P_{311} = P_{313} = P_{331} = P_{333} = m,\\
&P_{222} = 8m - 6.
\end{aligned}$$

Then, as in the proof of Lemma 16, we obtain $\langle Y_{bc}, Y_{bd}\rangle = 2q^{-8}(1+q^4)(1-q^8) \neq 0$. $\square$

Lemma 18 *For (b, c) and (d, e) in R_2 with $\partial(b, d) = \partial(c, e) = 1$ and $\partial(b, e) = \partial(c, d) = 3$, $\langle Y_{bc},\ Y_{de}\rangle \neq 0$.*

Proof: We have

$$\langle Y_{bc},\ Y_{de}\rangle = \sum_{x\in X} \frac{W(x, b)W(x, d)}{W(x, c)W(x, e)} = \sum_{i,j,k,\ell\in\{0,\dots,4\}} P_{ijk\ell}\frac{t_it_k}{t_jt_\ell},$$

where $P_{ijk\ell}$ denotes the size of $\Gamma_i(b) \cap \Gamma_j(c) \cap \Gamma_k(d) \cap \Gamma_\ell(e)$. The values of $P_{ijk\ell}$ can be determined in a similar way as in [29]. The non-zero values are given as

$$\begin{aligned}
&P_{0213} = P_{2031} = P_{1302} = P_{1324} = P_{3120} = P_{3142} = P_{2413} = P_{4231} = 1,\\
&P_{1322} = P_{3122} = P_{2213} = P_{2231} = 2m - 2,\\
&P_{1122} = P_{2211} = P_{2233} = P_{3322} = 2m.
\end{aligned}$$

Then we obtain $\langle Y_{bc},\ Y_{de}\rangle = 4q^{-4}(q^4 + 1)^2 \neq 0$. $\square$

Now we consider the graph $\Gamma^{(2)} = (X, R_2)$. Clearly, $\Gamma^{(2)}$ has two connected components X_1, X_2.

We claim that $R_2 \cap (X_i \times X_i)$ is connected in G ($i = 1, 2$). Since X_1 and X_2 are connected in $\Gamma^{(2)}$, it is enough to show that (b, c) and (b, d) are in the same connected component of G for all (b, c), (b, d) in $R_2 \cap (X_i \times X_i)$ with $c \neq d$. When $\partial(c, d) = 2$, (b, c) and (b, d) are adjacent in G by Lemma 17. When $\partial(c, d) = 4$, there is a vertex $e \in X_i$ such that $(b, e) \in R_2$ and $\partial(c, e) = \partial(e, d) = 2$, since $|\Gamma_2(c) \cap \Gamma_2(d)| = 8m - 2$ and hence $\Gamma_2(c) \cap \Gamma_2(d)$ contains $(8m - 2) - 2$ vertices e with $\partial(b, e) \neq 0, 4$. Then, by Lemma 17, (b, e) is adjacent to both (b, c) and (b, d) in G.

It is clear that there exist four vertices b, c, d, e which satisfy the conditions of Lemma 18. Hence, by Lemma 18, (b, c) and (d, e) are adjacent in G, so that there is an edge connecting $R_2 \cap (X_1 \times X_1)$ and $R_2 \cap (X_2 \times X_2)$. Therefore, R_2 is connected in G.

It is not difficult to show that the graph $\Gamma^{(3)} = (X,\ R_3)$ is also a Hadamard graph, and that the relations $R_i^{(3)} = \{(x, y) \mid \partial^{(3)}(x, y) = i\}$ defined by its distance function $\partial^{(3)}$ are $R_i^{(3)} = R_i$ if i is even, $R_1^{(3)} = R_3$, $R_3^{(3)} = R_1$. Thus the connectedness of R_3 is implied by the connectedness of R_1 by exchanging t_1 and t_3.

Finally, the connectedness of R_4 is implied by the following lemma.

Lemma 19 *For (b, c) and (d, e) in R_4 with b, c, d, e distinct,* $\langle Y_{bc}, Y_{de}\rangle \neq 0$.

Proof: Remark that $\partial(b, d) = \partial(c, e)$. We proceed as in Lemma 18.

When $\partial(b, d) = 1$, the non-zero values of $P_{ijk\ell}$ are

$$P_{0413} = P_{1304} = P_{3140} = P_{4031} = 1,$$
$$P_{1322} = P_{3122} = P_{2213} = P_{2231} = 4m - 1,$$

and we obtain $\langle Y_{bc}, Y_{de}\rangle = -16m \neq 0$.

When $\partial(b, d) = 2$, the non-zero values of $P_{ijk\ell}$ are

$$P_{0422} = P_{4022} = P_{2204} = P_{2240} = 1,$$
$$P_{1313} = P_{1331} = P_{3113} = P_{3131} = 2m,$$
$$P_{2222} = 8m - 4,$$

and we obtain $\langle Y_{bc}, Y_{de}\rangle = 16m \neq 0$.

When $\partial(b, d) = 3$, the non-zero values of $P_{ijk\ell}$ are

$$P_{0431} = P_{1340} = P_{3104} = P_{4013} = 1,$$
$$P_{1322} = P_{3122} = P_{2213} = P_{2231} = 4m - 1,$$

and we obtain $\langle Y_{bc}, Y_{de}\rangle = -16m \neq 0$. □

This completes the proof of the equality $\mathcal{A} = N(\widetilde{W}) = N(W)$.

6. Conclusion

We have given a construction which associates a dual pair of Bose-Mesner algebras with every type II matrix, and we have worked out some examples. But we are very far from understanding the power and applicability of this construction. Could we use it to obtain new association schemes, and possibly solve some open questions on the existence of such objects or related ones? Are there natural necessary conditions for a dual pair to come from a type II matrix? The general question is thus: what dual pairs are of the form $(N(W),\ N({}^tW))$ for some type II matrix W? It leads to more specific questions, such as: what self-dual Bose-Mesner algebras are of the form $N(W)$ with W symmetric?

Some progress on the above question could shed new light on the classification problems for type II matrices and for spin models.

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